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JEE

ADVANCED

MATHEMATICS

Algebra

1
1
1
3,14
2+2=4
MATH
(X+Y)
2
y
x

Pearson

Om Sharma

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I dedicate this book to my respected parents
Mr. Giriraj Prasad and Sampati Sharma
and family for their love, endless support,
encouragement and sacrifices

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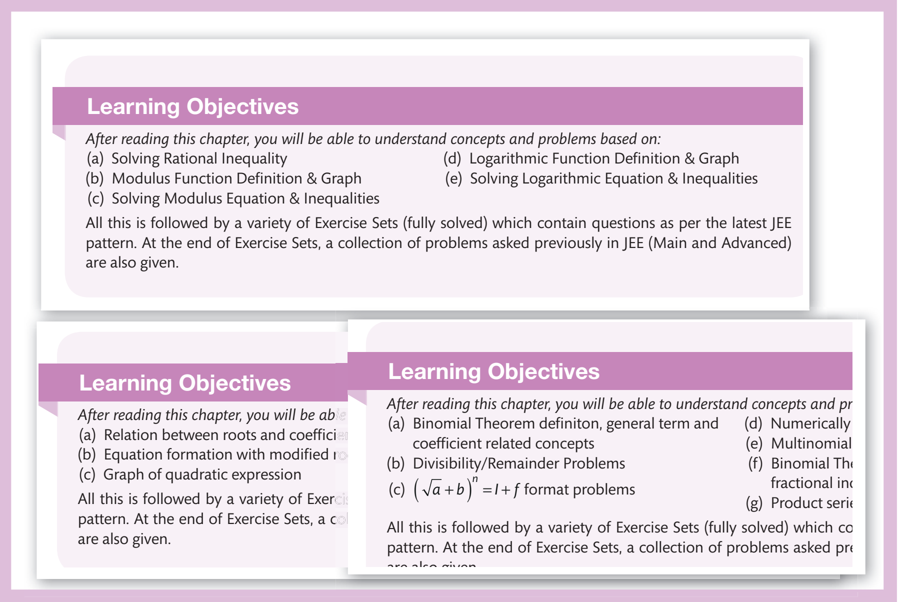
CHAPTER INSIGHT

Learning Objectives

Help the students set an aim to achieve the major take-aways from a particular chapter.



Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Solving Rational Inequality
(b) Modulus Function Definition & Graph
(c) Solving Modulus Equation & Inequalities
(d) Logarithmic Function Definition & Graph
(e) Solving Logarithmic Equation & Inequalities

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

Learning Objectives

After reading this chapter, you will be abl

(a) Relation between roots and coeffici
(b) Equation formation with modified r
(c) Graph of quadratic expression

All this is followed by a variety of Exerci pattern. At the end of Exercise Sets, a co are also given.

Learning Objectives

After reading this chapter, you will be able to understand concepts and pr

(a) Binomial Theorem definiton, general term and coefficient related concepts
(b) Divisibility/Remainder Problems
(c) $\left(\sqrt{a}+b\right)^n = I + f$ format problems
(d) Numerically
(e) Multinomial
(f) Binomial The fractional inc
(g) Product serie

All this is followed by a variety of Exercise Sets (fully solved) which co pattern. At the end of Exercise Sets, a collection of problems asked pr

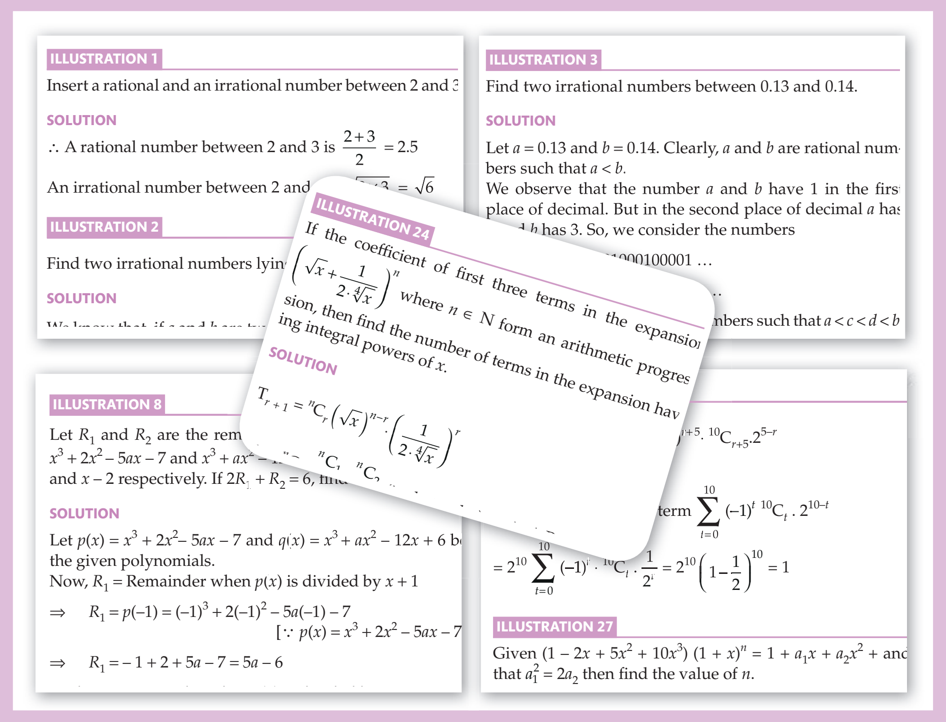


ILLUSTRATION 1

Insert a rational and an irrational number between 2 and 3

SOLUTION

∴ A rational number between 2 and 3 is $\frac{2+3}{2} = 2.5$

An irrational number between 2 and $= \sqrt{6}$

ILLUSTRATION 2

Find two irrational numbers lyin

SOLUTION

ILLUSTRATION 3

Find two irrational numbers between 0.13 and 0.14.

SOLUTION

Let $a = 0.13$ and $b = 0.14$. Clearly, a and b are rational numbers such that $a < b$.

We observe that the number a and b have 1 in the first place of decimal. But in the second place of decimal a has h has 3. So, we consider the numbers

1000100001 …

mbers such that $a < c < d < b$

ILLUSTRATION 24

If the coefficient of first three terms in the expansion $\left(\sqrt{x}+\frac{1}{2\cdot\sqrt[4]{x}}\right)^n$ where $n \in N$ form an arithmetic progression, then find the number of terms in the expansion having integral powers of x.

SOLUTION

$$T_{r+1} = {}^nC_r\left(\sqrt{x}\right)^{n-r}\cdot\left(\frac{1}{2\cdot\sqrt[4]{x}}\right)^r$$

${}^nC_1$ ${}^nC_2$

ILLUSTRATION 8

Let R_1 and R_2 are the rem $x^3+2x^2-5ax-7$ and x^3+ax^2 and $x-2$ respectively. If $2R_1+R_2=6$, fin

SOLUTION

Let $p(x) = x^3+2x^2-5ax-7$ and $q(x) = x^3+ax^2-12x+6$ b the given polynomials.

Now, R_1 = Remainder when $p(x)$ is divided by $x+1$

$\Rightarrow \quad R_1 = p(-1) = (-1)^3 + 2(-1)^2 - 5a(-1) - 7$ $\quad [\because p(x) = x^3+2x^2-5ax-7$

$\Rightarrow \quad R_1 = -1+2+5a-7 = 5a-6$

${}^{10}C_{r+5}\cdot 2^{5-r}$

term $\sum_{t=0}^{10}(-1)^t\,{}^{10}C_t\cdot 2^{10-t}$

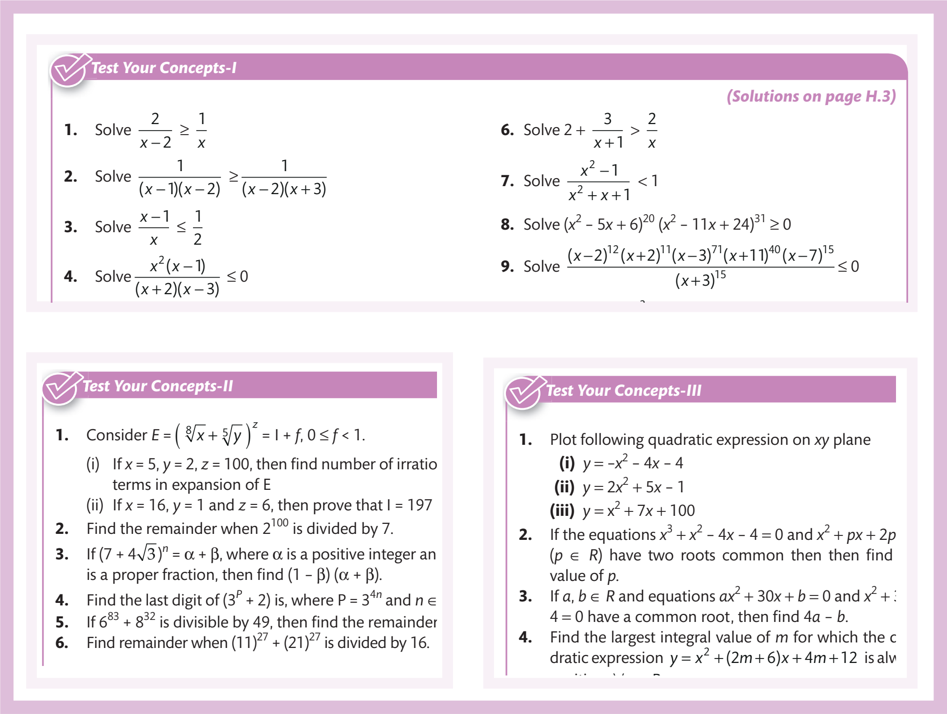
$= 2^{10}\sum_{t=0}^{10}(-1)^t\cdot{}^{10}C_t\cdot\frac{1}{2^t} = 2^{10}\left(1-\frac{1}{2}\right)^{10} = 1$

ILLUSTRATION 27

Given $(1-2x+5x^2+10x^3)(1+x)^n = 1+a_1x+a_2x^2+$ and that $a_1^2 = 2a_2$ then find the value of n.

Theory with Illustrations

Elaborative and simple theory helps the students to understand the illustrations supporting the theory. Please note that theory and problem solving techniques are based on simple learning program IF → THEN → ELSE. I would suggest you not to attempt the illustrations without going through the theory of that section.



Test Your Concepts-I

(Solutions on page H.3)

1. Solve $\frac{2}{x-2} \geq \frac{1}{x}$
2. Solve $\frac{1}{(x-1)(x-2)} \geq \frac{1}{(x-2)(x+3)}$
3. Solve $\frac{x-1}{x} \leq \frac{1}{2}$
4. Solve $\frac{x^2(x-1)}{(x+2)(x-3)} \leq 0$
6. Solve $2+\frac{3}{x+1} > \frac{2}{x}$
7. Solve $\frac{x^2-1}{x^2+x+1} < 1$
8. Solve $(x^2 - 5x + 6)^{20} (x^2 - 11x + 24)^{31} \geq 0$
9. Solve $\frac{(x-2)^{12}(x+2)^{11}(x-3)^{71}(x+11)^{40}(x-7)^{15}}{(x+3)^{15}} \leq 0$

Test Your Concepts-II

1. Consider $E = \left(\sqrt[8]{x}+\sqrt[5]{y}\right)^z = \text{I} + f, 0 \leq f < 1$.
   (i) If $x = 5$, $y = 2$, $z = 100$, then find number of irratio terms in expansion of E
   (ii) If $x = 16$, $y = 1$ and $z = 6$, then prove that I = 197
2. Find the remainder when 2^{100} is divided by 7.
3. If $(7 + 4\sqrt{3})^n = \alpha + \beta$, where α is a positive integer an is a proper fraction, then find $(1 - \beta)(\alpha + \beta)$.
4. Find the last digit of $(3^P + 2)$ is, where $P = 3^{4n}$ and $n \in$
5. If $6^{83} + 8^{32}$ is divisible by 49, then find the remainder
6. Find remainder when $(11)^{27} + (21)^{27}$ is divided by 16.

Test Your Concepts-III

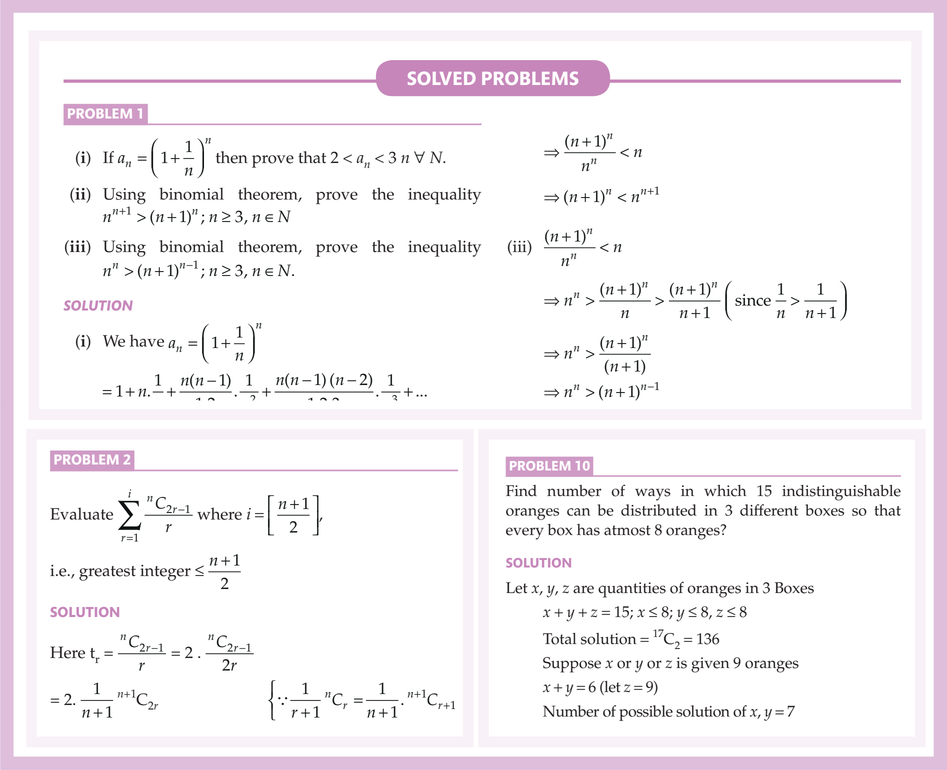
1. Plot following quadratic expression on *xy* plane
   (i) $y = -x^2 - 4x - 4$
   (ii) $y = 2x^2 + 5x - 1$
   (iii) $y = x^2 + 7x + 100$
2. If the equations $x^3 + x^2 - 4x - 4 = 0$ and $x^2 + px + 2p$ $(p \in R)$ have two roots common then then find value of p.
3. If $a, b \in R$ and equations $ax^2 + 30x + b = 0$ and $x^2 +$ $4 = 0$ have a common root, then find $4a - b$.
4. Find the largest integral value of m for which the c dratic expression $y = x^2 + (2m+6)x + 4m + 12$ is alw

Test Your Concepts

These topic based exercise sets are based on simple, single concept classification technique. These are meant for students practice after they study a particular topic and want to practice more on that topic learnt. Finally, in case of any difficulty they can refer to the hints and solutions to these exercise sets given at the end of the book.

Chapter End Solved Problems

These are based on multiple concept usage in a single problem approach so as to expose a student's brain to the ultimate throttle required to take the JEE examination.



SOLVED PROBLEMS

PROBLEM 1

(i) If $a_n = \left(1+\frac{1}{n}\right)^n$ then prove that $2 < a_n < 3\ n\ \forall\ N$.

(ii) Using binomial theorem, prove the inequality $n^{n+1} > (n+1)^n;\ n \geq 3, n \in N$

(iii) Using binomial theorem, prove the inequality $n^n > (n+1)^{n-1};\ n \geq 3, n \in N$.

SOLUTION

(i) We have $a_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + n.\frac{1}{\;} + \frac{n(n-1)}{\;}.\frac{1}{\;} + \frac{n(n-1)(n-2)}{\;}.\frac{1}{\;} + \ldots$$

$$\Rightarrow \frac{(n+1)^n}{n^n} < n$$

$$\Rightarrow (n+1)^n < n^{n+1}$$

(iii) $\frac{(n+1)^n}{n^n} < n$

$$\Rightarrow n^n > \frac{(n+1)^n}{n} > \frac{(n+1)^n}{n+1} \left(\text{since } \frac{1}{n} > \frac{1}{n+1}\right)$$

$$\Rightarrow n^n > \frac{(n+1)^n}{(n+1)}$$

$$\Rightarrow n^n > (n+1)^{n-1}$$

PROBLEM 2

Evaluate $\sum_{r=1}^{i} \frac{{}^nC_{2r-1}}{r}$ where $i = \left[\frac{n+1}{2}\right]$,

i.e., greatest integer $\leq \frac{n+1}{2}$

SOLUTION

Here $t_r = \frac{{}^nC_{2r-1}}{r} = 2 . \frac{{}^nC_{2r-1}}{2r}$

$= 2. \frac{1}{n+1}\,{}^{n+1}C_{2r}$ $\left\{\because \frac{1}{r+1}\,{}^nC_r = \frac{1}{n+1}.\,{}^{n+1}C_{r+1}\right.$

PROBLEM 10

Find number of ways in which 15 indistinguishable oranges can be distributed in 3 different boxes so that every box has atmost 8 oranges?

SOLUTION

Let x, y, z are quantities of oranges in 3 Boxes

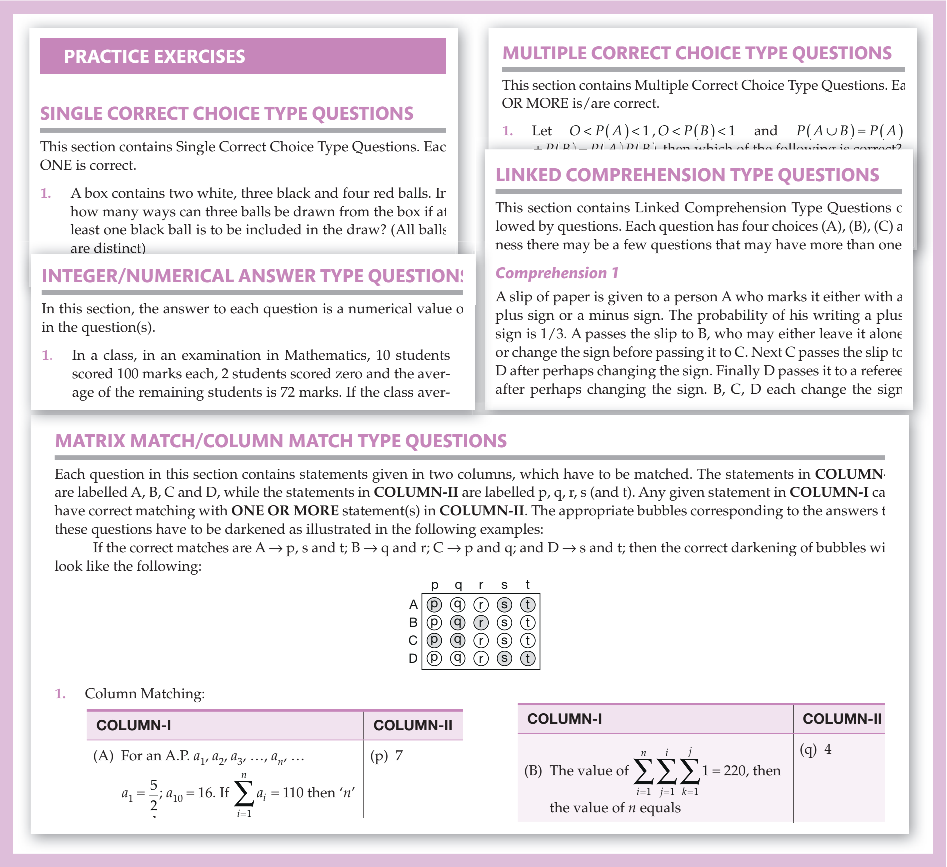
$x + y + z = 15;\ x \leq 8;\ y \leq 8, z \leq 8$

Total solution = ${}^{17}C_2 = 136$

Suppose x or y or z is given 9 oranges

$x + y = 6$ (let $z = 9$)

Number of possible solution of $x, y = 7$



PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Eac
ONE is correct.

1. A box contains two white, three black and four red balls. Ir how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? (All balls are distinct)

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Ea
OR MORE is/are correct.

1. Let $O<P(A)<1, O<P(B)<1$ and $P(A \cup B)=P(A)$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions c
lowed by questions. Each question has four choices (A), (B), (C) a
ness there may be a few questions that may have more than one

Comprehension 1

A slip of paper is given to a person A who marks it either with a plus sign or a minus sign. The probability of his writing a plus sign is 1/3. A passes the slip to B, who may either leave it alone or change the sign before passing it to C. Next C passes the slip tc D after perhaps changing the sign. Finally D passes it to a referee after perhaps changing the sign. B, C, D each change the sigr

INTEGER/NUMERICAL ANSWER TYPE QUESTION

In this section, the answer to each question is a numerical value o
in the question(s).

1. In a class, in an examination in Mathematics, 10 students scored 100 marks each, 2 students scored zero and the average of the remaining students is 72 marks. If the class aver-

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** ca have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers t these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles wi look like the following:

	p	q	r	s	t
A	p	q	r	s	t
B	p	q	r	s	t
C	p	q	r	s	t
D	p	q	r	s	t

1. Column Matching:

COLUMN-I	COLUMN-II
(A) For an A.P. $a_1, a_2, a_3, \ldots, a_n, \ldots$ $a_1=\frac{5}{2}; a_{10}=16$. If $\sum_{i=1}^{n} a_i = 110$ then 'n'	(p) 7

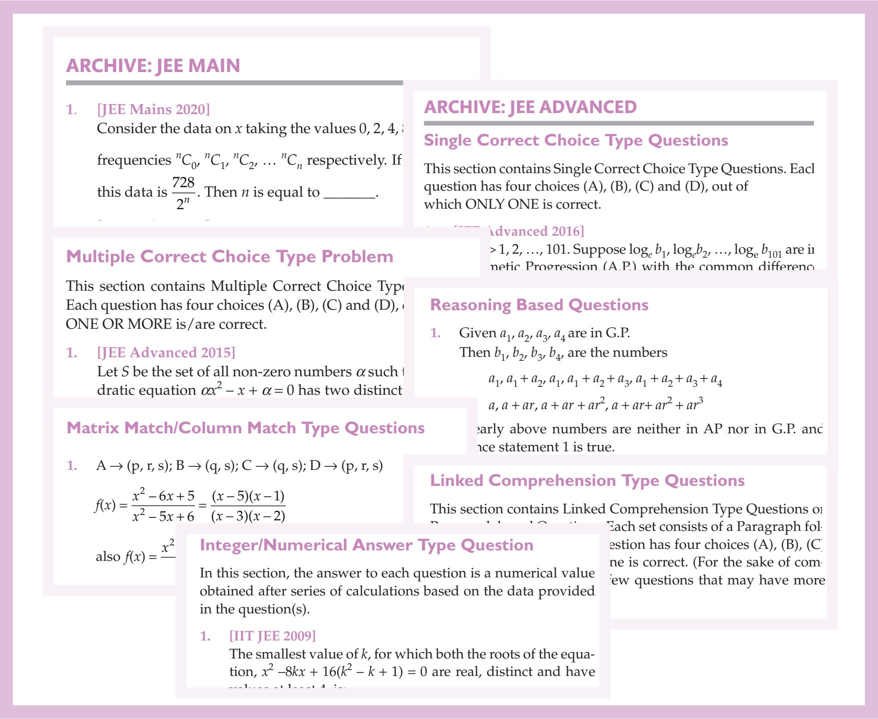
COLUMN-I	COLUMN-II
(B) The value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = 220$, then the value of n equals	(q) 4

Practice Exercise

Inclusion of all types of questions asked in JEE Advanced in adequate numbers helps you with enough practice

Archive JEE Main and Advanced

From this fully updated section, students get to know the actual pattern of the problems asked in the past examinations.



ARCHIVE: JEE MAIN

1. [JEE Mains 2020]
Consider the data on x taking the values 0, 2, 4,
frequencies ${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ respectively. If
this data is $\frac{728}{2^n}$. Then n is equal to _______.

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

This section contains Single Correct Choice Type Questions. Eacl
question has four choices (A), (B), (C) and (D), out of
which ONLY ONE is correct.

Advanced 2016]
> 1, 2, …, 101. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are ir

Multiple Correct Choice Type Problem

This section contains Multiple Correct Choice Typ
Each question has four choices (A), (B), (C) and (D),
ONE OR MORE is/are correct.

1. [JEE Advanced 2015]
Let S be the set of all non-zero numbers α such
dratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct

Reasoning Based Questions

1. Given a_1, a_2, a_3, a_4 are in G.P.
Then b_1, b_2, b_3, b_4, are the numbers
$a_1, a_1+a_2, a_1, a_1+a_2+a_3, a_1+a_2+a_3+a_4$
$a, a+ar, a+ar+ar^2, a+ar+ar^2+ar^3$
early above numbers are neither in AP nor in G.P. and
nce statement 1 is true.

Matrix Match/Column Match Type Questions

1. A → (p, r, s); B → (q, s); C → (q, s); D → (p, r, s)

$$f(x)=\frac{x^2-6x+5}{x^2-5x+6}=\frac{(x-5)(x-1)}{(x-3)(x-2)}$$

also $f(x) = \frac{x^2}{}$

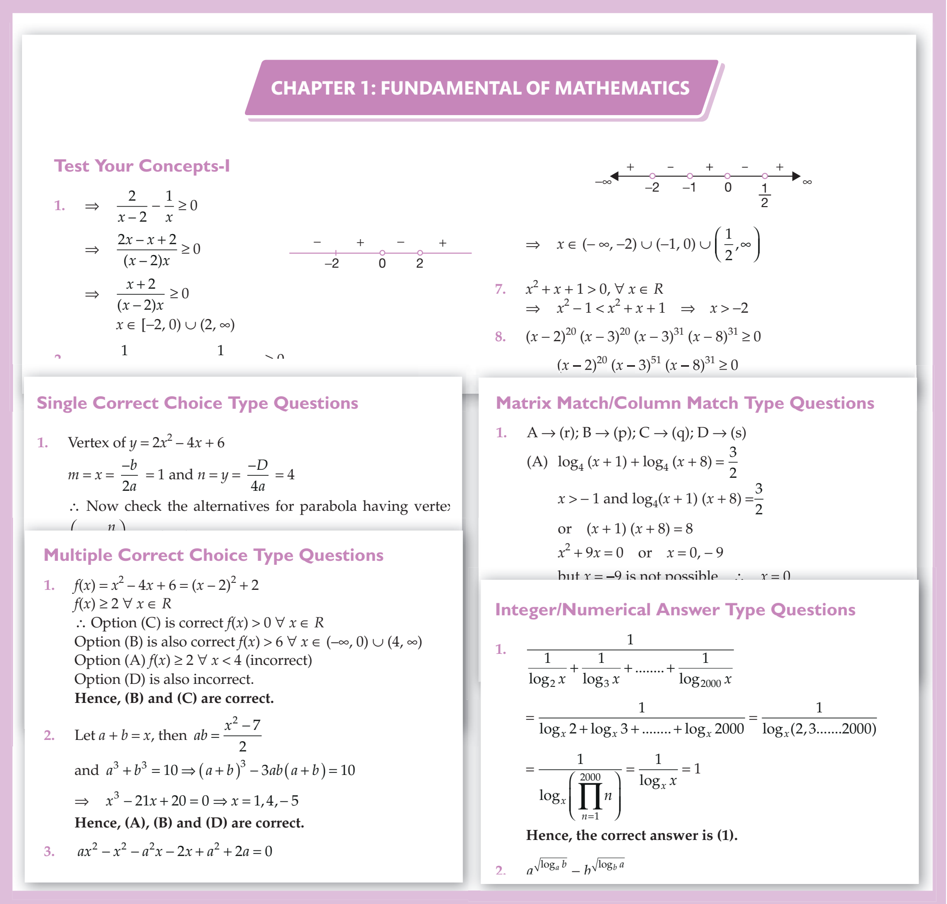
Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or
Each set consists of a Paragraph fol
estion has four choices (A), (B), (C
ne is correct. (For the sake of com
few questions that may have more

Integer/Numerical Answer Type Question

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [IIT JEE 2009]
The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have



CHAPTER 1: FUNDAMENTAL OF MATHEMATICS

Test Your Concepts-I

1. $\Rightarrow \frac{2}{x-2} - \frac{1}{x} \geq 0$

$\Rightarrow \frac{2x-x+2}{(x-2)x} \geq 0$

$\Rightarrow \frac{x+2}{(x-2)x} \geq 0$

$x \in [-2, 0) \cup (2, \infty)$

$\Rightarrow x \in (-\infty, -2) \cup (-1, 0) \cup \left(\frac{1}{2}, \infty\right)$

7. $x^2 + x + 1 > 0, \forall x \in R$
$\Rightarrow x^2 - 1 < x^2 + x + 1 \Rightarrow x > -2$

8. $(x-2)^{20} (x-3)^{20} (x-3)^{31} (x-8)^{31} \geq 0$

$(x-2)^{20} (x-3)^{51} (x-8)^{31} \geq 0$

Single Correct Choice Type Questions

1. Vertex of $y = 2x^2 - 4x + 6$

$m = x = \frac{-b}{2a} = 1$ and $n = y = \frac{-D}{4a} = 4$

∴ Now check the alternatives for parabola having verte

Matrix Match/Column Match Type Questions

1. A → (r); B → (p); C → (q); D → (s)

(A) $\log_4 (x+1) + \log_4 (x+8) = \frac{3}{2}$

$x > -1$ and $\log_4 (x+1)(x+8) = \frac{3}{2}$

or $(x+1)(x+8) = 8$

$x^2 + 9x = 0$ or $x = 0, -9$

but $x = -9$ is not possible ∴ $x = 0$

Multiple Correct Choice Type Questions

1. $f(x) = x^2 - 4x + 6 = (x-2)^2 + 2$
$f(x) \geq 2 \ \forall x \in R$
∴ Option (C) is correct $f(x) > 0 \ \forall x \in R$
Option (B) is also correct $f(x) > 6 \ \forall x \in (-\infty, 0) \cup (4, \infty)$
Option (A) $f(x) \geq 2 \ \forall x < 4$ (incorrect)
Option (D) is also incorrect.
Hence, (B) and (C) are correct.

2. Let $a + b = x$, then $ab = \frac{x^2 - 7}{2}$

and $a^3 + b^3 = 10 \Rightarrow (a+b)^3 - 3ab(a+b) = 10$

$\Rightarrow x^3 - 21x + 20 = 0 \Rightarrow x = 1, 4, -5$

Hence, (A), (B) and (D) are correct.

3. $ax^2 - x^2 - a^2x - 2x + a^2 + 2a = 0$

Integer/Numerical Answer Type Questions

1. $\frac{1}{\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \ldots\ldots + \frac{1}{\log_{2000} x}}$

$= \frac{1}{\log_x 2 + \log_x 3 + \ldots\ldots + \log_x 2000} = \frac{1}{\log_x (2, 3 \ldots\ldots 2000)}$

$= \frac{1}{\log_x \left(\prod_{n=1}^{2000} n \right)} = \frac{1}{\log_x x} = 1$

Hence, the correct answer is (1).

2. $a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}}$

Hints and Explanations

Exhaustive solutions with shortcuts (where ever needed), help students enhance their problem-solving skills.

PREFACE

In the past few years, the IIT-JEE has evolved as an examination designed to check a candidate's true scientific skills. The examination pattern needs one to see those little details which others fail to see. These details tell us how much in-depth we should know to explain a concept in the right direction. Keeping the present-day scenario in mind, this series is written for students, to allow them not only to learn the tools but also to see why they work so nicely in explaining the beauty of ideas behind the subject.

The central goal of this series is to help the students develop a thorough understanding of Mathematics as a subject. This series stresses on building a rock-solid technical knowledge based on firm foundation of the fundamental principles followed by a large collection of formulae. The primary philosophy of this book is to guide the aspirants towards detailed groundwork for strong conceptual understanding and development of problem-solving skills like mature and experienced mathematicians.

Each book in this new Mathematics series is to help the aspirants prepare for both Advanced and Main levels of JEE conducted for IITs and other elite engineering institutions in India. The books in this series is to strengthen the basic understanding of the concepts and their applications. The books will be equally useful for the students preparing for Mathematics Olympiads.

This book is enriched with detailed exhaustive theory that introduces the concepts of Mathematics in a clear, concise, thorough and easy-to-understand language. A large collection of relevant problems is provided in eight major categories (including updated archive for JEE Advanced and JEE Main), for which the solutions are demonstrated in a logical and stepwise manner. These not only make students exam-ready but also helps them gain analytical skills.

We have carefully divided the series into five parts to make the learning of different topics seamless for the students. These parts are

- Coordinate Geometry
- Algebra
- Trigonometry
- Calculus
- 3-D Vector

Finally, I would like to thank all my teacher friends who had been a guiding source of light throughout the entire journey of writing this book.

I apologise in advance for the errors (if any) that might have inadvertently crept in the text. I would be grateful to the readers who bring errors of any kind to my attention. I truly welcome all comments, critiques and suggestions. I hope this book will nourish you with the concepts involved so that you get a great rank at JEE.

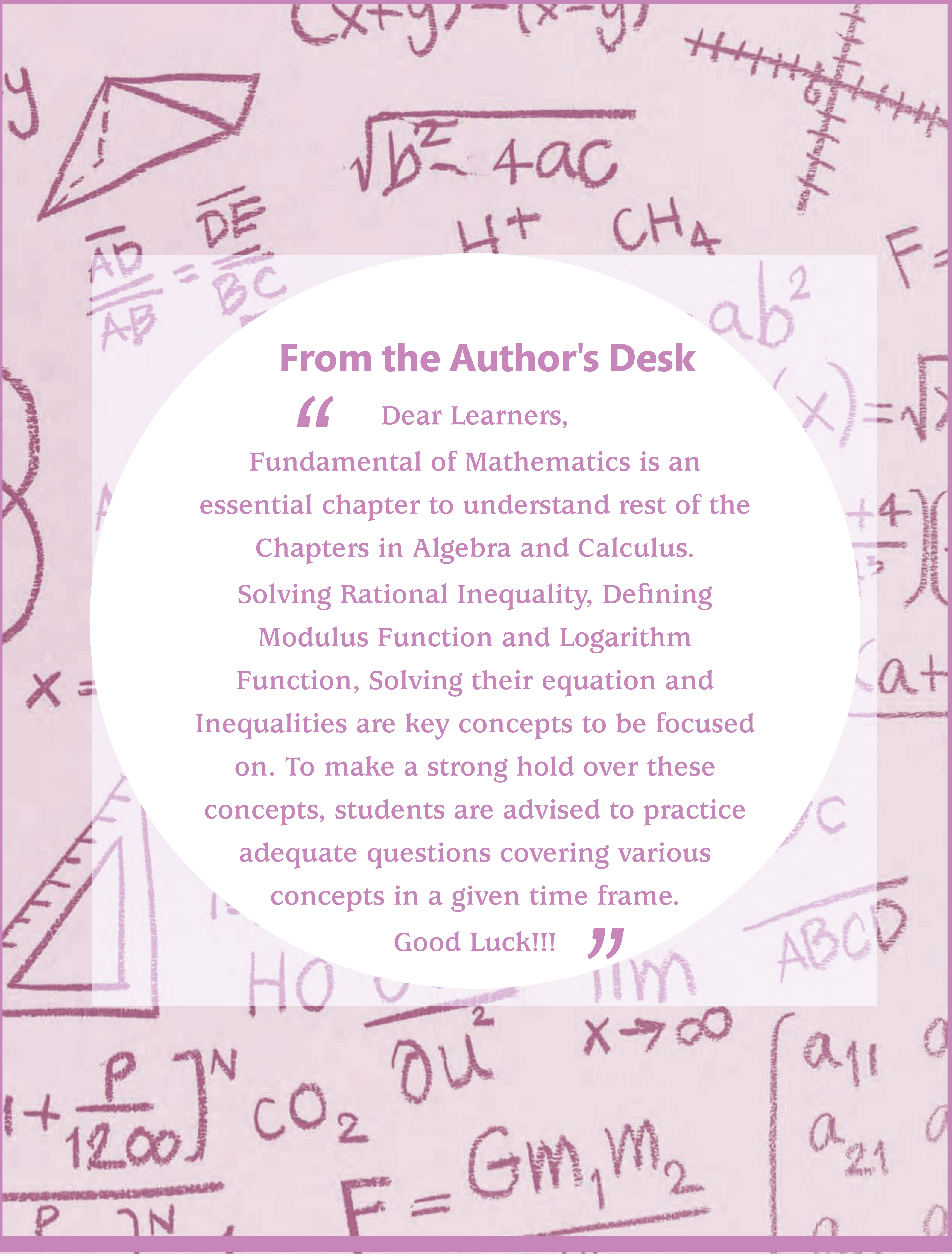
Wish you all the best!

Om Sharma

ABOUT THE AUTHOR

Om Sharma, an ex Research Fellow at BARC (Bhabha Atomic Research Centre), Mumbai, is an IIT Bombay (2007) Graduate. He has been a mathematics expert in the field of IIT JEE Mains and Advanced for more than 14 years from "Shiksha Ki Kaashi - Kota", Rajasthan. He has great exposure to teaching students in the offline and online mode of education.

He is also a motivational speaker and exceptional mentor, having skills to motivate students and ignite the spark in them for achieving success in targeting their goal. Throughout this journey, by the Grace of God, under his guidance and mentorship, many of his students have secured top ranks and become successful engineers and technocrats.



From the Author's Desk

> Dear Learners,
>
> Fundamental of Mathematics is an essential chapter to understand rest of the Chapters in Algebra and Calculus. Solving Rational Inequality, Defining Modulus Function and Logarithm Function, Solving their equation and Inequalities are key concepts to be focused on. To make a strong hold over these concepts, students are advised to practice adequate questions covering various concepts in a given time frame.
>
> Good Luck!!!

CHAPTER

1 Fundamental of Mathematics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Solving rational inequality
(b) Modulus function definition & graph
(c) Solving modulus equation & inequalities
(d) Logarithmic function definition & graph
(e) Solving logarithmic equation & inequalities

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

NUMBER SYSTEM

Natural Numbers

The counting numbers 1, 2, 3, 4, … are called natural numbers. The set of natural numbers is denoted by N.

$N = \{1, 2, 3, 4, \ldots\}$.

N is also denoted by I^+ or Z^+

Whole Numbers

Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by W.
Thus $W = \{0, 1, 2, \ldots\}$

Integers

The numbers … −3, −2, −1, 0, 1, 2, 3 … are called integers and the set of integers is denoted by I or Z.
Thus I (or Z) $= \{\ldots. -3, -2, -1, 0, 1, 2, 3 \ldots\}$

(a) Set of negative integers is denoted by I^- and consists of $\{\ldots., -3, -2, -1\}$
(b) Set of non-negative integers is denoted by w.
(c) Set of non-positive integers $\{\ldots, -3, -2, -1, 0\}$.

Note:
Zero is neither positive integer nor negative integer.

Even Integers

Integers which are divisible by 2 are called even integers. e.g. 0, ±2, ± 4, …

Odd Integers

Integers which are not divisible by 2 are called odd integers. e.g. ±1, ±3, ±5, ±7 …

Prime Number

A natural number (except unity) is said to be a prime number if it is exactly divisible by unity and itself only.
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, …

Composite Numbers

Natural numbers which are not prime (except unity) are called composite numbers.

Note:

(a) '1' is neither prime nor composite.
(b) '2' is the only even prime number.
(c) '4' is the smallest composite number.

Co-prime Numbers

Two natural numbers (not necessarily prime) are said to be co-prime, if their H.C.F. (Highest common factor) is one.
e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (4, 9), (5, 6), (7, 8) etc.
These numbers are also called as **relatively prime** numbers.

Note:

(a) Two distinct prime numbers are always co-prime but converse need not be true.
(b) Any two consecutive natural numbers are always co-prime numbers.

Twin Prime Numbers:

If the difference between two prime numbers is two, then the numbers are called as twin prime numbers.
e.g. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

Rational Numbers:

All the numbers those can be represented in the form p/q, where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q.

e.g. $\frac{1}{2}, 2, 0, -5, \frac{22}{7}, 2.5, 0.3333, 0.\overline{12793}$ etc.

Thus $Q = \left\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\right\}$. It may be noted that every integer is a rational number since it can be written as $p/1$. The decimal part of rational numbers is either terminating or recurring.

Irrational Numbers

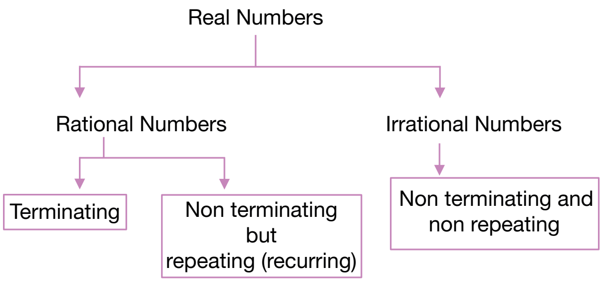
There are real numbers which cannot be expressed in p/q form. These numbers are called irrational numbers and their set is denoted by Q^C or $\bar{Q}$ (i.e. complementary set of Q). The decimal part of irrational numbers is neither terminating nor recurring.

e.g. $\sqrt{2}$, $1 + \sqrt{3}$, e, π, 0.327172398 … etc.

Note:
$e \approx 2.71$ is called Napier's constant and $\pi \approx 3.14$

Real Numbers

The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^C$.


Real Numbers
Rational Numbers
Irrational Numbers
Terminating
Non terminating but repeating (recurring)
Non terminating and non repeating

Note:

(a) Integers are rational numbers, but converse need not be true.
(b) Negative of an irrational number is an irrational number.
(c) Sum or difference of a rational number and an irrational number is always an irrational number e.g. $2 + \sqrt{3}$, $3 - \sqrt{5}$.
(d) The product or quotient of a non-zero rational number and an irrational number will always be an irrational number.
(e) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if $a = 0$.
(f) Sum, difference, product and quotient of two irrational numbers need not be an irrational number (it may be a rational number also).

Insertion of Rational and Irrational Number Between Two Rational Numbers

If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between a and b. Also, if a, b are rational numbers, then $\frac{a+b}{2}$ is a rational number between them.

ILLUSTRATION 1

Insert a rational and an irrational number between 2 and 3.

SOLUTION

A rational number between 2 and 3 is $\frac{2+3}{2} = 2.5$

An irrational number between 2 and 3 is $\sqrt{2 \times 3} = \sqrt{6}$

ILLUSTRATION 2

Find two irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$.

SOLUTION

We know that, if a and b are two distinct positive irrational numbers and if $\sqrt{ab}$ is irrational then $\sqrt{ab}$ is an irrational number lying between a and b.

$\therefore$ Irrational number between $\sqrt{2}$ and $\sqrt{3}$ is $\sqrt{\sqrt{2} \times \sqrt{3}} = \sqrt{\sqrt{6}} = \sqrt[4]{6}$

Irrational number between $\sqrt{2}$ and $6^{1/4}$ is

$$\sqrt{\sqrt{2} \times 6^{1/4}} = 2^{1/4} \times 6^{1/8} = 2^{2/8} \times 6^{1/8}$$

$$= 4^{1/8} \times 6^{1/8} = (24)^{1/8} = \sqrt[8]{24}$$

Hence required irrational number are $\sqrt[4]{6}$ and $\sqrt[8]{24}$.

ILLUSTRATION 3

Find two irrational numbers between 0.13 and 0.14.

SOLUTION

Let $a = 0.13$ and $b = 0.14$. Clearly, a and b are rational numbers such that $a < b$.
We observe that the number a and b have 1 in the first place of decimal. But in the second place of decimal a has 2 and b has 3. So, we consider the numbers

$$c = 0.1301001000100001\ \ldots$$

and, $d = 0.13101001000100001\ \ldots$

Clearly, c and d are irrational numbers such that $a < c < d < b$.

Complex Number

A number of the form $a + ib$ is called a Complex Number, where $a, b \in R$ and $i = \sqrt{-1}$. A Complex Number is usually denoted by 'z' and a set of complex numbers is denoted by C.

Note:

It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

Conjugate of a Complex Number

If $z = a + ib$, where $a, b \in R$, be a complex number then its conjugate complex number is represented by $\bar{z}$ and $\bar{z} = a - ib$.

Factors

A number is a factor of another, if the former exactly divides the latter without leaving a remainder (remainder is zero) 3 and 5 are factors of 12 and 25 respectively.

Multiples

A number is a multiple of another, if the former is exactly divisible by the latter without leaving a remainder (remainder is zero). 36 is a multiple of 2, 3, 4, 9 and 12.

SOME IMPORTANT FORMULAE

(i) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(ii) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(iii) $a^2 - b^2 = (a + b)(a - b)$

(iv) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(v) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(vi) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(vii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(viii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

(ix) $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(x) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(xi) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(xii) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

DIVISIBILITY TEST

(i) A number will be divisible by 2 iff the digit at the unit place is divisible by 2.

(ii) A number will be divisible by 3 iff the sum of all its digits is divisible by 3.

(iii) A number will be divisible by 4 iff last two digits of the number together are divisible by 4.

(iv) A number will be divisible by 5 iff digit at the unit place is either 0 or 5.

(v) A number will be divisible by 6 iff the digit at the unit place of the number is divisible by 2 and sum of all digits of the number is divisible by 3.

(vi) A number will be divisible by 8 iff the last 3 digits, all together, is divisible by 8.

(vii) A number will be divisible by 9 iff sum of all its digits is divisible by 9.

(viii) A number will be divisible by 10 iff its last digit is 0.

(ix) A number will be divisible by 11 iff the difference between the sum of the digits at even places and sum of the digits at odd places is a multiple of 11. e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321, 795432

DEFINITION OF INDICES

If 'a' is any non-zero real or imaginary number and 'm' is a positive integer, then $a^m = a.a.a\ldots a$ (m times). Here 'a' is called the base and m is the index, power or exponent.

Law of Indices

(i) $a^0 = 1, \quad (a \neq 0)$

(ii) $a^{-m} = \frac{1}{a^m}, \quad (a \neq 0)$

(iii) $a^{m+n} = a^m . a^n$, where m and n are real numbers

(iv) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are real numbers, $a \neq 0$

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

RATIO AND PROPORTION

Ratio

(i) If A and B be two quantities of the same kind, then their ratio is $A : B$; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction).

(ii) A ratio may represent in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots$ where $m, n, \ldots$ are non- zero numbers.

(iii) To compare two or more ratios, reduce them to common denominator.

Proportion

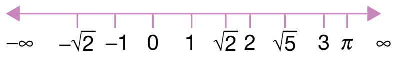
When two ratios are equal, then the four quantities composing them are said to be proportional.

If $\frac{a}{b} = \frac{c}{d}$, then it is written as $a : b = c : d$ or $a : b :: c : d$

(i) 'a' and 'd' are known as extremes and 'b' and 'c' are known as means.
(ii) An important property of proportion: Product of extremes = product of means.
(iii) If $a : b = c : d$, then $b : a = d : c$ (invertendo)
(iv) If $a : b = c : d$, then $a : c = b : d$ (Alternando)
(v) If $a : b = c : d$, then $\frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)
(vi) If $a : b = c : d$, then $\frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)
(vii) If $a : b = c : d$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo and dividendo).

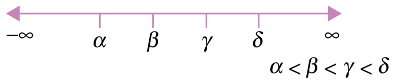
REAL NUMBER LINE

(i) Every real number either rational or irrational number, can be represented on real number line at their exact position uniquely.



(ii) Two unequal real numbers lie on real number line as greater number (β) is on right side of lesser number (α).

(iii) Any real number present on real number line is always greater than all the number lying left side of it and lesser than all the numbers lying right of it.



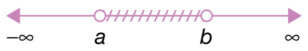
(iv) Imaginary numbers do not follow order property.
(v) Square of a real number is always a non-negative real number.
$\forall\, x \in R$ then $x^2 \geq 0$ (Always)

TYPES OF INTERVALS

Intervals are basically subsets of R. If there are two numbers $a, b \in R$ such that $a < b$, we can define four types of intervals as follows:

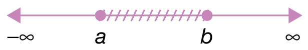
Open Interval

(a, b) or $]a, b[= \{x : a < x < b\}$ i.e. end points are not included as they are shown as open points



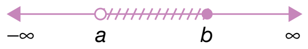
Closed Interval

$[a, b] = \{x : a \leq x \leq b\}$ i.e. end points are also included as they are shown as dark points
This is possible only when both a and b are finite.

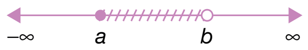


Semi Open or Semi Closed Interval

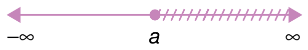
(i) $(a, b]$ or $]a, b] = \{x : a < x \leq b\}$ can be represented as where a is open point (not included) and b is dark point (included)



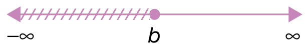
(ii) $[a, b)$ or $[a, b\,[= \{x : a \leq x < b\}$ can be represented as



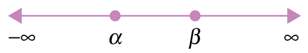
(iii) $[a, \infty) = \{x : x \geq a\}$ can be represented as



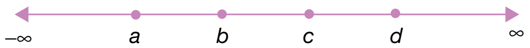
(iv) $(-\infty, b] = \{x : x \leq b\}$ can be represented as



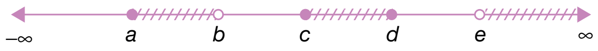
(v) For some particular values of x, we use symbol { } e.g.
If $x = \alpha, \beta$ we can write it as $x \in \{\alpha, \beta\}$
only α and β are included, no intermediate values



(vi) If $x = a, b, c, d$ we can write it as $x \in \{a, b, c, d\}$
Only four values a, b, c, d are included



(vii) If there is no values of x, then we say $x \in \phi$ (null set)
(viii) $x \in [a, b) \cup [c, d] \cup (e, \infty)$ can be represented as

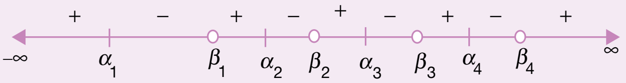


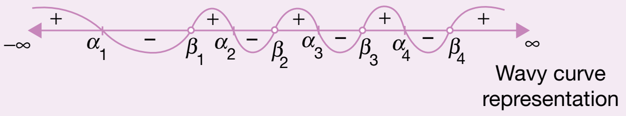
SOLVING RATIONAL INEQUALITY (WAVY CURVE METHOD)

Let $f(x) = \frac{(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)}{(x-\beta_1)(x-\beta_2)(x-\beta_3)(x-\beta_4)}$

Zeroes: Real values of x where $f(x)$ becomes zero where zeroes of given inequality are $x = \alpha_1, \alpha_2, \alpha_3, \alpha_4$

Poles: Real values of x where $f(x)$ becomes not defined where poles of given inequality $x \neq \beta_1, \beta_2, \beta_3, \beta_4$
Let $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \alpha_4 < \beta_4$
Sign of $f(x)$ are shown in two formats as following on real number line.





Now we can observe the sign of $f(x)$ in different interval of x on real number line.

$f(x) \geq 0$ we get $x \in (-\infty, \alpha_1] \cup (\beta_1, \alpha_2] \cup (\beta_2, \alpha_3] \cup (\beta_3, \alpha_4] \cup (\beta_4, \infty)$

$f(x) > 0$ we get $x \in (-\infty, \alpha_1) \cup (\beta_1, \alpha_2) \cup (\beta_2, \alpha_3) \cup (\beta_3, \alpha_4) \cup (\beta_4, \infty)$

(Zeroes are not part of solution as equality sign is not present)

$f(x) \leq 0$ we get $x \in [\alpha_1, \beta_1) \cup [\alpha_2, \beta_2) \cup [\alpha_3, \beta_3) \cup [\alpha_4, \beta_4)$

$f(x) < 0$ we get $x \in (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \cup (\alpha_3, \beta_3) \cup (\alpha_4, \beta_4)$

Note:

(i) Poles can never be the part of solution of any inequality.

(ii) Zeroes are part of solution only if equality sign is present with given inequality.

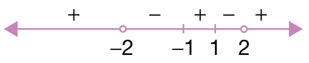
(iii) + or − sign are written according to the sign of $f(x)$ in given interval.

ILLUSTRATION 4

Solve $\dfrac{(x-1)(x+1)}{(x-2)(x+2)} \geq 0$

SOLUTION

$$\frac{(x-1)(x+1)}{(x-2)(x+2)} \geq 0$$



zeroes $x = 1, -1$; poles $x \neq -2, 2$

$$x\ (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$$

ILLUSTRATION 5

Solve $\dfrac{(5-x)(2-x)(2+x)}{(x+2)(x-1)x} < 0$

SOLUTION

$$\frac{(5-x)(2-x)(2+x)}{(x+2)(x-1)x} < 0$$

poles $x \neq -2, 0, 1$

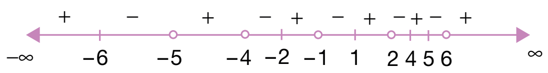
on simplification $\dfrac{(5-x)(2-x)}{(x-1)x} < 0$

$$x \in (0, 1) \cup (2, 5)$$

ILLUSTRATION 6

Solve $\dfrac{(x-1)(x+2)(x-4)(x-5)(x+6)}{(x+1)(x-2)(x+4)(x+5)(x-6)} \leq 0$

SOLUTION



$$\frac{(x-1)(x+2)(x-4)(x-5)(x+6)}{(x+1)(x-2)(x+4)(x+5)(x-6)} \leq 0$$

Zeroes $x = 1, -2, 4, 5, -6$

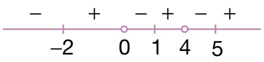
Poles $x \neq -1, 2, -4, -5, 6$

$$x \in [-6, -5) \cup (-4, -2] \cup (-1, 1] \cup (2, 4] \cup [5, 6)$$

ILLUSTRATION 7

Solve $\dfrac{(5-x)(x+2)(1-x)}{x(x-4)} \geq 0$

SOLUTION



$$\frac{(5-x)(x+2)(1-x)}{x(x-4)} \geq 0$$

zeroes $x = -2, 1, 5$; Poles $x \neq 0, 4$

$$x \in [-2, 0) \cup [1, 4) \cup [5, \infty)$$

ILLUSTRATION 8

Find all real values of x, which satisfy

$x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$

SOLUTION

If $a < b$, $(x - a)(x - b) > 0 \;\Rightarrow\; x < a; x > b$

and $(x - a)(x - b) \leq 0 \;\Rightarrow\; a \leq x \leq b$

Now $x^2 - 3x + 2 > 0 \;\Rightarrow\; (x - 1)(x - 2) > 0$

$\Rightarrow\; x < 1; x > 2$ …(1)

and $x^2 - 3x - 4 \leq 0 \;\Rightarrow\; (x + 1)(x - 4) \leq 0$

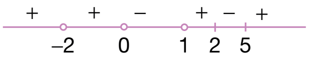
$\Rightarrow\; -1 \leq x \leq 4$ …(2)

Combining (1) and (2) we get real values of x, which satisfy

$x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$ and are given by $-1 \leq x < 1$; $2 < x \leq 4$ or $x \in [-1, 1) \cup (2, 4]$.

ILLUSTRATION 9

Solve: $\dfrac{x^2-3x+2}{x^2+3x+2} \geq 1$

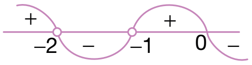


SOLUTION

$$\frac{x^2-3x+2}{x^2+3x+2} \geq 1 \;\Rightarrow\; \frac{x^2-3x+2}{x^2+3x+2} - 1 \geq 0 \qquad \text{…(1)}$$

$$\Rightarrow\; \frac{-6x}{x^2+3x+2} \geq 0 \;\Rightarrow\; \frac{-6x}{(x+2)(x+1)} \geq 0$$

On the real line, mark the points $x = -2$, $x = -1$ and $x = 0$



$\therefore$ The solution of the inequality (1) is $x < -2$ and $-1 < x \le 0$

$\therefore x \in (-\infty, -2) \cup (-1, 0]$.

ILLUSTRATION 10

Solve: $\dfrac{8x^2+16x-51}{(2x-3)(x+4)} > 3$

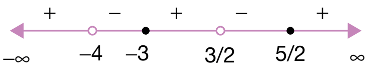
SOLUTION

$$\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3 \Rightarrow \frac{8x^2+16x-51}{(2x-3)(x+4)} - 3 > 0$$

$$\Rightarrow \quad \frac{8x^2+16x-51-3(2x-3)(x+4)}{(2x-3)(x+4)} > 0$$

$$\Rightarrow \quad \frac{2x^2+x-15}{(2x-3)(x+4)} > 0 \Rightarrow \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$$

Critical points are: $x = -4, -3, 3/2, 5/2$



$\Rightarrow \quad x \in (-\infty, -4) \cup (-3, 3/2) \cup (5/2, \infty)$

Working Rule

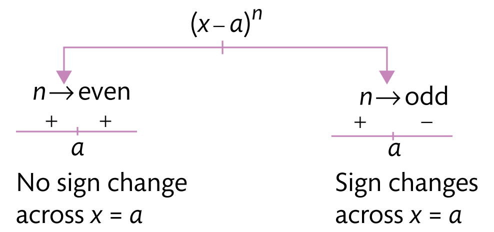
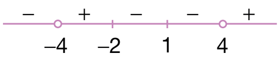


ILLUSTRATION 11

Solve $\dfrac{(x-1)^2(x+2)}{(x^2-16)} \ge 0$

SOLUTION

$$\frac{(x-1)^2(x+2)}{(x-4)(x+4)} \ge 0$$



Poles $x \ne -4, 4$; zeroes $x = 1, -2$

Required $x \in (-4, -2] \cup \{1\} \cup (4, \infty)$

ILLUSTRATION 12

Solve $\dfrac{(x-2)^4(x+2)^5}{(x-3)^{11}} \le 0$

SOLUTION

$$\frac{(x-2)^4(x+2)^5}{(x-3)^{11}} \le 0$$

poles $x \ne 3$

Zeroes $x = 2, -2$

$x \in [-2, 3)$

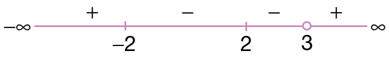


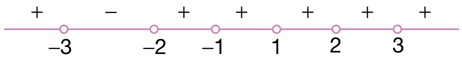
ILLUSTRATION 13

Solve $\dfrac{(x-1)^{10}(x+2)^{103}(x-3)^{98}}{(x+1)^{12}(x-2)^{36}(x+3)^{51}} > 0$

SOLUTION

$$\frac{(x-1)^{10}(x+2)^{103}(x-3)^{98}}{(x+1)^{12}(x-2)^{36}(x+3)^{51}} > 0$$

$x \ne 1, -2, 3, -1, 2, -3$



Zeroes are not part of solution as equality sign is not present in question

$x \in (-\infty, -3) \cup (-2, -1) \cup (-1, 1) \cup (1, 2) \cup (2, 3) \cup (3, \infty)$

or $\quad x \in (-\infty, -3) \cup (-2, \infty) - \{-1, 1, 2, 3\}$

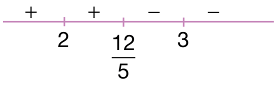
ILLUSTRATION 14

Solve $(x-2)^{20}(x-3)^{20}(24-10x)^{31} \ge 0$

SOLUTION

$$(x-2)^{20}(x-3)^{20}(24-10x)^{31} \ge 0$$

zeroes $x = 2, 3, \dfrac{12}{5}$



$$x \in \left(-\infty, \frac{12}{5}\right] \cup \{3\}$$

ILLUSTRATION 15

Solve: $\dfrac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} > 1$

SOLUTION

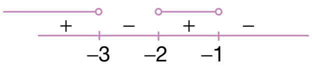
$$\frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} > 1$$

Do not cross multiply here

$$\frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} - 1 > 0$$

$$\frac{(x-1)(x-2)(x-3)-(x+1)(x+2)(x+3)}{(x+1)(x+2)(x+3)} > 0$$

$$\Rightarrow \quad \frac{-12(x^2+1)}{(x+1)(x+2)(x+3)} > 0$$



$\Rightarrow \quad (-\infty, -3) \cup (-2, -1)$

ILLUSTRATION 16

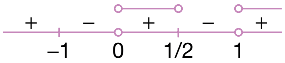
Solve: $\dfrac{x-1}{x} - \dfrac{x+1}{x-1} < 2$

SOLUTION

$$\frac{(x-1)^2 - x(x+1)}{x(x-1)} - 2 < 0$$

$$\Rightarrow \quad \frac{x^2 - 2x + 1 - x^2 - x - 2x^2 + 2x}{x(x-1)} < 0$$

$$\Rightarrow \quad \frac{-2x^2 - x + 1}{x(x-1)} < 0$$



$$\Rightarrow \quad \frac{2x^2 + 2x - x - 1}{x(x-1)} > 0$$

$$\Rightarrow \quad \frac{(2x-1)(x+1)}{x(x-1)} > 0$$

$$x \in (-\infty, -1) \cup \left(0, \frac{1}{2}\right) \cup (1, \infty)$$

ILLUSTRATION 17

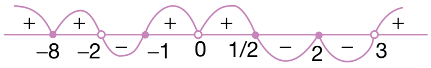
Solve the inequality if

$$f(x) = \frac{(x-2)^{10}(x+1)^3\left(x - \frac{1}{2}\right)^5 (x+8)^2}{x^{24}(x-3)^3(x+2)^5} > 0$$

SOLUTION

$$\text{Let } f(x) = \frac{(x-2)^{10}(x+1)^3\left(x - \frac{1}{2}\right)^5 (x+8)^2}{x^{24}(x-3)^3(x+2)^5}$$

the poles and zeros are 0, 3, –2, 2, –1, $\frac{1}{2}$, –8



If $f(x) > 0 \; \forall \; x \in (-\infty, -8) \cup (-8, -2) \cup (-1, 0) \cup \left(0, \frac{1}{2}\right) \cup (3, \infty)$

Test Your Concepts-I

(Solutions on page H.3)

1. Solve $\frac{2}{x-2} \geq \frac{1}{x}$
2. Solve $\frac{1}{(x-1)(x-2)} \geq \frac{1}{(x-2)(x+3)}$
3. Solve $\frac{x-1}{x} \leq \frac{1}{2}$
4. Solve $\frac{x^2(x-1)}{(x+2)(x-3)} \leq 0$
5. Solve $\frac{x^2-1}{2x+5} < 3$
6. Solve $2 + \frac{3}{x+1} > \frac{2}{x}$
7. Solve $\frac{x^2-1}{x^2+x+1} < 1$
8. Solve $(x^2 - 5x + 6)^{20}(x^2 - 11x + 24)^{31} \geq 0$
9. Solve $\frac{(x-2)^{12}(x+2)^{11}(x-3)^{71}(x+11)^{40}(x-7)^{15}}{(x+3)^{15}} \leq 0$
10. Solve $\frac{(x-22/7)^3}{(x-\pi)^4} \leq 0$

Answer keys

1. $x \in [-2, 0) \cup (2, \infty)$ **2.** $x \in (-3, 1) \cup (2, \infty)$ **3.** $x \in (0, 2]$ **4.** $x \in (-\infty, -2) \cup [1, 3) \cup \{0\}$

5. $(-\infty, -5/2) \cup (-2, 8)$ **6.** $(-\infty, -2) \cup (-1, 0) \cup (1/2, +\infty)$ **7.** $(-2, \infty)$ **8.** $x \in (-\infty, 3] \cup [8, \infty)$

9. $x \in (-3, -2] \cup [3, 7] \cup \{-11, 2\}$ **10.** $x \in \left(-\infty, \frac{22}{7}\right] - \{\pi\}$

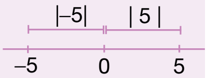
MODULUS FUNCTION (ABSOLUTE VALUE FUNCTION)

By definition square root of a non-negative real number is always a non-negative real number

$\sqrt{x^2} = |x|$ where $|x| \geq 0$,

Geometrically it represents distance between number x and origin.

$|-5| = 5; |5| = 5$



(i) $|-x| = |x|$

(ii) $|x|^2 = x^2$

(iii) $|xy| = |x||y|$

(iv) $|x - a| = |a - x|$

(v) Modulus function can not be a negative quantity

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

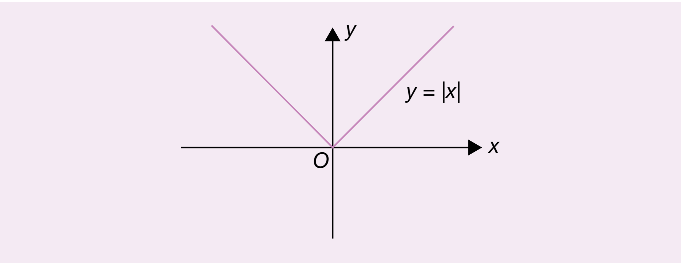
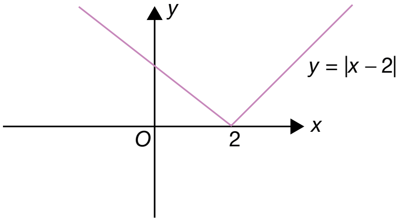


ILLUSTRATION 18

Define and plot $y = |x-2|$.

SOLUTION



$$y = f(x) = |x-2| = \begin{cases} x-2 & x \ge 2 \\ -(x-2) & x < 2 \end{cases}$$

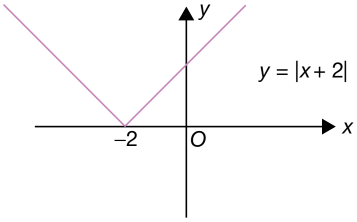
using $f(0) = 2, f(2) = 0, f(4) = 2$

we can plot it as given

ILLUSTRATION 19

Define and plot $y = |x+2|$.

SOLUTION



$$y = |x+2| = \begin{cases} x+2 & x \ge -2 \\ -(x+2) & x < -2 \end{cases}$$

using $f(0) = 2, f(-2) = 0, f(-4) = 2$

we can plot it as given

ILLUSTRATION 20

Make the following expressions free from modulus sign for all $x \in R$

(i) $|x^2 - 3x - 4|$

(ii) $|x^2 - 7x + 10|$

(iii) $|x^3 + 8|$

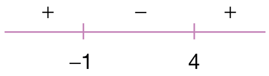
(iv) $|x+3| + |x| + |x-1|$

SOLUTION

(i) $|x^2 - 3x - 4|$

$f(x) = x^2 - 3x - 4 = (x-4)(x+1)$

sign of $f(x)$ is shown in figure



$f(x) \ge 0$ in $x \in (-\infty, -1] \cup [4, \infty)$

$f(x) < 0$ in $x \in (-1, 4)$

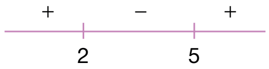
$y = |x^2 - 3x - 4|$

$$= \begin{cases} -(x^2 - 3x - 4) & \text{if } x \in (-1, 4) \\ x^2 - 3x - 4 & \text{if } x \in (-\infty, -1] \cup [4, \infty) \end{cases}$$

(ii) $|x^2 - 7x + 10|$

$f(x) = x^2 - 7x + 10 = (x-2)(x-5)$

sign of $f(x)$ is shown in figure



$f(x) \ge 0$ in $x \in (-\infty, 2] \cup [5, \infty)$

$f(x) < 0$ in $x \in (2, 5)$

$$y = |x^2 - 7x + 10| = \begin{cases} (x^2 - 7x + 10) & \text{if } x \le 2 \\ -(x^2 - 7x + 10) & \text{if } 2 < x < 5 \\ (x^2 - 7x + 10) & \text{if } x \ge 5 \end{cases}$$

(iii) $|x^3 + 8|$

$f(x) = x^3 + 8,$

$f(x) \ge 0 \quad \Rightarrow \quad x^3 + 8 \ge 0 \Rightarrow x^3 \ge -8 \Rightarrow x \ge -2$

$f(x) < 0, x^3 + 8 < 0$ for $x < -2$

$$y = |x^3 + 8| = \begin{cases} -(x^3 + 8) & \text{if } x < -2 \\ (x^3 + 8) & \text{if } x \ge -2 \end{cases}$$

(iv) $y = |x+3| + |x| + |x-1|$

critical points –3, 0, 1

For $x < -3$, we have $y = -3 - x - x - x + 1 = -3x - 2$,

For $-3 \le x < 0$, we have $y = x + 3 - x - x + 1 = -x + 4$,

For $0 \le x < 1$, we have $y = x + 3 + x + 1 - x = x + 4$,

For $x \ge 1$, we have $y = x + 3 + x + x - 1 = 3x + 2$,

$$y = |x+3| + |x| + |x-1| = \begin{cases} -3x - 2, & x < -3 \\ -x + 4, & -3 \le x < 0 \\ x + 4, & 0 \le x < 1 \\ 3x + 2, & x \ge 1 \end{cases}$$

ILLUSTRATION 21

Plot it $y = |x-2| + |x+4|$

SOLUTION

Two modulus functions $|x-2|$ and $|x+4|$ are present so we have two critical points $x = 2, -4$

As we can see that linear terms are present with modulus sign. So just need few points in different interval of x

$f(2) = 0 + 6 = 6$

$f(-4) = 6 + 0 = 6$

$f(3) = 1 + 7 = 8$

$f(-5) = 7 + 1 = 8$

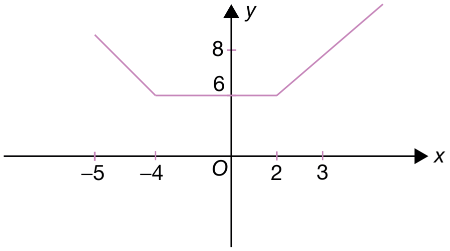


ILLUSTRATION 22

Plot $y = |x-3| - 2|2x+8|$.

SOLUTION

$$y = |x-3| - 2|2x+8| = \begin{cases} 3x+19, & x < -4 \\ -5x-13, & -4 \le x \le 3 \\ -3x-19, & x > 3 \end{cases}$$

As we can see that linear terms are present with modulus sign. So just need few points in different interval of x

$f(-4) = 7$

$f(3) = 0 - 28 = -28$

$f(4) = 1 - 2|16| = -31$

$f(-5) = 8 - 4 = 4$

$f(0) = 3 - 16 = -13$

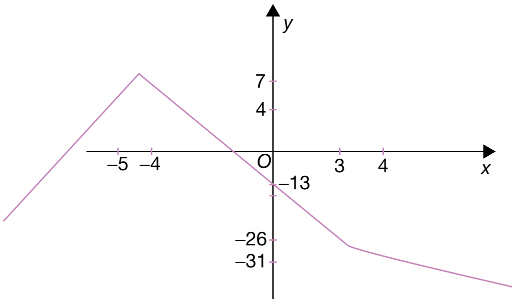


ILLUSTRATION 23

$y = |x-2| - 2|x+4| + |6-x|$

SOLUTION

$$y = \begin{cases} 16, & x < -4 \\ -4x, & -4 \le x \le 2 \\ -2x-6, & 2 < x \le 6 \\ -16, & x > 6 \end{cases}$$

As we can see that linear terms are present with modulus sign. So just need few points in different interval of x

$f(-5) = 7 - 2 + 11 = 16$

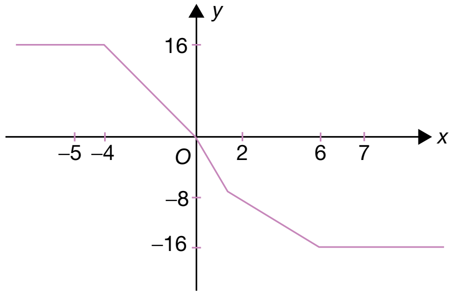
$f(7) = 5 - 22 + 1 = -16$

$f(2) = 0 - 12 + 4 = -8$

$f(-4) = 6 - 0 + 10 = 16$

$f(6) = 4 - 20 + 0 = -16$

$f(0) = 2 - 8 + 6 = 0$



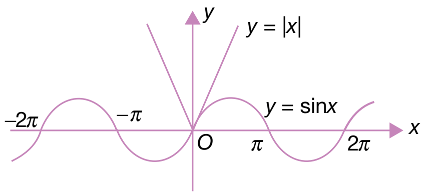
Working Rule

Number of real solution of equation $(f(x) = g(x))$ = number of point of intersection of curves $y = f(x)$ and $y = g(x)$

ILLUSTRATION 24

Final number of solutions of equation $|x| = \sin x$.

SOLUTION

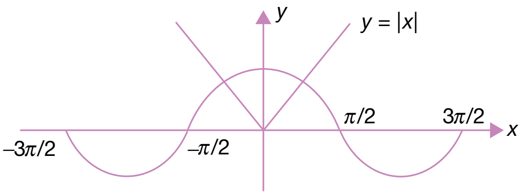


As we can observe, they both intersect at only one point $x = 0$
Hence only one real solution

ILLUSTRATION 25

Final number of solutions of equation $|x| = \cos x$.

SOLUTION



As we can observe, they both intersect at two points
Hence two real solutions

ILLUSTRATION 26

Find number of solution of equation

$$|x-3|-2|x-6|=-10 \text{ graphically}$$

SOLUTION

$$y=f(x)=|x-3|-2|x-6|$$

As we can see that linear terms are present with modulus sign so just need few points in different interval of x

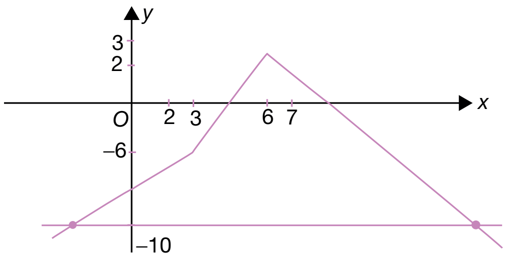
$$f(3)=0-2|3|=-6$$

$$f(6)=3-6=3$$

$$f(2)=1-8=-7$$

$$f(7)=4-2=2$$

using above points we can plot the graph of $f(x)$ as shown



now we draw a line $y=-10$

both curves intersect at two points

ILLUSTRATION 27

If $|x-2|+2|x+2|=P$ then find P if given equation has

(i) no solution

(ii) 1 solution

(iii) 2 solutions

(iv) more than 2 solutions

SOLUTION

Lets first plot $y=|x-2|+2|x+2|$

$y=p$ is a line parallel to x-axis

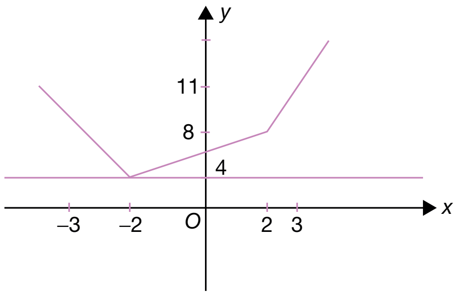
$$f(-2)=4$$

$$f(2)=8$$

$$f(3)=1+2\cdot 5=11$$

$$f(-3)=5+2=7$$

$$f(0)=2+4=6$$



(i)	No solution	$P<4$
(ii)	1 solution	$P=4$
(iii)	2 solutions	$P>4$
(iv)	more than 2 solutions	$P\in\phi$

ILLUSTRATION 28

Find value of k if equation

$$|x+2|-2|x-7|+3|x-4|=k \text{ has}$$

(i) one solution **(ii)** two solutions

(iii) 3 solutions **(iv)** 4 solutions

(v) no solution **(vi)** ∞ solutions

SOLUTION

$$y=|x+2|-2|x-7|+3|x-4|=f(x)$$

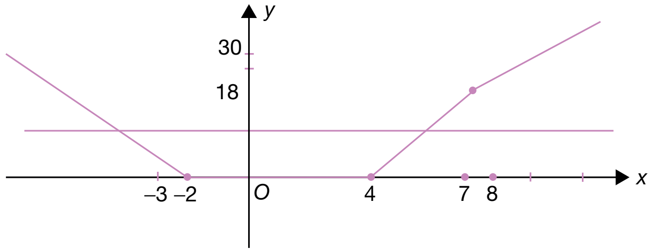
$$f(-2)=0-18+18=0$$

$$f(4)=6-6+0=0$$

$$f(7)=9-0+3=18$$

$$f(-3)=1-20+21=2$$

$$f(8)=10-2+12=20$$



Now for equation $f(x)=k$ we have plotted $y=f(x)$ and we will observe it point of intersection with line $y=k$ for different values of k

(i)	one solution	$K\in\phi$
(ii)	2 solution	$K>0$
(iii)	3 solution	$K\in\phi$
(iv)	4 solution	$K\in\phi$
(v)	no solution	$K<0$
(vi)	∞ solution	$K=0$

SOLVING MODULUS EQUATION

Working Rule

TYPE-I

$$|f(x)|=a$$

$a\geq 0$

$f(x)=\pm a$

$a<0$

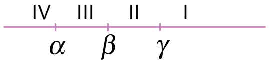
$x\in\phi$

No solution

TYPE-II

$$|f(x)|+\lambda|g(x)|=\mu h(x)$$

First we get critical points where modulus function present in question becomes zero



$f(x) = 0 \quad \Rightarrow \quad x = \alpha$ (let)

$g(x) = 0 \quad \Rightarrow \quad x = \beta, \gamma$ (let)

Let $\alpha < \beta < \gamma$

Now form cases and solve given equation

ILLUSTRATION 29

Solve $||x-4|-2| = 6$

SOLUTION

$||x-4|-2| = 6$

$|x-4| - 2 = \pm 6$

⊕ $|x-4| - 2 = +6$ or Θ $|x-4| - 2 = -6$

$|x-4| = 8$ $\qquad$ $|x-4| = -4$

$x \in \phi$ (no solution)

$x - 4 = \pm 8$

$x - 4 = 8$ or $x - 4 = -8$

$x = 12$ or $x = -4$

final answer $x \in \{-4, 12\}$

ILLUSTRATION 30

Solve $||x+6|-2| = 1$

SOLUTION

$||x+6|-2| = 1$

$|x+6| - 2 = \pm 1$

$|x+6| - 2 = 1$ or $|x+6| - 2 = -1$

$|x+6| = 3$ or $|x+6| = 1$

$x + 6 = \pm 3$ or $x + 6 = \pm 1$

$x + 6 = 3$ or $x + 6 = -3$ or $x + 6 = +1$ or $x + 6 = -1$

$x = -3$ or $x = -9$ or $x = -5$ or $x = -7$

final values of $x \in \{-9, -7, -5, -3\}$

ILLUSTRATION 31

Solve $||\,|x+2|-2|+2| = 4$.

SOLUTION

$||\,|x+2|-2|+2| = 4$

$|\,|x+2| - 2| + 2 = \pm 4$

⊕ $|\,|x+2| - 2| = 2$

Θ $|\,|x+2| - 2| = -6$ (No Solution)

$\Rightarrow$ $|x+2| - 2 = \pm 2$

⊕ $|x+2| - 2 = +2$

$\Rightarrow$ $|x+2| = 4$ or $x + 2 = \pm 4$ or $x = -6, 2$

Θ $|x+2| - 2 = -2$

$\Rightarrow$ $|x+2| = 0$ or $x = -2$

Final values $x \in \{-6, -2, 2\}$

ILLUSTRATION 32

Solve $|x+3| - 4x - 6 = 0$.

SOLUTION

CASE-1: $x \geq -3$

$+(x+3) - 4x - 6 = 0$

$\Rightarrow$ $-3x - 3 = 0 \quad \Rightarrow \quad x + 1 = 0$

Or $x = -1$

CASE-2: $x < -3$

$-(x+3) - 4x - 6 = 0$

$\Rightarrow$ $-5x - 9 - 0$

$\Rightarrow$ $x = -\frac{9}{5} = -1{\cdot}87 > -3$ (Not Acceptable)

$x \in \phi$

Final answer CASE-1 ∪ CASE-2, we have $x = -1$ only

ILLUSTRATION 33

Solve $|x^2 - 4x| - 3x = 2$.

SOLUTION

$x(x-4) = 0 \Rightarrow x = 0, 4$ critical points

CASE-1: $x \in (-\infty, 0) \cup (4, \infty)$

$+(x^2 - 4x) - 3x = 2$

$x^2 - 7x - 2 = 0$

$$x = \frac{7 \pm \sqrt{49+8}}{2}$$

$$x = \frac{7 \pm \sqrt{57}}{2}$$ both acceptable

CASE-2: $0 \leq x \leq 4$

$-(x^2 - 4x) - 3x = 2$

$-x^2 + x - 2 = 0$

$x^2 - x + 2 = 0$ No real solution

Final values of x, CASE-1 ∪ CASE-2.

We have $x = \dfrac{7 \pm \sqrt{57}}{2}$

ILLUSTRATION 34

Solve $|x-2|-x=5$.

SOLUTION

$|x-2|$ has critical point $x=2$, we need to solve it by forming two cases

CASE-1: when $x \geq 2$

$$+(x-2)-x=5$$

$$-2=5 \text{ (Not Possible)} \quad \therefore \quad x \in \phi$$

CASE-2: when $-\infty < x < 2$

$$-(x-2)-x=5$$

$$-x+2-x=5$$

$$x=-\frac{3}{2} \quad \text{as } x<2$$

Final values of x, CASE-1 $\cup$ CASE-2, we have $x=-\dfrac{3}{2}$

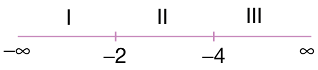
ILLUSTRATION 35

Solve $|x-4|+2|x+2|-3x=0$

SOLUTION

we have $|x-4|$ and $|x+2|$ critical point $x=-2, 4$,
we are solving it by forming three cases

$$\left.\begin{aligned}|x-4|=0 &\Rightarrow x=4\\ |x+2|=0 &\Rightarrow x=-2\end{aligned}\right\}$$



CASE-1: $-\infty < x < -2$

$$-(x-4)+2(-(x+2))-3x=0$$

$\Rightarrow$ $-x+4-2x-4-3x=0$

$\Rightarrow$ $x=0$ but $x<-2 \quad \therefore x \in \phi$

CASE-2: $-2 \leq x \leq 4$

$$-(x-4)+2(+(x+2))-3x=0$$

$\Rightarrow$ $-x+4+2x+4-3x=0$

$\Rightarrow$ $8-2x=0 \quad \Rightarrow \quad x=4$

CASE-3: $x>4$

$$+(x-4)+2(+(x+2))-3x=0$$

$\Rightarrow$ $x-4+2x+4-3x=0 \quad \Rightarrow \quad 0=0$

always true $\quad \therefore x>4$

Final set of values of x, we have CASE-1 $\cup$ CASE-2 $\cup$ CASE-3

$$x \in [4, \infty)$$

ILLUSTRATION 36

Solve $|x+4|-2|x-6|+6x-7=0$

SOLUTION

two modulus are present hence we have critical points $x=-4, x=6$, we are solving by forming three cases

CASE-1: $x<-4$

$$-(x+4)-2(-(x-6))+6x-7=0$$

$$-x-4+2x-12+6x-7=0$$

$\Rightarrow$ $7x=23$

$$x=\frac{23}{7} \text{ not acceptable as } x<-4$$

CASE-2: $-4 \leq x \leq 6$

$$+(x+4)-2(-(x-6))+6x-7=0$$

$\Rightarrow$ $x+4+2x-12+6x-7=0$

$\Rightarrow$ $9x=15$

$$x=\frac{5}{3}$$

CASE-3: $x>6$

$$+(x+4)-2(+(x-6))+6x-7=0$$

$\Rightarrow$ $x+4-2x+12+6x-7=0$

$\Rightarrow$ $5x+9=0$

$$x=-\frac{9}{5} \text{ not acceptable as } x>6$$

Final values of x we have from CASE-1 $\cup$ CASE-2 $\cup$ CASE-3

$$x=\frac{5}{3} \text{ only}$$

ILLUSTRATION 37

Solve $|x^2-3x-4|=9-|x^2-1|$.

SOLUTION

Here

$$|(x-4)(x+1)|=9-|(x+1)(x-1)|$$

CASE-1: If $x<-1$,

$$x^2-3x-4=9-(x^2-1)$$

or $\quad 2x^2-7x+4x-14=0$

or $\quad (2x-7)(x+2)=0;$

$\therefore \quad x=\dfrac{7}{2}, -2.$

But $x<-1$. So $x=-2$.

CASE-2: If $-1 \leq x \leq 1$,

the equation is $(-(x-4)(x+1))=9+(x+1)(x-1)$

or $\quad -(x-4)(x+1)=9+(x^2-1)$

or $\quad -x^2+3x+4=8+x^2$

or $\quad 2x^2-3x+4=0$

or $x = \dfrac{3 \pm \sqrt{9-32}}{4}$ which are not real.

So there is no solution in the interval.

CASE-3: If $1 \le x < 4$,

the equation is $-(x^2 - 3x - 4) = 9 - (x^2 - 1)$

or $3x + 4 = 10$;

$\therefore$ $x = 2$ which satisfies $1 \le x < 4$.

CASE-4: If $x \ge 4$,

the then equation is

$$x^2 - 3x - 4 = 9 - (x^2 - 1)$$

or $2x^2 - 3x - 14 = 0$

or $(2x - 7)(x + 2) = 0$;

$\therefore$ $x = \dfrac{7}{2}, -2.$

But $x \ge 4$; so x has no solution in this interval.

Hence, final solutions are case (1) ∪ (2) ∪ (3) ∪ (4) $x = -2$, 2 only.

SOLVING MODULUS INEQUALITY

Working Rule

TYPE-I

$|f(x)| \le K$

$K \ge 0$

$-K \le f(x) \le K$

$-K \le f(x)$ and $f(x) \le K$

(We take common of above two inequality)

$K < 0$

$x \in \phi$

No solution

TYPE-II

$|f(x)| \ge K$

$K \ge 0$

$f(x) \ge K$ or $f(x) \le -K$

(We take union of above two inequality)

$K < 0$

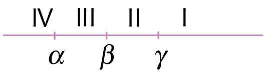
Always True

$\therefore$ x is set of all values where $f(x)$ is defined

TYPE-III

$|f(x)| + \lambda|g(x)| \ge \text{ or } \le \mu h(x)$

First we get critical points where modulus functions present in question become zero



$f(x) = 0 \Rightarrow x = \alpha$ (let)

$g(x) = 0 \Rightarrow x = \beta, \gamma$ (let)

Let $\alpha < \beta < \gamma$

Now form cases and solve given equation

ILLUSTRATION 38

Solve $|x - 4| \le 6$.

SOLUTION

$$|x - 4| \le 6$$

$\Rightarrow$ $-6 \le x - 4 \le 6$

$\Rightarrow$ $-6 + 4 \le x \le 4 + 6$

$\Rightarrow$ $-2 \le x \le 10$

$x \in [-2, 10]$

ILLUSTRATION 39

Solve $|x + 10| \le 2$.

SOLUTION

$$|x + 10| \le 2$$

$\Rightarrow$ $-2 \le x + 10 \le 2$

$\Rightarrow$ $-2 - 10 \le x \le 2 - 10$

$\Rightarrow$ $-12 \le x \le -8$

$x \in [-12, -8]$

ILLUSTRATION 40

Solve $||x + 2| + 4| \le 6$.

SOLUTION

$$-6 \le |x + 2| + 4 \le 6$$

$\Rightarrow$ $-10 \le |x + 2| \le 2$

$-10 \le |x + 2|$ and $|x + 2| \le 2$

$-10 \le |x + 2|$ is always true for $\forall\, x \in R$ …(1)

and $|x + 2| \le 2$

or $-2 \le x + 2 \le 2$

or $-4 \le x \le 0$ …(2)

common of both conditions we have $x \in [-4, 0]$

ILLUSTRATION 41

Solve $|x - 4| \ge 6$.

SOLUTION

$$|x - 4| \ge 6$$

$x - 4 \ge 6$ or $x - 4 \le -6$

$\Rightarrow$ $x \ge 4 + 6$ $\Rightarrow$ $x \le 4 - 6$

$\Rightarrow$ $x \ge 10$ ∪ $x \le -2$

$x \in (-\infty, -2] \cup [10, \infty)$

ILLUSTRATION 42

Solve $|x+8| \geq 2$.

SOLUTION

$|x+8| \geq 2$

$x+8 \geq 2 \quad \text{or} \quad x+8 \leq -2$

$\Rightarrow \quad x \geq -6 \quad \Rightarrow \quad x \leq -10$

$x \in (-\infty, -10] \cup [-6, \infty)$

ILLUSTRATION 43

Solve $||x-2|+4| \geq 6$.

SOLUTION

$||x-2|+4| \geq 6$

$|x-2|+4 \geq 6 \quad \text{or} \quad |x-2|+4 \leq -6$

$|x-2| \geq 2 \quad\quad |x-2| \leq -10$ (not possible)

$x-2 \geq 2$ or $x-2 \leq -2 \quad\quad x \in \phi$

$x \geq 4 \cup x \leq 0$

$x \in (-\infty, 0] \cup [4, \infty)$

ILLUSTRATION 44

Solve $||x-2|-6| \geq 3$.

SOLUTION

$||x-2|-6| \geq 3$

$\Rightarrow \quad |x-2|-6 \geq 3 \quad \text{or} \quad |x-2|-6 \leq -3$

$\Rightarrow \quad |x-2| \geq 3+6 \quad\quad |x-2| \leq 6-3$

$\Rightarrow \quad |x-2| \geq 9 \quad\quad |x-2| \leq 3$

$x-2 \geq 9$ or $x-2 \leq -9 \quad\quad -3 \leq x-2 \leq 3$

$x \geq 11 \cup x \leq -7 \quad\quad -1 \leq x \leq 5$

$x \in (-\infty, -7] \cup [-1, 5] \cup [11, \infty)$

ILLUSTRATION 45

Solve $||x-2|-2| \leq 6$.

SOLUTION

$||x-2|-2| \leq 6$

$\Rightarrow \quad -6 \leq |x-2|-2 \leq 6$

$\Rightarrow \quad -6+2 \leq |x-2| \leq 2+6$

$\Rightarrow \quad -4 \leq |x-2| \leq 8$

$-4 \leq |x-2| \quad \cap \quad |x-2| \leq 8$

Always true for all real $x \quad\quad -8 \leq x-2 \leq 8$

$\because \quad |x-2| \geq 0 \quad\quad -6 \leq x \leq 10$

$x \in R \quad \ldots(1) \quad \cap \quad x \in [-6, 10] \quad \ldots(2)$

final common values (1) and (2)

we get $x \in [-6, 10]$

ILLUSTRATION 46

Solve $||x-3|+1| \leq 7$.

SOLUTION

$||x-3|+1| \leq 7$

$\Rightarrow \quad -7 \leq |x-3|+1 \leq 7$

$\Rightarrow \quad -8 \leq |x-3| \leq 6$

$-8 \leq |x-3| \quad \cap \quad |x-3| \leq 6$

Always true for $\forall\, x \in R$ ...(1)

$-6 \leq x-3 \leq 6$

$-3 \leq x \leq 9$ or $x \in [-3, 9]$ ...(2)

Final common values of x, we have (1) $\cap$ (2)

$x \in [-3, 9]$

ILLUSTRATION 47

Solve $||x+4|-4| < 4$.

SOLUTION

$-4 < |x+4|-4 < 4$

$0 < |x+4| < 8$

$0 < |x+4| \quad \cap \quad |x+4| < 8$

$x \neq -4 \quad \cap \quad -12 < x < 4$

$x \in (-12, 4) - \{-4\}$

ILLUSTRATION 48

Solve $|x-4| - 4x + 8 \leq 0$.

SOLUTION

CASE-1: $x \geq 4$

$+(x-4) - 4x + 8 \leq 0$

$\Rightarrow \quad -3x + 4 \leq 0$

$\Rightarrow \quad 4 \leq 3x \quad \Rightarrow \quad x \geq \dfrac{4}{3}$ but $x \geq 4$

$x \in [4, \infty)$ ...(1)

CASE-2: $x < 4$

$-(x-4) - 4x + 8 \leq 0$

$\Rightarrow \quad -5x + 12 \leq 0 \quad \Rightarrow \quad 12 \leq 5x$

$\Rightarrow \quad x \geq \dfrac{12}{5}$ but $x < 4$

$$x \in \left[\frac{12}{5}, 4\right) \qquad \ldots(2)$$

Final solution case(1) ∪ (2) $x \in \left[\frac{12}{5}, 4\right) \cup [4, \infty)$

$$x \in \left[\frac{12}{5}, \infty\right)$$

ILLUSTRATION 49

Solve $||x+1|-4| \le 2$.

SOLUTION

$$||x+1|-4| \le 2$$

$$-2 \le |x+1| - 4 \le 2$$

$$-2+4 \le |x+1| \le 2+4$$

$$2 \le |x+1| \quad \cap \quad |x+1| \le 6$$

$$-6 \le x+1 \le 6$$

$$-7 \le x \le 5 \qquad \ldots(2)$$

$$|x+1| \ge 2$$

$$x+1 \ge 2 \text{ or } x+1 \le -2$$

$$x \ge 1 \quad \cup \quad x \le -3$$

$$x \in (-\infty, -3] \cup [1, \infty) \qquad \ldots(1)$$

Final values (1) ∩ (2)

$$x \in [-7, -3] \cup [1, 5]$$

ILLUSTRATION 50

Solve $|||x+1|-1|-1| \le 1$.

SOLUTION

$$|||x+1|-1|-1| \le 1$$

$$\Rightarrow \quad -1 \le ||x+1|-1| - 1 \le 1$$

$$\Rightarrow \quad 0 \le ||x+1|-1| \le 2$$

$$0 \le ||x+1|-1| \text{ and } ||x+1|-1| \le 2$$

$0 \le ||x+1|-1|$ is always true for all real x ...(1)

and $||x+1|-1| \le 2$

or $-2 \le |x+1| - 1 \le 2$ or $-1 \le |x+1| \le 3$

$-1 \le |x+1|$ and $|x+1| \le 3$

$-1 \le |x+1|$ is true for $\forall\, x \in R$ and $|x+1| \le 3$

or $-3 \le x+1 \le 3$ or $-4 \le x \le 2$ ...(2)

Final common values of (1) and (2) we get $x \in [-4, 2]$

ILLUSTRATION 51

Solve $\left|\frac{3x}{x-1}\right| \ge 2$.

SOLUTION

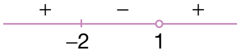
$$\left|\frac{3x}{x-1}\right| \ge 2$$

$$\Rightarrow \quad \frac{3x}{x-1} \ge 2 \quad \text{or} \quad \frac{3x}{x-1} \le -2$$

$$\Rightarrow \quad \frac{3x}{x-1} - 2 \ge 0$$

$$\Rightarrow \quad \frac{3x-2x+2}{x-1} \ge 0$$

$$\Rightarrow \quad \frac{x+2}{x-1} \ge 0$$

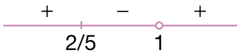


$$x \in (-\infty, -2] \cup (1, \infty) \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{3x}{x-1} + 2 \ge 0$$

$$\Rightarrow \quad \frac{5x-2}{x-1} \le 0$$

$$x \in \left[\frac{2}{5}, 1\right) \qquad \ldots(2)$$



Final solution case (1) ∪ (2)

$$x \in (-\infty, -2] \cup \left[\frac{2}{5}, 1\right) \cup (1, \infty)$$

ILLUSTRATION 52

Solve $\left|\frac{2x-1}{x}\right| \le 2$.

SOLUTION

$$\left|\frac{2x-1}{x}\right| \le 2$$

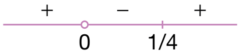
$$\Rightarrow \quad -2 \le \frac{2x-1}{x} \le 2$$

$$-2 \le \frac{2x-1}{x} \quad \cap \quad \frac{2x-1}{x} \le 2$$

$$\Rightarrow \quad 0 \le \frac{2x-1}{x} + 2$$

$$\Rightarrow \quad \frac{2x-1+2x}{x} \ge 0$$

$$\Rightarrow \quad \frac{4x-1}{x} \ge 0$$

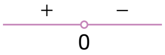


$$x \in (-\infty, 0) \cup \left[\frac{1}{4}, \infty\right) \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{2x-1}{x} - 2 \le 0$$

$$\Rightarrow \quad \frac{2x-1-2x}{x} \le 0$$

$$\Rightarrow -\frac{1}{x} \le 0$$



$$x \in (0, \infty) \qquad \ldots(2)$$

Common values of (1) and (2)

Final values $x \in \left[\frac{1}{4}, \infty\right)$

ILLUSTRATION 53

Solve $\left|\frac{3x}{x^2-4}\right| \le 1$.

SOLUTION

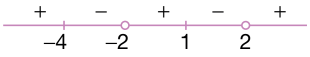
$$\left|\frac{3x}{x^2-4}\right| \le 1 \quad \Rightarrow \quad -1 \le \frac{3x}{x^2-4} \le 1$$

$$-1 \le \frac{3x}{x^2-4} \cap \frac{3x}{x^2-4} \le 1$$

$$0 \le \frac{3x}{x^2-4}+1 \quad \text{and} \quad \frac{3x}{x^2-4}-1 \le 0$$

$$\Rightarrow \frac{x^2+3x-4}{x^2-4} \ge 0$$

$$\Rightarrow \frac{(x+4)(x-1)}{(x+2)(x-2)} \ge 0$$



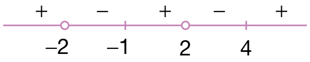
$x \in (-\infty, -4] \cup$

$(-2, 1] \cup (2, \infty)$ …(1)

$$\Rightarrow \frac{3x-x^2+4}{x^2-4} \le 0$$

$$\Rightarrow -\frac{(x^2-3x-4)}{x^2-4} \le 0$$

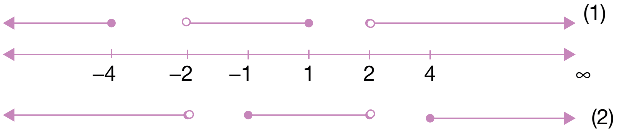
$$\Rightarrow \frac{(x-4)(x+1)}{(x+2)(x-2)} \ge 0$$



$x \in (-\infty, -2)$

$\cup [-1, 2) \cup [4, \infty)$ …(2)

final common values of (1) and (2)



$$x \in (-\infty, -4] \cup [-1, 1] \cup [4, \infty)$$

ILLUSTRATION 54

Solve $\left|\frac{2x-1}{x-1}\right| > 2.$

SOLUTION

$$\left|\frac{2x-1}{x-1}\right| > 2$$

$$|2x-1| > 2\,|x-1| \text{ but } x \ne 1$$

$$\Rightarrow (2x-1)^2 - (2x-2)^2 > 0$$

$$\Rightarrow (2x-1+2x-2)\,(2x-1-2x+2) > 0$$

$$\Rightarrow (4x-3)\,(1) > 0 \quad \Rightarrow \quad x > \frac{3}{4} \text{ and } x \ne 1$$

$$x \in \left(\frac{3}{4}, 1\right) \cup (1, \infty)$$

ILLUSTRATION 55

Solve $\left|\frac{2x}{x-4}\right| \le 1.$

SOLUTION

$$\left|\frac{2x}{x-4}\right| \le 1$$

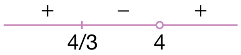
$$-1 \le \frac{2x}{x-4} \le 1$$

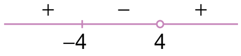
$$-1 \le \frac{2x}{x-4} \quad \cap \quad \frac{2x}{x-4} \le 1$$

$$\Rightarrow 0 \le \frac{2x}{x-4}+1 \quad \cap \quad \Rightarrow \frac{2x}{x-4}-1 \le 0$$

$$\Rightarrow \frac{x-4+2x}{x-4} \ge 0 \quad \cap \quad \Rightarrow \frac{2x-x+4}{x-4} \le 0$$

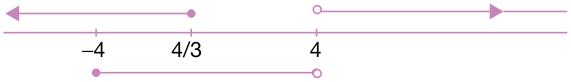
$$\Rightarrow \frac{3x-4}{x-4} \ge 0 \quad \cap \quad \Rightarrow \frac{x+4}{x-4} \le 0$$





$$x \in \left(-\infty, \frac{4}{3}\right] \cup (4, \infty) \;\ldots(1) \cap x \in [-4, 4) \quad \ldots(2)$$

Final values of x, (1) ∩ (2)



Solution $x \in \left[-4, \frac{4}{3}\right]$

ILLUSTRATION 56

Solve $|x-2| - 4x + 6 \ge 0.$

SOLUTION

It is a Type III profile of Modulus inequality

We have $|x-2|$ so we get critical point $x = 2$ and we should solve it by forming cases

CASE-1: $x \ge 2$

$$+(x-2) - 4x + 6 \ge 0$$

$$\Rightarrow -3x + 4 \ge 0$$

$$\Rightarrow x \le \frac{4}{3} \quad \text{but } x \ge 2 \quad \therefore \quad x \in \phi$$

CASE-2: $x < 2$

$$-(x-2) - 4x + 6 \ge 0$$

$$\Rightarrow -5x + 8 \ge 0$$

$$\Rightarrow x \le \frac{8}{5} \quad \text{but } x < 2 \quad \therefore \quad x \le \frac{8}{5}$$

final CASE-1 ∪ CASE-2 we get $x \in \left(-\infty, \frac{8}{5}\right]$

ILLUSTRATION 57

Solve the inequality $|x^2 - x - 6| < x + 2.$

SOLUTION

When $x^2 - x - 6 = 0$, $x = -2, 3$

CASE-1: $x \le -2$ and $x \ge 3$

$\therefore \quad x^2 - x - 6 \geq 0$

Now $|x^2 - x - 6| < x + 2$

$\Rightarrow \quad x^2 - x - 6 < x + 2 \Rightarrow x^2 - 2x - 8 < 0$

or $\quad (x - 4)(x + 2) < 0$

we get $x \in (-2, 4)$ but $x \leq -2$ and $x \geq 3$

$\therefore \quad 3 \leq x < 4$

CASE-2: $-2 < x < 3$

When $-2 < x < 3$ then $x^2 - x - 6 < 0$

$\therefore \quad |x^2 - x - 6| = -(x - x - 6)$

Now $|x^2 - x - 6| < x + 2$

$\Rightarrow \quad -(x^2 - x - 6) < x + 2$

$\Rightarrow \quad x^2 - 4 > 0$ or $(x - 2)(x + 2) > 0$

we get $x \in (-\infty, -2) \cup (2, \infty)$ but $-2 < x < 3$

we get $2 < x < 3$

Thus solution set case (1) $\cup$ (2) $x \in (2, 3) \cup [3, 4)$ or $x \in (2, 4)$

ILLUSTRATION 58

Solve $|3x - 6| - 10x + 7 \leq 0.$

SOLUTION

We have $|3x - 6|$ so we get critical point $x = 2$ and we should solve it by forming cases

CASE-1: $x \geq 2$

$$+ (3x - 6) - 10x + 7 \leq 0$$

$\Rightarrow \quad x \geq \dfrac{1}{7}$ but $x \geq 2$

We get $x \geq 2$ …(1)

CASE-2: $x < 2$

$$- (3x - 6) - 10x + 7 \leq 0$$

$\Rightarrow \quad -13x + 13 \leq 0 \quad \Rightarrow \quad x \geq 1$ but $x < 2$

we get $x \in [1, 2)$ …(2)

final CASE-1 $\cup$ CASE-2

we have $x \in [1, 2) \cup [2, \infty)$ or $x \in [1, \infty)$

ILLUSTRATION 59

Solve $|x - 1| + |x - 2| + |x - 3| \geq 6.$

SOLUTION

$$|x - 1| + |x - 2| + |x - 3| \geq 6$$

We are having three critical points $x = 1, 2, 3$

CASE-1: $x \geq 3$

$$x - 1 + x - 2 + x - 3 \geq 6$$

$\Rightarrow \quad 3x \geq 12$ or $x \geq 4$ we get $x \in [4, \infty)$

CASE-2: $2 \leq x < 3$

$$x - 1 + x - 2 - x + 3 \geq 6$$

$\Rightarrow \quad x \geq 6$ but $2 \leq x < 3$

$\therefore \quad x \in \phi$

CASE-3: $1 \leq x < 2$

$$x - 1 - x + 2 - x + 3 \geq 6$$

$\Rightarrow \quad x \leq -2$ but $1 \leq x < 2$

$\therefore \quad x \in \phi$

CASE-4: $x < 1$

$$-x + 1 - x + 2 - x + 3 \geq 6$$

$\Rightarrow \quad x \leq 0$

$\therefore \quad x \in (-\infty, 0]$

Final answer case (1) $\cup$ (2) $\cup$ (3) $\cup$ (4)
We have $x \in (-\infty, 0] \cup [4, \infty)$

MODULUS INEQUALITY (TYPE IV)

Working Rule

For any two real number a and be following modulus inequality always holds

$$||a| - |b|| \leq |a + b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a - b| \leq |a| + |b|$$

Lets discuss when equality holds

CASE-1:

$$|a + b| = |a| + |b|$$

$$||a| - |b|| = |a - b|$$

a and b are of same sign, it means it is valid for $ab \geq 0$

CASE-2:

$$|a - b| = |a| + |b|$$

$$||a| - |b|| = |a + b|$$

a and b are of opposite sign, it means it is valid for $ab \leq 0$

ILLUSTRATION 60

Solve $2|x| = |x - 1| + |x + 1|.$

SOLUTION

$$2|x| = |x - 1| + |x + 1|$$

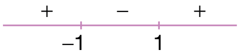
As we can see that there are three modulus functions are present, solving it using forming four cases, it is very time taking

Hence we are using $||a|-|b|| \le |a+b| \le |a| + |b|$

let $a = x - 1$ and $b = x + 1$ then $|x-1+x+1| = |x-1| + |x+1|$

$$|a+b| = |a| + |b| \text{ is satisfied hence } ab \ge 0$$

$$(x-1)(x+1) \ge 0$$



Final solution set $x \in (-\infty, -1] \cup [1, \infty)$

ILLUSTRATION 61

Solve if $|x-4| + |x+4| = 8$.

SOLUTION

$$||a| - |b|| \le |a+b| \le |a| + |b|$$

$$a = x - 4;\ b = x + 4,\ a - b = x - 4 - x - 4 = -8$$

given equation is of form $|a| + |b| = |a-b|$
it is true for $ab \le 0$

$$(x-4)(x+4) \le 0$$

Final solution set $x \in [-4, 4]$

ILLUSTRATION 62

Solve $|x^2 - 4x + 3| + |x^2 - 11x + 24| = 7|x-3|$.

SOLUTION

$$a = x^2 - 4x + 3,\ b = x^2 - 11x + 24;$$

$$a - b = x^2 - 4x + 3 - x^2 + 11x - 24$$

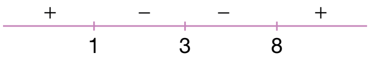
$$= 7x - 21 = 7(x-3)$$

given equation is of the form $|a| + |b| = |a-b|$

it is true for $ab \le 0 \Rightarrow (x^2 - 4x + 3)(x^2 - 11x + 24) \le 0$

$$(x-1)(x-3)(x-3)(x-8) \le 0$$

$$(x-1)(x-3)^2(x-8) \le 0$$



Final solution set $x \in [1, 8]$

ILLUSTRATION 63

Solve: $\big||x^2-2x+6|-|x+6|\big| = |x^2-3x|$

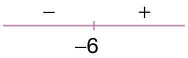
SOLUTION

$$||a|-|b|| \le |a \pm b| \le |a|+|b|$$

let $x^2 - 2x + 6 = a$ and $x + 6 = b$
given equation is of the form $||a| - |b|| = |a-b|$
it is true for $ab \ge 0$

$$(x^2 - 2x + 6)(x + 6) \ge 0$$

$$[\underbrace{(x-1)^2 + 5}_{\text{Always} \oplus}]\ (x+6) \ge 0$$



$$x \ge -6$$

Final solution set $x \in [-6, \infty)$

ILLUSTRATION 64

Solve $|x^2 - 3x + 2| + |x + 5| > |x^2 - 2x + 7|$

SOLUTION

$\because\quad |a+b| \le |a| + |b|$ is always true

For given question $|a| + |b| > |a+b|$

for $a = x^2 - 3x + 2,\ b = x + 5$

It is true for $ab < 0$

or $\quad (x-1)(x-2)(x+5) < 0$

$$x \in (-\infty, -5) \cup (1, 2)$$

Test Your Concepts-II

(Solutions on page H.3)

1. Draw graph of the following expressions. Also find maximum/minimum value if it exists.
  - **(i)** $y = |x-2| + |x-1| + |x+1| + |x+2|$
  - **(ii)** $y = |2x-5| - 2|2x+5|$
  - **(iii)** $y = |2x-1| + |x-1|$
  - **(iv)** $y = |x-1| - |x-6|$

2. Plot $y = |x+2| - |x+4|$

3. Solve for x: $|x-2| = \sqrt{36}$

4. Solve $||x+2| - 4| = 10$

5. Solve $|5 - |x-1|| = 2$

6. Solve $||x-4| - 3| = 100$

7. If $|x-4| - 2|x| + |x+4| = \lambda$ then find λ when the given equation has
  - **(i)** no solution
  - **(ii)** 1 solution
  - **(iii)** 2 solution
  - **(iv)** infinite solution

8. Solve $|x-4| - 2|x+2| - 7x + 5 = 0$

9. Solve $||x-2| - 2| = 8$

10. Solve $||x-2| - 4| \le 6$

11. Solve $||x+2| + 4| \ge 6$

12. Solve $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

13. If $\dfrac{x^2-7|x|+10}{x^2-6x+9} < 0$ then find x.

14. Solve for x: $|x^2+5x+9| < |x^2+2x+2| + |3x+7|$

Answer keys

1. (i) Min. value = 6 (ii) Max. value = 10 (iii) Min. value = 1/2 (iv) Min. value = –5, Max. value = 5

3. $x = 8, -4$ **4.** $x = 12, -16$ **5.** $x \in \{-6, -2, 4, 8\}$ **6.** $x = 107, -99$

7. (i) $\lambda \in (-\infty, 0) \cup (8, \infty)$ (ii) $\lambda = 8$ (iii) $0 < \lambda < 8$ (iv) $\lambda = 0$

8. $x = \dfrac{1}{2}$ **9.** $x = 12, -8$ **10.** $x \in [-8, 12]$ **11.** $x \in (-\infty, -4] \cup [0, \infty)$

12. $[0, 8/5] \cup [5/2, \infty)$ **13.** $(-5, -2) \cup (2, 3) \cup (3, 5)$ **14.** $x < -\dfrac{7}{3}$

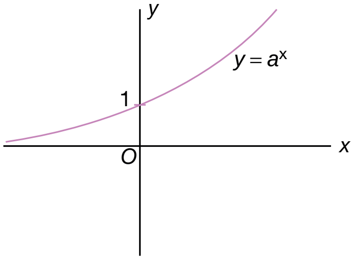
EXPONENTIAL FUNCTION

$y = a^x$ is defined for every real x where $a > 0, a \ne 1$ is known as exponential function. It has two type of behavior depending upon the values of a

CASE-1

$y = a^x$ where $a > 1$ for $\forall\ x \in R$

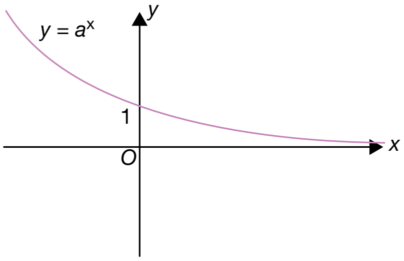
As we can see it has an increasing pattern, as we increase the values of x, a^x also increases



CASE-2:

$y = a^x$ where $0 < a < 1$ for $\forall\ x \in R$

As we can see it has a decreasing pattern, as we increase the values of x, a^x decreases.



EXPONENTIAL INEQUALITY

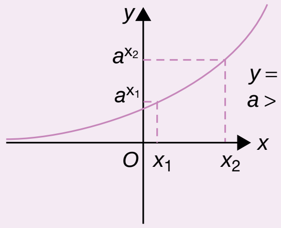
$y = a^x$

CASE-1:

base = $a > 1$

$x_1 < x_2 \quad \Rightarrow \quad a^{x_1} < a^{x_2}$

sign of inequality does not change



CASE-2:

$0 < a < 1$ (base a lies between 0 and 1)

$x_1 < x_2 \quad \Rightarrow \quad a^{x_1} > a^{x_2}$

sign of inequality changes

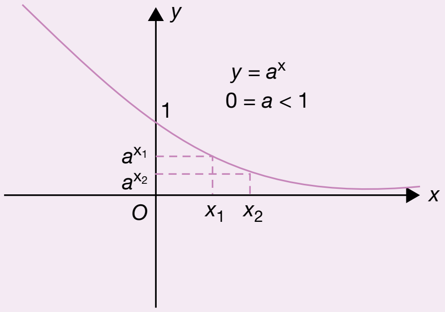


ILLUSTRATION 65

Solve for x, $2^{\frac{x-1}{2}} > 4$.

SOLUTION

$$2^{\frac{x-1}{2}} > 4 \quad \Rightarrow \quad 2^{\frac{x-1}{2}} > 2^2$$

Base = 2 > 1

$$\Rightarrow \quad \frac{x-1}{2} > 2 \quad \Rightarrow \quad x - 1 > 4 \quad \Rightarrow \quad x > 5$$

ILLUSTRATION 66

Solve $\left(\frac{1}{3}\right)^{\frac{2x-1}{7}} < 9$.

SOLUTION

$$\left(\frac{1}{3}\right)^{\frac{2x-1}{7}} < 9 \quad \Rightarrow \quad \left(\frac{1}{3}\right)^{\frac{2x-1}{7}} < \left(\frac{1}{3}\right)^{-2}$$

$$\text{Base} = \frac{1}{3} < 1$$

$$\Rightarrow \quad \frac{2x-1}{7} > -2 \quad \Rightarrow \quad 2x-1 > -14$$

$$\Rightarrow \quad 2x > -13 \quad \Rightarrow \quad x > -13/2$$

ILLUSTRATION 67

Solve $5^{|x-1|} \le \frac{1}{25}$.

SOLUTION

$$5^{|x-1|} \le \frac{1}{25} \quad \Rightarrow \quad 5^{|x-1|} \le 5^{-2}$$

$$\text{Base} = 5 > 1$$

$$\Rightarrow \quad |x-1| \le -2$$

but modulus cannot be negative

$$\therefore \quad x \in \phi$$

ILLUSTRATION 68

Solve $3^{2|x-1|} \ge \frac{1}{81}$.

SOLUTION

$$3^{2|x-1|} \ge \frac{1}{81} \quad \Rightarrow \quad 3^{2|x-1|} \ge 3^{-4}$$

$$\text{Base} = 3 > 1$$

$$\Rightarrow \quad 2|x-1| \ge -4 \quad \Rightarrow \quad |x-1| \ge -2 \text{ Always true}$$

$$\therefore \quad x \in R$$

ILLUSTRATION 69

Solve $2^{2|x-1|} \le 4$.

SOLUTION

$$2^{2|x-1|} \le 4 \quad \Rightarrow \quad 2^{2|x-1|} \le 2^2$$

$$\text{Base} = 2 > 1$$

$$\Rightarrow \quad |x-1| \le 1 \quad \Rightarrow \quad -1 \le x-1 \le 1 \quad \Rightarrow \quad 0 \le x \le 2$$

ILLUSTRATION 70

Solve $3^{||x-2|-2|} \ge 9$.

SOLUTION

$$3^{||x-2|-2|} \ge 9 \Rightarrow 3^{||x-2|-2|} \ge 3^2$$

$$\text{Base} = 3$$

$$\Rightarrow \quad ||x-2|-2| \ge 2$$

$$|x-2|-2 \ge 2 \qquad\qquad |x-2|-2 \le -2$$

$$|x-2| \ge 4 \qquad\qquad |x-2| \le 0$$

$$x-2 \ge 4 \text{ or } x-2 \le -4 \qquad\qquad x = 2 \text{ only}$$

$$x \ge 6 \text{ or } x \le -2$$

final answer $x \in (-\infty, -2] \cup \{2\} \cup [6, \infty)$

ILLUSTRATION 71

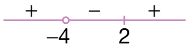
Solve $\frac{(3-\sin x)(x-2)}{(x+4)} \ge 0$.

SOLUTION

Zeroes $x = 2$, Pole $x \ne -4$

$3 - \sin x > 0$ always positive

$$\frac{(3-\sin x)(x-2)}{(x+4)} \ge 0$$



Final solution $x \in (-\infty, -4) \cup [2, \infty)$

ILLUSTRATION 72

Solve $\frac{2^x(\sin x - 2)}{(x^2-4)} \ge 0$.

SOLUTION

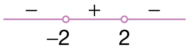
$\sin x \ne 2$ as $-1 \le \sin x \le 1$

$\sin x - 2$ is always negative for $\forall\, x \in R$

and 2^x is always positive for all real x

Poles $x \ne 2, -2$

$$\frac{2^x(\sin x - 2)}{(x-2)(x+2)} \ge 0 \text{ or } \frac{-1}{(x-2)(x+2)} \ge 0$$



Required $x \in (-2, 2)$

ILLUSTRATION 73

Solve $\frac{(2^x-4)(x-9)}{(2^x+4)(x+9)} \ge 0$

SOLUTION

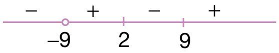
$$\frac{(2^x-4)(x-9)}{(2^x+4)(x+9)} \ge 0$$

$$2^x - 4 = 0 \quad \Rightarrow \quad 2^x = 4 = 2^2 \quad \Rightarrow \quad x = 2$$

Zeroes $x = 2, 9$

$2^x + 4$ is always positive

Poles $x \neq -9$



$x \in (-9, 2] \cup [9, \infty)$

ILLUSTRATION 74

Solve $\dfrac{(2^x - 16)(3^x - 9)}{(2^x + 16)(3^x + 9)} \geq 0.$

SOLUTION

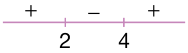
$2^x = 16 = 2^4 \Rightarrow x = 4$

$3^x = 9 = 3^2 \Rightarrow x = 2$

Zeroes $x = 2, 4$

$2^x + 16$ and $3^x + 9$ are always positive

$\dfrac{(2^x - 16)(3^x - 9)}{(2^x + 16)(3^x + 9)} \geq 0$ or $(2^x - 16)(3^x - 9) \geq 0$



Required $x \in (-\infty, 2] \cup [4, \infty)$

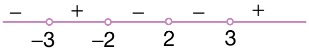
ILLUSTRATION 75

Solve $\dfrac{(x^2 - 4)(x - 3)}{(7^x - 49)(x + 3)} < 0.$

SOLUTION

Clearly $x \neq -3, 2, -2, 3$

$\dfrac{(x - 2)(x + 2)(x - 3)}{(7^x - 49)(x + 3)} < 0$



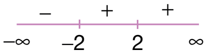
Required $x \in (-\infty, -3) \cup (-2, 3) - \{2\}$

ILLUSTRATION 76

Solve $(3^x - 9)\ (x^2 - 4)\ (7^x - 49)\ (5^x - 25) \geq 0.$

SOLUTION

$(3^x - 3^2)\ (x - 2)\ (x + 2)\ (7^x - 7^2)\ (5^x - 5^2) \geq 0$



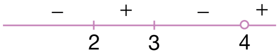
Required $x \in [-2, \infty)$

ILLUSTRATION 77

Solve $\dfrac{(5^x - 125)(7^x - 49)}{(4^x - 256)} \geq 0.$

SOLUTION

$\dfrac{(5^x - 5^3)(7^x - 7^2)}{(4^x - 4^4)} \geq 0$



We get $x \in [2, 3] \cup (4, \infty)$

ILLUSTRATION 78

Solve $\dfrac{(9 - x^2)(\sin^2 x - 4)}{(x - 1)} \geq 0.$

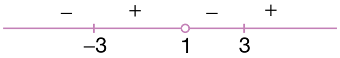
SOLUTION

$-1 \leq \sin x \leq 1$ for all real x

$\sin x - 2 < 0$ and $\sin x + 2 > 0$ always for all real x

$\dfrac{(3 - x)(3 + x)(\sin x - 2)(\sin x + 2)}{(x - 1)} \geq 0$

or $\dfrac{-(3 - x)(3 + x)}{(x - 1)} \geq 0$



We get, $x \in [-3, 1) \cup [3, \infty)$

ILLUSTRATION 79

Solve $\dfrac{(\sin^2 x - 4)(\sin^2 x - 9)}{(e^x + 4)} \geq 0.$

SOLUTION

$-1 \leq \sin x \leq 1$ for all real x

$\sin x - 2 < 0$, $\sin x - 3 < 0$, $\sin x + 2 > 0$ and $\sin x + 3 > 0$ always for all real x

$\dfrac{(\sin x - 2)(\sin x + 2)(\sin x - 3)(\sin x + 3)}{(e^x + 4)}$ is always a positive term for all real x

Final values $\forall\ x \in R$

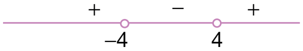
ILLUSTRATION 80

Solve $\dfrac{(3^x - 81)(2^x - 16)}{(x - 4)(x + 4)} \geq 0.$

SOLUTION

$\dfrac{(3^x - 81)(2^x - 16)}{(x - 4)(x + 4)} \geq 0$

$\Rightarrow \dfrac{(3^x - 3^4)(2^x - 2^4)}{(x - 4)(x + 4)} \geq 0$



$x \neq 4, -4$ (Poles)

We get, $x \in (-\infty, -4) \cup (4, \infty)$

DEFINITION OF LOGARITHMIC FUNCTION

As we know that exponential function $a^x = N$ where $a > 0$, $a \neq 1$ and $N > 0$

$$x = \log_a N \text{ (It is logarithmic function)}$$

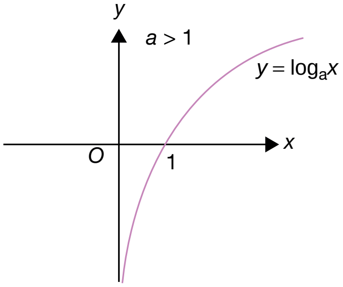
$y = \log_a x$ is defined for positive real x and It is read as 'Logarithm of x to the base a'

$\log_a x$ it tells about the index power given to base 'a' to become 'x'

CASE-1

$y = \log_a x$ where $x > 0$ for $a > 1$

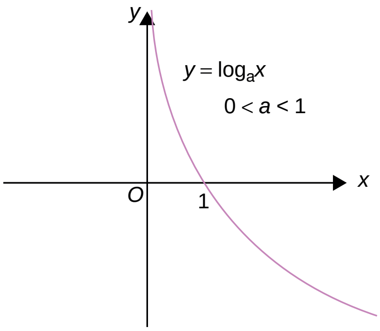
As we can see it has an increasing pattern, as we increase the values of x, $\log_a x$ also increases



CASE-2

$y = \log_a x$ where $x > 0$ for $0 < a < 1$

As we can see it has a decreasing pattern, as we increase the values of x, $\log_a x$ decreases.



We can conclude following conclusions from above graphs

If $\log_a x > 0$ then the x and the a (base) are on one side of the unity, then the logarithm is positive.

If $\log_a x < 0$ then the x and the a (base) are on opposite side of unity, then the logarithm is negative.

ILLUSTRATION 81

Determine the sign of following quantity

(i) $\log_5(1/7)$ **(ii)** $\log_{1/3} 1/8$

(iii) $\log_{1/6} 11$ **(iv)** $\log_{1/3} \pi$

SOLUTION

(i) $1/7 < 1 < 5$, we have $\log_5(1/7) < 0$.

(ii) $1/8 < 1/3 < 1$, we have $\log_{1/3}(1/8) > 0$.

(iii) $1/6 < 1 < 11$, we have $\log_{1/6} 11 < 0$.

(iv) $1/3 < 1 < \pi$, we have $\log_{1/3} \pi < 0$.

Note:

(i) $\log_a x$ is defined for $x > 0$, base $= a > 0$, base $= a \neq 1$

(ii) From the definition of the Logarithm of the number to a given base 'a'. $a^{\log_a N} = N$, $a > 0$, $a \neq 1$, and $N > 0$ is known as the fundamental logarithmic identity.

(iii) The base of log can be any positive number other than 1, but basically two bases are mostly used, they are 10 and e (=2.718 approx.).

Logarithm of numbers to the base 10 are named as common logarithms; where as the logarithms of the numbers to the base e are called as Natural or Napierian logarithms.

$$\log_e a = \log_{10} a \,.\, \log_e 10$$

or $$\log_{10} a = \frac{\log_e a}{\log_e 10} = 0.434 \log_e a.$$

Let M and N arbitrary positive number such that $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ then.

(i) $a^{\log_a x} = x$

(ii) $e^{\ell n a^x} = a^x$

(iii) $\log_a 1 = 0$

(iv) $\log_a a = 1$

(v) $\log_a MN = \log_a M + \log_a N$

(vi) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(vii) $\log_b a = \dfrac{1}{\log_a b}$

(viii) $\log_a N^{\alpha} = \alpha \log_a N$ (α any real no.)

(ix) $\log_{a^\beta} N^{\alpha} = \dfrac{\alpha}{\beta} \log_a N$ ($\alpha \neq 0$, $\beta \neq 0$)

(x) Base Change Theorem

$$\log_a b = \frac{\log_c b}{\log_c a} = \frac{\log_d b}{\log_d a} = \frac{\log_e b}{\log_e a}$$ where a, b, c, d are positive and not 1

(xi) $a^{\log_b c} = c^{\log_b a}$

Proof:

Let $\log_b c = t \Rightarrow c = b^t$

$$\text{RHS} = c^{\log_b a} = \left(b^t\right)^{\log_b a} = b^{t \log_b a} = b^{\log_b a^t} = a^t = \text{LHS}$$

ILLUSTRATION 82

Prove that $(zx)^{\log\frac{z}{x}}\,.\,(xy)^{\log\frac{x}{y}}\,.\,(zy)^{\log\frac{y}{z}} = 1$

SOLUTION

Let $A = (zx)^{\log\frac{z}{x}}\,.\,(xy)^{\log\frac{x}{y}}\,.\,(zy)^{\log\frac{y}{z}}$

then $\log A = \log(yz)^{\log\frac{y}{z}} + \log(zx)^{\log\frac{z}{x}} + \log\,(xy)^{\log\frac{x}{y}}$

$\Rightarrow \quad \log A = \log\frac{y}{z}\log\,(yz) + \log\frac{z}{x}\log\,(zx)$

$+ \log\frac{x}{y}.\log\,(xy)$

$= (\log y - \log z)\,(\log y + \log z) + (\log z - \log x)\,(\log z + \log x) + (\log x - \log y)\,(\log x + \log y)$

$= (\log y)^2 - (\log z)^2 + (\log z)^2 - (\log x)^2 + (\log x)^2 - (\log y)^2 = 0$

$\therefore \quad \log A = 0$ so $A = 1$, i.e. L.H.S. = 1

ILLUSTRATION 83

Compute the following

(i) $2^{\log_3 5} - 5^{\log_3 2}$

(ii) $\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \ldots + \log_{10}\tan 89^\circ$

SOLUTION

(i) $2^{\log_3 5} = 2^{\log_2 5\,\log_3 2} = \left(2^{\log_2 5}\right)^{\log_3 2} = 5^{\log_3 2} \qquad \ldots(1)$

$\therefore \quad 2^{\log_3 5} - 5^{\log_3 2} = 5^{\log_3 2} - 5^{\log_3 2} = 0$ by (1)

(ii) $\log_{10}\tan 1^\circ + \log_{10}\tan 2^\circ + \ldots + \log_{10}\tan 89^\circ$

$= \log_{10}(\tan 1^\circ\,\tan 2^\circ \ldots \tan 44^\circ\,\tan 45^\circ\,\tan 46^\circ \ldots \tan 89^\circ)$

$= \log_{10}[(\tan 1^\circ\tan 89^\circ)\,(\tan 2^\circ\tan 88^\circ)\ldots(\tan 44^\circ\tan 46^\circ)\,\tan 45^\circ]$

$= \log_{10}[(\tan 1^\circ\,\cot 1^\circ)\,(\tan 2^\circ\,\cot 2^\circ)\,\ldots\,(\tan 44^\circ\,\cot 44^\circ)\,\tan 45^\circ]$

$= \log_{10}[(\tan 1^\circ\,\cot 1^\circ)\,(\tan 2^\circ\,\cot 2^\circ)\,\ldots\,\tan 45^\circ]$

$= \log_{10}1 = 0$

ILLUSTRATION 84

Simplify: $\sqrt[3]{5^{\frac{1}{\log_7 5}} + \dfrac{1}{\sqrt{-\log_{10}(0.1)}}}$

SOLUTION

$$\left(5^{\log_5 7} + 1\right)^{1/3} = (8)^{1/3} = 2$$

ILLUSTRATION 85

Find

$\log_3\left(1+\frac{1}{3}\right) + \log_3\left(1+\frac{1}{4}\right) + \log_3\left(1+\frac{1}{5}\right) + \ldots + \log_3\left(1+\frac{1}{242}\right)$

SOLUTION

$\log_3\left(1+\frac{1}{3}\right) + \log_3\left(1+\frac{1}{4}\right) + \log_3\left(1+\frac{1}{5}\right) + \ldots +$

$\log_3\left(1+\frac{1}{242}\right) = \log_3\left(\frac{4}{3}\,\frac{5}{4}\ldots\ldots\frac{243}{242}\right) = \log_3\left(\frac{243}{3}\right)$

$\Rightarrow \quad \log_3 81 = 4$

ILLUSTRATION 86

Find the value of 'x' satisfying the equation,

$$4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$$

SOLUTION

$4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$

$\Rightarrow \quad 4^{\frac{1}{2}\log_3 3} + 9^2 = 10^{\log_x 83}$

$\Rightarrow \quad 83 = 10^{\log_x 83}$

This gives $x = 10$

ILLUSTRATION 87

If $\log 15 = a$ and $\log 75 = b$, then $\log_{75}45 = \dfrac{3a-b}{b}$, Mark True or False.

SOLUTION

$\log 15 = a \;\Rightarrow\; \log 5 + \log 3 = a \qquad \ldots(1)$

$\log 75 = b \;\Rightarrow\; \log 15 + \log 5 = b$

$\Rightarrow \quad \log 3 + 2\log 5 = b \qquad$ (from (1))

$\Rightarrow \quad -b + 2a = \log 3$

$\log_{75}45 = \dfrac{\log 15 + \log 3}{\log 75} = \dfrac{a - b + 2a}{b} = \dfrac{3a - b}{b}$

It is true.

ILLUSTRATION 88

If $\log_{12}18 = \alpha$ and $\log_{24}54 = \beta$.

Prove that $\alpha\beta + 5(\alpha - \beta) = 1$

SOLUTION

$$\alpha = \log_{12}18 = \frac{\log 18}{\log 12} = \frac{\log(3^2.2)}{\log(3.2^2)}$$

$$= \frac{2\log 3+\log 2}{\log 3+2\log 2} \text{ and}$$

$$\beta = \log_{24} 54 = \frac{\log 54}{\log 24} = \frac{\log(3^3.2)}{\log(3.2^3)}$$

$$= \frac{3\log 3+\log 2}{\log 3+3\log 2}$$

If $x = \log 2$ and $y = \log 3$ then

$$\alpha = \frac{x+2y}{2x+y},\ \beta = \frac{x+3y}{3x+y}$$

$$\therefore \quad a\beta + 5(\alpha-\beta) \qquad \ldots(1)$$

by substituting the value of α and β in the equation (1) and simplifying

we get $a\beta + 5(\alpha-\beta) = \dfrac{6x^2+5xy+y^2}{6x^2+5xy+y^2} = 1$

SOLVING LOG EQUATION

While solving logarithmic equation we have to always consider the domain conditions for which log function is defined. We can initially note down conditions for which log function is defined or we can cross check the values which we get after solving log equation and validate the correct solution with given question.

ILLUSTRATION 89

Solve $3^{3\log_3 x} = 27$.

SOLUTION

$$3^{\log_3 x^3} = 3^3 \Rightarrow x^3 = 3^3 \text{ we get } x = 3$$

ILLUSTRATION 90

Solve $\log_x(4x-3) = 2$.

SOLUTION

$$\log_x(4x-3) = 2$$

definition condition $4x-3>0$

$$\Rightarrow \quad x > \frac{3}{4} \text{ and } x>0 \text{ and } x \neq 1$$

$$\Rightarrow \quad 4x-3 = x^2 \Rightarrow x^2-4x+3 = 0 \Rightarrow (x-1)(x-3) = 0$$

$$x = 1, 3$$

$\therefore$ acceptable solution $x = 3$

ILLUSTRATION 91

Solve $\log_2(\log_3(\log_5(x^2+4))) = 0$.

SOLUTION

$$\log_2(\log_3(\log_5(x^2+4))) = 0$$

$$\Rightarrow \quad \log_3(\log_5(x^2+4)) = 2^{o} = 1$$

$$\Rightarrow \quad \log_5(x^2+4) = 3^1$$

$$\Rightarrow \quad x^2+4 = 5^3 = 125$$

$$\Rightarrow \quad x^2 = 121 \quad \Rightarrow \quad x = \pm 11$$

ILLUSTRATION 92

Solve $\log_{10}{}^2 x - \log_{10} x - 6 = 0$.

SOLUTION

$$\log_{10}{}^2 x - \log_{10} x - 6 = 0$$

$$\Rightarrow \quad (\log_{10} x - 3)(\log_{10} x + 2) = 0$$

$$\Rightarrow \quad \log_{10} x = 3 \quad \text{or} \quad \log_{10} x = -2$$

$$\Rightarrow \quad x = 10^3 \quad \text{or} \quad x = 10^{-2}$$

$$\Rightarrow \quad x = 1000 \quad \text{or} \quad \frac{1}{100}$$

ILLUSTRATION 93

Solve $3(\log_7 x + \log_x 7) = 10$.

SOLUTION

put $\log_7 x = t$

$$3\left(\log_7 x + \frac{1}{\log_7 x}\right) = 10$$

$$\Rightarrow \quad 3\left(t+\frac{1}{t}\right) = 10 \quad \Rightarrow \quad 3(t^2+1) = 10t$$

$$\Rightarrow \quad 3t^2+3-10t = 0 \quad \Rightarrow \quad 3t^2-10t+3 = 0$$

$$\Rightarrow \quad (t-3)(3t-1) = 0 \quad \Rightarrow \quad t = 3, \frac{1}{3}$$

$$t = \log_7 x = 3 \quad \Rightarrow \quad x = 7^3 = 343$$

$$t = \log_7 x = \frac{1}{3} \quad \Rightarrow \quad x = 7^{1/3}$$

Final answer $x = \left\{7^{\frac{1}{3}}, 343\right\}$

ILLUSTRATION 94

Solve $(x+2)^{\log_2(x+2)} = 8(x+2)^2$.

SOLUTION

$$(x+2)^{\log_2(x+2)} = 8(x+2)^2$$

Taking log both sides

$$\log_2 (x+2)^{\log_2(x+2)} = \log_2(8(x+2)^2)$$

$$\Rightarrow \quad \log_2(x+2).\log_2(x+2) = \log_2 8 + \log_2(x+2)^2$$

$$\Rightarrow \quad (\log_2(x+2))^2 = 3 + 2\log_2(x+2)$$

Let $\log_2(x+2) = t$,

we have $t^2 = 3 + 2t$ or $t^2 - 2t - 3 = 0$

$\Rightarrow \quad (t-3)\,(t+1) = 0$

$t = \log_2(x+2) = 3 \Rightarrow x + 2 = 2^{+3} = 8$ or $x = 6$

$t = \log_2(x+2) = -1 \Rightarrow x + 2 = 2^{-1} = \frac{1}{2}$ or $x = -\frac{3}{2}$

Cross check for domain constraint, we have $\log_2(x+2)$ is defined for both values

Final values $x \in \left\{-\frac{3}{2}, 6\right\}$

ILLUSTRATION 95

Solve $\log_2 x^2 + \log_2(x+2) = 4$.

SOLUTION

$\log_2 x^2 + \log_2(x+2) = 4$

$\Rightarrow \quad \log_2 x^2(x+2) = 4 \quad \Rightarrow \quad x^2(x+2) = 2^4 = 16$

$\Rightarrow \quad x^3 + 2x^2 - 16 = 0$ or $x^3 - 8 + 2x^2 - 8 = 0$

$\Rightarrow \quad (x-2)\,(x^2 + 2x + 4) + 2(x-2)(x+2) = 0$

or $\quad (x-2)[x^2 + 2x + 4 + 2x + 4] = 0$

or $\quad (x-2)\,(x^2 + 4x + 8) = 0$

$x = 2$ or $(x+2)^2 + 4 \neq 0$

Final answer $x = 2$

ILLUSTRATION 96

Solve $\log_2(\log_3(\log_5(x^2 - 19))) = 0$.

SOLUTION

$\log_2(\log_3(\log_5(x^2 - 19))) = 0$

$\Rightarrow \quad \log_3(\log_5(x^2 - 19)) = 2^{o} = 1$

$\Rightarrow \quad \log_5(x^2 - 19) = 3^1$

$\Rightarrow \quad x^2 - 19 = 5^3 = 125$

$\Rightarrow \quad x^2 = 144$ or $x = \pm 12$, given logarithm are defined for it.

$\therefore$ Final solution $x = \pm 12$

ILLUSTRATION 97

Solve $\log_4(\log_2 x) + \log_2(\log_4 x) = 2$.

SOLUTION

$\log_4(\log_2 x) + \log_2(\log_4 x) = 2$

$\Rightarrow \quad \log_{2^2}(\log_2 x) + \log_2(\log_{2^2} x) = 2$

$\Rightarrow \quad \frac{1}{2}\left(\log_2(\log_2 x)\right) + \log_2\left(\frac{1}{2}\log_2 x\right) = 2$

$\Rightarrow \quad \frac{1}{2}(\log_2(\log_2 x)) + \log_2(\log_2 x) - \log_2 2 = 2$

$\Rightarrow \quad \frac{3}{2}\log_2(\log_2 x) = 3 \quad \Rightarrow \quad \log_2(\log_2 x) = 2$

$\Rightarrow \quad \log_2 x = 2^2 = 4$

We get $x = 2^4 = 16$

$\log_2 x$ is defined for it

Final answer $x = 16$

ILLUSTRATION 98

Solve $2\log_4(4-x) = 4 - \log_2(-2-x)$.

SOLUTION

$2\log_4(4-x) = 4 - \log_2(-2-x)$

or $\quad 2\log_{2^2}(4-x) = \log_2 16 - \log_2(-2-x)$

or $\quad \log_2(4-x) = \log_2\left(\frac{16}{-2-x}\right)$

$\Rightarrow \quad 4 - x = \frac{16}{-2-x} \quad \Rightarrow \quad (4-x)(-2-x) = 16$

$\Rightarrow \quad (x-4)(x+2) = 16 \quad \Rightarrow \quad x^2 - 4x + 2x - 8 = 16$

$\Rightarrow \quad x^2 - 2x - 24 = 0$

$\Rightarrow \quad (x-6)\,(x+4) = 0$ we get $x = 6, -4$

Cross check

$\log_4(4-x)$ is not defined for $x = 6$

for $x = -4$, both log functions are defined

Final answer we have $x = -4$

ILLUSTRATION 99

If x_1, x_2 and x_3 are the three real solutions of the equation

$$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\frac{1}{\sqrt{x+1}-1} - \frac{1}{\sqrt{x+1}+1}}, \text{ where } x_1 > x_2 > x_3,$$

then prove that

(i) $x_1.x_3 = x_2{}^2$

(ii) $x_3{}^{-1} + 1 = x_1{}^{-1} + x_2{}^{-2}$

SOLUTION

$$x^{\log_{10}^2 x + \log_{10} x^3 + 3} = \frac{2}{\frac{1}{\sqrt{x+1}-1} - \frac{1}{\sqrt{x+1}+1}}$$

$$x^{\log_{10}^2 x + 3\log_{10} x^3 + 3} = \frac{2((x+1)-1)}{(\sqrt{x+1}+1) - (\sqrt{x+1}-1)} = x$$

So, $x^{\log_{10}^2 x + 3\log_{10} x + 3} = x$

Take log on both sides

$$(\log_{10}^2 x + 3\log_{10} x + 3)\log_{10} x = \log_{10} x$$

$$\log_{10}x((\log_{10}^2 x + 3\log_{10}x + 2) = 0$$

put $\log_{10}x = y$

$$y(y+1)(y+2) = 0 \quad \Rightarrow \quad y = 0, -1, -2$$

$$\log_{10}x = 0 \quad \Rightarrow \quad x = 1$$

$$\log_{10}x = -1 \quad \Rightarrow \quad x = \frac{1}{10}$$

$$\log_{10}x = -2 \quad \Rightarrow \quad x = \frac{1}{100}$$

Given $x_1 > x_2 > x_3$

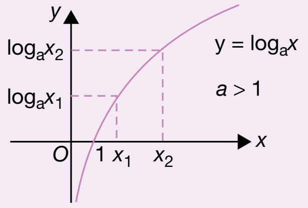
$$x_1 = 1 \quad x_2 = \frac{1}{10} \quad x_3 = \frac{1}{100}$$

(i) $x_1.x_3 = x_2^{\,2} = \dfrac{1}{100}$

(ii) $x_3^{-1} + 1 = x_1^{-1} + x_2^{-2} = 101$ hence proved

SOLVING LOGARITHMIC INEQUALITY

Base > 1

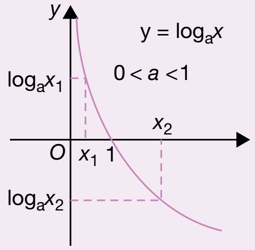

y
$\log_a x_2$
$\log_a x_1$
y = $\log_a x$
a > 1
O
1
x_1
x_2
x

$x_1 < x_2$

$\Rightarrow$ $\log_a x_1 < \log_a x_2$

No sign change in inequality

0 < Base < 1


y
y = $\log_a x$
0 < a <1
$\log_a x_1$
x_2
O
x_1
1
$\log_a x_2$

$x_1 < x_2$

$\Rightarrow$ $\log_a x_1 > \log_a x_2$

Sign changes in inequality

Now we will discuss the comparison of logarithmic function for different values of base

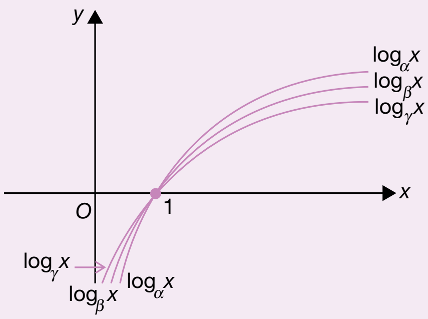
Let we have three function $y = \log_\alpha x$, $y = \log_\beta x$, $y = \log_\gamma x$

CASE-1

Base of logarithmic function are as $1 < \alpha < \beta < \gamma$, behaviour of all logarithmic functions is shown in figure

For $x > 1$ $\qquad \log_\alpha x > \log_\beta x > \log_\gamma x$

For $0 < x < 1$ $\qquad \log_\alpha x < \log_\beta x < \log_\gamma x$


y
$\log_\alpha x$
$\log_\beta x$
$\log_\gamma x$
x
O
1
$\log_\gamma x$
$\log_\beta x$
$\log_\alpha x$

CASE-2: $0 < \gamma < \beta < \alpha < 1$

For $0 < x < 1$ $\qquad \log_\alpha x > \log_\beta x > \log_\gamma x$

For $x > 1$ $\qquad \log_\alpha x < \log_\beta x < \log_\gamma x$

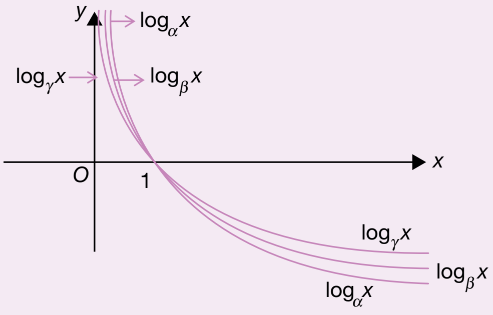

y
$\log_\alpha x$
$\log_\gamma x$
$\log_\beta x$
x
O
1
$\log_\gamma x$
$\log_\beta x$
$\log_\alpha x$

ILLUSTRATION 100

Solve $3^{\log_2(x^2+x+1)} > 1$

SOLUTION

$$3^{\log_2(x^2+x+1)} > 1$$

$$\Rightarrow \quad 3^{\log_2(x^2+x+1)} > 3^{o}$$

Base = 3 > 1

$$\log_2(x^2+x+1) > 0;$$

$$(\because\ x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} > 0 \quad \forall\, x \in R)$$

$$\Rightarrow \quad x^2+x+1 > 2^{o} = 1$$

$$\Rightarrow \quad x^2+x > 0 \quad \Rightarrow \quad x(x+1) > 0$$

We get $x\ (-\infty, -1) \cup (0, \infty)$

ILLUSTRATION 101

Solve $\log_x(x^2 - 2x + 2) \geq 1$.

SOLUTION

$$\log_x(x^2 - 2x + 2) \geq 1$$

$$x^2 - 2x + 2 = (x-1)^2 + 1 > 0 \quad \forall\, x \in R$$

CASE-1: $0 < x < 1$; $0 <$ Base < 1

after antilog

$$x^2 - 2x + 2 \leq x$$

$$\Rightarrow \quad x^2 - 3x + 2 \leq 0$$

$$\Rightarrow \quad (x-1)(x-2) \leq 0$$

$$x \in [1, 2] \text{ but } x \in (0, 1)$$

$$\therefore \quad x \in \phi$$

CASE-2: $x > 1$; Base > 1

$$\log_x(x^2 - 2x + 2) \geq 1$$

after antilog

$$x^2 - 2x + 2 \ge x$$

$$\Rightarrow \quad x^2 - 3x + 2 \ge 0$$

$$\Rightarrow \quad (x-1)(x-2) \ge 0$$

$$x \in (-\infty, 1] \cup [2, \infty) \text{ but } x > 1$$

$$\therefore \quad x \in [2, \infty)$$

Final answer CASE-1 ∪ CASE-2

$$x \in [2, \infty)$$

ILLUSTRATION 102

Solve $\dfrac{\sqrt{(x-8)(2-x)}}{\log_{0.3}\left(\dfrac{10}{7}(\log_2 5 - 1)\right)} \ge 0$ and $2^{x-3} - 31 > 0$.

SOLUTION

$$\therefore \quad \log_2 5 - 1 > 1 \quad \Rightarrow \quad \frac{10}{7}(\log_2 5 - 1) > \frac{10}{7} > 1$$

$$\log_{0.3}\left(\frac{10}{7}(\log_2 5 - 1)\right) < \log_{0.3} 1 = 0$$

$$\log_{0.3}\left(\frac{10}{7}(\log_2 5 - 1)\right) < 0$$

$$\frac{\sqrt{(x-8)(2-x)}}{\log_{0.3}\left(\dfrac{10}{7}(\log_2 5 - 1)\right)} \ge 0$$

$$\Rightarrow \quad \sqrt{(x-8)(2-x)} \le 0$$

$$\therefore \quad (x-8)(2-x) = 0 \quad \Rightarrow \quad x = 2, 8$$

$$2^{x-3} - 31 \ge 0 \quad \Rightarrow \quad 2^{x-3} \ge 31$$

$x = 2$, we get $\dfrac{1}{2} \ge 31$ False

$x = 8$, we get $2^5 \ge 31$

$\Rightarrow \quad 32 \ge 31$ True

$x = 8$ only solution

ILLUSTRATION 103

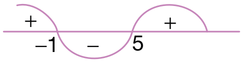
Solve the inequality $\log_{10}(x^2 - 16) \le \log_{10}(4x - 11)$.

SOLUTION

$$\log_{10}(x^2 - 16) \le \log_{10}(4x - 11)$$

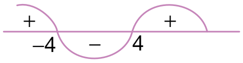
$$\Rightarrow \quad x^2 - 16 \le 4x - 11 \quad \Rightarrow \quad x^2 - 4x - 5 \le 0$$

$$\Rightarrow \quad -1 \le x \le 5 \qquad \ldots(1)$$



Also $\log_{10}(x^2 - 16)$ is defined if and only if $x^2 - 16 > 0$

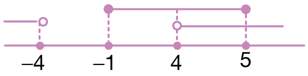
Now $x^2 - 16 > 0 \quad \Rightarrow \quad x < -4$ or $x > 4$ $\qquad \ldots(2)$



Again for $\log_{10}(4x - 11)$

to be defined if $4x - 11 > 0 \quad \Rightarrow \quad x > 11/4$ $\qquad \ldots(3)$

(1), (2) and (3) will simultaneously hold if $4 < x \le 5$



$\therefore$ Final solution set $x \in (4, 5]$

ILLUSTRATION 104

Solve $\log_3\left(\dfrac{1+2x}{1+x}\right) < 1$.

SOLUTION

$$\log_3\left(\frac{1+2x}{1+x}\right) < 1$$

$$\Rightarrow \quad \frac{1+2x}{1+x} < 3^1 \quad \Rightarrow \quad \frac{1+2x}{1+x} - 3 < 0$$

$$\Rightarrow \quad \frac{1+2x-3-3x}{x+1} < 0 \quad \Rightarrow \quad \frac{-2-x}{x+1} < 0$$

$$x \in (-\infty, -2) \cup (-1, \infty) \qquad \ldots(1)$$

Also, $\dfrac{1+2x}{1+x} > 0$

$$\Rightarrow \quad x \in (-\infty, -1) \cup \left(\frac{-1}{2}, \infty\right) \qquad \ldots(2)$$

Common values from (1) and (2)

We get $x \in (-\infty, -2) \cup \left(\dfrac{-1}{2}, \infty\right)$

ILLUSTRATION 105

Solve $\log_{x+3}(x^2 - x) < 1$.

SOLUTION

CASE-1: $0 < x + 3 < 1 \quad \Rightarrow \quad -3 < x < -2$ $\qquad \ldots(1)$

and $x^2 - x > 0 \quad \Rightarrow \quad x(x-1) > 0$

$$\Rightarrow \quad x \in (-\infty, 0) \cup (1, \infty) \qquad \ldots(2)$$

and $\log_{x+3}(x^2 - x) < 1 \quad \Rightarrow \quad x^2 - x > x + 3$

$$\Rightarrow \quad x^2 - 2x - 3 > 0$$

$$\Rightarrow \quad (x-3)(x+1) > 0$$

$$\Rightarrow \quad x \in (-\infty, -1) \cup (3, \infty) \qquad \ldots(3)$$

(1) ∩ (2) ∩ (3) we get $(-3, -2)$

CASE-2

(i) $x + 3 > 1 \quad \Rightarrow \quad x > -2$

(ii) $x^2 - x > 0 \quad \Rightarrow \quad x \in (-\infty, 0) \cup (1, \infty)$

(iii) $\log_{x+3}(x^2 - x) < 1 \quad x^2 - x < x - 3$

$\Rightarrow x^2 - 2x + 3 < 0 \quad \Rightarrow \quad (x - 3)(x + 1) < 0$

$\Rightarrow x \in (-1, 3)$

$(1) \cap (2) \cap (3)$

We get $(-1, 0) \cup (1, 3)$

Final solution set case $(1) \cup (2)$

$x \in (-3, -2) \cup (-1, 0) \cup (1, 3)$

ILLUSTRATION 106

Solve: $\log_{0\cdot5} \sqrt{\dfrac{x-4}{x+3}} < \log_{0\cdot5} 2.$

SOLUTION

$$\log_{0.5} \sqrt{\frac{x-4}{x+3}} < \log_{0\cdot5} 2$$

$$\sqrt{\frac{x-4}{x+3}} > 0, \quad x \in (-\infty, -3) \cup (4, \infty) \qquad \ldots(1)$$

and $\sqrt{\dfrac{x-4}{x+3}} > 2$

$$\frac{x-4}{x+3} > 4 \quad \text{(on squaring)}$$

$$\Rightarrow \quad \frac{x-4}{x+3} - 4 > 0$$

$$\Rightarrow \quad \frac{3x+16}{x+3} < 0$$

$$\Rightarrow \quad x \in \left(\frac{-16}{3}, -3\right) \qquad \ldots(2)$$

Common values of x, $(1) \cap (2)$

We get $x \in \left(\dfrac{-16}{3}, -3\right)$

CHARACTERISTIC AND MANTISSA

Given a number N,

Logarithm can be $\log_{10} N = \text{Integer} + \text{Fraction}$

Integer part is Characteristic (C) and fractional part is Mantissa (m)

(a) The mantissa part of the $\log_{10}N$ is always kept non-negative, it ranges from [0, 1]

(b) If $N > 1$ and the characteristic of $\log_{10}N$ is C then the number of digits in $N = C + 1$

(c) If $0 < N < 1$ and the characteristic of $\log_{10}N$ is C then number of zeros after decimal point of N = magnitude of the characteristic $-1 = |C| - 1$

Note:

Let $y = \log(N)$ when $0 < N < 1$

If N lies between 0 and 1, then the characteristic is negative

$N = 0.1 = 1/10$, $\log_{10}N = \log_{10}(1/10) = -1$, $C = -1$, $M = 0$

$N = 0.01$, $\log_{10}N = \log_{10}(10)^{-2} = -2$, $C = -2$, $M = 0$

$N = 0.001$, $\log_{10}N = \log_{10} 10^{-3} = -3$, $C = -3$, $M = 0$

No. of zeros after decimal $= |-3| - 1 = 2$

$N = 0.002$, $\log_{10}N = \log(2 \times 10^{-3}) = \log 2 + \log 10^{-3} = 0.3010 + (-3) = -2.699$

$C = -3$, $M = 0.3010$

Number of zeros after decimal = magnitude of the characteristic $-1 = |C| - 1 = |-3| - 1 = 2$

ILLUSTRATION 107

Find the number of digits in the following:

(i) 2^{100} **(ii)** 3^{10}

SOLUTION

By considering $x = 2^{100}$ and 3^{10} respectively and applying log on both sides we can solve the problems given above.

(i) Let $x = 2^{100}$

$\log_{10}x = \log_{10}2^{100} = 100\log_{10}2 = 100 \times 0.3010 = 30.10$

Characteristic = 30, Mantissa = 0.10

Number of digits $= C + 1 = 30 + 1 = 31$

(ii) Let $x = 3^{10}$

$\log_{10}x = 10 \log 3 = 10 \times 0.4771 = 4.771$

$C = 4$, $M = 0.771$

Number of digits $= C + 1 = 4 + 1 = 5$

ILLUSTRATION 108

Find the number of zeros after decimal before a significant figure in

(i) 3^{-50} **(ii)** 2^{-100} **(iii)** 7^{-100}

SOLUTION

Similar to the illustration above, we can solve these too.

(i) $\boldsymbol{N = 3^{-50}}$

$\log_{10} N = \log_{10} 3^{-50} = -50 \log_{10}3 = -50 \times (0.4771)$

$\Rightarrow \log_{10} N = -23.855$

Now to find the characteristic and mantissa many would say that ($c = -23$, $m = -0.855$) (which is wrong) because mantissa is always non-negative.

$\log_{10}N = -23.855 = \; = -24 + 0.145$

$C = -24$, $M = 0.145$.

Number of zeroes after decimal $= |-24| - 1 = 23$

(ii) $N = 2^{-100}$

$\log_{10} N = -100 \log_{10} 2 = -30.10 = -30 - 0.10 = -31 + 0.90.$

Number of zeroes after decimal $= |-31| - 1 = 30$

(iii) $N = 7^{-100}$

$\log_{10} N = -100 \log_{10} 7 = -100 \times 0.8451 = -84.51$
$= -85 + 0.49$
$C = -85, M = 0.49.$
Number of zeroes after decimal $= |-85| - 1 = 84$

ILLUSTRATION 109

Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10.

Assume $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$

SOLUTION

Simply by applying log on both sides and using various logarithm formulas we can solve the above illustration.

$$\log_{10} x = \log_{10}(0.15)^{20} = 20 \log_{10}(0.15)$$
$$= 20[\log 15 - 2] = 20[\log 3 + \log 5 - 2]$$
$$= 20[\log 3 + 1 - \log 2 - 2]$$
$$= 20[-1 + \log 3 - \log 2] = -20 \times 0.824$$
$$= -16.48 = \overline{17}.52$$

Hence, characteristic $= -17$ and mantissa $= 0.52$

Number of zeroes after decimal point of $(0.15)^{20}$ are 16

ILLUSTRATION 110

Find the number of positive integers which have the characteristic 2, when base of log is 6.

SOLUTION

If any number x has the characteristic a, when base of log is b then $x = ba$. By using the given condition we can solve the problem above.

$$x = 6^2 = 36; \log_6 x = \log_6 6^2 = 2 \log_6 6 = 2$$

The smallest natural number which has characteristic 3 with base 6 is 6^3

$$x = 6^3 = 216; \log_6 x = \log_6 6^3 = 3$$

Hence $x = 215$ will give characteristic 2

Natural numbers ranging from 36 to 215 will give characteristic 2, when taken log with base 6.

Number of positive integers $= 215 - 35 = 180$

Test Your Concepts-III

(Solutions on page H.5)

1. If solution set of the equation $3^{2x^2} - 2.3^{x^2+x+6} + 3^{2(x+6)} = 0$ is $\{\alpha, \beta\}$, then find $\alpha^2 + \beta^2$ is prime number.

2. Find the complete solution set of the inequation $3^{72} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^{\sqrt{x}} > 1.$

3. Find the number of real solutions of equation $x^{\log_x 2} + x^2 = 3x.$

4. If $\log_{10}(x-1)^3 - 3\log_{10}(x-3) = \log_{10} 8$, then find $\log_x 625.$

5. Find the value of the expression,

$$\log_4\left(\frac{x^2}{4}\right) - 2\log_4(4x^4) \text{ when } x = -2.$$

6. If $x = \log_k b = \log_b c = \frac{1}{2}\log_c d$, then find $\log_k d$ in terms of x.

7. Simplify $N = \frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \log_6^2 2.$

8. Find x satisfying, $\sqrt{5 - \log_2 x} = 3 - \log_2 x.$

9. Find x which satisfies system of equations $\log_{10}(3^x - 2^{4-x}) = 2 + \frac{1}{4}\log_{10} 16 - \frac{x}{2}\log_{10} 4$ and $\log_3\left(3^{x^2-13x+28} + \frac{2}{9}\right) = \log_5(0.2)$

10. Find the number of solution of the equation, $\log(-2x) = 2\log(x+1).$

11. Solve $\left(\frac{1}{2}\right)^{\log x^2} + 2 > 3.2^{-\log(-x)}$

12. If $\log_6 \log_2\left[\sqrt{4x+2} + 2\sqrt{x}\right] = 0$, then $x = \ldots$.

13. Let the solution of the equation $5^{1+\log_4 x} + 5^{-1-\log_4 x} = \frac{26}{5}$ be a and b, $a > b$, then find the value of $\frac{a}{b}$.

14. Solve for x:

$$\log_2(4(4^x + 1)).\log_2(4^x + 1) = \log_{1/\sqrt{2}}\left(1/\sqrt{8}\right).$$

Answer keys

1. 13 **2.** [0, 64) **3.** 1 **4.** 4 **5.** –6 **6.** $2x^3$ **7.** 1 **8.** 2

9. $x = 3$ **10.** 1 **11.** $x \in (-\infty, -1) \cup \left(\frac{-1}{10}, 0\right)$ **12.** $x = \frac{1}{16}$ **13.** 16 **14.** 0

SOLVED PROBLEMS

PROBLEM 1

Solve $\dfrac{10000^{-x}\cdot(\cos x+3)(\sin x-1\cdot 1)(x-2)}{(x+4)(x-4)}\geq 0$

SOLUTION

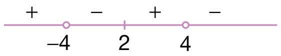
$$\frac{10000^{-x}\cdot(\cos x+3)(\sin x-1\cdot 1)(x-2)}{(x+4)(x-4)}\geq 0$$

1000^{-x} is always positive

sin x and cos x both lie between –1 and 1 hence $\cos x+3$ is always positive and $\sin x-1.1$ is always negative

Solving $\dfrac{10000^{-x}\cdot(\cos x+3)(\sin x-1\cdot 1)(x-2)}{(x+4)(x-4)}\geq 0$

or $\dfrac{-(x-2)}{(x+4)(x-4)}\geq 0$



Poles $x\neq -4, 4$ and zeroes $x=2$
Required $x\in(-\infty,-4)\cup[2,4)$

PROBLEM 2

Solve the equation for x:

$$\log 4+\left(1+\frac{1}{2x}\right)\log 3=\log\left(\sqrt[x]{3}+27\right)$$

SOLUTION

$$\log 4+\left(1+\frac{1}{2x}\right)\log 3=\log\left(3^{\frac{1}{x}}+27\right)$$

$$\log\left(4.3^{1+\frac{1}{2x}}\right)=\log\left(3^{\frac{1}{x}}+27\right)$$

$\Rightarrow\quad 12.3^{1/2x}=3^{1/x}+27$

Let $3^{1/2x}=t$

$\therefore\quad 12t=t^2+27$

$\Rightarrow\quad t^2-12t+27=0$

$\Rightarrow\quad t=3, 9$

$$3^{1/2x}=3\ \Rightarrow\ \frac{1}{2x}=1\ \Rightarrow\ x=\frac{1}{2}$$

$$3^{1/2x}=9\ \Rightarrow\ \frac{1}{2x}=2\ \Rightarrow\ x=\frac{1}{4}$$

but x has to a natural number (Since, $\sqrt[x]{3}$ is only defined, when x is natural number ≥ 2)

$\therefore\ x\in\phi$ (No solution)

PROBLEM 3

Find all integral solutions of the equation

$$4\log_{x/2}\sqrt{x}+2\log_{4x}(x^2)=3\log_{2x}(x^3)$$

SOLUTION

$$4\log_{x/2}\sqrt{x}\ +2\log_{4x}x^2=3\log_{2x}x^3$$

$$2\log_{x/2}x+4\log_{4x}x=9\log_{2x}x$$

$$\frac{2}{\log_x(x/2)}+\frac{4}{\log_x 4x}=\frac{9}{\log_x 2x}\text{; Let }\log_x 2=t$$

$$\frac{2}{1-t}+\frac{4}{2t+1}=\frac{9}{t+1}$$

$6(t+1)=9(t-2t^2+1)$

$18t^2-3t-3=0\quad\Rightarrow\quad 6t^2-t-1=0$

$\Rightarrow\quad (3t+1)(2t-1)=0$

$$t=\frac{1}{2}=\log_x 2\quad\Rightarrow\quad x=4$$

$$\Rightarrow\quad t=-\frac{1}{3}=\log_x 2\quad\Rightarrow\quad x=\frac{1}{8}$$

Now, checking for $x=1$

$x=1$ satisfies the original equation

$\therefore$ integral solution are {1, 4}

PROBLEM 4

Find number of ordered pair(s) satisfying the system of equations

$$2\log(x^2+y^2)-\log 5=\log\{2(x^2+y^2)+75\}\text{ and}$$

$$\log\left(\frac{x}{3}\right)+\log(5y)=1+\log 2$$

SOLUTION

$$\log\left(\frac{x}{3}\right)+\log(5y)=1+\log 2$$

$$\Rightarrow\quad \log\left(\frac{5xy}{3}\right)=\log 20\ \Rightarrow\ xy=\frac{20\times 3}{5}=12$$

$xy=12$ and domain $x>0$ and $y>0$

$2\log(x^2+y^2)-\log 5=\log\{2(x^2+y^2)+75\}$

Let $x^2+y^2=t$

$\log t^2-\log 5=\log(2t+75)$

$$\frac{t^2}{5}=2t+75\quad\Rightarrow\quad t^2-10t-375=0$$

$(t+15)(t-25)=0\quad\Rightarrow\quad t=25, -15$ (rejected)

$\Rightarrow\quad x^2+y^2=25$ and $xy=12$

on solving $x = 3$ and $y = 4$

$x = 4$ and $y = 3$

Hence number of ordered pair is 2

PROBLEM 5

If $1 + \log_2 (1+ \cos 2x) > \log_4 2$ then find x.

SOLUTION

$$1 + \log_2 (2\cos^2 x) > \frac{1}{2}\log_2 2$$

or $$2 + \log_2 \cos^2 x > \frac{1}{2}$$

or $$2 \log_2 \cos x > \frac{-3}{2}$$

$\therefore$ $$\log_2 \cos x > \frac{-3}{4}$$

$\Rightarrow$ $$\cos x > 2^{\frac{-3}{4}} = \frac{1}{2^{3/4}}$$

Let $\cos \alpha = \frac{1}{2^{3/4}}$, i.e., $\alpha = \cos^{-1}\frac{1}{2^{3/4}}$

So, from the sign-scheme we get,

$$2n\pi - \alpha < x < 2n\pi + \alpha$$

$\therefore$ $$2n\pi - \cos^{-1}\frac{1}{2^{3/4}} < x < 2n\pi + \cos^{-1}\frac{1}{2^{3/4}}$$

where $n \in Z$.

Note:

$\log_2(1 + \cos 2x)$ is defined for all x obtained.

PROBLEM 6

Solve $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$.

SOLUTION

Here $$\frac{\log \sin x}{\log \cos x} + \frac{\log \cos x}{\log \sin x} = 2$$

or $$(\log \sin x)^2 + (\log \cos x)^2$$

$$-2 (\log \cos x) (\log \sin x) = 0$$

or $$(\log \sin x - \log \cos x)^2 = 0,$$

i.e., $$\log \sin x = \log \cos x$$

$\therefore$ $$\sin x = \cos x \quad \Rightarrow \quad \tan x = 1$$

$\Rightarrow$ $$x = n\pi + \frac{\pi}{4} \qquad \ldots(1)$$

But here the logarithms are real if $\sin x > 0$, $\cos x > 0$, $\sin x \neq 1$, $\cos x \neq 1$

$\therefore$ x belongs to the first quadrant and

$$x \neq n\pi + (-1)^n \frac{\pi}{2}, 2n\pi$$

from (1), $x = 2m\pi + \frac{\pi}{4}$, $m \in Z$

PROBLEM 7

Solve the inequality $|3^x - 3| + 9^x - 3 > 0$.

SOLUTION

When $3^x - 3 = 0$, $3^x = 3^1$

$\therefore$ $x = 1$

CASE-1: When $x \leq 1$, $3^x - 3 \leq 0$

$\therefore$ $|3^x - 3| = -(3^x - 3)$

Now $|3^x - 3| + 9^x - 3 > 0$

$\Rightarrow$ $-(3^x - 3) + 9^x - 3 > 0$

$\Rightarrow$ $3^{2x} - 3^x > 0 \quad \Rightarrow \quad 3^{2x} > 3^x$

$\Rightarrow$ $2x > x \quad \Rightarrow \quad x > 0$ But $x \leq 1$

$\therefore$ $0 < x \leq 1$

CASE-2: When $x > 1$,

$3^x - 3 > 0$, we have $|3^x - 3| = 3^x - 3$

Now $|3^x - 3| + 9^x - 3 > 0$

$\Rightarrow$ $3^x - 3 + 3^{2x} - 3 > 0$

$\Rightarrow$ $y^2 + y - 6 > 0$, where $y = 3^x$

$\Rightarrow$ $y < -3$ or $y > 2$

But in this case $x > 1$

$\therefore$ $3^x > 3$

$\therefore$ $y > 2$

$\Rightarrow$ $3^x > 2 \quad \Rightarrow \quad x > \log_3 2$

Also $\log_3 2 < 1$

Now as in this case $x > 1$

$\therefore$ $1 < x < \infty$

Thus final solution set, case (1) $\cup$ (2) we have $0 < x < \infty$

PROBLEM 8

If a and b are the lengths of the sides and c the length of the hypotenuse of right angled triangle $c - b \neq 1$ and $c + b \neq 1$, prove that

$$\log_{c+b}a + \log_{c-b}a = 2\log_{c+b} a \log_{c-b}a$$

SOLUTION

From the property of right angled triangle we get $a^2 + b^2 = c^2$

or $a^2 = (c - b)(c + b)$

$$\text{L.H.S.} = \log_{c+b}a + \log_{c-b}a$$

$$= \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)}$$

$$= \frac{\log_a(c-b)+\log_a(c+b)}{\log_a(c+b)\log_a(c-b)}$$

$$= \frac{\log_a(c^2-b^2)}{\log_a(c+b)\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b)\log_a(c-b)}$$

$$= \frac{2\log_a a}{\log_a(c+b)\log_a(c-b)} = 2\log_{(c+b)} a \log_{(c-b)} a$$

$\therefore$ $\log_a a = 1$ and $\dfrac{1}{\log_y x} = \log_x y$

PROBLEM 9

Find solution if $(x^2-3x-3)^{x+1} > 1$.

SOLUTION

Clearly, $x^2-3x-3>0$, because it is the base.

$\therefore$ by sign-scheme $\dfrac{3-\sqrt{21}}{2} > x$ or $x > \dfrac{3+\sqrt{21}}{2}$

$\therefore$ taking logarithm,

$$\log_{10}(x^2-3x-3)^{x+1} > \log_{10} 1$$

or $\quad (x+1)\log_{10}(x^2-3x-3) > 0.$

Any one of the following is possible:

Case (1) $x+1>0$ and $\log_{10}(x^2-3x-3)>0$

$\Rightarrow x>-1$ and $x^2-3x-3>1 \Rightarrow x^2-3x-4>0$

$\Rightarrow x<-1$ or $x>4$

So common values $x>4$.

Case (2) $x+1<0$ and $\log_{10}(x^2-3x-3)<0$ or $x^2-3x-4<0$

$x<-1$ and $-1<x<4$

There is no such x.

Final solution-set case (1) ∪ (2) $4<x<\infty$

PROBLEM 10

Find x if

$$\log_{(2x+3)}(6x^2+23x+21) = 4 - \log_{(3x+7)}(4x^2+12x+9)$$

SOLUTION

In this equation, the domain is defined by

$$2x+3>0,\ 2x+3\neq 1,\ 3x+7>0,\ 3x+7\neq 1,$$

$$\Rightarrow \quad x>-3/2,\ x\neq -1,\ x>-7/3,\ x\neq -2$$

$$6x^2+23x+21 = (2x+3)(3x+7) > 0,$$

$$\Rightarrow \quad x \in \left(-\infty, \frac{7}{3}\right)\cup\left(\frac{-3}{2}, \infty\right)$$

and $4x^2+12x+9 = (2x+3)^2 > 0 \Rightarrow x \neq \dfrac{-3}{2}$

The given equations can be re-written as

$$1 + \log_{(2x+3)}(3x+7) + 2\log_{(3x+7)}(2x+3) - 4 = 0$$

or $\quad z + \dfrac{2}{z} - 3 = 0 \quad \Rightarrow \quad z^2-3z+2=0 \quad \Rightarrow \quad z=1, 2$

(putting $\log_{(2x+3)}(3x+7) = z$).

If $z = \log_{(2x+3)}(3x+7) = 1$, then $3x+7 = 2x+3$ or $x=-4$,

If $z = \log_{(2x+3)}(3x+7) = 2$, $3x+7 = (2x+3)^2$

$\Rightarrow 4x^2+9x+2 = 0$ giving $x = -2, -1/4$.

Of these only $x=-1/4$ lies in the domain.

Hence $x=-1/4$ is the solution.

PROBLEM 11

Solve the equation $2^{|x+2|} - |2^{x+1}-1| = 2^{x+1}+1$.

SOLUTION

When $x+2=0 \quad \Rightarrow \quad x=-2;$

When $2^{x+1}-1=0$, $2^{x+1}=1 \quad \Rightarrow \quad x=-1$

CASE-1: $x \le -2$

When $x\le -2$, $x+2\le 0$ and $2^{x+1}-1<0$

$\therefore$ Given equation becomes

$$2^{-(x+2)} - [-(2^{x+1}-1)] = 2^{x+1}+1$$

or $\quad 2^{-(x+2)} + 2^{x+1} - 1 = 2^{x+1}+1$

or $\quad 2^{-(x+2)} = 2 = 2^1$

$\therefore \quad -(x+2) = 1$

$\therefore \quad x=-3$

CASE-2: $-2<x<-1$

When $-2<x<-1$, $x+2>0$ and $2^{x+1}-1<0$

$\therefore$ Given equation becomes

$$2^{(x+2)} - [-(2^{x+1}-1)] = 2^{x+1}+1$$

or $\quad 2^{x+2} + 2^{x+1} - 1 = 2^{x+1}+1$

or $\quad 2^{x+2} = 2$

$\Rightarrow \quad x+2 = 1 \Rightarrow x=-1$

which is not acceptable in this case

CASE-3: $x \ge -1$

When $x\ge -1$, $x+2>0$ and $2^{x+1}-1\ge 0$

$\therefore$ Given equation becomes

$$2^{x+2} - (2^{x+1}-1) = 2^{x+1}+1$$

or $\quad 2^{x+2} - 2^{x+1} = 2^{x+1}$

or $\quad 2^{x+2} - 2.2^{x+1} = 0$

or $\quad 2^{x+2} - 2^{x+2} = 0$ which is true for all real x

But in this case $x\ge -1$

$\therefore \quad x \ge -1$

Final solution set case (1) ∪ (2) ∪ (3)

Thus $x = -3$ or $x \geq -1$

$\therefore \quad x \in \{-3\} \cup [-1, \infty)$

PROBLEM 12

Solve the equation

$$\log_{x^2+6x+8} \log_{2x^2+2x+3} (x^2 - 2x) = 0$$

SOLUTION

Base of logarithm $> 0, \neq 1$

$\therefore \quad x^2 + 6x + 8 > 0$ and $2x^2 + 2x + 3 > 0$

$\therefore \quad (x+4)(x+2) > 0$ and $\left(x + \frac{1}{2}\right)^2 + \frac{5}{4} > 0$ (always true)

$\Rightarrow \quad x \in (-\infty, -4) \cup (-2, \infty) \qquad \ldots(1)$

and $\quad x^2 - 2x > 0$ or $x(x-2) > 0$

$\therefore \quad x \in (-\infty, 0) \cup (2, \infty) \qquad \ldots(2)$

Then the given equation reduced in the form

$$\log_{x^2+6x+8} \log_{2x^2+2x+3} (x^2 - 2x) = 0$$

$\Rightarrow \quad \log_{2x^2+2x+3} (x^2 - 2x) = (x^2 + 6x + 8)^0$

$\Rightarrow \quad \log_{2x^2+2x+3} (x^2 - 2x) = 1$

$\Rightarrow \quad x^2 - 2x = (2x^2 + 2x + 3)^1$

$\Rightarrow \quad x^2 + 4x + 3 = 0$

$\therefore \quad x = -1, -3$

$\because \quad x = -3$ is not satisfied (1) and satisfied (2)

$\therefore \quad x = -3$ is not a solution of the given equation.

But $x = -1$ satisfies both conditions (1) and (2)

$\therefore \quad x = -1$, is only solution of the given equation.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Number of integral values of x which satisfies the inequality $\frac{(x+6)^2(x+5)(x+1)^2}{(x-3)} \le 0$, is

(A) 10 (B) 6
(C) 8 (D) 9

2. If $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ if x is such that –

(A) $x < -4$
(B) $-3 < x < 3/2$
(C) $x > 5/2$
(D) All three correct

3. The equation $|x+1||x-1| = a^2 - 2a - 3$ can have real solutions for x, if a belongs to

(A) $(-\infty, -1] \cup [3, \infty)$

(B) $[1-\sqrt{5}, 1+\sqrt{5}]$

(C) $[1-\sqrt{5}, -1] \cup [3, 1+\sqrt{5}]$

(D) $[-1, 3]$

4. If $\frac{|x|-1}{|x|-2} \ge 0, x \in R$ then x does not belong to

(A) $(-\infty, -2)$ (B) $[-1, 1]$
(C) $(2, \infty)$ (D) $(1, 2)$

5. If S is the set of all real 'x' such that $\frac{x^2(5-x)(1-2x)}{(5x+1)(x+2)}$ is negative and $\frac{3x+1}{6x^3+x^2-x}$ is positive, then S contains

(A) $(1, 4)$ (B) $(5, 11)$
(C) $\left(\frac{-3}{2}, \frac{1}{2}\right)$ (D) $(-10, -4)$

6. If the solution of the inequality $1 < \frac{3x^2-7x+8}{x^2+1} \le 2$, is $[\alpha, \beta]$ then mark incorrect option

(A) α is least natural number
(B) $\beta - \alpha$ is a prime number
(C) $\beta + \alpha$ is a prime number
(D) $\beta + 3\alpha$ is a prime number

7. If $\log_7 \frac{2x-6}{2x-2} > 0$ then $x \in$

(A) $(-\infty, 1)$ (B) $[1, 2]$
(C) $(2, \infty)$ (D) $[0, 2)$

8. Solution set of inequality $\frac{(e^x-1)(2x-3)(x^2+x+2)}{(\sin x - 2)(x+1)^2 x} \le 0$ is

(A) $\left[\frac{3}{2}, \infty\right)$ (B) $(-\infty, -1) \cup \left[\frac{3}{2}, \infty\right)$

(C) $(-1, 0) \cup \left[\frac{3}{2}, \infty\right)$ (D) $R - \{0, 1\}$

9. The solution set of the inequality $\frac{3^x - 4^x}{x^2-3x-4} \ge 0$ is

(A) $(-\infty, 0]$ (B) $[0, \infty)$
(C) $(-\infty, -1) \cup [0, 4)$ (D) $(-\infty, 1) \cup [0, 4)$

10. Sum of all the values of x satisfying the equation $\log_{17}\log_{11}\left(\sqrt{x+11}+\sqrt{x}\right) = 0$ is

(A) 25 (B) 36
(C) 171 (D) 0

11. Number of integral values of x satisfying the inequality $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$ is

(A) 6 (B) 7
(C) 8 (D) infinite

12. The complete set of values of x for which the expression $y = \sqrt{\log_{\frac{1}{2}}\left(\frac{x-1}{x-2}\right)}$ is defined

(A) $(-\infty, 1) \cup (2, \infty)$ (B) $(-\infty, 2)$
(C) $(-\infty, 1)$ (D) ϕ

13. Number of integral solutions of the equation $\log_{x-3}\left(\log_{2x^2-2x+3}\left(x^2+2x\right)\right) = 0$ is

(A) 4 (B) 2
(C) 1 (D) 0

14. The complete solutions set of the inequality $\frac{3^x(2x-5)(x^2+x+2)}{(\cos x - 2)(x^2+x)} \le 0$ is

(A) $(-\infty, -1)$ (B) $\left(\frac{5}{2}, \infty\right)$

(C) $\left(-1, \frac{5}{2}\right]$ (D) $(-1, 0) \cup \left[\frac{5}{2}, \infty\right)$

15. Sum of all the roots of the equation $\log_7(2^x - 1) + \log_7(2^x - 7) = 1$, is

(A) 8 (B) 3
(C) 4 (D) 1

16. Number of natural numbers for which the number $\log_{4-x}(x^2 - 14x + 45)$ is defined is

(A) 4 (B) 3
(C) 2 (D) 1

17. The number of integral value of x is $5x - 1 < (x+1)^2 < 7x - 3$, is
(A) 0 (B) 1
(C) 2 (D) 3

18. The value of x if $\sqrt{x^2 + \sqrt{x^2 + \sqrt{x^2 + \ldots}}} = 9$, is
(A) $6\sqrt{2}$ (B) $3\sqrt{2}$
(C) $3\sqrt{6}$ (D) $2\sqrt{6}$

19. Number of rational roots of the equation $|x^2 - 2x - 3| + 4x = 0$ is
(A) 0 (B) 1
(C) 2 (D) 4

20. The solution set of the equation, $(x+1)\log_3^2 x + 4x\log_3 x - 16 = 0$ is :
(A) an empty set
(B) a singleton
(C) a set consisting of exactly two elements
(D) a set consisting of more than two elements.

21. Complete solution of inequality $||3x - 9| + 2| > 2$ is
(A) $(-\infty, \infty)$ (B) $\{3\}$
(C) $R - \{3\}$ (D) ϕ

22. Values of x satisfying the equality $|x^2 + 8x + 7| = |x^2 + 4x + 4| + |4x + 3|$ for $x \in R$ are
(A) $(-2, \infty)$ (B) $\left(-\infty, \frac{-3}{4}\right]$
(C) $\left[\frac{-3}{4}, \infty\right) \cup \{-2\}$ (D) None of these

23. Which of the following when simplified does not reduce to unity ?
(A) $\log_{\frac{3}{2}} \log_4 \log_{\sqrt{3}} 81$ (B) $\log_2 6 + \log_2 \sqrt{\frac{2}{3}}$
(C) $-\frac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right)$ (D) $\log_{\frac{\sqrt{3}}{2}}\left(\frac{\sqrt{3}}{2}\right)$

24. Value of $\left[\frac{2}{\log_4(2000)^6} + \frac{3}{\log_5(2000)^6}\right]$ is
(A) $4^{\frac{1}{3}}$ (B) $\frac{1}{6}$
(C) $\sqrt[3]{4}$ (D) None of these

25. If $\log_{10}\frac{1025}{1024} = \alpha$ and $\log_{10} 2 = \beta$, then the value of $\log_{10} 4100$ in terms of α and β is equal to
(A) $\alpha + 9\beta$
(B) $\alpha + 12\beta$
(C) $12\alpha + \beta$
(D) $9\alpha + \beta$

26. If $\frac{1}{3}\log_3 M + 3\log_3 N = 1 + \log_{0.008} 5$, then
(A) $M^9 = \frac{9}{N}$ (B) $N^9 = \frac{9}{M}$
(C) $M^3 = \frac{3}{N}$ (D) $N^9 = \frac{3}{M}$

27. If $\ell n^2 x + 3\ \ell n\ x - 4$ is non negative, then x must lie in the interval :
(A) $[e, \infty)$ (B) $(-\infty, e^{-4}) \cup [e, \infty)$
(C) $(1/e, e)$ (D) None of these

28. The value of $\frac{6a^{\ell n b}\log_{a^2} b.\log_{b^2} a}{e^{\ell na.\ell nb}}$
(A) Independent of a (B) dependent of b
(C) depends on a (D) none

29. Let $N = \frac{81^{\frac{1}{\log_5 9}} + 3^{\frac{1}{\log_{\sqrt{6}} 3}}}{409}\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}} - 125^{\log_{25} 6}\right)$. Which of the following statements(s) is/are incorrect?
(A) N is prime
(B) N is rational
(C) N is coprime with every other natural number
(D) $\log_2 N$ is an even number

30. The number of integral value(s) of x satisfying the equation $|x^4.3^{|x-2|}.5^{x-1}| = -x^4.3^{|x-2|}.5^{x-1}$ is
(A) 2 (B) 3
(C) 1 (D) infinite

31. For the equation $\log_{3\sqrt{x}} x + \log_{3x}\sqrt{x} = 0$, which of the following holds good?
(A) no real solution
(B) one prime solution
(C) one integral solution
(D) no irrational solution

32. The solution of the equation $7^{\log x} - 5^{\log x + 1} = 3 \cdot 5^{\log x - 1} - 13 \cdot 7^{\log x - 1}$ is k. then mark incorrect information about k
(A) an even number (B) rational number
(C) irrational number (D) composite number

33. If $\left|\frac{12x}{4x^2 + 9}\right| \leq 1$ then
(A) $x \in R$
(B) $x \in \phi$
(C) $x \in \{1\}$
(D) $x \in$ C where C is set of complex numbers

34. The value of x satisfying the equation $2^{\log_2 e^{\ell n 5^{\log_5 7^{\log_7 10^{\log_{10}(8x-3)}}}}} = 13$, is
(A) 2 (B) 3
(C) 4 (D) 7

35. Let $N = 10^{3\log 2 - 2\log(\log 10^3) + \log((\log 10^6)^2) + \log((\log 10^6)^2)}$ where base of the logarithm is 10. The characteristic of the logarithm of N to the base 3, is equal to
(A) 2 (B) 3
(C) 4 (D) 5

36. The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has
(A) no irrational solutions
(B) no composite solutions
(C) two real solutions
(D) one integral solutions

37. Number of positive integers which have characteristic 2 when base is 5, is equal to
(A) 25 (B) 32
(C) 100 (D) 101

38. The value of x satisfying the equation $4^{\log_9 3}+9^{\log_2 4}=10^{\log_x 83}$ is equal to
(A) 9 (B) 4
(C) 10 (D) 100

39. Suppose that x is real number such that $\frac{(27)(9^x)}{4^x}=\frac{3^x}{8^x}$, then the value of $(2)^{-(1+\log_2 3)x}$, is
(A) 6 (B) 9
(C) 18 (D) 27

40. Number of values of x in the interval (0, 5) satisfying the equation $\frac{\ell n\left(\sqrt{\sqrt{x^2+1}+x}\right)+\ell n\left(\sqrt{\sqrt{x^2+1}-x}\right)}{\ell nx}=x$, is
(A) 1 (B) 2
(C) 3 (D) 0

41. The number of value(s) of x satisfying the equation $4^{\log_2(\ell nx)}+\ell n^3 x-3\ell n^2 x-5\ell nx+7=0$
(A) 0 (B) 1
(C) 2 (D) 3

42. The number of value(s) of x satisfying the equation $\frac{1}{\log_{3\sqrt{2}} x}+\frac{1}{\log_{4\sqrt{3}} x}+\frac{1}{\log_{2\sqrt{6}} x}=2$ is/are
(A) 0 (B) 1
(C) 2 (D) 3

43. If $\log_6 16=k$, then $\log_{18} 24=$
(A) $\frac{2(k+2)}{k-8}$ (B) $\frac{2(k+2)}{8-k}$
(C) $\frac{k+8}{k}$ (D) $\frac{k}{k+8}$

44. The number of zeroes after decimal before the start of any significant digit in the number $N=(0.18)^{20}$ are
(A) 15 (B) 14
(C) 13 (D) 12

45. The product of all the real solution of the equation $2\log_9(x-1)=2+\log^2_{(x-1)}3-\log_{\sqrt{3}}(x-1)$ is
(A) 3 (B) 4
(C) 40 (D) $\frac{4}{3}$

46. The value of x satisfying the equation $\left(\sqrt{5}\right)^{\log_5 5^{\log_5 5^{\log_5(x)}}}=2$ is
(A) 1 (B) 2
(C) 4 (D) 8

47. If $A=\log_{\sqrt{5}}\left(\left(\left(5^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}\cdot^{\cdot^{\cdot}}\text{10 times}}\right)$, then value of $\log_{\sqrt{3}}(1024A+1)$, is equal to
(A) 1 (B) 3
(C) 5 (D) 2

48. The value of x satisfying the equation $3^{\log_{\sqrt{3}}\sqrt{x}}+9^{\log_3 x}+27^{\log_3 2x^2}=3$ is
(A) a natural number
(B) a prime number
(C) a rational number which is not integer
(D) an irrational number

49. The value of the expression $5^{\frac{\log_3(\log_2 81)}{\log_3 5}}$ always lies between.
(A) 5 and 6 (B) 7 and 8
(C) 6 and 7 (D) 8 and 9

50. The sum of all the solution of the equation $7^{3x^2}\cdot 5^x=11$, is
(A) $\ell n\,5$ (B) $\log_5 7$
(C) $\frac{\ell n(1/5)}{\ell n7}$ (D) None

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. The possible value of x, satisfying the equation $\log_2(x^2-x)\times\log_2\left(\frac{x-1}{x}\right)+(\log_2 x)^2=4$, is (are)
(A) $\frac{5}{4}$ (B) 5
(C) $\frac{25}{4}$ (D) $\frac{15}{4}$

2. Let L be the number of digits in 3^{40} before decimal and M be the number of zeroes in 3^{-40} after decimal before significant digit, then
(A) $L+M=39$ (B) $L-M=2$
(C) $L-M=1$ (D) $L+M=38$

3. If $x^{\frac{3}{4}(\log_3 x)^2+(\log_3 x)-\frac{5}{4}}=\sqrt{3}$, then x has
(A) one integral solutions
(B) two rational solutions
(C) two irrational solutions
(D) no prime solution

4. Which of the following statement(s) is (are) correct
(A) $\log(1+2+3)=\log 1+\log 2+\log 3$
(B) $\frac{\log_9 11}{\log_5 13}\div\frac{\log_3 11}{\log_{\sqrt{5}} 13}$ equals 1.
(C) The value of $\log_2 7$ is a rational number which is not an integer.
(D) The equation $\log_2 x+\log_4(x+2)=2$ has two real solutions.

5. The number $N = \sqrt{10^{2+\left(\frac{1}{2}\right)\log 16}}$ (where base of logarithm is 10)
(A) is coprime with 9
(B) is a simple surd
(C) is an odd composite
(D) forms the sides of a right triangle with 99 and 101.

6. Which of the following numbers are non-positive?
(A) $5^{\sqrt{\log_5 4}} - 4^{\sqrt{\log_4 5}}$
(B) $\log_{\cot^2 \frac{\pi}{6}}\left(\sqrt{27}-\sqrt{12}\right)$
(C) $\log_{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$
(D) $\log_{\sec^2 \frac{\pi}{3}}\left(\operatorname{cosec}\frac{\pi}{6}\right)$

7. The equation $\frac{\log_{27}\left(\frac{9}{x}\right)}{(\log_3 x)^2} = 2$ has
(A) no integral solution
(B) two real solution
(C) no real solution
(D) two natural solution

8. The equation $(\log_{10}x + 2)^3 + (\log_{10}x - 1)^3 = (2\log_{10}x + 1)^3$ has
(A) no natural solution
(B) two rational solutions
(C) no prime solution
(D) one irrational solution

9. If $N = (\log_2 125 + 1)(\log_2 4 + \log_4 625) - (\log_2 25)(\log_4(15625) + 4)$ then N is equal to
(A) prime number
(B) irrational number
(C) rational number which is not natural number
(D) natural number which is coprime with 5

10. Let $\log_M N = \alpha + \beta$, where α is an integer and β is non negative fraction, If M and α are prime numbers and $\alpha + M = 7$ then $N \in [a, b)$. The absolute value of $(b - 5a)$ can be
(A) 0
(B) 24
(C) 48
(D) 96

11. If $\log_{\sqrt[n]{2}}\sqrt[n]{x} + \log_{\sqrt[n-1]{2}}\sqrt[n-1]{x} + \ldots + \log_2 x = 8$ then x can be equal to (x and n are both integers)
(A) 2
(B) 4
(C) 16
(D) 256

12. Let $a = (\log_3\pi)(\log_2 3)(\log_\pi 2)$, $b = \frac{\log 576}{3\log 2 + \log 3}$ the base of the logarithm being 10, $c = 2$ (sum of the solution of the equation $(3)^{4x} - (3)^{(2x+\log_3(12))} + 27 = 0$ and $d = 7^{(\log_7 2 + \log_7 3)}$ then $\left(a + b + \frac{c}{d}\right)$ simplifies to
(A) rational which is not natural
(B) natural but not prime
(C) irrational
(D) even but not composite

13. The value of the expression $\frac{1}{\log_4(18)} + \frac{1}{2\log_6(3) + \log_6(2)} + \frac{5}{\log_3(18)}$, is
(A) odd
(B) an irrational
(C) even composite
(D) twin prime with 5

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Consider the equation

$$\sqrt{x^2+2x+1} - \sqrt{4x^2+4x+1} + \sqrt{4x^2} = 2$$

1. The number of integral solution(s) of equation is/are
(A) 0
(B) 2
(C) 3
(D) 4

2. The sum of all the solutions of the equation is
(A) 0
(B) $\frac{-2}{3}$
(C) $\frac{2}{3}$
(D) 1

3. If α be the largest integral value of x satisfying the given equation then the number of digits in α^{50} is
(A) 15
(B) 16
(C) 23
(D) 24

Comprehension 2

Let α and β are the solutions of the equation $\left(\sqrt{x}\right)^{-1+\log_5 x} = 5$ where $\alpha \in$ I and $\beta \in$ Q.

[Use : $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$]

4. The number of significant digits before decimal in $(\alpha)^{10}$ is
(A) 13
(B) 14
(C) 15
(D) 16

5. Number of zeroes after decimal before a significant digits in $(\beta)^{10}$ is
(A) 5
(B) 7
(C) 8
(D) 6

6. The value of $(\beta)^{\log_{25} 9}$ is
(A) $\frac{1}{3}$
(B) 5
(C) $\frac{1}{5}$
(D) 9

Comprehension 3

Let $f(x) = \log_{\left(\frac{25-x^2}{16}\right)}\left(\frac{24-2x-x^2}{14}\right)$

On the basis of above information, answer the following:

7. The values of x for which the $\sqrt{f(x)} \times \log_{\sec^2(8.5)} \frac{25-x^2}{16} > 0$

   (A) $(-3, 3)$ (B) $(-1-\sqrt{11}, 3)$
   (C) $(-3, -1+\sqrt{11})$ (D) $(-5-\sqrt{11}, -1+\sqrt{11})$

8. If $\left(\frac{25-x^2}{16}\right) \in (0, 1)$ then values of x for which $f(x) > 1$ will be

   (A) $(3, 4)$ (B) $(-5, -3)$
   (C) $(3, 5)$ (D) $(-6, -3) \cup (3, 4)$

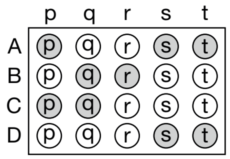
9. If $\left(\frac{25-x^2}{16}\right) > 1$ then values of x for which $f(x) > 1$ will be

   (A) $(-3, 0)$ (B) $(-3, 1)$
   (C) $(-3, 2)$ (D) $(-3, 3)$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following

COLUMN-I	COLUMN-II
(A) If $\log_4(x+1) + \log_4(x+8) = \frac{3}{2}$, then value(s) of x is (are)	(p) 1
(B) If $\ln x = 0$ then x is equal to	(q) 4
(C) The value of $4\left(3\log_2\frac{81}{80} + 5\log_2\frac{25}{24} + 7\log_2\frac{16}{15}\right)$ is	(r) 0
(D) The remainder when $2x^5 - x^3 + x^2 + 1$ is divided by $(2x+1)$ is k. Then $\frac{16k+11}{16}$ is equal to	(s) 2

2. Let $f(x) = \frac{x^2-11x+18}{x^2-7x+12}$

COLUMN-I	COLUMN-II
(A) Complete solution of $0 < f(x) < 1$	(p) $(3, 4)$
(B) Complete solution of $f(x) < 1$	(q) $\left(\frac{3}{2}, 2\right) \cup (9, \infty)$
(C) Complete solution of $f(x) < 0$	(r) $(2, 3) \cup (4, 9)$
(D) Complete solution of $f(x) > 2$	(s) $(-\infty, 2) \cup (3, 4) \cup (9, \infty)$
	(t) $\left(\frac{3}{2}, 3\right) \cup (4, \infty)$

3.

COLUMN-I	COLUMN-II
(A) The number of solution(s) of the equation $\log_4(x-1) = \log_2(x-3)$ is	(p) 0
(B) the smallest integer greater than $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}$ is coprime with	(q) 1
(C) The number of solution(s) of the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = \frac{1}{x^2}$ is	(r) 2
(D) The number of solution(s) of the equation $\log_3\log_5\left(\sqrt{x+5} - \sqrt{x}\right) = 0$, is	(s) 3
	(t) 4

4.

COLUMN-I	COLUMN-II
(A) The expression $x = \log_2\log_9\sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}$ simplifies to	(p) an integer
(B) The expression $N = 2^{(\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \ldots \cdot \log_{99} 100)}$ simplifies to	(q) a prime
(C) The expression $\frac{1}{\log_5 3} + \frac{1}{\log_6 3} - \frac{1}{\log_{10} 3}$ simplifies to	(r) a natural
(D) The number $N = \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{14-6\sqrt{5}}}}$ simplifies to	(s) a composite

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If $x = \prod_{n=1}^{2000} n$, then the value of the expression, $\dfrac{1}{\dfrac{1}{\log_2 x}+\dfrac{1}{\log_3 x}+\ldots\ldots+\dfrac{1}{\log_{2000} x}}$ is ______

2. The value of $a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}}$ (where $a, b > 0$ and $a, b \neq 1$) is equal to

3. Real x satisfying the equation $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$ is ______.

4. Find the product of all the solution(s) of the equation $\dfrac{3}{4}(\log_x 3) - 3(\log_{27} x) = 2(\log_3 x)$.

5. Number of solutions of the equation $\log_3\left(3+\sqrt{x}\right)+\dfrac{1}{\log_{\left(1+x^2\right)} 3} = 1$ is ________

6. The sum of all the real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is ______

7. Number of values of x satisfy the equation (is/are) $|x^2+4x+3| + 2x + 5 = 0$

8. If λ is the minimum value of the expression $|x-p| + |x-15| + |x-p-15|$ for x in the range $p \le x \le 15$ where $0 < p < 15$. Then $\dfrac{\lambda}{5} =$

9. Find number of solution of $|x-4| + 3|x+6| - 2x = 5$

10. Number of negative integer solution the equation : $|x+1| - |x| + 3\,|x-1| - 2\,|x-2| = x+2$

11. Find number of solution of $|x-6| + |x+6| < 12$

12. Find number of single digit positive integers satisfying $|x-5| + |x+5| = 2|x|$

13. Find number of single digit negative integers satisfying $||x-6| - |x+6|| = 12$

14. Find number of positive integers satisfying $|x^2-3x+2| + |x^2+7x+12| = 10|x+1|$

15. Find number of positive integers satisfying $|x^2-1| + |x^2-4| \le 3$

16. Find number of negative integers satisfying $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

17. Find number of single digit positive integers satisfying $\left|\dfrac{3x}{x^2-4}\right| \ge 1$

18. Find number of single digit prime numbers satisfying $\dfrac{(x-2)^{100}(2+x)^{101}(x-1)^{10}}{x^{41}(x+1)^{39}(4-x)^{11}} \le 0$

19. Let x, y, z be positive real number such that $x^{\log_2 7} = 8$; $y^{\log_3 5} = 9$ and $z^{\log_2 216} = 5^{1/3}$ then find the value of $x^{(\log_2 7)^2} + y^{(\log_3 5)^2} + z^{(\log_2 216)^2}$.

20. Find the sum of all values of x satisfying the equation $\log_4(x-1) = \log_2(x-3)$.

21. If p denotes the product of the two values of x satisfying the equation $\log_{2x}(2) + \log_4(2x) = \dfrac{-3}{2}$ then find the value of $\log_2\left(\dfrac{1}{p}\right)$.

22. If x_1 and x_2 are the two solution of the equation $3^{\log_2 x} - 12x^{\log_{16} 9} + 27 = 0$, then find the value of $x_1^2 + x_2^2$

23. Find the value of $\dfrac{\log_5 9\cdot\log_7 5\cdot\log_3 7}{\log_2\sqrt{6}} + \dfrac{1}{\log_9\sqrt{6}}$

24. If $A = \dfrac{(\log_3 1 - \log_3 4)(\log_3 9 - \log_3 2)}{(\log_3 1 - \log_3 9)(\log_3 8 - \log_3 4)}$, then find the value of $2(3^A)$.

25. Find the modulus of the sum of all solutions to $\left(x^2+3x+1\right)^{x^2+2x-8} = 1$.

26. Find the sum of all integral values of x satisfying $(\log_5 x)^2 + \log_{5x}\left(\dfrac{5}{x}\right) = 1$.

ARCHIVE: JEE MAIN

1. **[JEE Mains 2020]**
   Let S be the set of all real roots of the equation, $3^x(3^x-1)+2 = |3^x-1| + |3^x-2|$. Then S:
   (A) contains at least four elements.
   (B) is an empty set.
   (C) contains exactly two elements.
   (D) is a singleton.

2. **[JEE Mains 2020]**
   The value of $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\text{.to }\infty\right)}$ is equal to ________.

3. **[JEE Mains 2016]**
   The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60} = 1$ is
   (A) –4 (B) 6
   (C) 5 (D) 3

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problem

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. **[IIT 2011]**
Let (x_0, y_0) be the solution of the following equations $(2x)^{\ell n\,2} = (3y)^{\ell n\,3}$ and $3^{\ell n\,x} = 2^{\ell n\,y}$ then x_0 is
(A) $\frac{1}{6}$ (B) $\frac{1}{3}$
(C) $\frac{1}{2}$ (D) 6

2. **[IITSc.–2002]**
The set of all real numbers x for which $x^2 - |x+2| + x > 0$, is
(A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(C) $(-\infty, -1) \cup (1, \infty)$ (D) $(\sqrt{2}, \infty)$

3. **[IIT–2001]**
Find the no. of solution of $\log_4(x-1) = \log_2(x-3)$
(A) 3 (B) 1
(C) 2 (D) 0

4. **[IIT JEE 90]**
The number $\log_2 7$ is :
(A) an integer (B) a rational number
(C) an irrational number (D) a prime number

Multiple Correct Choice Type Problem

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[IIT JEE 2013]**
If $3^x = 4^{x-1}$, then $x =$
(A) $\frac{2\log_3 2}{2\log_3 2 - 1}$ (B) $\frac{2}{2 - \log_2 3}$
(C) $\frac{1}{1 - \log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3 - 1}$

2. **[IIT JEE 89]**
Indicate all correct alternatives, where base of the log is 2.
The equation $x^{(3/4)(\log x)^2 + \log x - (5/4)} = \sqrt{2}$ has
(A) at least one real solution
(B) exactly three real solutions
(C) exactly one irrational solution
(D) complex roots

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. **[IIT JEE 2007]**

Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

COLUMN-I	COLUMN-II
(A) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$

Integer/Numerical Answer Type Question

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[JEE Advanced 2018]**

The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is ________.

2. **[IIT JEE 2012]**
The value of

$$6 + \log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\cdots}}}\right)$$ is

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. D	**2.** D	**3.** A	**4.** D	**5.** A	**6.** D	**7.** A	**8.** A	**9.** C	**10.** A
11. B	**12.** C	**13.** D	**14.** D	**15.** B	**16.** C	**17.** B	**18.** A	**19.** B	**20.** C
21. C	**22.** C	**23.** B	**24.** B	**25.** B	**26.** B	**27.** D	**28.** A	**29.** B	**30.** C
31. C	**32.** C	**33.** A	**34.** A	**35.** B	**36.** C	**37.** C	**38.** C	**39.** D	**40.** D
41. C	**42.** B	**43.** B	**44.** B	**45.** B	**46.** C	**47.** D	**48.** A	**49.** C	**50.** D

Multiple Correct Choice Type Questions

1. A, B	**2.** A, C	**3.** A, B	**4.** A, B	**5.** A, D
6. A, C	**7.** A, B	**8.** B, C, D	**9.** A, D	**10.** A, D
11. A, B, C, D	**12.** A	**13.** A, D		

Linked Comprehension Type Questions

1. B	**2.** A	**3.** B	**4.** B	**5.** D	**6.** A	**7.** C	**8.** C	**9.** B

Matrix Match/Column Match Type Questions

1. A → (r)	B → (p)	C → (q)	D → (s)
2. A → (q)	B → (t)	C → (r)	D → (p)
3. A → (q)	B → (q, r, t)	C → (s)	D → (p)
4. A → (p)	B → (p, r, s)	C → (p, r)	D → (p, q, r)

Integer/Numerical Answer Type Questions

1. 1	**2.** 0	**3.** 2	**4.** 1	**5.** 0	**6.** 4	**7.** 2	**8.** 3	**9.** 0	**10.** 1
11. 0	**12.** 5	**13.** 4	**14.** 2	**15.** 2	**16.** 0	**17.** 3	**18.** 3	**19.** 374	**20.** 5
21. 5	**22.** 272	**23.** 4	**24.** 9	**25.** 7	**26.** 6				

ARCHIVE: JEE MAIN

1. D	**2.** 4	**3.** D

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. C	**2.** B	**3.** B	**4.** C

Multiple Correct Choice Type Problems

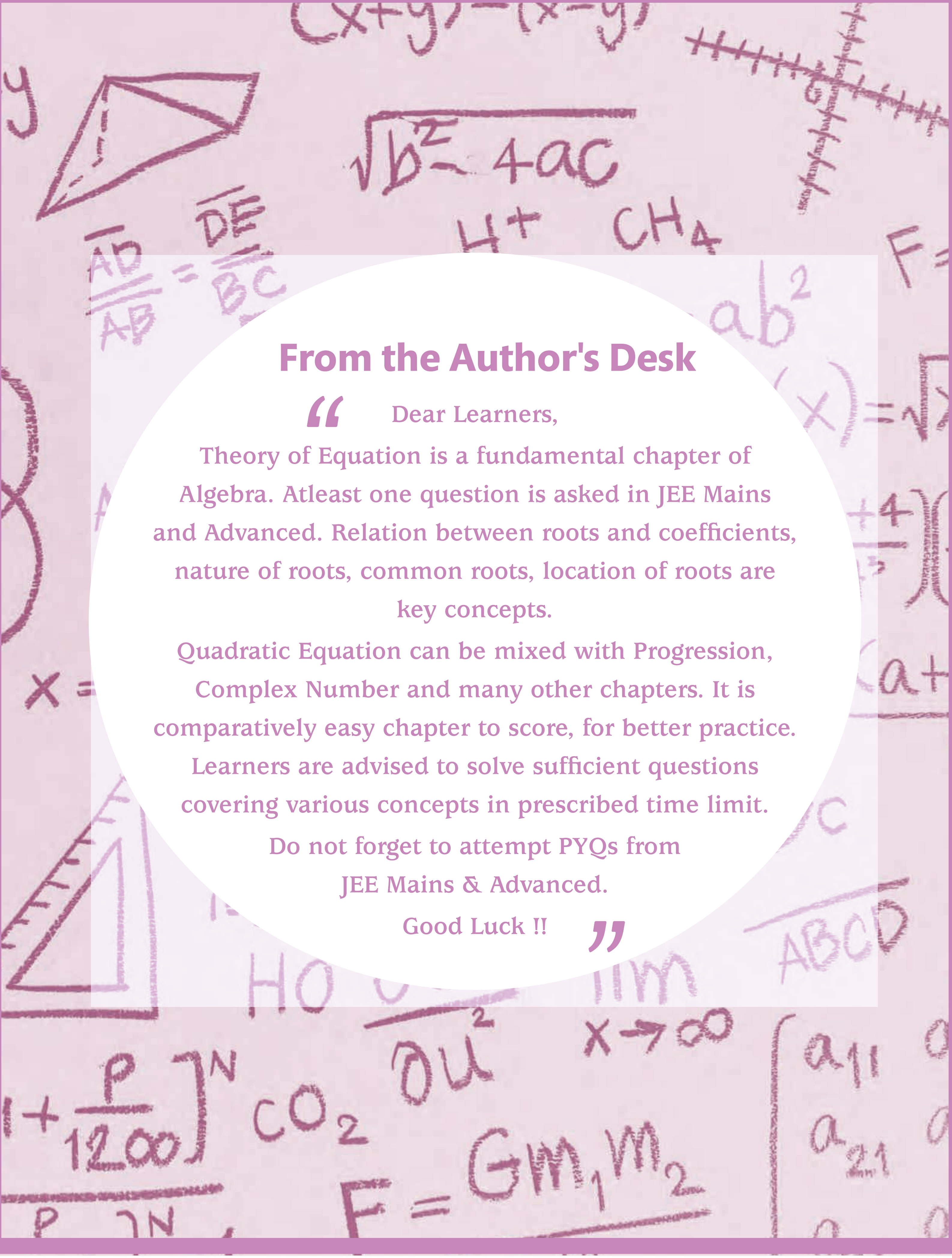
1. A, B, C	**2.** A, B, C, D

Matrix Match/Column Match Type Questions

1. A → (p, r, s)	B → (q, s)	C → (q, s)	D → (p, r, s)

Integer/Numerical Answer Type Questions

1. 8	**2.** 4



From the Author's Desk

> "Dear Learners,
>
> Theory of Equation is a fundamental chapter of Algebra. Atleast one question is asked in JEE Mains and Advanced. Relation between roots and coefficients, nature of roots, common roots, location of roots are key concepts.
>
> Quadratic Equation can be mixed with Progression, Complex Number and many other chapters. It is comparatively easy chapter to score, for better practice. Learners are advised to solve sufficient questions covering various concepts in prescribed time limit. Do not forget to attempt PYQs from JEE Mains & Advanced.
>
> Good Luck !!"

CHAPTER

2 Theory of Equation

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Relation between roots and coefficient
(b) Equation formation with modified roots
(c) Graph of quadratic expression
(d) Range of quadratic expression
(e) Common roots problems
(f) Location of roots

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

POLYNOMIAL FUNCTION

If $a_n \neq 0$ then $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n ($n \in W$) and $a_0, a_1, a_2, \ldots a_n$ are coefficients of polynomials. If coefficients are real ($a_0, a_1, a_2, \ldots, a_n \in R$) then it is known as **real polynomials**.

If coefficients are non-real ($a_0, a_1, a_2, \ldots, a_n \in C$) then it is known as **complex polynomial function**.

$ax^3 + bx^2 + cx + d$ is a cubic polynomial if $a \neq 0$.

$ax^4 + bx^3 + cx^2 + dx + e$ is degree 4 polynomial if $a \neq 0$.

$4x^8 + 4x^6 - 2x^2 + 3x - 7$ is real polynomial of degree 8.

$2x^6 + 4ix^4 - (2 - i)x^2 + (3 + i)x - 7$ is complex polynomial of degree 6 where $i = \sqrt{-1}$

RATIONAL FUNCTION/EXPRESSION

Ratio of two polynomials in x, $\dfrac{P(x)}{Q(x)}$ is known as rational function.

$$\frac{x^2 - 3x + 5}{x + 2}, \frac{x^4 + x^2 - 6}{3x + 2}, \frac{7x + 5}{x^3 - 4x^2 + 5}$$

Remainder Theorem

If a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is divided by $(x - \alpha)$ then remainder is $P(\alpha)$.

Explanation:

Let when $P(x)$ is divided by $(x - \alpha)$, we get quotient $Q(x)$ and remainder as R then we can rewrite it as $P(x) = Q(x)(x - \alpha) + R$

Put $x = \alpha$, we get $P(\alpha) = R$, hence remainder is $P(\alpha)$

If a polynomial $P(x)$ of degree n is divided by quadratic expression $ax^2 + bx + c$ then remainder can be as $\alpha x + \beta$.

If a polynomial $P(x)$ of degree n is divided by cubic $ax^3 + bx^2 + cx + d$ then remainder can be as $\alpha x^2 + \beta x + \gamma$.

If a polynomial $P(x)$ of degree n is divided by degree 4 polynomial $ax^4 + bx^3 + cx^2 + d$ then remainder can be as $\alpha x^3 + \beta x^2 + \gamma x + \delta$.

If a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is divided by polynomial of degree m, $A(x) = ax^m + bx^{m-1} + \ldots + cx + d$ then remainder is of form polynomial of degree $(m - 1)$ as

$$R(x) = \alpha x^{m-1} + \beta x^{m-2} + \ldots + \gamma x + \delta.$$

Factor Theorem

If $(x - \alpha)$ is a factor of polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where $n \in N$ then $P(\alpha) = 0$.

ILLUSTRATION 1

If $x - 2$ and x are two factors of the polynomial $f(x) = 2x^3 - 5x^2 + ax + b$. Find the values of a and b.

SOLUTION

$$f(2) = 2(2)^3 - 5(2)^2 + a(2) + b = 0$$

$$\Rightarrow \quad 16 - 20 + 2a + b = 0 \quad \Rightarrow \quad 2a + b = 4 \qquad \ldots(1)$$

$$f(0) = 2(0)^3 - 5(0)2 + a(0) + b = 0 \quad \Rightarrow \quad b = 0$$

Put $b = 0$ in eq. (1)

$$\Rightarrow \quad 2a + 0 = 4$$

So, $2a = 4 \quad \Rightarrow \quad a = 2.$

ILLUSTRATION 2

Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

SOLUTION

Here, $p(x) = x^4 + x^3 - 2x^2 + x + 1$, and the zero of $x - 1$ is 1.

So, $p(1) = (1)^4 + (1)^3 - 2(1)^2 + 1 + 1 = 2$

So, by the Remainder Theorem, 2 is the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

ILLUSTRATION 3

Find the remainder when the polynomial $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$ is divided by $x + 2$.

SOLUTION

We have, $x + 2 = x - (-2)$. So, by remainder theorem, when $f(x)$ is divided by $(x-(-2))$ the remainder is equal to $f(-2)$.

Now, $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$

$$\Rightarrow \quad f(-2) = 2\,(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$$

$$\Rightarrow \quad f(-2) = 2 \times 16 - 6 \times (-8) + 2 \times 4 + 2 + 2$$

$$\Rightarrow \quad f(-2) = 32 + 48 + 8 + 2 + 2 = 92$$

Hence, required remainder = 92.

ILLUSTRATION 4

If $ax^3 + bx^2 + x - 6$ has $x + 2$ as a factor and leaves a remainder 4 when divided by $(x - 2)$, find the values of a and b.

SOLUTION

Let $p(x) = ax^3 + bx^2 + x - 6$ be the given polynomial. Then, $(x + 2)$ is a factor of $p(x)$

$$\Rightarrow \quad p(-2) = 0 \qquad [x + 2 = 0 \Rightarrow x = -2]$$

$$\Rightarrow \quad a(-2)^3 + b(-2)^2 + (-2) - 6 = 0$$

$$\Rightarrow \quad -8a + 4b = 8 \quad \Rightarrow \quad -2a + b = 2 \qquad \ldots(1)$$

It is given that $p(x)$ leaves the remainder 4 when it is divided by $(x - 2)$. Therefore,

$$p(2) = 4 \qquad [x - 2 = 0 \Rightarrow x = 2]$$

$$\Rightarrow \quad a(2)^3 + b(2)^2 + 2 - 6 = 4$$

$$\Rightarrow \quad 8a + 4b - 4 = 4 \Rightarrow 8a + 4b = 8 \Rightarrow 2a + b = 2 \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2b = 4 \quad \Rightarrow \quad b = 2$$

Putting $b = 2$ in (1), we get

$$-2a + 2 = 2 \quad \Rightarrow \quad -2a = 0 \quad \Rightarrow \quad a = 0.$$

Hence, $a = 0$ and $b = 2$.

ILLUSTRATION 5

If $x^2 - 1$ is a factor of $ax^4 + bx^3 + cx^2 + dx + e$, show that $a + c + e = b + d = 0$.

SOLUTION

Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ be the given polynomial. Then, $(x^2 - 1)$ is a factor of $p(x)$

$\Rightarrow \quad (x - 1)\,(x + 1)$ is a factor of $p(x)$

$\Rightarrow \quad (x - 1)$ and $(x + 1)$ are factors of $p(x)$

$\Rightarrow \quad p(1) = 0$ and $p(-1) = 0$

$[x - 1 = 0 \Rightarrow x = 1$ and $x + 1 = 0 \Rightarrow x = -1]$

$\Rightarrow \quad a + b + c + d + e = 0$ and $a - b + c - d + e = 0$

Adding and subtracting these two equations, we get $2(a + c + e) = 0$ and $2(b + d) = 0$

$\Rightarrow \quad a + c + e = 0$ and $b + d = 0$

$$\Rightarrow \quad a + c + e = b + d = 0$$

ILLUSTRATION 6

A polynomial of degree $n > 3$ leaves remainders 2, 1, –1 when divided by $(x - 1)$, $(x + 2)$ and $(x + 1)$ respectively. What will be the remainder if the polynomial is divided by $(x^2 - 1)\,(x + 2)$?

SOLUTION

Since $(x^2 - 1)\,(x + 2)$ is a cubic, the remainder when $f(x)$ is divided by $(x^2 - 1)\,(x + 2)$ should be of the form $ax^2 + bx + c$.

Required polynomial $f(x) = (x^2 - 1)\,(x + 2)\,Q(x) + ax^2 + bx + c$.

Using remainder theorem

Putting $x = 1, -2, -1$ respectively, we get

$$f(1) = a + b + c = 2$$

$$f(-2) = 4a - 2b + c = 1$$

$$f(-1) = a - b + c = -1$$

Solving these, $a = \dfrac{7}{6},\ b = \dfrac{3}{2},\ c = -\dfrac{2}{3}$

$\therefore$ The required remainder is $\dfrac{7}{6}x^2 + \dfrac{3}{2}x - \dfrac{2}{3}$

ILLUSTRATION 7

If $P(x) = x^2 + bx + c$ and $P(x)$ is factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$ then find $P(1)$.

SOLUTION

Since $P(x)$ divides both, hence $P(x)$ divides

$$\left(3x^4+4x^2+28x+5\right)-3\left(x^4+6x^2+25\right)$$

$$=-14x^2+28x-70$$

$$=-14\left(x^2-2x+5\right) \Rightarrow P(x)=x^2-2x+5$$

$\therefore \quad P(1)=4$

ILLUSTRATION 8

Let R_1 and R_2 are the remainders when the polynomials $x^3+2x^2-5ax-7$ and $x^3+ax^2-12x+6$ are divided by $x+1$ and $x-2$ respectively. If $2R_1+R_2=6$, find the value of a.

SOLUTION

Let $p(x)=x^3+2x^2-5ax-7$ and $q(x)=x^3+ax^2-12x+6$ be the given polynomials.

Now, R_1 = Remainder when $p(x)$ is divided by $x+1$

$\Rightarrow \quad R_1=p(-1)=(-1)^3+2(-1)^2-5a(-1)-7$
$[\because p(x)=x^3+2x^2-5ax-7]$

$\Rightarrow \quad R_1=-1+2+5a-7=5a-6$

And, R_2 = Remainder when $q(x)$ is divided by $x-2$

$\Rightarrow \quad R_2=q(2)=(2)^3+a\times 2^2-12\times 2+6$
$[\because q(x)=x^3+ax^2-12x+6]$

$\Rightarrow \quad R_2=8+4a-24+6=4a-10$

Substituting the values of R_1 and R_2 in

$2R_1+R_2=6$, we get $2(5a-6)+(4a-10)=6$

$\Rightarrow \quad 10a-12+4a-10=6 \quad \Rightarrow \quad 14a-22=6$

$\Rightarrow \quad 14a=28 \quad \Rightarrow \quad a=2$

ILLUSTRATION 9

If $f(x)$ is a polynomial of degree four with leading coefficient one satisfying $f(1)=1$, $f(2)=2$, $f(3)=3$, then find $\left[\dfrac{f(-1)+f(5)}{f(0)+f(4)}\right]$ (where [.] represents greatest integer function).

SOLUTION

According to question

$$f(x)-x=(x-1)(x-2)(x-3)(x-\alpha)$$

$$\Rightarrow \quad f(-1)=24(1+\alpha)-1$$

$$f(0)=6\alpha$$

$$f(4)=6(4-\alpha)+4$$

$$f(5)=24(5-\alpha)+5$$

$$\Rightarrow \quad \left[\frac{f(-1)+f(5)}{f(0)+f(4)}\right]=\left[\frac{148}{28}\right]=5$$

ILLUSTRATION 10

Let $P(x)$ be a polynomial with degree 2009 and leading coefficient unity such that $P(0)=2008$, $P(1)=2007$, $P(2)=2006$, …, $P(2008)=0$ then find the value of $P(2009)$.

SOLUTION

$$P(x)-2008+x=x(x-1)(x-2)(x-3)\ldots(x-2008)$$

Put $x=2009$

$$P(2009)+1=(2009)!$$

We have $P(2009)=(2009)!-1$

ILLUSTRATION 11

If $a, b, c \in N$ then find total number of polynomials of the form $6x^3+ax^2+bx-c$ divisible by x^2+x+1.

SOLUTION

$$(6x-c)(x^2+x+1)=6x^3+ax^2+bx-\mathrm{c}$$

Equating the coefficients

$\Rightarrow \quad 6-c=a$ and $6-c=b$

$\Rightarrow \quad a+c=6$ and $b+c=6$ by taking difference we get $a=b$

$a=1=b, c=5$

or $\quad a=2=b, c=4$

or $\quad a=3=b=c$

or $\quad a=4=b, c=2$

or $\quad a=5=b, c=1$

Possible 5 polynomials.

EQUATION AND IDENTITY

Root

A value which satisfies the any given polynomial equation. Roots can be real or complex in nature. Roots can be distinct or repetitive.

Polynomial of degree n, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ has exactly n roots (real or non-real) is known as **Equation**.

If equation, $P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ is satisfied by more than n distinct values of x then this equation converts into an **Identity**.

In simple words for Identity, $P(x)$ is identically zero or we can say all the coefficients are zero.

i.e. $a_n = a_{n-1} = a_{n-2} = \ldots = a_0 = 0$

lets discuss it for quadratic and cubic form.

Equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ is called a quadratic equation has exactly two roots (real or imaginary).

If equation $ax^2 + bx + c = 0$ is satisfied for more than 2 distinct values of x then it becomes an Identity, hence $a = b = c = 0$.

Equation of the form $ax^3 + bx^2 + cx + d = 0$ where $a, b, c, d \in R$ and $a \neq 0$ is called a cubic equation has exactly three roots (real or imaginary).

If equation $ax^3 + bx^2 + cx + d = 0$ is satisfied for more than 3 distinct values of x then it becomes an Identity, hence $a = b = c = d = 0$.

$ax^4 + bx^3 + cx^2 + dx + e = 0$ represents different polynomial equations depending upon the values of a, b, c, d, e		
If $a \neq 0$	Degree 4 Polynomial Equation $[ax^4 + bx^3 + cx^2 + dx + e = 0]$	Exactly four roots (real or non-real)
If $a = 0, b \neq 0$	Cubic equation $[bx^3 + cx^2 + dx + e = 0]$	Exactly three roots (real or non-real)
If $a = 0, b = 0, c \neq 0$	Quadratic Equation $[cx^2 + dx + e = 0]$	Exactly two roots (real or non-real)
If $a = b = c = 0, d \neq 0$	Linear Equation $[dx + e = 0]$	Exactly one real root
If $a = b = c = d = 0, e \neq 0$	Non zero constant term [Contradiction]	No roots
If $a = b = c = d = e = 0$	Identity	Infinite solutions (True for all x)

ILLUSTRATION 12

If equation $(a - 1)\, x^2 + (a^2 + a - 2)\, x + (a^2 - 3a + 2) = 0$ has more than two distinct roots then find a.

SOLUTION

$(a-1)\, x^2 + (a+2)\,(a-1)\, x + (a-2)\,(a-1) = 0$ as it is an identity

Hence all coefficient must be zero

Coefficient of $x^2 = a - 1 = 0 \quad \Rightarrow \quad a = 1$

Coefficient of $x = (a + 2)\,(a - 1) = 0 \quad \Rightarrow \quad a = 1, -2$

Constant term $= (a - 2)\,(a - 1) = 0 \quad \Rightarrow \quad a = 1, 2$

So common value is $a = 1$

ILLUSTRATION 13

Find the number of values of 'p' for which the equation

$$\left(p^2 - 3p + 2\right)x^2 - \left(p^2 - 5p + 4\right)x + \left(p - p^2\right) = 0$$

possesses more than two roots.

SOLUTION

Making all coefficients as zero

$$\left(p^2 - 3p + 2\right) = 0 \Rightarrow p = 1, 2$$

We have $\left(p^2 - 5p + 4\right) = 0 \Rightarrow p = 1, 4$

$$\left(p - p^2\right) = 0 \Rightarrow p = 0, 1$$

Common value of $p = 1$

QUADRATIC EQUATION

Equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ is called a quadratic equation, where a, b, c are coefficients of this equation.

Quadratic equation has exactly two roots. It can be real or complex in nature and roots can be distinct or repetitive.

Roots of Quadratic Equation

To find roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, first divide it by a.

we get $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$\Rightarrow \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ these are the roots of the given equation.

Let the roots are α and β then $\alpha = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ $\beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

(a) sum of the roots $= \alpha + \beta = -\dfrac{b}{a}$

(b) product of the roots $= \alpha\beta = \dfrac{c}{a}$

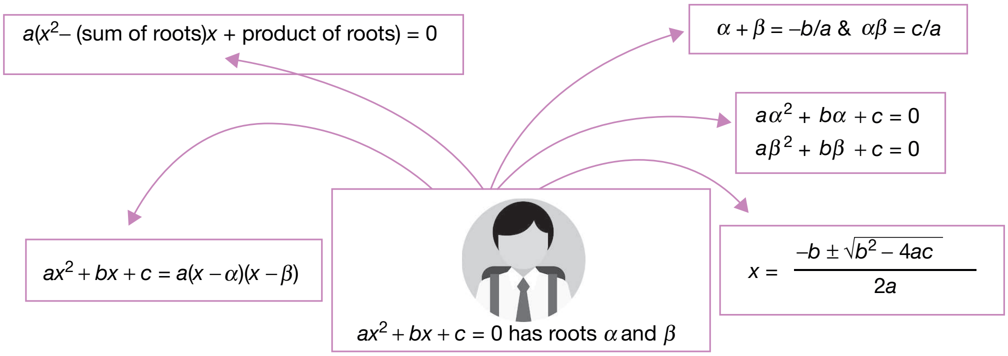
(c) If α and β are the roots of $ax^2 + bx + c = 0$ then, $ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta)$

(d) The equation whose roots are α and β is

$$(x - \alpha)\,(x - \beta) = 0$$

$$\Rightarrow \quad x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$$

$$\Rightarrow \quad x^2 - (\text{sum of roots})\, x + (\text{product of roots}) = 0$$



$a(x^2 -$ (sum of roots)x + product of roots$) = 0$

$\alpha + \beta = -b/a$ & $\alpha\beta = c/a$

$a\alpha^2 + b\alpha + c = 0$
$a\beta^2 + b\beta + c = 0$

$ax^2 + bx + c = a(x - \alpha)(x - \beta)$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$ax^2 + bx + c = 0$ has roots α and β

Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are α and β then

(i) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \left|\dfrac{\sqrt{b^2 - 4ac}}{a}\right| = \left|\dfrac{\sqrt{D}}{a}\right|$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \dfrac{b^2 - 2ac}{a^2}$

(iii) $|\alpha^2 - \beta^2| = |\alpha+\beta|\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \left|\dfrac{b\sqrt{b^2 - 4ac}}{a^2}\right|$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = -\dfrac{b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$
$= (\alpha - \beta)\{(\alpha+\beta)^2 - \alpha\beta\}$

(vi) $\alpha^4 + \beta^4 = \{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 - \beta^2)(\alpha^2 + \beta^2)$
$= (\alpha - \beta)(\alpha+\beta)((\alpha+\beta)^2 - 2\alpha\beta)$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha+\beta)^2 - \alpha\beta$

(ix) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha+\beta)$

(xi) $\left(\dfrac{\alpha}{\beta}\right)^2 + \left(\dfrac{\beta}{\alpha}\right)^2 = \dfrac{\alpha^4+\beta^4}{\alpha^2\beta^2} = \dfrac{(\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2}$

Formation of an Equation with Given Roots

A quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$

$\therefore\quad x^2 - \alpha x - \beta x + \alpha\beta = 0 \quad\Rightarrow\quad x^2 - (\alpha+\beta)x + \alpha\beta = 0$

In other words required equation is x^2 − (sum of roots)x + product of roots = 0

Equation in terms of the roots of another equation
If α, β are roots of the equation $ax^2 + bx + c = 0$ then the new equation whose roots are

(i) $-\alpha, -\beta \Rightarrow ax^2 - bx + c = 0$ (replace x by $-x$)
(ii) $1/\alpha, 1/\beta \Rightarrow cx^2 + bx + a = 0$ (replace x by $1/x$)
(iii) $\alpha^n, \beta^n; n \in N \Rightarrow a(x^{1/n})^2 + b(x^{1/n}) + c = 0$ (replace x by $x^{1/n}$)
(iv) $k\alpha, k\beta \Rightarrow ax^2 + kbx + k^2c = 0$ (replace x by x/k)
(v) $k + \alpha, k + \beta \Rightarrow a(x - k)^2 + b(x - k) + c = 0$ (replace x by $x - k$)
(vi) $\dfrac{\alpha}{k}, \dfrac{\beta}{k} \Rightarrow k^2ax^2 + kbx + c = 0$ (replace x by kx)
(vii) $\alpha^{1/n}, \beta^{1/n}; n \in N \Rightarrow a(x^n)^2 + b(x^n) + c = 0$ (replace x by x^n)

Note:
(i) This method can also be applied to higher degree polynomial equations.
(ii) Above method is valid when same transformation is applied to both new roots

ILLUSTRATION 14

The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be –2 and –15. Find the roots of the original equation.

SOLUTION

Here product of roots $= q = (-2) \times (-15) = 30$, incorrect value of $p = 17$ correct value of $p = 13$. Hence original equation is $x^2 + 13x + 30 = 0$ or $(x + 10)(x + 3) = 0$

$\therefore$ Roots are –10, –3

ILLUSTRATION 15

If α and β are roots of equation $x^2 + x + 1 = 0$, then find the value of $\alpha^4 + \beta^4 + \dfrac{1}{\alpha^2\beta^2} + \alpha^3 + \beta^3 + \alpha^6 + \beta^6 + \alpha^8 + \beta^8$.

SOLUTION

$$\alpha + \beta = -1,\ \alpha\beta = 1$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = -1$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = -1$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 2$$

$$\alpha^6 + \beta^6 = (\alpha^3 + \beta^3)^2 - 2\alpha^3\beta^3 = 2$$

$$\alpha^8 + \beta^8 = (\alpha^4 + \beta^4)^2 - 2\alpha^4\beta^4 = -1$$

$$(\alpha^4 + \beta^4) + \frac{1}{\alpha^2\beta^2} + (\alpha^3 + \beta^3) + (\alpha^6 + \beta^6) + (\alpha^8 + \beta^8)$$

$$= -1 + 1 + 2 + 2 - 1 = 3$$

ILLUSTRATION 16

Find the quadratic equation over rational coefficients whose one root is $2\cos^2\dfrac{\pi}{10}$.

SOLUTION

Let one root be α

$$\therefore \quad \alpha = 2\cos^2 18^\circ = 1 + \cos 36^\circ = 1 + \frac{(\sqrt{5} + 1)}{4}$$

$$= \frac{5 + \sqrt{5}}{4}$$

$$\therefore \quad \text{other root} = \beta = \frac{5 - \sqrt{5}}{4}$$

$$\therefore \quad \alpha + \beta = \frac{5}{2} \text{ and } \alpha\beta = \frac{20}{16} = \frac{5}{4}$$

$\therefore$ Quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \Rightarrow 4x^2 - 10x + 5 = 0.$$

ILLUSTRATION 17

If roots of the equation $x^2 - 2x \sec\dfrac{\pi}{8} + 1 = 0$ are a and b, then find the values of $(a^2 + b^2)$.

SOLUTION

$$x^2 - 2x \sec\frac{\pi}{8} + 1$$

$$\Rightarrow \quad x = \left(\sec\frac{\pi}{8} \pm \tan\frac{\pi}{8}\right)$$

$$\therefore \quad a^2 + b^2 = 2\left(\sec^2\frac{\pi}{8} + \tan^2\frac{\pi}{8}\right) = 2(1 + 2\tan^2 22.5^\circ)$$

$$= 2(1 + 2(\sqrt{2} - 1)^2) = 2(7 - 4\sqrt{2}) = 14 - 8\sqrt{2}$$

ILLUSTRATION 18

Find the sum of all values of p for which one root of the quadratic equation $x^2 + 3(1 - 3p)x + 2 = 0$ is twice as large as the other root.

SOLUTION

Given $x^2 + 3(1 - 3p)x + 2 = 0$ has roots α and 2α

$$\therefore \quad 3\alpha = 3(3p - 1) \quad \Rightarrow \quad \alpha = (3p - 1)$$

Also, $2\alpha^2 = 2 \Rightarrow \alpha^2 = 1 \Rightarrow \alpha = \pm 1$

So, for $\alpha = -1$, $p = 0$ and for $\alpha = 1$, $p = \dfrac{2}{3}$

sum of all possible values of $p = 0 + \dfrac{2}{3} = \dfrac{2}{3}$.

ILLUSTRATION 19

If α, β are the roots of equation $x^2 + x - 2 = 0$, then find the value of $\dfrac{\alpha\beta^4(\beta + 1)^4 + \beta\alpha^4(\alpha + 1)^4}{\alpha^2 + \beta^2 + \alpha + \beta}$.

SOLUTION

We have, $x^2 + x - 2 = 0$ has roots α and β.

$$\therefore \quad \alpha + \beta = -1;\ \alpha\beta = -2$$

$$\frac{\alpha\beta^4(\beta + 1)^4 + \beta\alpha^4(\alpha + 1)^4}{\alpha^2 + \beta^2 + \alpha + \beta}$$

$$= \frac{\alpha\beta^4(-\alpha)^4 + \beta\alpha^4(-\beta)^4}{(\alpha + \beta)^2 - 2\alpha\beta + \alpha + \beta}$$

$$= \frac{(\alpha\beta)^4(\alpha + \beta)}{-2\alpha\beta} = \frac{(\alpha\beta)^4}{2\alpha\beta} = \frac{16}{-4} = -4$$

ILLUSTRATION 20

Let α and β are the roots of $x^2 + 10x - 7 = 0$. If $a_n = \alpha^n + \beta^n$ for $n \geq 1$ then find the value of $\dfrac{a_{12} - 7a_{10}}{2a_{11}}$.

SOLUTION

$$x^2 + 10x - 7 = 0$$

$$\Rightarrow \quad x^{12} + 10x^{11} - 7x^{10} = 0$$

$$\left.\begin{array}{l}\alpha^{12}+10\alpha^{11}-7\alpha^{10}=0\\ \beta^{12}+10\beta^{11}-7\beta^{10}=0\end{array}\right\}\text{ adding we get}$$

$$\left(\alpha^{12}+\beta^{12}\right)+10\left(\alpha^{11}+\beta^{11}\right)-7\left(\alpha^{10}+\beta^{10}\right)=0$$

$$a_{12}+10a_{11}-7a_{10}=0$$

$$a_{12}-7a_{10}=-10a_{11}\quad\Rightarrow\quad\frac{a_{12}-7a_{10}}{2a_{11}}=-5$$

ILLUSTRATION 21

If α, β are the roots of $cx^2+bx+a=0$, then find the roots of equation $a(x+1)^2+b(x+1)+c=0$.

SOLUTION

Roots of the equation $cx^2+bx+a=0$ are α, β then roots of the equation $ax^2+bx+c=0$ are $\frac{1}{\alpha}, \frac{1}{\beta}$ then roots of the equation $a(x+1)^2+b(x+1)+c=0$ are $\frac{1}{\alpha}-1$, $\frac{1}{\beta}-1$ or $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$

ILLUSTRATION 22

If $(\alpha,\beta),(\beta,\gamma)$ and (γ,α) are respectively the roots of $x^2-2px+2=0$, $x^2-2qx+3=0$ and $x^2-2rx+6=0$, where α,β,γ are all positive, then find the value of $p+q+r$.

SOLUTION

We have,

$$\alpha\beta=2,\ \beta\gamma=3 \text{ and } \gamma\alpha=6$$

$$\Rightarrow\quad \alpha\beta.\beta\gamma.\gamma\alpha=2.3.6$$

$$\Rightarrow\quad (\alpha\beta\gamma)^2=36 \text{ or } \alpha\beta\gamma=6\left[\alpha,\beta,\gamma \text{ are all positive}\right]$$

Thus, we have,

$$\alpha=\frac{\alpha\beta\gamma}{\beta\gamma}=\frac{6}{3}=2$$

Similarly, $\beta=1$ and $\gamma=3$

Also, we have,

$$2p=\alpha+\beta=3,\ 2q=\beta+\gamma=4 \text{ and } 2r=\gamma+\alpha=5$$

Add all $2p+2q+2r=3+4+5$

$$\therefore\quad p+q+r=6$$

ILLUSTRATION 23

If α and β are the roots of the equation $ax^2+bx+c=0$ then find the equation whose roots are, $\alpha^2+\beta^2$, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

SOLUTION

Here let S be the sum and P be the product of the roots $\alpha^2+\beta^2$, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

Now $S=(\alpha^2+\beta^2)+\frac{1}{\alpha^2}+\frac{1}{\beta^2}=(\alpha^2+\beta^2)\left(1+\frac{1}{(\alpha\beta)^2}\right)$

$$=\left[(\alpha+\beta)^2-2\alpha\beta\right]\left(1+\frac{1}{(\alpha\beta)^2}\right)$$

$$=\left(\left(-\frac{b}{a}\right)^2-\frac{2c}{a}\right)\left(1+\frac{a^2}{c^2}\right)=(b^2-2ac)\left(\frac{a^2+c^2}{a^2c^2}\right)$$

As the product

$$P=(\alpha^2+\beta^2)\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)$$

$$=\frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2}=\left(\frac{b^2-2ac}{a^2}\right)^2\frac{1}{\frac{c^2}{a^2}}=\left(\frac{b^2-2ac}{ac}\right)^2$$

Hence equation is $(ac)^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2=0$

ILLUSTRATION 24

If $ax^{17}+bx^{16}+1$ is divisible by x^2-x-1, then find the integral value of a.

SOLUTION

Let roots of $x^2-x-1=0$ are $\alpha, \beta \Rightarrow \alpha+\beta=1,\ \alpha\beta=-1$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=3$$

$$\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=9-2=7$$

$$\alpha^8+\beta^8=(\alpha^4+\beta^4)^2-2\alpha^4\beta^4=49-2=47$$

Also $\left\{\begin{array}{l}a\alpha^{17}+b\alpha^{16}+1=0\\ a\beta^{17}+b\beta^{16}+1=0\end{array}\right.$

$$\Rightarrow\quad a\alpha+b=-\frac{1}{\alpha^{16}} \text{ and } a\beta+b=-\frac{1}{\beta^{16}}$$

$$\Rightarrow\quad a(\alpha-\beta)=\frac{\alpha^{16}-\beta^{16}}{(\alpha\beta)^{16}}$$

$$\Rightarrow\quad a=(\beta^8+\alpha^8)(\beta^4+\alpha^4)(\beta^2+\alpha^2)(\beta+\alpha)$$

$$\Rightarrow\quad a=47.7.3.1=987$$

ILLUSTRATION 25

If α, β are the roots of $x^2-p(x+1)-c=0$ then find $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$.

SOLUTION

Here the equation is $x^2-p(x+1)-c=0$

$$\therefore\quad \alpha+\beta=p,\ \alpha\beta=-(p+c)$$

$$\Rightarrow\quad (\alpha+1)(\beta+1)=1-c$$

Now given expression

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)},$$

Putting value of $1-c=(\alpha+1)(\beta+1)$

$$=\frac{\alpha+1}{\alpha-\beta}+\frac{\beta+1}{\beta-\alpha}=\frac{\alpha+1-\beta-1}{\alpha-\beta}=1$$

ILLUSTRATION 26

If α, β be the roots of $ax^2+bx+c=0$ and γ, δ those of $\ell x^2 + mx + n = 0$, then find the equation whose roots are $\alpha\gamma+\beta\delta$, and $\alpha\delta+\beta\gamma$.

SOLUTION

Here Sum of the roots = $(\alpha\gamma+\beta\delta)+(\alpha\delta+\beta\gamma)$

$$=\alpha(\gamma+\delta)+\beta(\gamma+\delta)=(\alpha+\beta)(\gamma+\delta)=\frac{bm}{a\ell}$$

Also product of roots = $(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)$

$$=(\alpha^2+\beta^2)\gamma\delta+\alpha\beta(\gamma^2+\delta^2)$$

$$=\left[(\alpha+\beta)^2-2\alpha\beta\right]\gamma\delta+\alpha\beta[(\gamma+\delta)^2-2\gamma\delta]$$

$$=\left[\frac{b^2}{a^2}-\frac{2c}{a}\right]\frac{n}{\ell}+\left[\frac{m^2}{\ell^2}-\frac{2n}{\ell}\right]\frac{c}{a}$$

$$=\frac{b^2n\ell+m^2ac-4acn\ell}{a^2\ell^2}$$

Hence from $x^2-Sx+P=0$

We get $a^2\ell^2x^2-ab\ell mx+(b^2-4ac)n\ell+m^2ac=0$

ILLUSTRATION 27

If α, β be the roots $x^2+px-q=0$ and γ, δ be the roots of $x^2+px+r=0$ then find $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}$.

SOLUTION

Here

$$\left.\begin{aligned}\alpha+\beta&=-p\\ \gamma+\delta&=-p\end{aligned}\right\}\Rightarrow \alpha+\beta=\gamma+\delta$$

Now $(\alpha-\gamma)(\alpha-\delta)=\alpha^2-\alpha(\gamma+\delta)+\gamma\delta=\alpha^2-\alpha(\alpha+\beta)+r$

$$=-\alpha\beta+r=-(-q)+r=q+r$$

By symmetry of the results

$$(\beta-\gamma)(\beta-\delta)=q+r$$

Hence the ratio $\dfrac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}=1$

ILLUSTRATION 28

If one root of the quadratic equation $ax^2+bx+c=0$ be n^{th} root of the other then find $(ac^n)^{1/n+1}+(a^nc)^{1/n+1}$.

SOLUTION

Let one root be α and other β, such that $\beta^{1/n}=\alpha\Rightarrow\beta=\alpha^n$

$$\text{Now } \alpha+\beta=-\frac{b}{a}\quad\Rightarrow\quad\alpha+\alpha^n=-\frac{b}{a} \qquad \ldots(1)$$

$$\text{and } \alpha\beta=\frac{c}{a}\Rightarrow\alpha\alpha^n=\frac{c}{a}\Rightarrow\alpha^{n+1}=\frac{c}{a}$$

$$\therefore \quad \alpha=\left(\frac{c}{a}\right)^{\frac{1}{n+1}} \qquad \ldots(2)$$

From (1) and (2)

$$\alpha+\alpha^n=\left(\frac{c}{a}\right)^{\frac{1}{n+1}}+\left(\frac{c}{a}\right)^{\frac{n}{n+1}}=-\frac{b}{a}$$

$$\Rightarrow \quad (a^nc)^{1/n+1}+(ac^n)^{1/n+1}=-b$$

ILLUSTRATION 29

If α, β are the roots of the equation $ax^2+bx+c=0$ and $S_n=\alpha^n+\beta^n$, then find $aS_{n+1}+cS_{n-1}$.

SOLUTION

Here α, β are roots

$$\therefore \quad a\alpha^2+b\alpha+c=0 \qquad \ldots(1)$$

$$a\beta^2+b\beta+c=0 \qquad \ldots(2)$$

Now $aS_{n+1}+cS_{n-1}=a[\alpha^{n+1}+\beta^{n+1}]+c\,[\alpha^{n-1}+\beta^{n-1}]$

$=\alpha^{n-1}[a\alpha^2+c]+\beta^{n-1}[a\beta^2+c]$ using (1) and (2)

$=-b\,\alpha\,\alpha^{n-1}-b\,\beta\,\beta^{n-1}=-b(\alpha^n+\beta^n)=-bS_n$

ILLUSTRATION 30

α, β are the roots of the equation $P(x^2-x)+x+5=0$. If P_1, P_2 are the two value of P for which α, β are connected by $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, find the value of $\frac{P_1}{P_2^2}+\frac{P_2}{P_1^2}$.

SOLUTION

$$Px^2-Px+x+5=0 \quad\Rightarrow\quad Px^2-x(P-1)+5=0$$

α and β are the roots of the equation

$$\Rightarrow \quad \alpha+\beta=\frac{P-1}{P} \text{ and } \alpha\beta=\frac{5}{P} \qquad \ldots(1)$$

Given $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$

$$\Rightarrow \quad \frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{4}{5}$$

$$\text{i.e.} \quad \frac{(\alpha+\beta)^2}{\alpha\beta}=\frac{4}{5}+2=\frac{14}{5}$$

$$\therefore \quad \frac{(P-1)^2}{P^2}\cdot\frac{P}{5}=\frac{14}{5} \text{ using (1)}$$

i.e. $P^2 - 16P + 1 = 0$

The roots of the equation are P_1, P_2

∴ $P_1 + P_2 = 16,\ P_1P_2 = 1$

Now $\dfrac{P_1}{P_2^2} + \dfrac{P_2}{P_1^2} = \dfrac{P_1^3 + P_2^3}{(P_1P_2)^2}$

$$= \frac{(P_1+P_2)^3 - 3P_1P_2(P_1+P_2)}{(P_1P_2)^2} = \frac{(16)^3 - 3\times 1\times 16}{1^2}$$

$$= 16[(16)^2 - 3] = 16\times 253 = 4048$$

ILLUSTRATION 31

If α, β are two real numbers satisfying $\alpha^2 + \beta^2 = 5$ and $3(\alpha^5 + \beta^5) = 11(\alpha^3 + \beta^3)$, then find $\alpha\beta$.

SOLUTION

$$\frac{(\alpha^5+\beta^5)}{(\alpha^3+\beta^3)} = \frac{11}{3}$$

$$\Rightarrow \quad \frac{1}{(\alpha^2+\beta^2-\alpha\beta)}(\alpha^4+\beta^4-\alpha\beta(\alpha^2+\beta^2-\alpha\beta)) = \frac{11}{3}$$

$$\Rightarrow \quad \frac{(\alpha^4+\beta^4)}{(\alpha^2+\beta^2-\alpha\beta)} - \alpha\beta = \frac{11}{3}$$

$$\Rightarrow \quad \frac{25-2(\alpha\beta)^2}{5-\alpha\beta} - \alpha\beta = \frac{11}{3}$$

Let $\alpha\beta = t$;

$$\frac{25-2t^2}{5-t} - t = \frac{11}{3} \Rightarrow 3t^2 + 4t - 20 = 0$$

$$\Rightarrow \quad (t-2)(3t+10) = 0$$

We get $t = 2$; $t = -\dfrac{10}{3} = \alpha\beta$; $\alpha^2 + \beta^2 = 5 \Rightarrow \alpha^2 + \dfrac{100}{9\alpha^2} = 5$

$$\Rightarrow \quad 9\alpha^4 - 45\alpha^2 + 100 = 0$$

No real α

∴ $\alpha\beta \neq -\dfrac{10}{3}$

Now, $t = 2 = \alpha\beta$ (acceptable)

$$\alpha^2 + \beta^2 = 5 \Rightarrow \alpha^2 + \frac{4}{\alpha^2} = 5$$

$$\Rightarrow \quad \alpha^4 - 5\alpha^2 + 4 = 0$$

$$\Rightarrow \quad (\alpha^2 - 1)(\alpha^2 - 4) = 0$$

$$\Rightarrow \quad \alpha^2 = 1, 4 \quad \therefore \quad \alpha\beta = 2$$

ILLUSTRATION 32

If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$, then find h.

SOLUTION

We have α, β are the roots of $ax^2 + bx + c = 0$ then $\alpha + \beta = -\dfrac{b}{a}$, $\alpha\beta = \dfrac{c}{a}$ and $\alpha + h$, $\beta + h$ are the roots of $px^2 + qx + r = 0$, then $\alpha + h + \beta + h = -\dfrac{q}{p}$, $(\alpha + h)(\beta + h) = \dfrac{r}{p}$.

$$\Rightarrow \quad \alpha + \beta + 2h = -\frac{q}{p} \quad \Rightarrow \quad -\frac{b}{a} + 2h = \frac{-q}{p}$$

$$\left[\because \ \alpha + \beta = -\frac{b}{a}\right]$$

$$\Rightarrow \quad h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$$

ILLUSTRATION 33

Let α, β be the roots of $x^2 + ax + 1 = 0$. If the equation whose roots are $-\left(\alpha + \dfrac{1}{\beta}\right)$ and $-\left(\dfrac{1}{\alpha} + \beta\right)$ is $x^2 - 2ax + k = 0$ then find k.

SOLUTION

$$\alpha + \beta = -a,\ \alpha\beta = 1$$

Let $y = -\left(\alpha + \dfrac{1}{\beta}\right) = -\left(\dfrac{\alpha\beta + 1}{\beta}\right) = -\dfrac{2}{\beta} \Rightarrow \beta = -\dfrac{2}{y}$

$$f(\beta) = 0 \Rightarrow \beta^2 + a\beta + 1 = 0 \Rightarrow \frac{4}{y^2} - \frac{2a}{y} + 1 = 0$$

$\Rightarrow \quad 4 - 2ay + y^2 = 0$ is the required equation

We get $k = 4$

ILLUSTRATION 34

If $e^{\lambda}, e^{-\lambda}$ are the roots of equation $3x^2 - (a + b)x + 2a = 0$, $a, b, \lambda \in R$, $\lambda \neq 0$ then find the least integral value of b.

SOLUTION

$f(x) = 3x^2 - (a + b)x + 2a = 0$ has roots e^{λ} and $e^{-\lambda}$

Product of the roots $= 1 \quad \Rightarrow \quad \dfrac{2a}{3} = 1 \quad \Rightarrow \quad a = \dfrac{3}{2}$

Now, sum of the roots $= e^{-\lambda} + e^{\lambda} = \dfrac{a+b}{3} > 2$

$(\because e^{\lambda} + e^{-\lambda} > 2$ for $\lambda \neq 0)$

$$\Rightarrow \quad b > 6 - a \quad \Rightarrow \quad b > 6 - \frac{3}{2} \quad \Rightarrow \quad b > \frac{9}{2}$$

∴ Least integral value of b is 5.

ILLUSTRATION 35

Find value of 'a' for which the sum of the square of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assumes the least value.

SOLUTION

Let α, β be the roots of the given equation. Then, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$.

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a - 2)^2 + 2(a + 1)$

$$= a^2 - 2a + 6 = (a - 1)^2 + 5$$

Clearly, $\alpha^2 + \beta^2 \geq 5$. So, the minimum value of $\alpha^2 + \beta^2$ is 5 which it attains at $a = 1$.

ILLUSTRATION 36

If α, β are the roots of $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of $x^n + 1 + (x + 1)^n = 0$ then show that n is an even integer.

SOLUTION

Here $\alpha + \beta = -p$, $\alpha\beta = q$

Let $t = x^n \Rightarrow$ when $x = \alpha$, $t = \alpha^n$

when $x = \beta$, $t = \beta^n$

$\therefore \quad t^2 + p^n t + q^n = 0$

$\therefore \quad \alpha^n + \beta^n = -p^n, \ \alpha^n\beta^n = q^n$

Again $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are roots of $x^n + 1 + (x + 1)^n = 0$

$$\Rightarrow \quad \left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

$\Rightarrow \quad \alpha^n + \beta^n + (\alpha + \beta)^n = 0$

$\Rightarrow \quad -p^n + (-p)^n = 0$ which is true only if n is even integer.

Hence proved.

ILLUSTRATION 37

Solve: $(x + 9)(x - 3)(x - 7)(x + 5) = 385$

SOLUTION

$$(x + 9)(x - 7)(x - 3)(x + 5) = 385$$

$\Rightarrow \quad (x^2 + 2x - 63)(x^2 + 2x - 15) = 385$

Put $x^2 + 2x = y$ …(1)

$\therefore \quad (y - 63)(y - 15) = 385$

$\therefore \quad y^2 - 78y + 560 = 0 \quad \Rightarrow \quad (y - 70)(y - 8) = 0$

$\therefore \quad y = 70$ or 8

When $y = 70$, (1) gives $x^2 + 2x - 70 = 0$

$$\therefore \quad x = \frac{-2 \pm \sqrt{4 + 4 \times 70}}{2} = -1 \pm \sqrt{71}$$

when $y = 8$, (1) gives $x^2 + 2x - 8 = 0$

$\Rightarrow \quad (x + 4)(x - 2) = 0 \quad \Rightarrow \quad x = -4$ or 2.

$\therefore \quad$ Solutions are -4, 2 and $-1 \pm \sqrt{71}$

ILLUSTRATION 38

Solve: $\sqrt{2x + 7} + \sqrt{3x - 18} = \sqrt{7x + 1}$

SOLUTION

Squaring both sides

$\Rightarrow \quad 2x + 7 + 3x - 18 + 2\sqrt{2x + 7}\sqrt{3x - 18} = 7x + 1$

$\Rightarrow \quad 5x - 11 + 2\sqrt{2x + 7}\sqrt{3x - 18} = 7x + 1$

$\Rightarrow \quad 2x + 12 = 2\sqrt{2x + 7}\sqrt{3x - 18}$

Again squaring both sides

(assuming $2x + 12 \geq 0 \Rightarrow x \geq -6$)

$$x^2 + 36 + 12x = (2x + 7)(3x - 18)$$

i.e. $\quad 5x^2 - 27x - 162 = 0$

$\Rightarrow \quad$ i.e. $(5x + 18)(x - 9) = 0$

$\therefore \quad x = 9$ or $x = \frac{-18}{5}$ …(1)

$\sqrt{2x + 7}$ is defined for $x \geq -7/2$

$\sqrt{3x - 18}$ is defined for $x \geq 6$

$\sqrt{7x + 1}$ is defined for $x \geq -1/7$

$x = 9$ is the only acceptable solution.

NOTE:

(i) When we square an equation, we have to check if the solution(s) thus obtained satisfied the given equation. The values which do not satisfy the equation are called **extraneous roots**.

(ii) Consider the equation $x = a$.
On squaring, we get $x^2 = a^2 \Rightarrow x = \pm a$ but we are given $x = a$, hence here $x = -a$ is an extraneous value.

Test Your Concepts-I

(Solutions on page H.19)

1. Find the remainder when $p(x) = 4x^3 - 12x^2 + 14x - 3$ is divided by $g(x) = x - \frac{1}{2}$.
2. Find the value of k, if $x + 3$ is a factor of $3x^2 + kx + 6$.
3. If both $x - 2$ and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that $p = r$.
4. If the polynomials $ax^3 + 4x^2 + 3x - 4$ and $x^3 - 4x + a$ leave the same remainder when divided by $(x - 3)$, find the value of a.

5. Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g\left(x^{12}\right)$ is divided by the polynomial $g(x)$?

6. If α, β be roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$ then find $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$.

7. If α, β are roots of $x^2 - 12x + 4 = 0$, then find $\frac{(\alpha-2)^4 + (\beta-2)^4}{17\alpha\beta}$.

8. Suppose that 'a' and 'b' are non zero real numbers and the equation $x^2 + ax + b = 0$ has solution a and b then find the pair (a, b).

9. Let α, β are the roots of the equation $x^2 + ax + b = 0$ such that $\frac{3\alpha - 5 - \alpha^2}{(\beta - 3)} = \beta$, then find the value of $|a^2 - 2b + 3a|$ (where, $\beta \neq 3$).

10. If α, β are roots of the equation $x^2 + ax + 2 = 0$ such that $\alpha^6\beta^5 = 32$, then find the value of 'a'.

11. If $a^2 = 10\alpha - 12$, $b^2 = 15\beta - 27$ and $\alpha > \beta$, then find the value of $\frac{2\beta + 3\alpha}{\alpha\beta}$.

12. Let α, β are the roots of the equation $x^2 + ax + b = 0$ and γ, δ are the roots of $x^2 - ax + b - 2 = 0$.

If $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{5}{12}$ and $\alpha\beta\gamma\delta = 24$, then find the value of 'a'.

13. If α, β are roots of equation $P(x^2 - x) + x + 5 = 0$ and P_1 and P_2 are two values of P obtained from $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then find value of $\frac{P_1}{P_2^3} + \frac{P_2}{P_1^3}$.

Answer keys

1. 3/2 **2.** 11 **4.** -1 **5.** 6 **6.** $(q^2 - p^2)$ **7.** 2^7

8. (1, -2) **9.** 5 **10.** -3 **11.** 5/3 **12.** 5 **13.** 64514

POLYNOMIAL EQUATION OF DEGREE *n*

(i) If $a_n \neq 0$ then $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ has exactly n roots (real or imaginary).

(ii) If all coefficients of equation are real ($a_0, a_1, a_2, \ldots, a_n \in R$) then imaginary roots always occur in conjugate pairs, In simple words if $p + iq$ ($q \neq 0$) is a root of $P(x) = 0$, then $p - iq$ is also a root of given equation where $p, q \in R$.

(iii) If all coefficients of equation are non-real ($a_0, a_1, a_2, \ldots, a_n \in C$) then imaginary roots may/may not occur in conjugate pairs.

(iv) A polynomial equation in x of odd degree has at least one real root (moreover it has an odd number of real roots)

(v) If α is a root of $P(x) = 0$, then $(x - \alpha)$ is a factor of $P(x)$ and vice-versa.

(vi) If $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are roots of $P(x) = 0$, then $P(x)$ can be written in the form,

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \quad \ldots(1)$$

$$= a_n(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$

$$= a_n[x^n - x^{n-1}(\Sigma\alpha_1) + x^{n-2}(\Sigma\alpha_1\alpha_2) + \ldots + (-1)^n\alpha_1\alpha_2\alpha_3 \ldots \alpha_{n]} \quad \ldots(2)$$

Now comparing the coefficients from above identity (1) and (2) then

Sum of the roots

$$= \Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} + \alpha_n = -\frac{a_{n-1}}{a_n}$$

Sum of product of roots taken two at a time =

$$\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \ldots + \alpha_{n-1}\alpha_n + \alpha_n\alpha_1$$

$$= (-1)^2\frac{a_{n-2}}{a_n} = +\frac{a_{n-2}}{a_n}$$

Sum of product of roots taken three at a time =

$$\Sigma\alpha_1\alpha_2\alpha_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_5 + \ldots + \alpha_{n-2}\alpha_{n-1}\alpha_n = (-1)^3\frac{a_{n-3}}{a_n} = -\frac{a_{n-3}}{a_n}$$

Product of the all roots =

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_{n-1}\alpha_n = (-1)^n\frac{a_0}{a_n}$$

1. Rewrite polynomial is descending degree of variable x. If few terms like $x, x^2, \ldots$ are not present, rewrite them with coefficient as zero

2. Apply +, – sign alternatively.

(a) If quadratic equation $ax^2 + bx + c = 0$ has roots α and β

$$ax^2 \overset{\ominus}{+} bx \overset{\oplus}{+} c = 0$$

sum of the roots $= \alpha + \beta = -\frac{b}{a}$

$= -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$;

Product of the roots $= \alpha\beta = +\frac{c}{a}$

$= \frac{\text{coefficient term}}{\text{coefficient of } x^2}$

(b) If cubic equation $ax^3+bx^2+cx+d=0$ has roots α, β, γ

$$ax^3 \overset{\ominus}{+} bx^2 \overset{\oplus}{+} cx \overset{\ominus}{+} d = 0$$

Sum of the roots $= \Sigma\alpha = \alpha+\beta+\gamma = -\dfrac{b}{a}$

$= -\dfrac{\text{coefficient of } x^2}{\text{coefficient of } x^3}$

sum of product of roots taken two at a time

$= \Sigma\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = +\dfrac{c}{a}$

$= \dfrac{\text{coefficient of } x}{\text{coefficient of } x^3}$

Product of roots $= \alpha\beta\gamma = -\dfrac{d}{a}$

$= -\dfrac{\text{constant term}}{\text{coefficient of } x^3}$

(c) Lets consider degree 4 polynomials $ax^4+bx^3+cx^2+dx+e=0$ has roots $\alpha, \beta, \gamma, \delta$

$$ax^4 \overset{\ominus}{+} bx^3 \overset{\oplus}{+} cx^2 \overset{\ominus}{+} dx \overset{\oplus}{+} e = 0$$

sum of roots $= \Sigma\alpha = \alpha+\beta+\gamma+\delta = -\dfrac{b}{a}$

$= -\dfrac{\text{coefficient of } x^3}{\text{coefficient of } x^4}$

sum of product of roots taken two at a time $= \Sigma\alpha\beta$

$= \alpha\beta+\beta\gamma+\gamma\delta+\alpha\gamma+\alpha\delta+\beta\delta = +\dfrac{c}{a}$

$= \dfrac{\text{coefficient of } x^2}{\text{coefficient of } x^4}$

sum of product of roots taken three at a time $= \Sigma\alpha\beta\gamma$

$= \alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\alpha\beta\delta = -\dfrac{d}{a}$

$= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^4}$

product of roots $= \alpha\beta\gamma\delta = +\dfrac{e}{a}$

$= \dfrac{\text{coefficient term}}{\text{coefficient of } x^4}$

Remark(s)

(a) If coefficients of $P(x)$ have 'm' changes in sign, then $P(x)=0$ have at most 'm' positive roots and if $P(-x)$ have 't' changes in sign, then $P(x)=0$ have at most 't' negative roots. Using this we can find maximum no. of real roots, of a polynomial equation.

(b) If $f(a)\, f(b) < 0$, then there will be atleast one root in (a, b) of $f(x)=0$.

EQUATION FORMATION WITH MODIFIED ROOTS

If $P(x)=0$ has roots $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ then the equation having roots

$$f(\alpha_1), f(\alpha_2), f(\alpha_3), \ldots, f(\alpha_n) \text{ is given by } P(f^{-1}(x))=0.$$

In particular if we have roots to be $\alpha_1+k, \alpha_2+k, \ldots, \alpha_n+k$, then the corresponding equation is $P(x-k)=0$ and if we have roots to be $k\alpha_1, k\alpha_2, \ldots, k\alpha_n$, then the corresponding equation is $P(x/k)=0$.

ILLUSTRATION 39

If one root or the equation $x^2-2x+k=0$ is $1+2i$ and $k \in R$ then find the value of k, where $i=\sqrt{-1}$.

SOLUTION

As coefficients are real so imaginary roots must be present with their conjugates.

$1+2i$ and $1-2i$ are roots

Now product of the roots $= (1+2i)(1-2i) = k$

$\Rightarrow \quad k=5$

ILLUSTRATION 40

If $x=2+i$ and $x=1-3i$ are the roots of the equation $x^4+ax^3+23x^2+bx+c=0$ $(a, b, c \in R)$ then find the value of $|a+b+c|$, where $i=\sqrt{-1}$.

SOLUTION

As coefficients of equation are real then

$x^4+ax^3+23x^2+bx+c=0$ has roots $2+i$, $2-i$, $1-3i$ and $1+3i$

$$\therefore \quad x^4+ax^3+23x^2+bx+c = (x^2-4x+5)(x^2-2x+10) \qquad \ldots(1)$$

Put $x=1$ on both sides of above equation (1), we get,

$a+b+c+24 = (2)\cdot(9)$

$\Rightarrow \quad a+b+c=-6$

$\Rightarrow \quad |a+b+c|=6$

ILLUSTRATION 41

If α, β, γ are roots of equation $x^3+qx+r=0$ then. Find equation whose roots are

(i) $2\alpha+2\beta+\gamma, \alpha+2\beta+2\gamma, 2\alpha+\beta+2\gamma$

(ii) $-\dfrac{r}{\alpha}, -\dfrac{r}{\beta}, -\dfrac{r}{\gamma}$

(iii) $-\alpha^3, -\beta^3, -\gamma^3$

SOLUTION

(i) New roots $2\alpha+2\beta+\gamma, \alpha+2\beta+2\gamma, 2\alpha+\beta+2\gamma$

given equation $x^3+qx+\gamma=0$

Rewriting as $x^3+0.x^2+qx+r=0$

$\alpha+\beta+\gamma=0;\ \Sigma\alpha\beta=q;\ \alpha\beta\gamma=-\gamma$

New roots $2\alpha+2\beta+\gamma, \alpha+2\beta+2\gamma, 2\alpha+\beta+2\gamma$

[using $\alpha+\beta+\gamma=0$]

New roots are $-\gamma, -\alpha, -\beta$

Now $x = -\alpha \Rightarrow \alpha = -x$

$$\alpha^3 + q\alpha + r = 0 \Rightarrow -x^3 - qx + r = 0$$

$\Rightarrow x^3 + qx - r = 0$ is the required equation whose roots are $2\alpha + 2\beta + \gamma, \alpha + 2\beta + 2\gamma, 2\alpha + \beta + 2\gamma$

(ii) New root $x = -\dfrac{r}{\alpha} \Rightarrow \alpha = -\dfrac{r}{x}$

($\because$ α is root of $x^3 + qx + r = 0$)

$$\alpha^3 + q\alpha + r = 0 \Rightarrow \left(\frac{-r}{x}\right)^3 + q\left(-\frac{r}{x}\right) + r = 0$$

$$\Rightarrow \frac{-r^3 - qrx^2 + rx^3}{x^3} = 0$$

$$\Rightarrow rx^3 - qrx^2 - r^3 = 0$$

$x^3 - qx^2 - r^2 = 0$ is the new equation whose roots are $-\dfrac{r}{\alpha}, -\dfrac{r}{\beta}, -\dfrac{r}{\gamma}$

(iii) New root $x = -\alpha^3 \Rightarrow \alpha^3 = (-x) \Rightarrow \alpha = (-x)^{1/3}$

α is a root of $x^3 + qx + r = 0$ then $\alpha^3 + q\alpha + r = 0$

$$\left(\left((-x)^{1/3}\right)^3 + q(-x)^{1/3}\right) + r = 0$$

$$\Rightarrow -x - qx^{1/3} + r = 0 \Rightarrow r - x = qx^{1/3}$$

$$\Rightarrow (r - x)^3 = (qx^{1/3})^3$$

$$\Rightarrow r^3 - x^3 - 3rx\,(r - x) = q^3x$$

$\Rightarrow x^3 - 3rx^2 + (q^3 + 3r^2)x - r^3 = 0$ has roots $-\alpha^3, -\beta^3, -\gamma^3$

ILLUSTRATION 42

If α, β, γ and δ are roots of equation $x^4 - 7x^2 + x - 5 = 0$, then find the value of $(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\beta + \gamma + \delta)(\alpha + \gamma + \delta)$.

SOLUTION

As we can observe, coefficient of $x^2 = 0$.

$\therefore$ sum of roots $= \alpha + \beta + \gamma + \delta = 0$

so, $(\alpha + \beta + \gamma)(\alpha + \gamma + \delta)(\beta + \gamma + \delta)(\alpha + \beta + \delta)$

$= (-\delta)(-\beta)(-\alpha)(-\gamma) = \alpha\beta\gamma\delta = -5$ (Product of roots)

ILLUSTRATION 43

Let $x^4 - x^2 + 200x - 7 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find new equation whose roots are $\alpha + \beta + \gamma, \beta + \gamma + \delta, \gamma + \delta + \alpha, \delta + \alpha + \beta$.

SOLUTION

Given equation $x^4 - x^2 + 200x - 7 = 0$

Rewriting it as $x^4 + 0.x^3 - x^2 + 200x - 7 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

$$\alpha + \beta + \gamma + \delta = 0$$

We are looking for new equation whose roots are $x = \alpha + \beta + \gamma = \alpha + \beta + \gamma + \delta - \delta = -\delta$

$\Rightarrow$ $\delta = -x$ is a root of equation $x^4 - x^2 + 200x - 7 = 0$

$$\delta^4 - \delta^2 + 200\,\delta - 7 = 0$$

$$(-x)^4 - (-x)^2 + 200(-x) - 7 = 0 \text{ or } x^4 - x^2 - 200\,x - 7 = 0$$

ILLUSTRATION 44

If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$ then find the value of $\sum \dfrac{1 + \alpha}{1 - \alpha}$.

SOLUTION

$x^3 - x^2 - 0.x - 1 = 0$ has roots α, β, γ

$$\sum \alpha = 1; \sum \alpha\beta = 0; \alpha\beta\gamma = 1$$

$$\sum \frac{1 + \alpha}{1 - \alpha} = -\sum \frac{-\alpha + 1 - 2}{1 - \alpha}$$

$$= \sum\left(\frac{2}{1 - \alpha} - 1\right) = 2\sum \frac{1}{1 - \alpha} - 3$$

$$f(x) = x^3 - x^2 - 1 = (x - \alpha)(x - \beta)(x - \gamma)$$

$$\ln f(x) = \ln(x^3 - x^2 - 1)$$

$$= \ln\,(x - \alpha)(x - \beta)(x - \gamma)$$

on differentiation

Now $\dfrac{1}{x - \alpha} + \dfrac{1}{x - \beta} + \dfrac{1}{x - \gamma} = \dfrac{f'(x)}{f(x)}$

$$\frac{1}{x - \alpha} + \frac{1}{x - \beta} + \frac{1}{x - \gamma} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$$

$$\Rightarrow \frac{1}{1 - \alpha} + \frac{1}{1 - \beta} + \frac{1}{1 - \gamma} = -1$$

$$\sum \frac{1 + \alpha}{1 - \alpha} = 2\sum \frac{1}{1 - \alpha} - 3 = -5.$$

Alternate

Let $x = \dfrac{1 + \alpha}{1 - \alpha} \Rightarrow \alpha = \dfrac{x - 1}{x + 1}$ is root of given Eq.

$$\alpha^3 - \alpha^2 - 1 = 0$$

$$\Rightarrow \left(\frac{x - 1}{x + 1}\right)^3 - \left(\frac{x - 1}{x + 1}\right)^2 - 1 = 0$$

$$\Rightarrow (x - 1)^3 - (x - 1)^2(x + 1) - (x + 1)^3 = 0$$

On solving

$$x^3 + 5x^2 - x + 3 = 0$$

Sum of roots $= \sum \dfrac{1 + \alpha}{1 - \alpha} = -5$

ILLUSTRATION 45

If α, β and γ are the roots of the equation $5x^3 - qx - 1 = 0$, $(q \in R)$ then find the value of

$$\frac{\alpha^2 - 3}{\beta\gamma} + \frac{\beta^2 - 3}{\gamma\alpha} + \frac{\gamma^2 - 3}{\alpha\beta}$$

SOLUTION

$5x^3 - qx - 1 = 0$ has roots α, β and γ, so $\alpha + \beta + \gamma = 0$

Now, $\dfrac{\alpha^2 - 3}{\beta\gamma} + \dfrac{\beta^2 - 3}{\gamma\alpha} + \dfrac{\gamma^2 - 3}{\alpha\beta}$

$$= \frac{\alpha^3 - 3\alpha + \beta^3 - 3\beta + \gamma^3 - 3\gamma}{\alpha\beta\gamma}$$

$$= \frac{\alpha^3 + \beta^3 + \gamma^3 - 3(\alpha + \beta + \gamma)}{\alpha\beta\gamma} \qquad (\because\ \alpha + \beta + \gamma = 0 \Rightarrow \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma)$$

$$= \frac{3\alpha\beta\gamma}{\alpha\beta\gamma} = 3$$

ILLUSTRATION 46

If α, β are the roots of the equation $\lambda\left(x^2 - x\right) + x + 5 = 0$ and if λ_1 and λ_2 are two values of λ for which the roots α, β are related by $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{5}$, then find the value of $\dfrac{\lambda_1}{\lambda_2} + \dfrac{\lambda_2}{\lambda_1}$.

SOLUTION

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$$

$$\Rightarrow \quad \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{14}{5} \Rightarrow \frac{\left(\dfrac{\lambda - 1}{\lambda}\right)^2}{\dfrac{5}{\lambda}} = \frac{14}{5}$$

$$\Rightarrow \quad (\lambda - 1)^2 = 14\lambda \Rightarrow \lambda^2 - 16\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 16\lambda + 1 = 0$$

Now $\dfrac{\lambda_1}{\lambda_2} + \dfrac{\lambda_2}{\lambda_1} = \dfrac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2} = \dfrac{(\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2}$

$$= \frac{16^2 - 2}{2} = 254$$

ILLUSTRATION 47

If α, β, γ are the roots of the equation $x^3 + 3x + c = 0$ then find the value of $\dfrac{1}{27}\sum(\alpha - \beta)^2(\beta - \gamma)^2$.

SOLUTION

$x^3 + 0.x^2 + 3x + c = 0$ has roots α, β, γ

then $\alpha + \beta + \gamma = 0$, $\Sigma\alpha\beta = 3$, $\alpha\beta\gamma = -c$

Let $\partial_1 = (\alpha - \beta)(\beta - \gamma)$, $\partial_2 = (\beta - \gamma)(\gamma - \alpha)$,

$$\partial_3 = (\gamma - \alpha)(\alpha - \beta)$$

$$\partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1 = (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)[\alpha - \beta + \beta - \gamma + \gamma - \alpha] = 0$$

and

$$\partial_1 + \partial_2 + \partial_3 = (\alpha - \beta)(\beta - \gamma) + (\beta - \gamma)(\gamma - \alpha) + (\gamma - \alpha)(\alpha - \beta)$$

$$= \alpha\beta + \beta\gamma + \gamma\alpha - \alpha^2 - \beta^2 - \gamma^2$$

$$= 3(\alpha\beta + \beta\gamma + \gamma\alpha) - (\alpha + \beta + \gamma)^2 = 3(3) - 0 = 9$$

$$\frac{1}{27}\sum(\alpha - \beta)^2(\beta - \gamma)^2 = \frac{\partial_1^2 + \partial_2^2 + \partial_3^2}{27}$$

$$= \frac{(\partial_1 + \partial_2 + \partial_3)^2 - 2(\partial_1\partial_2 + \partial_2\partial_3 + \partial_3\partial_1)}{27} = \frac{81}{27} = 3$$

NATURE OF ROOTS

Let discuss roots of quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$,

As we know roots are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

To understand the nature of roots we need to analyze the term written in square root as $b^2 - 4ac = D$ (let Discriminant), we can rewrite roots as $x = \dfrac{-b \pm \sqrt{D}}{2a}$

Discriminant ($D = b^2 - 4ac$)	**Nature of Roots**
$D > 0$	Real and Distinct $(\alpha, \beta \in R$ and $\alpha \neq \beta)$
$D = 0$	Real and Equal $(\alpha = \beta = -\dfrac{b}{2a})$ Quadratic expression can be written as perfect square of linear polynomial $ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2$
$D < 0$	Imaginary (non-real) and Distinct $(\alpha, \beta \notin R$ and $\alpha \neq \beta)$ The roots are complex conjugates of each other $\alpha = p + iq$ and $\beta = p - iq$ where p and $q \in R$

Special Cases

Let discuss some special cases of nature of roots of quadratic equation $ax^2 + bx + c = 0$
where $a, b, c \in R$,

(a) If a, b, c are rational numbers and $D = b^2 - 4ac$ is a perfect square of a rational number then the roots are rational.

Proof:

As we know roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b \pm \sqrt{D}}{2a}$$

$D = b^2 - 4ac = a$ square of a rational number $= p^2$ where p is a rational number

$$x = \frac{-b \pm p}{2a}$$

where b, a, p are all rational numbers and addition/subtraction or division of rational numbers are also a rational number.

Hence roots are rational numbers

(b) If a, b, c are rational numbers and $D = b^2 - 4ac$ is not a perfect square then the roots are irrational conjugates of each other. If $p + \sqrt{q}$ is one root then $p - \sqrt{q}$ is other root.

Proof:

As we know roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b \pm \sqrt{D}}{2a}$$

$D = b^2 - 4ac =$ Not a perfect square of a rational number, it means square root term is going to present in roots hence

roots are $x = \dfrac{-b + \sqrt{D}}{2a}$, $x = \dfrac{-b - \sqrt{D}}{2a}$

we can conclude that If $p + \sqrt{q}$ is one root then $p - \sqrt{q}$ is other root

(c) If $a = 1$ and $b, c \in I$ and $D = b^2 - 4ac$ is a perfect square of an integer then quadratic equation has integral roots

Proof:

As we know roots of quadratic equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b \pm \sqrt{D}}{2a}$$

Given $a = 1, b, c \in \text{I}$ and $D = b^2 - 4c = k^2$ where k is an integer

CASE-1

Let b is an odd integer, $b = 2n + 1$ where $n \in I$
then $D = b^2 - 4c = (\text{odd})^2 -$ even (square of an odd number is always odd)

$= (\text{odd}) - \text{even} = (\text{odd})^2 = k^2$ where k is an odd number

Now

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\text{odd} \pm \sqrt{(\text{odd})^2}}{2}$$
$$= \frac{\text{odd} \pm \text{odd}}{2} = \frac{\text{even}}{2} = \text{integer}$$

hence both roots are integers

CASE-2

Let b is an even integer, $b = 2n$ where $n \in I$
then $D = b^2 - 4c = (\text{even})^2 -$ even (square of an even number is always even)

$= (\text{even}) - \text{even} = (\text{even})^2 = k^2$ where k is an even number

Now

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{\text{even} \pm \sqrt{(\text{even})^2}}{2}$$
$$= \frac{\text{even} \pm \text{even}}{2} = \frac{\text{even}}{2} = \text{integer}$$

hence both roots are integers

(d) If $a + b + c = 0$ then 1 is a root of the equation and if a, b, c are rational numbers then both the roots are rational.

ILLUSTRATION 48

Comment on nature of roots of following quadratic equation

(i) $x^2 - x + 1 = 0$ **(ii)** $x^2 - 3x + 1 = 0$
(iii) $x^2 - 12x + 36 = 0$ **(iv)** $-x^2 - x + 2 = 0$

SOLUTION

(i) $x^2 - x + 1 = 0$
$D = b^2 - 4ac = (-1)^2 - 4 = -3 < 0$
Distinct and imaginary roots

(ii) $x^2 - 3x + 1 = 0$
$D = b^2 - 4ac = (-3)^2 - 4(1)(1) = 5 > 0$
Distinct and real roots

(iii) $x^2 - 12x + 36 = 0$
$D = b^2 - 4ac = (-12)^2 - 4(1)\,(36) = 144 - 144 = 0$
Equal and real roots

(iv) $-x^2 - x + 2 = 0$
$a = -1, b = -1, c = 2$
$D = b^2 - 4ac = (-1)^2 - 4(-1)2 = 1 + 8 = 9 > 0$
real and distinct roots

ILLUSTRATION 49

Prove that the roots of the equation

$$(b-c)\,x^2+2(c-a)\,x+(a-b)=0 \text{ are always real.}$$

SOLUTION

Given $(b-c)x^2+2(c-a)x+(a-b)=0$

Discriminant $D=[2(c-a)]^2-4(b-c)\,(a-b)$

$$=4[(c-a)^2-(b-c)\,(a-b)]$$

$$=4[(c^2+a^2-2ac)-(ab-ac-b^2+bc)]$$

$$=4[a^2+c^2-2ac-ab+ac+b^2-bc]$$

$$=4[a^2+b^2+c^2-ac-ab-bc]$$

$$=2[(a^2+b^2+c^2-ac-ab-bc)]$$

$$=2[(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ac+a^2)]$$

$$=2[(a-b)^2+(b-c)^2+(c-a)^2]\ge 0, \text{ which is always true.}$$

$\because \quad D\ge 0 \quad \therefore \quad$ roots are real.

Hence proved

ILLUSTRATION 50

If equation $\dfrac{x^2-bx}{ax-c}=\dfrac{K-1}{K+1}$ has equal roots in magnitude but opposite in sign. Find k.

SOLUTION

$$\frac{x^2-bx}{ax-c}=\frac{K-1}{K+1}\Rightarrow (K+1)\,(x^2-bx)=(ax-c)\,(K-1)$$

$$\Rightarrow \quad (K+1)x^2-b(K+1)x=a(K-1)\,x-c(K-1)$$

$$\Rightarrow \quad (K+1)x^2-x[b(K+1)+a(K-1)]+c(K-1)=0$$

Let roots are α and $-\alpha$

$\therefore \quad$ Sum of roots $=0\Rightarrow b(K+1)+a(K-1)=0$

$$K(b+a)+b-a=0\Rightarrow K=\frac{a-b}{a+b}$$

ILLUSTRATION 51

If $c\ne 0$ and the equation $\dfrac{p}{2x}=\dfrac{a}{x+c}+\dfrac{b}{x-c}$ has two equal roots , then find p.

SOLUTION

We can write the given equation as

$$\frac{p}{2x}=\frac{(a+b)x+c(b-a)}{x^2-c^2}$$

or $\quad p(x^2-c^2)=2(a+b)x^2-2c(a-b)\,x$

or $\quad (2a+2b-p)x^2-2c(a-b)\,x+pc^2=0$

For this equation to have equal roots

Discriminant $D=B^2-4AC=0$

$$\Rightarrow \quad 4[c^2\,(a-b)^2-pc^2\,(2a+2b-p)]=0$$

$$\Rightarrow \quad (a-b)^2-2p\,(a+b)+p^2=0 \qquad [\because c^2\ne 0]$$

$$\Rightarrow \quad [p-(a+b)]^2=(a+b)^2-(a-b)^2=4ab$$

$$\Rightarrow \quad p-(a+b)=\pm 2\sqrt{ab}$$

$$\Rightarrow \quad p=a+b\pm 2\sqrt{ab}=(\sqrt{a}\pm\sqrt{b})^2$$

ILLUSTRATION 52

Let $p,\,q\in\{1,2,3,4\}$. find the number of equations of the form $px^2+qx+1=0$ having real roots.

SOLUTION

For the equation $px^2+qx+1=0$ to have real roots

$$D\ge 0\Rightarrow q^2\ge 4p$$

If $p=1$ then $q^2\ge 4\Rightarrow q=2,3,4$

If $p=2$ then $q^2\ge 8\Rightarrow q=3,4$

If $p=3$ then $q^2\ge 12\Rightarrow q=4$.

If $p=4$ then $q^2\ge 16\Rightarrow q=4$

No. of required equation = 7.

ILLUSTRATION 53

Find all the integral values of a for which the quadratic equation $(x-a)\,(x-10)+1=0$ has integral roots.

SOLUTION

Here the equation is $x^2-(10+a)\,x+10a+1=0 \qquad \ldots(1)$

Since integral roots will always be rational it means D should be a perfect square

Form (1)

$$D=(10+a)^2-4(10a+1)=a^2-20a+96$$

$$\Rightarrow \quad D=(a-10)^2-4$$

$$\Rightarrow \quad 4=(a-10)^2-D$$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a-10)^2=4$ and $D=0$

$$\Rightarrow \quad (a-10)=\pm 2 \quad \Rightarrow \quad a=12,\,8$$

Alternative Method

$(x-a)\,(x-10)=-1$

Here $(x-a)$ and $(x-10)$ are integers. So $(x-a)$ and $(x-10)$ have to be opposite in sign and their magnitude will be unity.

CASE-1: If $(x-a)=1$ then $(x-10)=-1\Rightarrow x=9,\ a=8$

CASE-2: If $(x-a)=-1$ then $(x-10)=1\Rightarrow x=11,\ a=12$

Final values are $a=8,\,12$

ILLUSTRATION 54

Find total number of values of a so that $x^2-x-a=0$ has integral roots, where $a\in N$ and $6\le a\le 100$.

SOLUTION

$$D=1+4a=\text{odd, Let } D=(2\lambda+1)^2$$

$$1+4a=1+4\lambda^2+4\lambda\Rightarrow a=\lambda(\lambda+1)$$

Since $a\in[6,100]$

$\therefore$ $a=6,12,\ldots 90$ then $D=5^2,7^2,9^2,\ldots,19^2$

$\therefore$ Thus 'a' can attain 8 different values.

ILLUSTRATION 55

If $m\in Z$ and the equation $mx^2+(2m-1)x+(m-2)=0$ has rational roots, then find m.

SOLUTION

The discriminant D of the given equation is given by

$$D=(2m-1)^2-4m\,(m-2)=4m+1$$

(odd integer as $m\in Z$)

If the given equation has rational roots, then the discriminant should be a perfect square of a rational number, say a. i.e., $4m+1=a^2\Rightarrow a$ is an odd integer

[$\because$ $4m+1$ is an odd integer]

Let $a=2n+1$, where $n\in Z$ then,

$$a^2=4m+1\Rightarrow(2n+1)^2=4m+1$$

$\Rightarrow$ $m=n\,(n+1)$, where $n\in Z$

ILLUSTRATION 56

If $3^{\sqrt{\log_3 7}}$ and $7^{\sqrt{\log_7 3}}$ are the roots of the equation $ax^2+bx+c=0$ then find the value of b^2-4ac.

SOLUTION

Let α and β are two roots of $ax^2+bx+1=0$

$$\alpha=3^{(\log_3 7)^{1/2}} \text{ and } \beta=7^{\sqrt{\log_7 3}}$$

$$\log_3\alpha=(\log_3 7)^{\frac{1}{2}}\ \Rightarrow\ \frac{\log_e\alpha}{\log_e 3}=\sqrt{\frac{\log_e 7}{\log_e 3}}$$

$\Rightarrow$ $\log_e\alpha=\sqrt{\log_e 7.\log_e 3}$

$$\log_7\beta=\sqrt{\log_7 3}\ \Rightarrow\ \frac{\log_e\beta}{\log_e 7}=\sqrt{\frac{\log_e 3}{\log_e 7}}$$

$\Rightarrow$ $\log_e\beta=\sqrt{\log_e 3\times\log_e 7}$

since α and β are equal roots of $ax^2+bx+c=0$ so $b^2-4ac=0$.

ILLUSTRATION 57

Let α,β be the roots of $ax^2+bx+c=0$; γ,δ be the roots of $px^2+qx+r=0$; and D_1, D_2 the respective discriminants of the equations. If α,β,γ and δ are in A.P., then find D_1/D_2.

SOLUTION

We have α,β be the roots of $ax^2+bx+c=0$ then $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$,

and γ,δ be the roots of $px^2+qx+r=0$ then $\gamma+\delta=-\frac{q}{p}$ and $\gamma\delta=\frac{r}{p}$.

Now as $\alpha,\beta,\gamma,\delta$ are in AP then $\beta-\alpha=\delta-\gamma$

$\Rightarrow$ $(\beta-\alpha)^2=(\delta-\gamma)^2\Rightarrow(\beta+\alpha)^2-4\alpha\beta=(\gamma+\delta)^2-4\,\gamma\delta$

$\Rightarrow$ $\frac{b^2}{a^2}-\frac{4c}{a}=\frac{q^2}{p^2}-\frac{4r}{p}\ \Rightarrow\ \frac{b^2-4ac}{a^2}=\frac{q^2-4rp}{p^2}$

$\Rightarrow$ $\frac{D_1}{a^2}=\frac{D_2}{p^2}\ \Rightarrow\ \frac{D_1}{D_2}=\frac{a^2}{p^2}$

ILLUSTRATION 58

Find real roots of $(6-x)^4+(8-x)^4=16$.

SOLUTION

Let $y=7-x$. Then the given equation become.

$$(y+1)^4+(y-1)^4=16\Rightarrow y^4+6y^2-7=0$$

$\Rightarrow$ $(y^2-1)\,(y^2+7)=0\Rightarrow y^2-1=0$ [$\because$ $y^2+7\ne 0$]

$\Rightarrow$ $y=\pm 1$

$\Rightarrow$ $7-x=\pm1\Rightarrow x=6, 8$ only

Test Your Concepts-II

(Solutions on page H.20)

1. Let a,b,c be the three roots of the equation $x^3+x^2-333x-1002=0$. If $P=a^3+b^3+c^3$ then find the value of $\frac{P}{2006}$.
2. If product of two roots of the equation $x^4-18x^3+kx^2+200x-1984=0$ is -32 then find the value of $90-k$.
3. Equation $(1+m)x^2-2(1+3m)x+(1+8m)=0$ has equal roots find m?
4. If equation $2x^2-4x+K=0$ can be written as perfect square of a linear term find K.
5. If $a,b,c\in R$, $a\ne0$ and the quadratic equation $ax^2+bx+c=0$ has no real root, then show that $(a+b+c)\,c>0$.

6. If α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then form the equation whose roots are $\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}$ and $\gamma - \frac{1}{\alpha\beta}$.

7. If α, β, γ and δ are roots of the equation $x^4 - 3x^3 + x^2 - 2x + 1 = 0$, then find the value of $\sum \alpha^3$.

Answer keys

1. 1 **2.** 4 **3.** 0,3 **4.** 2 **6.** $r^2x^3 + pr(r+1)x^2 + q(r+1)^2 x + (r+1)^3 = 0.$ **7.** 24

COMMON ROOT(S) OF QUADRATIC EQUATIONS

Consider two quadratic equations,

$$a_1x^2 + b_1x + c_1 = 0 \text{ and } a_2x^2 + b_2x + c_2 = 0$$

For Two Common Roots

- If two quadratic equation has both roots common then two equations should be identical for that, and coefficients of equations are proportional then the ratio of coefficients must be same. hence we get

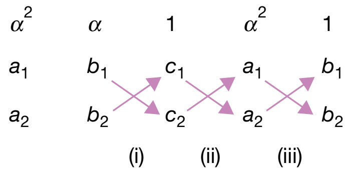
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

- Two quadratic equations with real coefficients has a non-real root as common root then both roots of given quadratic equations are common as imaginary roots always occur in pair.
- Two quadratic equations with rational coefficients has an irrational root as a common root then both roots of given quadratic equations are common as surd roots always occur in pair.

Let $x = \alpha$ be the common root of two equations. So α should satisfy the two equations.

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \text{ and } a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving the two equations



$$\text{We get } \frac{\alpha^2}{(b_1c_2 - b_2c_1)} = \frac{\alpha}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)} \quad \ldots(1)$$

$$\Rightarrow \quad \alpha^2 = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) \text{ and } \alpha = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

$$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)\,(a_1b_2 - a_2b_1)$$

It is a condition for one common root we can rewrite as

(Middle Term)^2 = (First Term) (Last Term) in reference with relation (1)

Note:

To find the common root between the two equations, remove the term containing x^2 term from both the equations and find the value of x, this is the common root.

ILLUSTRATION 59

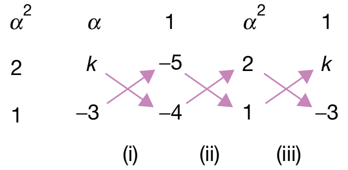
Find the value of k, so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have one root in common.

SOLUTION

Let α be the common root of the two equations
Then,

$$2\alpha^2 + k\alpha - 5 = 0$$

$$\alpha^2 - 3\alpha - 4 = 0$$



Solving these two equations, we get

$$\frac{\alpha^2}{-4k - 15} = \frac{\alpha}{-5 + 8} = \frac{1}{-6 - k}$$

$$\Rightarrow \quad \alpha^2 = \frac{4k + 15}{k + 6} \text{ and } \alpha = \frac{-3}{k + 6}$$

$$\Rightarrow \quad \left(\frac{-3}{k+6}\right)^2 = \frac{4k + 15}{k + 6} \qquad [\because\ \alpha^2 = (\alpha)^2]$$

$$\Rightarrow \quad (4k + 15)(k + 6) = 9 \Rightarrow 4k^2 + 39k + 81 = 0$$

$$\Rightarrow \quad k = -3 \text{ or } k = -\frac{27}{4}.$$

Alternate

$$x^2 - 3x - 4 = 0 \Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = -1, 4$$

let common root is $x = -1$, then is also satisfies other equation $2x^2 + kx - 5 = 0$, $2 - k - 5 = 0 \Rightarrow k = -3$
let common root is $x = 4$, then is also satisfies other equation $2x^2 + kx - 5 = 0$, $32 + 4k - 5 = 0 \Rightarrow k = -\frac{27}{4}$

ILLUSTRATION 60

If the expression $x^2 - 11x + a$ and $x^2 - 14x + 2a$ must have a common factor then find the common factor.

SOLUTION

Here let $x - \alpha$ is the common factor
then $x = \alpha$ is root of the corresponding equation

$$\therefore \quad \alpha^2 - 11\alpha + a = 0$$

$$\alpha^2 - 14\alpha + 2a = 0$$

Subtracting $3\alpha - a = 0 \Rightarrow \alpha = a/3$

Hence $\frac{a^2}{9} - 11\frac{a}{3} + a = 0$, $a = 0$ or $a = 24$

For $a = 0$, we have equations

$$x^2 - 11x = x(x - 11) \text{ and } x^2 - 14x = x(x - 14)$$

we get common factor as x
for $a = 24$,

we have equation as $\left\{\begin{array}{l} x^2 - 11x + 24 = (x-3)(x-8) \\ x^2 - 14x + 48 = (x-6)(x-8) \end{array}\right.$

we get common factor as $x - 8$
so we have solution as $x(x - 8)$

Alternate

For making calculation simple we should eliminating a from both

$$\left.\begin{array}{r} 2x^2 - 22x + 2a \\ x^2 - 14x + 2a \end{array}\right\} x^2 - 8x = 0 \quad \Rightarrow \quad x(x-8) = 0$$

Common factors are x and $(x - 8)$

ILLUSTRATION 61

If the equations $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root and $b \neq c$, then show that equation formed by non-common roots of given equations is $x^2 + ax + bc = 0$.

SOLUTION

Let α, β be the roots of $x^2 + bx + ca = 0$ and α, γ be the roots of $x^2 + cx + ab = 0$
Then $\alpha^2 + b\alpha + ca = 0$ and $\alpha^2 + c\alpha + ab = 0$

$$\Rightarrow \quad (b - c)\alpha + a(c - b) = 0 \Rightarrow \alpha = a$$

Also $\alpha\beta = ca$ and $\alpha\gamma = ab$. Therefore, $\beta = c$ and $\gamma = b$. Since $\alpha = a$ is a root of the equation $x^2 + bx + ca = 0$ we get

$$a^2 + ba + ca = 0 \Rightarrow a + b + c = 0$$

Thus $\beta + \gamma = b + c = -a$ and $\beta\gamma = bc$
Hence, β and γ are roots of the equation $x^2 + ax + bc = 0$
Hence proved.

ILLUSTRATION 62

If equations $x^2 - x - b = 0$ and $x^2 - (b + 1)x + 2 = 0$ have a common root, then find b.

SOLUTION

Let α be a common root

$$\Rightarrow \quad \alpha^2 - \alpha - b = 0 \text{ and } \alpha^2 - (b+1)\alpha + 2 = 0$$

We get $\alpha = \frac{b+2}{b}$ now from first equation

$$\Rightarrow \quad \left(\frac{b+2}{b}\right)^2 - \left(\frac{b+2}{b}\right) - b = 0$$

$$\Rightarrow \quad \frac{b^2 + 4 + 4b - b^2 - 2b - b^3}{b^2} = 0$$

$$\Rightarrow \quad b^3 - 2b - 4 = 0 = (b-2)(b^2 + 2b + 2) = 0 \Rightarrow b = 2$$

ILLUSTRATION 63

If $x^2 + 3x + 5$ is the greatest common divisor of $(x^3 + ax^2 + bx + 1)$ and $(2x^3 + 7x^2 + 13x + 5)$ then find the value of $[a + b]$.

SOLUTION

$$2(x^3 + ax^2 + bx + 1) - (2x^3 + 7x^2 + 13x + 5)$$

$$= (2a - 7)x^2 + (2b - 13)x - 3$$

Now, $(2a - 7)x^2 + (2b - 13)x - 3 = 0$ and $x^2 + 3x + 5 = 0$

have both roots common $\frac{2a-7}{1} = \frac{2b-13}{3} = \frac{-3}{5}$

$$2a - 7 = \frac{-3}{5} \Rightarrow a = \frac{35-3}{10} = 3.2$$

$$2b - 13 = \frac{-9}{5} \Rightarrow b = \frac{56}{10} = 5.6$$

Hence, $[a + b] = 8$

ILLUSTRATION 64

If equations $x^3 + 3px^2 + 3qx + r = 0$ and $x^2 + 2px + q = 0$ have a common root, show that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$.

SOLUTION

Let α be the common root, then

$$\alpha^3 + 3p\alpha^2 + 3q\alpha + r = 0 \qquad \ldots(1)$$

and $\alpha^2 + 2p\alpha + q = 0$ …(2)

From (1) − α (2),

we get $p\alpha^2 + 2q\alpha + r = 0$ …(3)

From (2) and (3),

$$\frac{\alpha^2}{2(pr - q^2)} = \frac{\alpha}{pq - r} = \frac{1}{2(q - p^2)}$$

From first and second $\alpha = \frac{2(pr - q^2)}{pq - r}$ …(4)

From second and third, $\alpha = \frac{pq - r}{2(q - p^2)}$ …(5)

From (4) and (5),

we get $4(q^2 - pr)(p^2 - q) = (pq - r)^2$

ILLUSTRATION 65

A quadratic polynomial $P(x)$ is such that $P(x) = 0$ and $P(P(P(x))) = 0$ have a common root, then find the value of $P(0) \cdot P(1)$.

SOLUTION

Let $P(x) = x^2 + ax + b$, and let r be a common root.

$\therefore \quad P(r) = 0$ and $P(P(P(r))) = 0$ i.e., $P(P(0)) = 0$, $P(0) = b$

$\therefore \quad P(b) = 0 \Rightarrow b^2 + ab + b = 0$

$\Rightarrow \quad b(a + b + 1) = 0 \Rightarrow P(0) \times P(1) = 0$

ILLUSTRATION 66

If the equations $x^3 - x - 6 = 0$ and $x^2 - bx - c = 0$ $(b, c \in R)$ have exactly one root in common then find the value of $(2b + c)$.

SOLUTION

As, $x^3 - x - 6 = 0$

$$\Rightarrow \quad (x - 2)\underbrace{(x^2 + 2x + 3)}_{\text{Non-real roots}} = 0 \qquad \dots(1)$$

and $x^2 - bx - c = 0$ $(b, c \in R)$ $\qquad \dots(2)$

As, exactly one root is common, so common roots can't be non-real, other wise two roots will be common.
So, common root = 2
Hence, $(2)^2 - b(2) - c = 0 \Rightarrow (2b + c) = 4$.

ILLUSTRATION 67

If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + 2 = 0$ have a common root and $a, b \in R$, then find the value of $a + b$.

SOLUTION

Discriminant of $x^2 + 3x + 5 = 0$ is negative.

Therefore the roots of $x^2 + 3x + 5 = 0$ are imaginary conjugate pairs. Since the equations have a common root, the common root is imaginary and hence both the roots are common. Therefore the coefficients of the equations are proportional

$$\therefore \quad \frac{a}{1} = \frac{b}{3} = \frac{2}{5}$$

$$\therefore \quad a = \frac{2}{5} \text{ and } b = \frac{6}{5}$$

$$\therefore \quad a + b = \frac{8}{5}$$

ILLUSTRATION 68

If the equation $x^2 + 2\alpha x + \alpha^2 - 1 = 0$ and $x^2 + 2\beta x + \beta^2 - 1 = 0$ have a common root $(\alpha \neq \beta)$ then find the value of the expression $2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2$.

SOLUTION

Subtracting the two equation we get the common root as $x = -\frac{1}{2}(\alpha + \beta)$ substituting this in any equation we get $(\alpha - \beta)^2 = 4$ i.e. $|\alpha - \beta| = 2$

Now

$$2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2 = 2\alpha^2 - 4\alpha\beta + 2\beta^2 - 2$$
$$= 2(\alpha - \beta)^2 - 2 = -2.4 - 2 = 8 - 2 = 6$$

ILLUSTRATION 69

Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$), then find the value of p.

SOLUTION

Since cubic is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$. Therefore, $x^2 + ax + b$ and $x^2 + bx +$ a must have a common roots

$$x^2 + ax + b = 0$$
$$x^2 + bx + a = 0$$

subtract

$\Rightarrow \quad x(a - b) = (a - b) \Rightarrow x = 1$

Therefore, common roots

for $x^2 + ax + b = 0$ are 1 and $\alpha \Rightarrow 1.\alpha = b \Rightarrow \alpha = b$

for $x^2 + bx + a = 0$ are 1 and $\beta \Rightarrow 1.\beta = a \Rightarrow \beta = a$

roots of cubic $x^3 + px^2 + qx + 72 = 0$ be $1, a, b$

$\Rightarrow \quad 1 \times a \times b = -72 \qquad \dots(1)$

and $\quad a + b + 1 = 0 \qquad \dots(2)$

(from $x^2 + ax + b = 0$ put $x = 1$)

$$\therefore \quad a - \frac{72}{a} = -1$$

$\Rightarrow \quad a^2 + a - 72 = 0 \Rightarrow (a + 9)(a - 8) = 0 \Rightarrow a = -9, 8$

Therefore roots are $1, -9, 8$

Now sum of roots $= -p$

$\Rightarrow \quad 1 - 9 + 8 = -p$

$\Rightarrow \quad p = 0$

GREATEST AND LEAST VALUE OF QUADRATIC EXPRESSION

Expression of the form $ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$ is called a quadratic expression (quadratic polynomial) in x so in general quadratic expression is represented by

$$y = f(x) = ax^2 + bx + c$$

$$= a\left\{\left(x+\frac{b}{2a}\right)^2+\left(\frac{4ac-b^2}{4a^2}\right)\right\} = a\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a}$$

where $D = b^2 - 4ac$

$$\Rightarrow \quad \left(y+\frac{D}{4a}\right) = a\left(x+\frac{b}{2a}\right)^2$$

Let $x+\frac{b}{2a} = X$ and $y+\frac{D}{4a} = Y$

$\Rightarrow \quad Y = aX^2 \Rightarrow X^2 = \frac{Y}{a}$ hence the shape of $y = f(x)$ is a parabola

For vertex

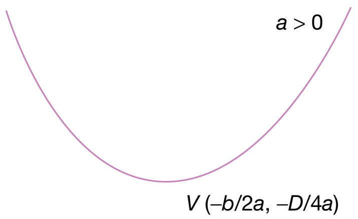
$$X = 0 \Rightarrow x+\frac{b}{2a} = 0 \Rightarrow x = -\frac{b}{2a} \text{ and}$$

$$Y = 0 \Rightarrow y+\frac{D}{4a} = 0 \Rightarrow y = -\frac{D}{4a}$$

Vertex $V\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is an extreme point

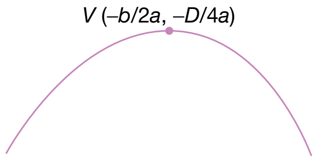
(a) If $a > 0$, the parabola opens upwards and vertex is minimum point of quadratic expression.
[least value of Quadratic Expression]

$f(x)$ is least at vertex hence when at $x = -\frac{b}{2a}$ then $y = -\frac{D}{4a}$ is minimum value of quadratic expression.



(b) If $a < 0$, then the parabola opens downwards and vertex is maximum point of quadratic expression
[Greatest value of Quadratic Expression]

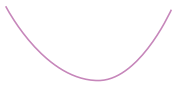
$f(x)$ is greatest at vertex hence when at $x = -\frac{b}{2a}$ then $y = -\frac{D}{4a}$ is maximum value of quadratic expression.



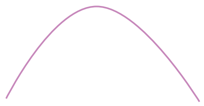
Graph of Quadratic Expression

Let $y = f(x) = Ax^2 + Bx + C$ where $A, B, C \in R$ and $A \neq 0$, four plotting any quadratic expression we just need four steps

(I) Observe the sign of A (coefficient of x^2)
if $A > 0$ (upward Parabola)



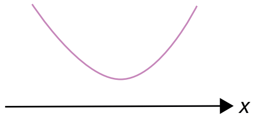
If $A < 0$ (Downward Parabola)



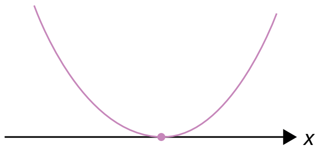
(II) Get discriminant $D = B^2 - 4AC$ then exactly one case can happen among following 6 cases depending upon the value of a and D

CASE-1: ($A > 0$)

(i) $D < 0$ (Imaginary Roots)

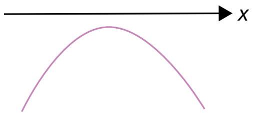


(ii) $D = 0$ (Real and Equal Roots)

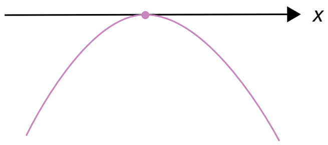


CASE-2: ($A < 0$)

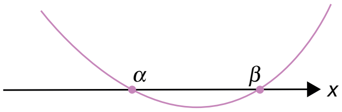
(i) $D < 0$ (Imaginary Roots)



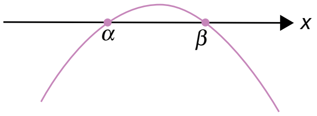
(ii) $D = 0$ (Real and Equal Roots)



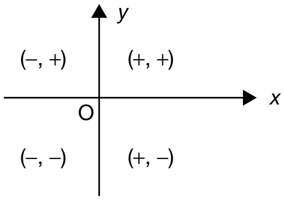
(iii) $D > 0$ (Real and Distinct Roots)



(iii) $D > 0$ (Real and Distinct Roots)



(III) Get vertex $V\left(-\frac{B}{2A}, -\frac{D}{4A}\right)$ and locate it in xy plane and identify the quadrant.



(IV) Quadratic expression $y = Ax^2 + Bx + C$ cuts y-axis at point $(0, C)$

ILLUSTRATION 70

Plot following quadratic Expression

(i) $y = 2x^2 - 4x + 7$

(ii) $y = 2x^2 - 3x - 1$

(iii) $y = -x^2 + x - 1$

(iv) $y = -x^2 + 2\sqrt{2}\,x - 2$

SOLUTION

(i) $y = 2x^2 - 4x + 7$

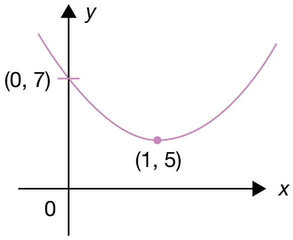
$A = 2 > 0$ upward parabola

$D = B^2 - 4AC = (-4)^2 - 4(2)7 = -40 < 0$ imaginary roots

Vertex $\left(\frac{-B}{2A}, -\frac{D}{4A}\right) = \left(\frac{4}{4}, -\frac{(-40)}{8}\right) = (1, 5)$

It lies in Ist quadrant

It cuts y-axis at point $(0, 7)$



(ii) $y = 2x^2 - 3x - 1$

$A = 2, B = -3, C = -1$

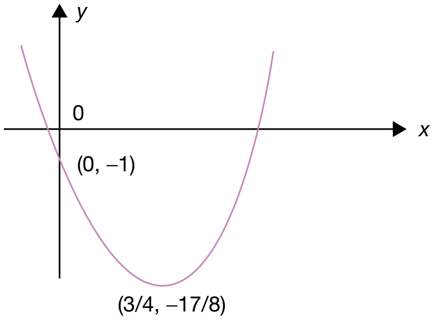
$A = 2 > 0$ upward parabola.

$D = B^2 - 4AC = (-3)^2 - 4(2)(-1) = 17 > 0$ real and distinct roots

Vertex $\left(\frac{-B}{2A}, -\frac{D}{4A}\right) = \left(\frac{3}{4}, -\frac{17}{8}\right)$

It lies in IVth quadrant

It cuts y-axis at point $(0, -1)$



(iii) $y = -x^2 + x - 1$

$A = -1 < 0$ downward parabola.

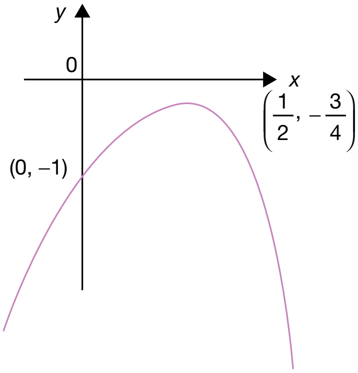
$B = 1, C = -1$

$D = B^2 - 4AC = 1 - 4 = -3 < 0$ imaginary roots

Vertex $\left(\frac{-B}{2A}, -\frac{D}{4A}\right) = \left(\frac{1}{2}, -\frac{3}{4}\right)$

It lies in IVth quadrant

It cuts y-axis at $(0, -1)$



(iv) $y = -x^2 + 2\sqrt{2}\,x - 2$

$A = -1 < 0$ downward parabola.

$B = 2\sqrt{2}, C = -2$

$D = B^2 - 4AC = 8 - 8 = 0$ real and equal roots

Vertex $\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = \left(\sqrt{2}, 0\right)$

It cuts y-axis at $(0, -2)$

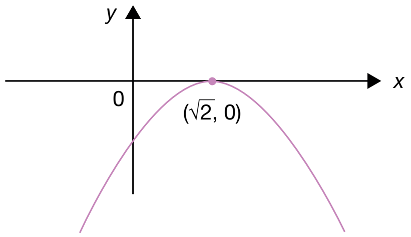
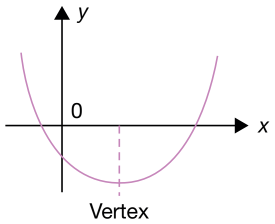


ILLUSTRATION 71

Observe following given graph of quadratic expression $y = ax^2 + bx + c$ and comment on sign of a, b, c and D (discriminant).

SOLUTION

(i)



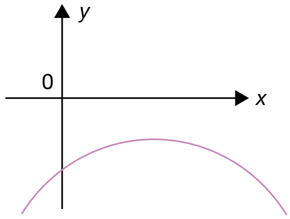
$a > 0$ as it is upward parabola.
$c < 0$ It cuts y-axis below origin

abscissa (x-coordinate) of vertex $> 0 \Rightarrow \dfrac{-b}{2a} > 0$

$\Rightarrow b < 0$

It cuts x-axis at two distinct real points it means $D >$ 0(real and distinct roots)

(ii)



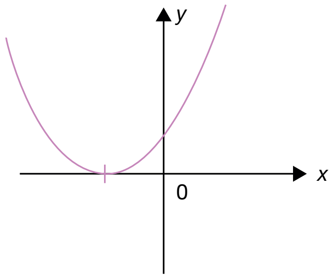
$a < 0$ as it is downward parabola.
$c < 0$ It cuts y-axis below origin

abscissa (x -coordinate) of vertex $> 0 \Rightarrow \dfrac{-b}{2a} > 0$

$\Rightarrow b > 0$

Graph is below the x-axis, it means $D < 0$ (Imaginary roots)

(iii)



$a > 0$ as it is upward parabola.
$c > 0$ It cuts y-axis above origin

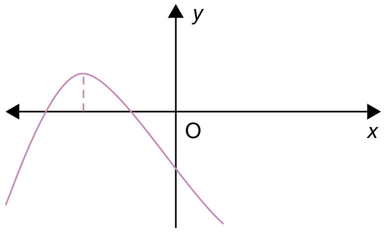
abscissa (x -coordinate) of vertex $< 0 \Rightarrow \dfrac{-b}{2a} < 0$

$\Rightarrow b > 0$

It touches x-axis, it means $D = 0$ (real and equal roots)

ILLUSTRATION 72

Observe the given graph of expression $y = ax^2 + bx + c$, comment on sign of abc?



SOLUTION

As given curve is downward parabola so $a < 0$
Graph cuts curve below origin on y axis so $c < 0$
abscissa of vertex < 0

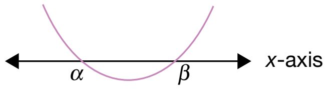
$\Rightarrow \dfrac{-b}{2a} < 0 \Rightarrow b < 0$ Now $abc < 0$

Analysis of Quadratic Expression

$$y = f(x) = ax^2 + bx + c (a \neq 0)$$

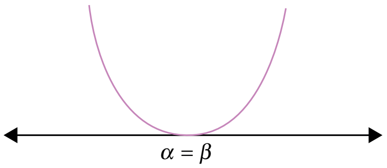
(a) $a > 0, D > 0$ (real and distinct)

$f(x) > 0$ for $x \in (-\infty, \alpha) \cup (\beta, \infty)$
$f(x) < 0$ for $x \in (\alpha, \beta)$



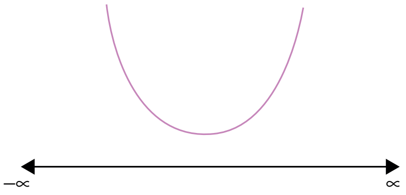
(b) $a > 0, D = 0$ (real and equal roots)

$f(x) \geq 0$ for all $x \in R$
$f(x) < 0$ for $x \in \phi$



(c) $a > 0; D < 0$ (imaginary roots)

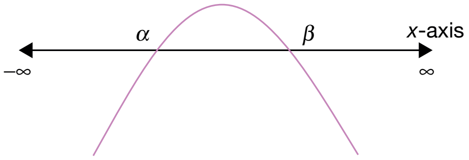
$f(x) > 0$ for $\forall\ x \in R$
$f(x) < 0$ $x \in \phi$



(d) $a < 0; D > 0$ (real and distinct roots)

$f(x) > 0$ for $x \in (\alpha, \beta)$

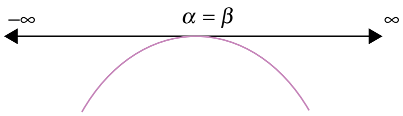
$f(x) < 0$ for $x \in (-\infty, \alpha) \cup (\beta, \infty)$



(e) $a < 0, D = 0$ (real and equal)

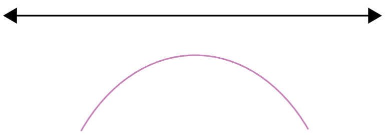
$f(x) > 0$ for $x \in \phi$

$f(x) \le 0$ for $\forall\ x \in R$



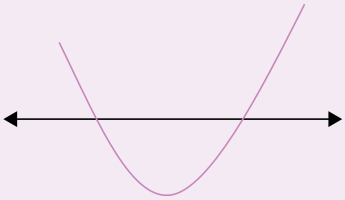
(f) $a < 0; D = 0$ (Imaginary roots)

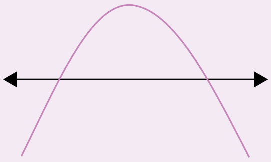
$f(x) < 0$ for $\forall\ x \in R$



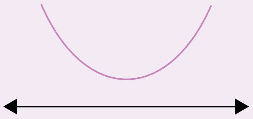
Remark(s)

(A) Quadratic expression $y = ax^2 + bx + c$ changes its sign then following two cases are possible
Condition: Discriminant $D > 0$

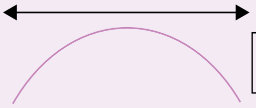




(B) Quadratic expression $y = ax^2 + bx + c$ does not change its sign then following cases are possible



$y = ax^2 + bx + c > 0$ for $\forall\ x \in R$
Condition $a > 0$ and $D < 0$



$y = ax^2 + bx + c > 0$ for $\forall\ x \in R$
Condition $a < 0$ and $D < 0$

ILLUSTRATION 73

Find integer k for which the inequality

$x^2 - 2(4k - 1)\,x + 15k^2 - 2k - 7 > 0$ is valid for any real x.

SOLUTION

Let $f(x) = x^2 - 2\,(4k - 1)x + 15k^2 - 2k - 7$

If $f(x) > 0$ for all real x then Discriminant $D < 0$ as coefficient of $x^2 > 0$

$\Rightarrow\quad 4\,(4k - 1)^2 - 4\,(15k^2 - 2k - 7) < 0$

$\Rightarrow\quad k^2 - 6k + 8 < 0 \quad\Rightarrow\quad 2 < k < 4.$

Integral $k = 3$

ILLUSTRATION 74

Find the least integral value of 'a' such that $(a-3)x^2 + 12x + (a+6) > 0\ \forall x \in R$.

SOLUTION

$$ax^2 + bx + c > 0\ \forall x \in R \Rightarrow a > 0,\ D < 0$$

$\Rightarrow\quad$ (i) $a - 3 > 0$ (ii) $D < 0 \Rightarrow 144 - 4(a - 3)(a + 6) < 0$

$\Rightarrow\quad a^2 + 3a - 54 > 0 = (a + 9)(a - 6) > 0,\ a \in (-\infty, -9) \cup (6, \infty)$

Common values $a > 6$
Least integral value of $a = 7$

ILLUSTRATION 75

If $(a - 1)x^2 - (a + 1)x + (a + 1) > 0\ \forall\ x \in R$, then find the range of a.

SOLUTION

$$(a - 1)x^2 - (a + 1)x + (a + 1) > 0\ \forall x \in R$$

$$A = a - 1 > 0 \text{ and } D < 0$$

$\Rightarrow\quad a > 1$ and $(a + 1)^2 - 4(a + 1)\,(a - 1) < 0$

$\Rightarrow\quad a > 1$ and $(a + 1)\,(-3a + 5) < 0$

$\Rightarrow\quad a > 1$ and $(a + 1)\,(3a - 5) > 0$

$\Rightarrow\quad a > 1$ and $a \in (-\infty, -1) \cup \left(\dfrac{5}{3}, \infty\right)$

$\Rightarrow\quad a \in \left(\dfrac{5}{3}, \infty\right)$

ILLUSTRATION 76

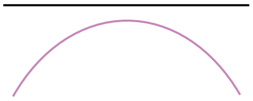
If $c < 0$ and quadratic equation $ax^2 + bx + c = 0$ does not have real roots then find sign of

(i) $a - b + c$; **(ii)** $4a - 2b + c$

SOLUTION

As quadratic equation $ax^2 + bx + c = 0$ does not have real roots it means it is either entirely above x axis or below x axis.

Given that $f(0) = 0 + 0 + c < 0$, it means it below x axis as shown in graph



It means $f(x) = ax^2 + bx + c < 0$ always $\forall\ x \in R$

put $x = -1, f(-1) = a - b + c < 0$

$$f(-2) = 4a - 2b + c < 0$$

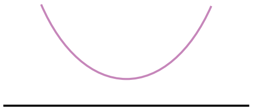
ILLUSTRATION 77

If $c > 0$ and $ax^2 + 2bx + 3c = 0$ does not have any real root then comment on sign of $4a - 4b + 3c$.

SOLUTION

As quadratic equation $ax^2 + 2bx + 3c = 0$ does not have real roots it means it is either entirely above x axis or below x axis.

Given that $f(0) = 0 + 0 + c > 0$, it means it is above x axis as shown in graph



It means $f(x) = ax^2 + 2bx + 3c > 0$ always $\forall\ x \in R$

$$f(-2) = 4a - 4b + 3c > 0$$

ILLUSTRATION 78

Find the number of integral value of 'b' for which inequality $(a^2 + 1)x^2 + 4(a + b)x + 2 < 0$ is true for atleast one 'x' and $\forall\ a \in R$.

SOLUTION

$(a^2 + 1)x^2 + 4(a + b)x + 2 = 0$ has real roots.

$\therefore$ $D > 0 \Rightarrow 8(2(a + b)^2 - (a^2 + 1)) > 0$

$\Rightarrow$ $a^2 - (4b)a + 2b^2 - 1 > 0$ which is a quadratic in a and true for every real a $\quad \forall\ a \in R$

$D' < 0$

$\Rightarrow$ $16b^2 - 4(2b^2 - 1) < 0 \Rightarrow 8b^2 + 4 < 0$ no solution

ILLUSTRATION 79

Find the smallest integral value of λ for which

$(\lambda - 2)x^2 + 4(2x + 1) + \lambda > 0$ for all real values of x.

SOLUTION

$(\lambda - 2)x^2 + 4(2x + 1) + \lambda > 0$

$\Rightarrow$ $(\lambda - 2)x^2 + 8x + \lambda + 4 > 0\ \forall\ x \in R$

(i) $\lambda - 2 > 0 \Rightarrow \lambda > 2$ ...(1)

(ii) $D < 0 \Rightarrow 64 - 4(\lambda - 2)(\lambda + 4) < 0$

$\Rightarrow$ $16 - (\lambda^2 + 4\lambda - 2\lambda - 8) < 0$

$\Rightarrow$ $-\lambda^2 - 2\lambda + 24 < 0 \Rightarrow \lambda^2 + 2\lambda - 24 > 0$

$\Rightarrow$ $(\lambda - 4)(\lambda + 6) > 0$

$\Rightarrow$ $\lambda \in (-\infty, -6) \cup (4, \infty)$ ...(2)

$\therefore$ $\lambda \in (4, \infty)$

$\therefore$ Smallest integer value of λ is 5.

ILLUSTRATION 80

Find the set of real values of 'a' for which the graph of the function $f(x) = \dfrac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always below the x-axis.

SOLUTION

$f(x) < 0$ and since $x^2 - 8x + 32 > 0, \forall\ x \in R$

$\Rightarrow$ $ax^2 + 2(a+1)x + 9a + 4 < 0, \forall\ x \in R$

$\Rightarrow$ $a < 0$ and $D = 4(a+1)^2 - 4a(9a+4) < 0$

$\Rightarrow$ $8a^2 + 2a - 1 > 0$

$\Rightarrow$ $(2a+1)(4a-1) > 0$

$\Rightarrow$ $a \in \left(-\infty, -\dfrac{1}{2}\right) \cup \left(\dfrac{1}{4}, \infty\right)$

So acceptable values of 'a' are;

$a \in \left(-\infty, -\dfrac{1}{2}\right)$ (since $a < 0$)

ILLUSTRATION 81

Let $x^2 + 2y^2 - 2ky - 2 \ge x(k + 2y)\ \forall\ x \in R$ then find the number of integral values of k.

SOLUTION

$x^2 - (2y + k)x + 2y^2 - 2ky - 2 \ge 0\ \forall\ x \in R$

$D \le 0 \Rightarrow (2y + k)^2 - 4(2y^2 - 2ky - 2) \le 0$

$\Rightarrow$ $4y^2 + k^2 + 4ky - 8y^2 + 8ky + 8 \le 0$

$\Rightarrow$ $-4y^2 + 12ky + k^2 + 8 \le 0$

$\Rightarrow$ $4y^2 - 12ky - (k^2 + 8) \ge 0\ \forall\ x \in R$

$D \le 0 \Rightarrow 144k^2 + 4.4.(k^2 + 8) \le 0$ (not True)

$\therefore$ $k \in \phi$.

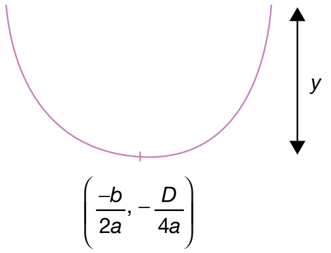
RANGE OF QUADRATIC EXPRESSION

Let the quadratic expression is $y = ax^2 + bx + c\ (a \ne 0)$
As x varies then set of values of y is known as range

Range of Quadratic Expression for all Real Values of *x*

CASE-1 for $a > 0$

If all real x, complete range of quadratic expression as we can see from graph is $y \in \left[-\frac{D}{4a}, \infty\right)$



CASE-2 for $a < 0$

If all real x, complete range of quadratic expression as we can see from graph is $y \in \left(-\infty, -\frac{D}{4a}\right]$

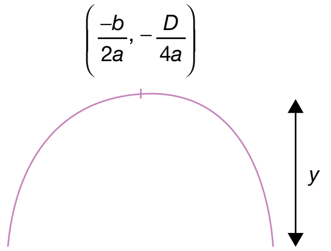


ILLUSTRATION 82

For quadratic expression $y = x^2 - 2x + 7$, find range for all real x.

SOLUTION

We have $a = 1, b = -2, c = 7$

$$D = b^2 - 4ac = 4 - 4(1)7 = 4(-6) = -24$$

vertex of quadratic is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right) = \left(\frac{2}{2}, \frac{24}{4}\right) = (1, 6)$

Range of quadratic expression is $y \in [6, \infty)$

ILLUSTRATION 83

For quadratic expression $y = -2x^2 + x - 4$, find complete range for all real x?

SOLUTION

We have $a = -2 < 0; b = 1; c = -4$

$$x = -\frac{b}{2a} = \frac{-1}{-4} = \frac{1}{4}$$

$$D = b^2 - 4ac = 1 - 4(-2)(-4) = -31$$

vertex is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right) = \left(\frac{1}{4}, -\frac{31}{8}\right)$

Required range $y \in \left(-\infty, \frac{-31}{8}\right]$

Range of Quadratic Expression in Bounded Interval of *x*

A quadratic expression $y = f(x) = ax^2 + bx + c$ where $x \in [x_1, x_2]$

x-coordinate of vertex $x_0 = -\frac{b}{2a}$

Range of y if $x \in [x_1, x_2]$

$x_0 = \frac{-b}{2a} \in [x_1, x_2]$

Range = $\left[\text{min of }\left\{\frac{-D}{4a}, f(x_1), f(x_2)\right\}, \text{max of }\left\{\frac{-D}{4a}, f(x_1), f(x_2)\right\}\right]$

$x_0 = \frac{-b}{2a} \notin [x_1, x_2]$

Range = $\left[\text{min of}\left\{f(x_1), f(x_2)\right\}, \text{max of }\left\{f(x_1), f(x_2)\right\}\right]$

ILLUSTRATION 84

Quadratic expression $y = 2x^2 - 3x + 2$. Find range if

(i) $x \in [-2, 2]$ **(ii)** $x \in [1, 2]$

SOLUTION

(i) $x \in [-2, 2]$

$$x_0 = -\frac{b}{2a} = -\frac{(-3)}{2(2)} = \frac{3}{4} \in [-2, 2]$$

$$f(-2) = 8 + 6 + 2 = 16$$

$$f(2) = 8 - 6 + 2 = 4$$

$$f\left(\frac{3}{4}\right) = 2\left(\frac{9}{16}\right) - \frac{9}{4} + 2 = 2 - \frac{9}{8} = \frac{7}{8}$$

Required range $y \in \left[\frac{7}{8}, 16\right]$

(ii) $x \in [1, 2]$

$$f(x) = 2x^2 - 3x + 2$$

$$x_0 = -\frac{b}{2a} = \frac{3}{4} \notin [1, 2]$$

$$f(1) = 2 - 3 + 2 = 1$$

$$f(2) = 8 - 6 + 2 = 4$$

Required range $y \in [1, 4]$

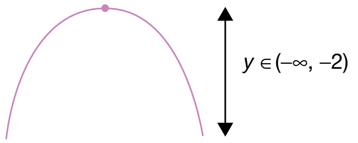
ILLUSTRATION 85

Quadratic expression $y = -x^2 + 4x - 6$ find range if

(i) $x \in R$ **(ii)** $x \in (0, 4]$ **(iii)** $x \in [-3, -1)$

SOLUTION

(i)



$x \in R$

$$D = B^2 - 4AC = 16 - 4\,(-1)\,(-6) = -8$$

$$\text{Vertex}\left(-\frac{b}{2a}, -\frac{D}{4a}\right) = \left(\frac{-4}{-2}, \frac{8}{-4}\right) = (2, -2)$$

Range $y \in (-\infty, -2]$

(ii) $x_0 = -\dfrac{b}{2a} = 2 \in (0, 4]$

$$f(2) = -2$$

$$f(0) = -6$$

$$f(4) = -16 + 16 - 6 = -6$$

Required range $y \in [-6, -2]$

(iii) $x \in [-3, -1)$

$$f(x) = -x^2 + 4x - 6$$

$$x_0 = -\frac{b}{2a} = 2 \notin [-3, -1)$$

$$f(-3) = -9 - 12 - 6 = -27$$

$$f(-1) = -1 - 4 - 6 = -11$$

Required range $y \in [-27, -11)$

ILLUSTRATION 86

If the quadratic polynomial $f(x) = (a - 3)x^2 - 2ax + 3a - 7$ ranges from $[-1, \infty)$ for every $x \in R$, then find the value of a.

SOLUTION

$$f(x) = (a - 3)x^2 - 2ax + 3a - 7$$

Since range of $f(x) \in [-1, \infty]\ \forall\ x \in R$

Hence range of $(a - 3)x^2 - 2ax + 3a - 6 \in [0, \infty]$

$\therefore \quad a - 3 > 0$ and $D = 0$

$\Rightarrow \quad 4a^2 - 4(a - 3)(3a - 6) = 0$

$\Rightarrow \quad 2a^2 - 15a + 18 = 0 \quad \Rightarrow \quad (a - 6)(2a - 3) = 0$

$\Rightarrow \quad a = 6, \dfrac{3}{2}$ but $a > 3$

We get $a = 6$ only

ILLUSTRATION 87

Let $f(x) = x^2 + ax + b$. If the maximum and the minimum values of $f(x)$ are 3 and 2 respectively for $0 \le x \le 2$ then find the possible value of b if $a = -2$.

SOLUTION

$$f(x) = x^2 + ax + b$$

Vertex of $f(x)$ is at $x = \dfrac{-a}{2} = 1$ it lies in given interval of x

$$f(x)]_{\min} = f\left(-\frac{a}{2}\right) = \frac{a^2}{4} - \frac{a^2}{2} + b = b - \frac{a^2}{4} = 2 \text{ (given)} \quad \ldots(1)$$

we get $b = 3$

and $f(x)]_{\max} = f(0)$ or $f(2)$

But $f(0) = b = 3$ and $f(2) = 4 + 2a + b = 4 - 4 + 3 = 3$

Final $b = 3$.

ILLUSTRATION 88

If α and β (where $\alpha, \beta \in R$) are the roots of equation $x^2 + 4x + b = 0$ and $|\alpha - \beta| \le 2$, then find the range of b.

SOLUTION

α and β are real

$\therefore \quad D \ge 0 \quad \Rightarrow \quad 16 - 4b \ge 0 \quad \Rightarrow \quad b \le 4 \quad \ldots(1)$

$\quad |\alpha - \beta| \le 2 \quad \Rightarrow \quad (\alpha - \beta)^2 \le 4$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta \le 4$

$\Rightarrow \quad 16 - 4b \le 4$

$\Rightarrow \quad 4b \ge 12 \quad \Rightarrow \quad b \ge 3 \quad \ldots(2)$

$\therefore$ From (1) and (2) $b \in [3, 4]$

ILLUSTRATION 89

Find the range of $y = x + \dfrac{1}{x}$ where $x \in R_0$.

SOLUTION

$$y = x + \frac{1}{x} = \frac{x^2 + 1}{x} \quad \Rightarrow \quad yx = x^2 + 1$$

$\Rightarrow \quad x^2 - yx + 1 = 0$

where x is a non-zero real roots real

$$D \ge 0 \quad \Rightarrow \quad y^2 - 4 \ge 0 \quad \Rightarrow \quad (y + 2)\,(y - 2) \ge 0$$

Range $y \in (-\infty, -2] \cup [2, \infty)$

ILLUSTRATION 90

Let α, β are two real roots of equation $x^2 + ax + b = 0$ ($a, b \in R$ and $b \neq 0$). If the quadratic equation $g(x) = 0$ has two roots $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$ such that sum of roots is equal to product of roots, then find the range of b.

SOLUTION

$x^2 + ax + b = 0$ has root α and β

$$\alpha + \beta = -a;\ \alpha\beta = b \qquad \ldots(1)$$

$g(x) = 0$ has two roots $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$

sum of roots = product of roots

$$\alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$$

$$\Rightarrow \quad \alpha + \beta + \frac{(\alpha+\beta)}{\alpha\beta} = \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$$

Using equation (1)

$$\Rightarrow \quad -a - \frac{a}{b} = b + \frac{a^2 - 2b}{b} + \frac{1}{b}$$

$$\Rightarrow \quad a^2 + a(b+1) + (b^2 - 2b + 1) = 0$$

$D \geq 0$ (Quadratic in a)

$$(b+1)^2 - 4(b^2 - 2b + 1) \geq 0$$

$$\Rightarrow \quad (3b-1)(b-3) \leq 0 \Rightarrow b \in \left[\frac{1}{3}, 3\right]$$

ILLUSTRATION 91

Find the values of a for which the expression $\frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$ assumes all real values for real values of x.

SOLUTION

Let $y = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$. Then,

$$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$$

Since x is real. So, roots of the above equation are real.

$\therefore$ Discriminant $D \geq 0$ for all $y \in R$

$$\Rightarrow \quad 9(1-y)^2 + 4(a+4y)(4+ay) \geq 0 \text{ for all } y \in R$$

$$\Rightarrow \quad (9 + 16a)y^2 + (4a^2 + 46)y + (9 + 16a) \geq 0 \text{ for } y \in R$$

$$\Rightarrow \quad A = 9 + 16a > 0 \text{ and Discriminant} \leq 0$$

$$\Rightarrow \quad (4a^2 + 46)^2 - 4(9 + 16a)(9 + 16a) \leq 0$$

$$\Rightarrow \quad 4(2a^2 + 23)^2 - 4(9 + 16a)^2 \leq 0$$

$$\Rightarrow \quad (2a^2 + 23)^2 - (9 + 16a)^2 \leq 0$$

$$\Rightarrow \quad (2a^2 + 16a + 32)(2a^2 - 16a + 14) \leq 0$$

$$\Rightarrow \quad 4(a+4)^2(a^2 - 8a + 7) \leq 0$$

$$\Rightarrow \quad a^2 - 8a + 7 \leq 0$$

$$\Rightarrow \quad (a-1)(a-7) \leq 0 \quad \Rightarrow \quad 1 \leq a \leq 7.$$

We get $9 + 16a > 0$ and $1 \leq a \leq 7$. final values $1 \leq a \leq 7$.
Thus given expression assumes all real values for all real values of x, if $a \in [1, 7]$.

ILLUSTRATION 92

If x is real, prove that $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ cannot have value between 5 and 9.

SOLUTION

Let $y = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$

$$\Rightarrow \quad x^2(y-1) + 2x(y-17) - (7y-71) = 0$$

CASE-1: $y = 1$

Then $0 + 2x(-16) - (-64) = 0 \quad \Rightarrow \quad x = 2$ real

CASE-2: $y \neq 1$

$$x^2(y-1) + 2x(y-17) - (7y-71) = 0$$

Then it is a quadratic equation with x as real, it means above equation has real roots So,
Discriminant $D \geq 0$

$$\Rightarrow \quad 4(y-17)^2 + 4(y-1)(7y-71) \geq 0$$

$$\Rightarrow \quad 4(8(y^2 - 14y + 45)) \geq 0 \quad \Rightarrow \quad (y-5)(y-9) \geq 0$$

$$\Rightarrow \quad y \leq 5 \text{ or } y \geq 9 \text{ but } y \neq 1$$

final values from both cases are $y \leq 5$ or $y \geq 9$, hence the values of the given expression cannot lies between 5 and 9.

ILLUSTRATION 93

If x is real then find the values the expression $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$ can take.

SOLUTION

Let $y = \frac{x^2 + 14x + 9}{x^2 + 2x + 3}$

$$\Rightarrow \quad x^2(1-y) + 2x(7-y) + 3(3-y) = 0$$

CASE-1: ($y = 1$)

$$0 + 12x + 6 = 0 \quad \Rightarrow \quad x = -1/2$$

CASE-2: ($1 - y \neq 0 \quad \Rightarrow \quad y \neq 1$)

It is a quadratic equation with real x

Hence $D = 4(7-y)^2 - 12(1-y)(3-y) \geq 0$

$$\Rightarrow \quad 4(-2y^2 - 2y + 40) \geq 0$$

$$\Rightarrow \quad y^2 + y - 20 \leq 0$$

$\Rightarrow \quad (y+5)(y-4) \le 0 \quad \Rightarrow \quad -5 \le y \le 4$, but $y \ne 1$

final answer from both the cases
we have $-5 \le y \le 4$

ILLUSTRATION 94

If $x \in R$ then find the range of $f(x) = \dfrac{x^2+2x-3}{2x^2+3x-9}$.

SOLUTION

$$f(x) = \frac{(x+3)(x-1)}{(2x-3)(x+3)} \quad x \ne -3, \frac{3}{2}$$

$$f(x) = y = \frac{x-1}{2x-3};\ 2xy - 3y = x - 1$$

$$x = \frac{3y-1}{2y-1} \quad \therefore \ y \ne \frac{1}{2}$$

but since $x \ne -3$ hence $y \ne \dfrac{4}{9}$.

$$\therefore \quad y \in R - \left\{\frac{4}{9}, \frac{1}{2}\right\}$$

ILLUSTRATION 95

If x is real, then find the maximum value of $\dfrac{3x^2+9x+17}{3x^2+9x+7}$

SOLUTION

Let $y = \dfrac{3x^2+9x+17}{3x^2+9x+7}$

$\Rightarrow \quad 3x^2(y-1) + 9x(y-1) + (7y-17) = 0$

Case-1: $y = 1$

$0 + 7 - 17 = 0$ not true

$\therefore \quad y \ne 1$

Case-2: $y \ne 1$

As x is real, so $D \ge 0$

$\Rightarrow \quad 81(y-1)^2 - 12(y-1)(7y-17) \ge 0$

$\Rightarrow \quad (y-1)(y-41) \le 0$

$\Rightarrow \quad 1 \le y \le 41$

But $y \ne 1$, so $1 < y \le 41$

$\Rightarrow \quad y_{\max} = 41.$

Aliter:

$$y = \frac{3x^2+9x+17}{3x^2+9x+7} = \frac{(3x^2+9x+7)+10}{3x^2+9x+7}$$

$$y = 1 + \frac{10}{(3x^2+9x+7)} = 1 + \frac{10}{3\left[\left(x+\frac{3}{2}\right)^2 + \frac{1}{12}\right]}$$

$$y_{\max} = 1 + \frac{10}{3\left(0+\frac{1}{12}\right)} = 41$$

ILLUSTRATION 96

Find the number of integral values of a in the interval [0, 100] so that the range of function $f(x) = \dfrac{x+a}{x^2+1}$ contain the interval [0, 1].

SOLUTION

Let $y = \dfrac{x+a}{x^2+1} \quad \Rightarrow \quad yx^2 - x + (y-a) = 0$

Now we must have $D \ge 0$

$\Rightarrow \quad 1 - 4(y^2 - ay) \ge 0 \quad \Rightarrow \quad 4y^2 - 4ay - 1 \le 0$

Let $f(y) = 4y^2 - 4ay - 1 \le 0\ \forall\ y \in [0, 1]$

This possible if $f(0) \le 0$ is true $\forall\ y \in R$.

$$f(1) \le 0 \Rightarrow 3 - 4a \le 0 \Rightarrow a \ge \frac{3}{4}$$

Hence, $a \in \left[\dfrac{3}{4}, \infty\right)$. So, number of integral values of a in [0, 100] are 100.

ILLUSTRATION 97

Let $\dfrac{y+3}{2y+5} = \sin^2 x + 2\cos x + 1$, where $x \in R$. If the range of y is $(-\infty, a] \cup [b, \infty)$, then find the value of $(3a - 5b)$.

SOLUTION

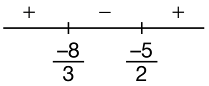
Clearly, $\dfrac{y+3}{2y+5} = 3 - (1-\cos x)^2$, where $x \in R$

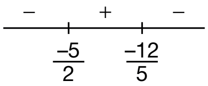
But $0 \le (1-\cos x)^2 \le 4 \quad \Rightarrow \quad -1 \le 3 - (1-\cos x)^2 \le 3$

$$\Rightarrow \quad -1 \le \frac{y+3}{2y+5} \le 3$$

$$\Rightarrow \quad -1 \le \frac{y+3}{2y+5} \text{ and } \frac{y+3}{2y+5} \le 3$$

$$\Rightarrow \quad \frac{3y+8}{2y+5} \ge 0 \text{ and } \frac{-5y-12}{2y+5} \le 0$$





$$y \in \left(-\infty, -\frac{8}{3}\right] \cup \left(-\frac{5}{2}, \infty\right) \qquad \ldots(1)$$

$$\cap\ y \leftarrow \left(-\infty, -\frac{5}{2}\right) \cup \left[-\frac{12}{5}, \infty\right) \qquad \ldots(2)$$

$$\Rightarrow \quad y \in \left(-\infty, \frac{-8}{3}\right] \cup \left[\frac{-12}{5}, \infty\right)$$

$$\therefore \quad a = \frac{-8}{3} \text{ and } b = \frac{-12}{5}$$

Hence, $(3a - 5b) = 3\left(\frac{-8}{3}\right) - 5\left(\frac{-12}{5}\right) = -8 + 12 = 4.$

Test Your Concepts-III

(Solutions on page H.20)

1. Plot following quadratic expression on xy plane

- **(i)** $y = -x^2 - 4x - 4$
- **(ii)** $y = 2x^2 + 5x - 1$
- **(iii)** $y = x^2 + 7x + 100$

2. If the equations $x^3 + x^2 - 4x - 4 = 0$ and $x^2 + px + 2p = 0$ ($p \in R$) have two roots common then then find the value of p.

3. If $a, b \in R$ and equations $ax^2 + 30x + b = 0$ and $x^2 + 3x + 4 = 0$ have a common root, then find $4a - b$.

4. Find the largest integral value of m for which the quadratic expression $y = x^2 + (2m+6)x + 4m + 12$ is always positive, $\forall\ x \in R$.

5. Find the number of integral values of k for which $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 \geq 0$ hold for all x.

6. Find the largest integral value of k for which the quadratic trinomial $P(x) = (k - 2)x^2 + 8x + k + 4$ is non-positive for all real values of x.

7. If the graph of $f(x) = x^2 + (3 - k)x + k$, (where $k \in R$) lies above and below x-axis, then find k.

8. Find the least integral value of k for the quadratic equation $x^2 - 2(k + 2) + k^2 + 12 = 0$ has two distinct solution.

9. Quadratic expression $y = 2x^2 - 4x + 3$. Find range if

- **(i)** $x \in [0, 4]$
- **(ii)** $x \in (2, 3]$

10. For real x, find the set of value of the rational function $f(x) = \dfrac{x^2 - 2x + 2}{2x - 2}$ can take.

11. Let $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$, then find the range of $f(x)$.

12. Find the range of function $y = \dfrac{2x^2 + 4x - 3}{x^2 + 2x + 3}$.

13. Find the number of integral values of x satisfying the inequality $\dfrac{x^2 + 6x - 7}{|x + 4|} < 0$.

14. If $(y^2 - 5y)(x^2 + 2x + 4) < 2$, for all $x \in R$, then find the number of integers in the range of y.

15. Find the number of integers in the range of a for which $\dfrac{x^2 - ax - 2}{x^2 - 3x + 4} > -1\ \forall\ x \in R$.

16. If $x^2 + 2ax + 10 - 3a > 0$ for each $x \in R$, then find a.

17. If every pair from among the equations $x^2 + px + qr = 0$, $x^2 + qx + rp = 0$ and $x^2 + rx + pq = 0$ has a common root, then find the square of product of three common roots.

Answer keys

2. -1 **3.** 0 **4.** 0 **5.** 3 **6.** $k \in (-\infty, -6]$ **7.** $k \in (-\infty, 1) \cup (9, \infty)$ **8.** 3

9. (i) $y \in (1, 19)$ (ii) $y \in (3, 9]$ **10.** $y \geq 1$ or $y \leq -1$ **11.** $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

12. $y \in \left[-\frac{5}{2}, 2\right)$ **13.** 6 **14.** (6) **15.** 7 **16.** $-5 < a < 2$ **17.** $(pqr)^2$

LOCATION OF ROOTS

Position of Roots Relative to a Scalar k

Let $f(x) = Ax^2 + Bx + C$ where $A, B, C \in R$ and $A \neq 0$ and α, β be the roots of $f(x) = 0$.

(I) If both the roots are less than k then three necessary conditions need to be satisfied

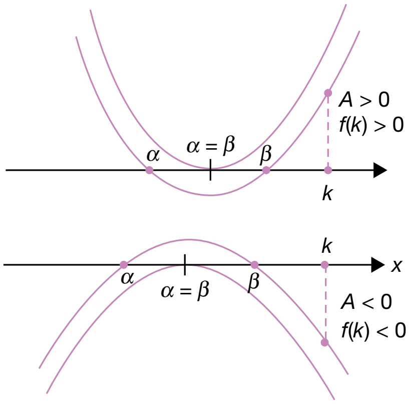
(i) $D \geq 0$ (ii) $A\,f(k) > 0$ (iii) $k > -\dfrac{B}{2A}$

Explanation:

As both roots are less than k, It means we are comparing roots with a real number k then both roots must be real as imaginary cannot be compared then $D \geq 0$.

For a quadratic equation to have both roots less than k following cases are possible.

$Af(k) > 0$ summarizes both the cases in one condition



x coordinate of vertex is mid-point of roots and both roots are less than k then abscissa of vertex is also less than k hence we get condition $k > -\frac{B}{2A}$

(II) If both the roots are greater than k then three necessary conditions need to be satisfied

(i) $D \geq 0$ (ii) $Af(k) > 0$ (iii) $k < -\frac{B}{2A}$

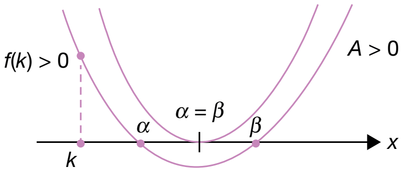
Explanation:

As both roots are greater than k, It means we are comparing roots with a real number k then both roots must be real as imaginary cannot be compared then $D \geq 0$.

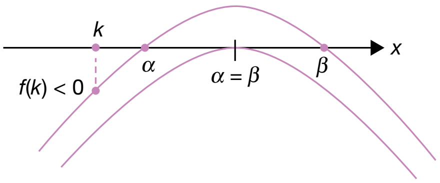
For a quadratic equation to have both roots greater than k following cases are possible

$af(k) > 0$ summarizes both the cases in one condition

CASE-1: $A > 0$



CASE-2: $A < 0$



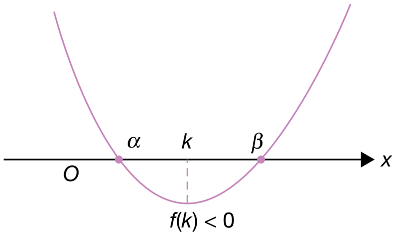
x coordinate of vertex is mid-point of roots and both roots are greater than k then abscissa of vertex is also greater than k hence we get condition $k < -\frac{B}{2A}$

(III) If k lies in between the roots then necessary and sufficient condition is $Af(k) < 0$.

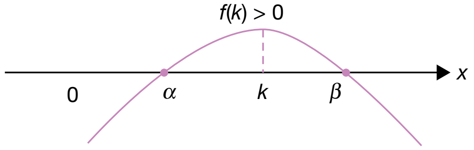
Explanation:

Following two possibilities exist, condition for both possibilities can be summed up in one condition as $Af(k) < 0$.

CASE-1: $A > 0$



CASE-2: $A < 0$



As we can see that graph in both possibilities are taking positive and negative values hence no condition on D (discriminant) is required.

Position of Roots Relative to an Interval

(I) Exactly one root lie in the interval $x \in (k_1, k_2)$ then necessary and sufficient condition is $f(k_1).f(k_2) < 0$

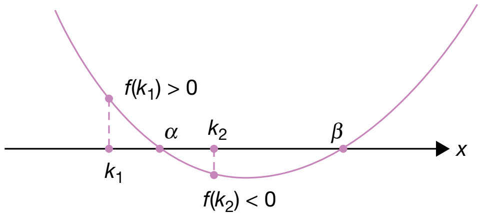
Note: k_1 and k_2 are not roots of $f(x)$

Explanation:

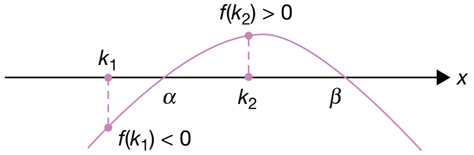
CASE-1

Let smaller root lies in the interval $x \in (k_1, k_2)$ then following two possibilities exists which gives us a condition $f(k_1).f(k_2) < 0$

For $A > 0$,



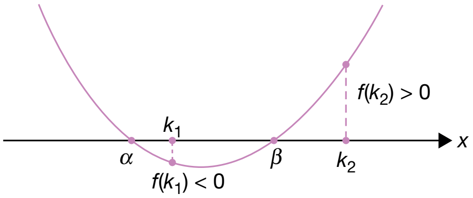
For $A < 0$



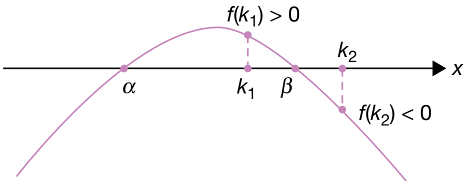
CASE-2

Let the bigger root lies in the interval $x \in (k_1, k_2)$ then following two possibilities exists which gives us a condition $f(k_1).f(k_2) < 0$

For $A > 0$



For $A < 0$



From both the case we are getting necessary and sufficient condition as $f(k_1).f(k_2) < 0$.

(II) If both the roots lie in the interval $x \in (k_1, k_2)$ then four conditions are necessary to be satisfied

(i) $D \geq 0$ (ii) $Af(k_1) > 0$

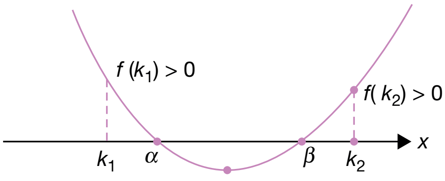
(iii) $A f(k_2) > 0$ (iv) $k_1 < -\frac{B}{2A} < k_2$

Explanation:

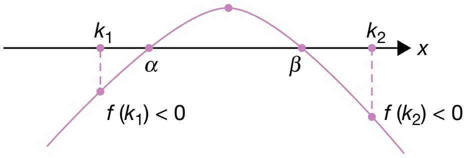
As both roots lie in a real valued interval hence roots of quadratic has to be real so we get $D \geq 0$.

Following two possibilities exist for different values of A, both cases are giving us conditions as $Af(k_1) > 0$ and $Af(k_2) > 0$.

CASE-1: $A > 0$



CASE-2: $A < 0$



We can also observe that abscissa (x-coordinate) of vertex is lying in between k_1 and k_2.

(III) If k_1 and k_2 lie between then roots then $A f(k_1) < 0$ and $A f(k_2) < 0$.

Explanation:

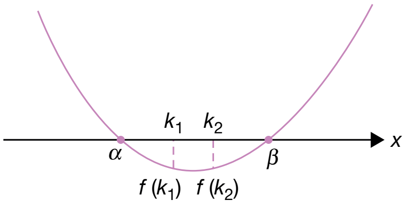
Following two possibilities exists, for first case $A > 0$, we get $f(k_1) < 0$ and $f(k_2) < 0$

For second case $A < 0$, we get $f(k_1) > 0$ and $f(k_2) > 0$

We can combined these conditions in one effective condition as $Af(k_1) < 0$ and $Af(k_2) < 0$.

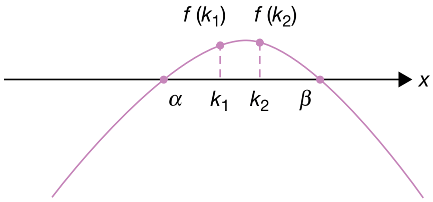
CASE-1

For $A > 0$



CASE-2

For $A < 0$



As we can see that graph in both possibilities are taking positive and negative values hence no condition on D (discriminant) is required.

ILLUSTRATION 98

Find the constant m so that '1' lies between the roots of the quadratic equation

$$(2m + 1)x^2 - mx + m - 2 = 0$$

SOLUTION

Given equation is $(2m + 1)x^2 - mx + (m - 2) = 0$ quadratic, we can rewrite it after dividing by $(2m + 1)$ as

$$f(x) = x^2 - \frac{m}{(2m+1)}x + \frac{(m-2)}{(2m+1)} = 0 \qquad (\text{As } m \neq -1/2)$$

If '1' lies between the roots, so $f(1) < 0$

$$\Rightarrow \quad 1 - \frac{m}{2m+1} + \frac{m-2}{2m+1} < 0 \Rightarrow \frac{2m+1-m+m-2}{2m+1} < 0$$

$$\Rightarrow \quad -\frac{1}{2} < m < \frac{1}{2}$$

Alternate:

$$Af(k_1) < 0 : A = 2m+1;\ k_1 = 1$$

$$(2m+1)(2m+1-m+m-2) < 0$$

$$\Rightarrow \quad (2m+1)(2m-1) < 0$$

$$\Rightarrow \quad m \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

ILLUSTRATION 99

Find the smallest integral value of 'a' for which the in equation $(a - 1)x^2 - (a + 1)x + (a - 1) \geq 0$ for all $x \geq 2$.

SOLUTION

$$a(x^2-x+1)-(x^2+x+1)\ge 0$$

$$\Rightarrow \quad a \ge \frac{x^2+x+1}{x^2-x+1}; a \ge \frac{(x^2-x+1)+2x}{x^2-x+1}$$

$$\Rightarrow \quad a \ge 1+\frac{2x}{x^2-x+1}; a \ge 1+\frac{2}{\left(x+\frac{1}{x}\right)-1};$$

$$a \ge 1+\left.\frac{2}{\left(x+\frac{1}{x}\right)-1}\right]_{\max}$$

$$\because \quad \left.x+\frac{1}{x}\right]_{\min} = \frac{5}{2} \text{ for } x \ge 2$$

Now

$$a \ge 1+\frac{2\cdot 2}{3}; a \ge 1+\frac{4}{3} \text{ or } a \ge \frac{7}{3}$$

$\therefore$ Smallest integral value = 3.

ILLUSTRATION 100

Find all the values of a, so that 6 lies between the roots of the equation $x^2+2(a-3)x+9=0$.

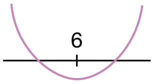
SOLUTION

Let $f(x) = x^2 + 2(a-3)x + 9$. If 6 lies between the roots of $f(x) = 0$, then we must have $Af(6) < 0$

($\because$ coefficient of x^2 is positive, $A = 1$)

$$f(6) < 0 \Rightarrow 36 + 12(a-3) + 9 < 0$$

$$\Rightarrow \quad 12a + 9 < 0 \Rightarrow a < -\frac{3}{4}$$



we get $a < -3/4$

i.e. $a \in (-\infty, -3/4)$.

ILLUSTRATION 101

Find the real values of 'a' for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite signs.

SOLUTION

The roots of the given equation will be of opposite signs, if they are real and their product is negative $f(0) < 0$.

$$\Rightarrow \quad 0 - 0 + (a^2 - 4a) < 0$$

$$\Rightarrow \quad a(a-4) < 0 \quad \Rightarrow \quad 0 < a < 4.$$

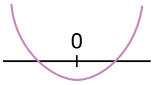


ILLUSTRATION 102

If one root of the equation $x^2 - (p+1)x + p^2 + p - 8 = 0$ is greater than 2 and the other root is smaller than 2, then find p.

SOLUTION

$$f(2) = 4 - 2(p+1) + p^2 + p - 8 < 0$$

$$\Rightarrow \quad p^2 - p - 6 < 0$$

$$\Rightarrow \quad (p-3)(p+2) < 0$$

$$\Rightarrow \quad -2 < p < 3$$

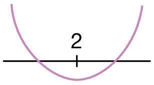


ILLUSTRATION 103

If the equation $x^2 + 2(k+1)x + 9k - 5 = 0$ has only negative roots, then find k.

SOLUTION

Let $f(x) = x^2 + 2(k+1)x + 9k - 5$. Let α, β be the roots of $f(x) = 0$. The equation $f(x) = 0$ will have both negative roots, if

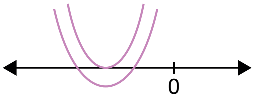
(i) Discriminant $D \ge 0$

$\Rightarrow 4(k+1)^2 - 36k + 20 > 0$

$\Rightarrow 4(k^2 - 7k + 6) \ge 0$

$\Rightarrow (k-1)(k-6) \ge 0$

$\Rightarrow k \le 1$ or $k \ge 6$ …(1)



(ii) $Af(0) > 0 \Rightarrow 9k - 5 > 0 \Rightarrow k > 5/9$ …(2)

(iii) $-\frac{B}{2A} < 0 \Rightarrow -\frac{2(k+1)}{2} < 0$

$\Rightarrow k + 1 > 0 \Rightarrow k > -1$ …(3)

So common part of (1), (2), (3) is

$$k \in \left(\frac{5}{9}, 1\right] \cup [6, \infty)$$

ILLUSTRATION 104

If α, β are roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ then find the number of integral value of m for which $\alpha, \beta \in (-2, 4)$.

SOLUTION

$$(x-m)^2 - 1 = 0 \Rightarrow (x-m-1)(x-m+1) = 0$$

$$x = m+1 \text{ or } m-1$$

Now $-2 < m+1 < 4 \Rightarrow -3 < m < 3$

and $-2 < m-1 < 4 \Rightarrow -1 < m < 5$

Hence $m \in (-1, 3) \Rightarrow m = 0, 1, 2$

ILLUSTRATION 105

Find the number of non-negative integral values of k for which the equation $5x^2 + (13-k)x - 3k - 6 = 0$ has atleast one real solution in $(-2, 2)$.

SOLUTION

$$5x^2 + (13-k)x - 3k - 6 = 0$$

$$(x+3)(5x-k-2)=0$$

$$\Rightarrow \quad x=-3,\ \frac{k+2}{5}$$

Since, $x \in (-2, 2)$

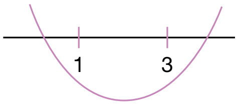
$$\therefore \quad -2<\frac{k+2}{5}<2 \quad \Rightarrow \quad -10<k+2<10$$

$$\Rightarrow \quad -12<k<8.$$

ILLUSTRATION 106

Find the possible set of values of a for which one roots of equation $x^2+2(a-1)x+a+5=0$ is smaller than 1 and other root is greater than 3.

SOLUTION



$$f(1)<0$$

$$\Rightarrow \quad 1+2a-2+a+5<0$$

$$\Rightarrow \quad 3a+4<0 \Rightarrow a<-\frac{4}{3} \qquad \text{...(1)}$$

$$f(3)<0$$

$$\Rightarrow \quad 9+6a-6+a+5<0$$

$$\Rightarrow \quad 7a+8<0 \Rightarrow a<-\frac{8}{7} \qquad \text{...(2)}$$

$(1) \cap (2)$

$$a \in \left(-\infty, -\frac{4}{3}\right)$$

ILLUSTRATION 107

If exactly one root of equation $(k^2+3k+2)x^2+(k^2+k-2)x+k+5=0$ is infinity then find possible value of 'k'.

SOLUTION

If exactly one root of $ax^2+bx+c=0$ is infinite then $a=0$ but $b \neq 0$

$$k^2+3k+2=0 \text{ and } k^2+k-2 \neq 0$$

$$(k+1)(k+2)=0 \text{ and } (k+2)(k-1) \neq 0$$

$$\therefore \quad k=-1 \text{ only.}$$

ILLUSTRATION 108

Find the value of 'a' for which exactly one root of $2t^2+(a-2)t+a=0$ lies in $(1, 3)$.

SOLUTION

$$f(1)\,f(3)<0$$

$$\Rightarrow \quad (2a)\,(4a+12)<0$$

$$\Rightarrow \quad -3<a<0$$

If $a=0$, then equation reduces to $2t^2-2t=0$ gives $t=0, 1$

If $a=-3$, then equation reduces to $2t^2-5t-3=0$

$$\Rightarrow \quad t=-\frac{1}{2}, 3$$

Hence required $a \in (-3, 0)$

ILLUSTRATION 109

Find the number of integral values of a for which the numbers 1 and a lie between the numbers x_0 and y_0, where (x_0, y_0) is a solution of the system of equations $x+y-1=2a$, $2xy=a^2-a$.

SOLUTION

Given, $x+y-1=2a$ ...(1)

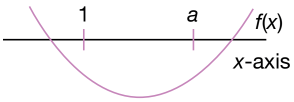
and $2xy=a^2-a$ ...(2)

$\therefore$ From equation (1), put $y=(1+2a-x)$ in equation (2), we get

$$2x(1+2a-x)=a^2-a$$

$$\Rightarrow \quad 2x^2-4ax-2x+a^2-a=0$$

Let $f(x)=2x^2-4ax-2x+a^2-a$



Clearly, $f(1)<0$ and $f(a)<0$ must be satisfied simultaneously.

Now, $f(1)<0 \ \Rightarrow \ a^2-5a<0 \ \Rightarrow \ 0<a<5$ ...(3)

and $f(a)<0 \ \Rightarrow \ a^2+3a>0$

$\Rightarrow \quad a<-3$ or $a>0$ ...(4)

$\therefore$ (3) $\cap$ (4) $\ \Rightarrow \ a \in (0, 5)$

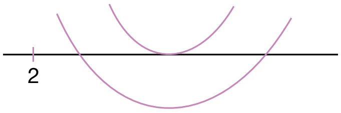
Hence, number of integral values of a are four i.e., $a=1, 2, 3, 4$.

ILLUSTRATION 110

If the quadratic equation $mx^2-nx+12=0$, where m and n are positive integer not exceeding 10, has both roots greater than 2, then find the number of possible ordered pairs (m, n).

SOLUTION

$1 \leq m, n \leq 10$



(i) $D \geq 0$

$\Rightarrow n^2-48m \geq 0$ ...(1)

(ii) $\dfrac{-b}{2a}>2 \ \Rightarrow \ \dfrac{n}{2m}>2 \ \Rightarrow \ n>4m$ ...(2)

(iii) $f(2)>0 \Rightarrow 2m-n+6>0 \ \Rightarrow \ 2n+6>n$ ...(3)

$\therefore$ From (1) $\cap$ (2) $\cap$ (3)

Since m, n are positive integer which are less than or equal to 10

Hence, $m = 1$ and $n = 7$ only possible.

ILLUSTRATION 111

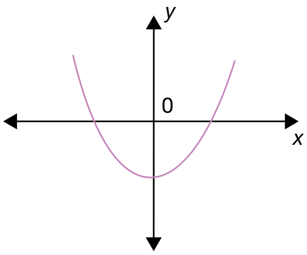
If the inequality $\cot^2 x + (k+1)\cot x - (k-3) < 0$ is true for atleast one $x \in \left[0, \frac{\pi}{2}\right]$, then find the least positive integral value of k.

SOLUTION

Let $\cot x = t;\ t \in (0, \infty)$

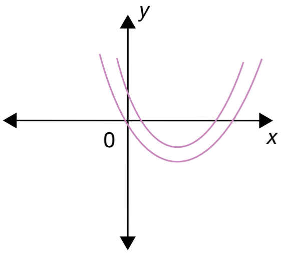
$$f(t) = t^2 + (k+1)t - (k-3)$$

CASE-1:



$f(0) < 0 \quad \Rightarrow \quad k > 3$

CASE-2:



$$D > 0 \Rightarrow (k+1)^2 + 4(k-3) > 0 \Rightarrow k^2 + 6k - 11 > 0$$

$\Rightarrow \quad k \in (-\infty, -3-2\sqrt{5}) \cup (-3+2\sqrt{5}, \infty) \quad \ldots(1)$

$-\frac{B}{2A} > 0 \quad \Rightarrow \quad -\left(\frac{k+1}{2}\right) > 0 \quad \Rightarrow \quad k < -1 \quad \ldots(2)$

$f(0) \geq 0 \quad \Rightarrow \quad k \leq 3 \quad \ldots(3)$

(1) $\cap$ (2) $\cap$ (3)

$$k \in (-\infty, -3-2\sqrt{5})$$

From CASE-1 and CASE-2

$\therefore \quad k \in (-\infty, -3-2\sqrt{5}) \cup (3, \infty)$

$\therefore \quad k_{\text{least positive integer}} = 4$

ILLUSTRATION 112

If the range of the value of a for which the roots of the equation $x^2 - 2x - a^2 + 1 = 0$ lie between the roots of the equation $x^2 - 2(a+1)x + a(a-1) = 0$ is (p, q), find the value of $\left(q + \frac{1}{p^2}\right)$.

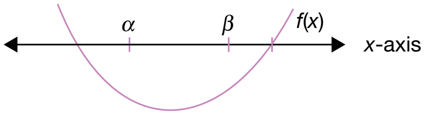
SOLUTION

We have

$x^2 - 2x - (a^2 - 1) = 0$ has roots α and β.

$$\therefore \quad \alpha, \beta = \frac{2 \pm \sqrt{4 + 4(a^2-1)}}{2} = 1 \pm a$$

Let $f(x) = x^2 - 2(a+1)x + a(a-1)$



Now, $f(\alpha) < 0$ and $f(\beta) < 0$ must hold simultaneously where $\alpha = 1 + a;\ \beta = 1 - a$.

So, $f(\alpha) < 0$

$\Rightarrow \quad (a+1)^2 - 2(a+1)^2 + a^2 - a < 0 \quad \Rightarrow \quad -3a - 1 < 0$

$\Rightarrow \quad a > \frac{-1}{3} \quad \ldots(1)$

and $f(\beta) < 0$

$\Rightarrow \quad (1-a)^2 - 2(1-a^2) + a^2 - a < 0$

$\Rightarrow \quad 4a^2 - 3a - 1 < 0 \quad \Rightarrow \quad (4a+1)(a-1) < 0$

$\Rightarrow \quad \frac{-1}{4} < a < 1 \quad \ldots(2)$

$\therefore$ From (1) and (2), we get

$$a \in \left(-\frac{1}{4}, 1\right) \quad \Rightarrow \quad p = \frac{-1}{4} \text{ and } q = 1$$

$$\Rightarrow \quad \left(q + \frac{1}{p^2}\right) = 1 + 16 = 17$$

ILLUSTRATION 113

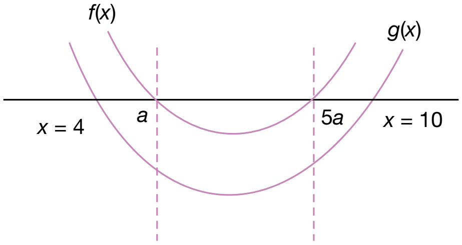
If all the solution of the inequality $x^2 - 6ax + 5a^2 \leq 0$ are also the solutions of inequality $x^2 - 14x + 40 \leq 0$ then find the number of possible integral value of a.

SOLUTION

Let $f(x) = x^2 - 6ax + 5a^2 = (x-a)(x-5a)$

and $g(x) = x^2 - 14x + 40 = (x-4)(x-10)$

Clearly, $g(a) \leq$ and $g(5a) \leq 0$ must simultaneously.



Now, $g(a) \leq 0 \quad \Rightarrow \quad a^2 - 14a + 40 \leq 0$

$\Rightarrow \quad 4 \leq a \leq 10 \qquad \ldots(1)$

Also, $g(5a) \leq 0 \quad \Rightarrow \quad 25a^2 - 70a + 40 \leq 0$

$\Rightarrow \quad (5a-4)(a-2) \leq 0$

$\Rightarrow \quad \dfrac{4}{5} \leq a \leq 2 \qquad \ldots(2)$

$\therefore$ (1) $\cap$ (2) gives $a \in \phi$.

Hence no integral value of a can satisfy it.

ILLUSTRATION 114

Find number of integral values of m for which

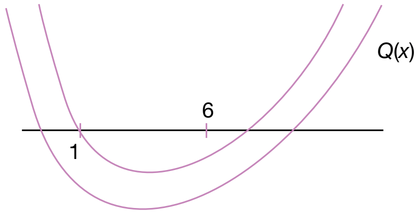
$$x^2 - 2mx + 6m - 41 < 0, \ \forall\ x \in (1, 6]$$

SOLUTION

Let $Q(x) = x^2 - 2mx + 6m - 41$

As, $Q(x) < 0$ for all values of $x \in (1, 6]$. So

$\left.\begin{array}{ll}(1) & Q(1) \leq 0 \\ (2) & Q(6) < 0\end{array}\right\}$ must be satisfied simultaneously.



Now, $Q(1) \leq 0 \quad \Rightarrow \quad 1 - 2m + 6m - 41 \leq 0$

$\Rightarrow \quad m \leq 10 \qquad \ldots(1)$

and $Q(6) < 0 \quad \Rightarrow \quad 36 - 12m + 6m - 41 < 0$

$\Rightarrow \quad m > \dfrac{-5}{6} \qquad \ldots(2)$

Now, (1) $\cap$ (2) $\quad \Rightarrow \quad m \in \left(\dfrac{-5}{6}, 10\right]$.

$$m = \{0,1,2,3,4,5,6,7,8,9,10\}$$

Number of integral values of $m = 11$.

ILLUSTRATION 115

The equation $2(\log_3 x)^2 - |\log_3 x| + a = 0$ has exactly four real solutions if $a \in \left(0, \dfrac{1}{K}\right)$, then find the value of K.

SOLUTION

On putting $\log_3 x = t$, we get

$$2t^2 - |t| + a = 0$$

If $t > 0$, then $2t^2 - t + a = 0 \qquad \ldots(1)$

If $t < 0$, then $2t^2 + t + a = 0 \qquad \ldots(2)$

If given equation has four roots then Eq. (1) must have both roots positive and Eq. (2) has both roots negative. Now, Eq. (1) has both roots positive, if $D > 0$

and $\quad a/2 > 0$

$\Rightarrow \quad 1 - 8a > 0$ and $a > 0$

$\Rightarrow \quad a \in \left(0, \dfrac{1}{8}\right)$ on taking intersection.

Again, Eq. (2) has both roots negative, if $D > 0$ and $a/2 > 0$.

We again get $a \in \left(0, \dfrac{1}{8}\right) \quad \Rightarrow \quad K = 8$

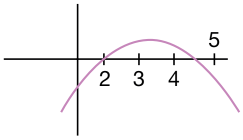
ILLUSTRATION 116

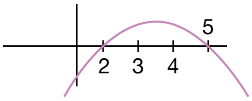
The quadratic expression $ax^2 + |2a-3|x - 6$ is positive for exactly two integral values of x then find $2 + [a]$ (where [.] denotes the greatest integer function).

SOLUTION

For above condition $a < 0$ now we have equation as

$$f(x) = ax^2 + (3-2a)x - 6 = 0$$





$f(0) = -6 < 0$ and $f(2) = 0$

$f(3) = 9a + 9 - 6a - 6 > 0 \quad \Rightarrow \quad a > -1$

$f(4) = 16a + 12 - 8a - 6 > 0 \quad \Rightarrow \quad a > -3/4$

$f(5) = 25a + 15 - 10a - 6 \leq 0 \quad \Rightarrow \quad a \leq -3/5$

we get $-3/4 < a \leq -3/5 \quad \Rightarrow \quad [a] = -1$

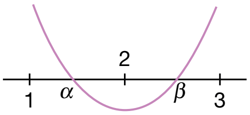
Now $2 + [a] = 1$

ILLUSTRATION 117

If α, β be the roots of the equation $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the integral solutions of λ.

SOLUTION

Clearly, roots are real and distinct.



$$f(1) > 0 \Rightarrow 4 - 16 + \lambda > 0 \Rightarrow \lambda > 12$$
$$f(2) < 0 \Rightarrow 16 - 32 + \lambda < 0 \Rightarrow \lambda < 16$$
$$f(3) > 0 \Rightarrow 36 - 48 + \lambda > 0 \Rightarrow \lambda > 12$$

$\therefore$ Solution is $12 < \lambda < 16$

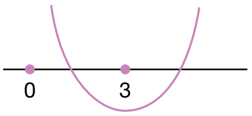
$\therefore$ Integral value of $\lambda = 13, 14, 15$

ILLUSTRATION 118

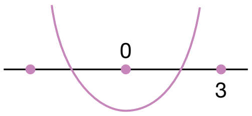
Find all possible values of a for which exactly one root of the quadratic equation $x^2 - (a+1)x + 2a = 0$ lies in the interval (0, 3).

SOLUTION

$x^2 - (a+1)x + 2a = 0$ in (0, 3)



or



$$f(0)\, f(3) < 0$$
$$2a(9 - (a+1)3 + 2a) < 0 \Rightarrow a(a-6) > 0$$
$$a \in (-\infty, 0) \cup (6, \infty)$$

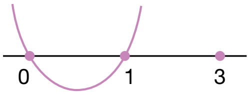
(i) Check for $x = 0$ is a root

$0^2 - (a+1)\,0 + 2a = 0 \Rightarrow a = 0$

Quadratic equation is $x^2 - (0+1)x + 0 = 0$

$\Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$

So, there is a root between (0, 3) so $a = 0$ is included in range of a.



(ii) check for $x = 3$

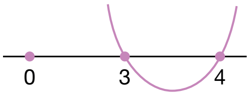
$3^2 - (a+1)3 + 2a = 0 \Rightarrow 9 - 3a - 3 + 2a = 0$

$\Rightarrow a = 6$

Quadratic is $x^2 - (6+1)x + 12 = 0$

$\Rightarrow x^2 - 7x + 12 = 0$

$x = 3$ and 4



there is no root between (0, 3)

So, $a = 6$ is not included in range of a.

So, find answer is $a \in (-\infty, 0] \cup (6, \infty)$

ILLUSTRATION 119

If the root of $x^2 - ax + b = 0$ are real and differ by a quantity which is less than $c(c > 0)$. Prove that b lies between $\left(\frac{a^2 - c^2}{4}\right)$ and $\left(\frac{a^2}{4}\right)$.

SOLUTION

As roots of $x^2 - ax + b = 0$ are Real and different

$$D = a^2 - 4b > 0 \text{ or } b < \frac{a^2}{4} \qquad \ldots(1)$$

Let the root are a and b given $|\alpha - \beta| < c$

$$\Rightarrow \quad \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} < c \quad \Rightarrow \quad \sqrt{a^2 - 4b} < c$$

$$\Rightarrow \quad a^2 - 4b < c^2 \qquad (c > 0, a^2 - 4b > 0)$$

$$b > \frac{1}{4}(a^2 - c^2) \qquad \ldots(2)$$

From (1) and (2)

$$\frac{1}{4}(a^2 - c^2) < b < \frac{a^2}{4}$$

Hence proved

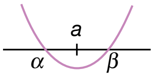
ILLUSTRATION 120

The values of 'a' for which $2x^2 - 2\,(2a+1)\,x + a\,(a+1) = 0$ may have one root less than a and other root greater than a then find a.

SOLUTION

Let $f(x) = 2x^2 - 2(2a+1)\,x + a(a+1)$. For a to lie between the roots,

we must have $f\,(a) < 0$



$$\Rightarrow \quad 2a^2 - 2a(2a+1) + a(a+1) < 0$$
$$\Rightarrow \quad -a^2 - a < 0 \quad \Rightarrow \quad a^2 + a > 0$$
$$\Rightarrow \quad a > 0 \text{ or } a < -1.$$

ILLUSTRATION 121

If the roots of the equation $ax^2 + bx + c = 0$ be $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, prove that $(a+b+c)^2 = b^2 - 4ac$.

SOLUTION

Given equation is $ax^2 + bx + c = 0$

Roots of equation are be $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$

$$\therefore \quad \frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a}$$

$$\text{and} \quad \frac{k+2}{k} = \frac{c}{a} \quad \Rightarrow \quad k = \frac{2a}{c-a}$$

Putting the value of k in (2), we get

$$\frac{k+1}{k}+\frac{k+2}{k+1}$$

$$=\frac{c+a}{2a}+\frac{2c}{c+a}=-\frac{b}{a} \text{ or } \frac{(c+a)^2+4ac}{2a(c+a)}=-\frac{b}{a}$$

or $(c+a)^2+4ac=-2bc-2ab$

or $(c+a)^2+2b(c+a)=-4ac$

or $(c+a)\,(c+a+2b)=-4ac$

or $(a+b+c-b)\,(a+b+c+b)=-4ac$

or $(a+b+c)^2-b^2=-4ac$

or $(a+b+c)^2=b^2-4ac$

Hence proved.

ILLUSTRATION 122

For what values of m can the expression $2x^2+mxy+3y^2-5y-2$ be expressed as the product of two linear factors?

SOLUTION

Comparing the given expression with $ax^2+2hxy+by^2+2gx+2fy+c$, we have

$$a=2,\ h=m/2,\ b=3,\ c=-2,\ f=-5/2,\ g=0$$

$$ax^2+2x(hy+g)+by^2+2fy+c=0$$

$$D=4\left[(hy+g)^2-(a)(by^2+2fy+c)\right]$$

$$=4\left[(h^2-ab)y^2+2(hg-af)y+g^2-ac\right]$$

It is a quadratic in y

For linear factors its discriminant should be zero.

$$4(hg-af)^2-4(h^2-ab)(g^2-ac)=0$$

$\Rightarrow$ $abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow$ $-12-\frac{25}{2}+2\left(\frac{m^2}{4}\right)=0$

$\Rightarrow$ $m^2=49 \quad \Rightarrow \quad m=\pm 7.$

ILLUSTRATION 123

If the expression $ax^2+by^2+cz^2+2ayz+2bzx+2cxy$ can be resolved into rational factors. Prove that $a^3+b^3+c^3=3abc$.

SOLUTION

$$ax^2+by^2+cz^2+2ayz+2bzx+2cxy$$

$$=z^2\left[a\left(\frac{x}{z}\right)^2+b\left(\frac{y}{z}\right)^2+c+2a\left(\frac{y}{z}\right)+2b\left(\frac{x}{z}\right)+2c\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)\right]$$

$$=z^2\,[aX^2+bY^2+2cXY+2bX+2aY+c] \qquad \ldots(1)$$

where $X=\frac{x}{z}$ and $Y=\frac{y}{z}$.

The given expression can be resolved into rational factors if the expression within brackets in (1) is expressible into rational factors, the condition for which is

$$abc+2fgh-af^2-bg^2-ch^2=0$$

$\Rightarrow$ $abc+2abc-a\,.\,a^2-b.\,b^2-c\,.\,c^2=0$

$\Rightarrow$ $a^3+b^3+c^3=3\,abc$

ILLUSTRATION 124

Find the number of real roots of the equation

$$e^{\sin x}-e^{-\sin x}-4=0.$$

SOLUTION

Let $e^{\sin x}=y$ then given equation reduces to

$$y-\frac{1}{y}-4=0$$

$$y^2-4y-1=0$$

$$y=\frac{4\pm\sqrt{20}}{2}=2\pm\sqrt{5}=4.23,-0.23$$

But $y=e^{\sin x}$ is never negative.

So $y=e^{\sin x}=4.23$

$\Rightarrow$ $\sin x=\log 4.23>1$

which is not possible. Hence the equation has no real root.

ILLUSTRATION 125

If $0\le x\le 2\pi$, then find the solution of the equation $16^{\sin^2 x}+16^{\cos^2 x}=10$.

SOLUTION

Let $16^{\sin^2 x}=y$, then $16^{\cos^2 x}=16^{1-\sin^2 x}=\frac{16}{y}$

Hence $y+\frac{16}{y}=10 \quad \Rightarrow \quad y^2-10y+16=0$

$\Rightarrow$ $(y-2)(y-8)=0$

or $y=2,8$

Now $16^{\sin^2 x}=2 \quad \Rightarrow \quad 2^{4\sin^2 x}=(2)^1$

$\Rightarrow$ $4\sin^2 x=1$

$\therefore$ $\sin x=\pm\frac{1}{2} \quad \Rightarrow \quad x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

and $16^{\sin^2 x}=8 \quad \Rightarrow \quad 2^{4\sin^2 x}=2^3$

$\Rightarrow$ $\sin x=\pm\frac{\sqrt{3}}{2} \quad \Rightarrow \quad x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$

Hence final solutions are

$$x \in \left\{\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}\right\}$$

ILLUSTRATION 126

Find the number of real solutions of

$$\sin(e^x) = 5^x + 5^{-x}$$

SOLUTION

We have $5^x + 5^{-x} = (5^{x/2} - 5^{-x/2})^2 + 2 \geq 2$

If $\sin(e^x) = 5^x + 5^{-x}$ has a solution , we will get $\sin(e^x) \geq 2$, which is not possible for any real x, as

$$|\sin\theta| \leq 1 \text{ for all } \theta \in R.$$

No solution

Test Your Concepts-IV

(Solutions on page H.22)

1. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then find a.
2. If quadratic equation $x^2 - 3ax + a^2 = 9$ has roots of opposite sign, then find the sum of all possible values of a.
3. Find the least positive integral value of 'k' for which $(x + k - 1)(x - 2k - 7) < 0 \; \forall \; x \in [-1, 2]$.
4. If quadratic equation $ax^2 + bx + c = 0$ ($a, b, c, \in R$ and $a \neq 0$) has non real roots, such that $a + b + c > 0$, then consider the statements.
   (I) $a - b + c < 0$
   (II) $4a + 2b + c > 0$
   (III) $a - 2b + 4c > 0$
   Find true statements among above.
5. The expression $x^2 - mx + 1$ is negative for values of x in (1, 2) then find m.
6. Find the number of integral values of k for which the equation $x^2 - 4|x| + 3 - |k - 1| = 0$ has four distinct real roots.
7. If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then find a.
8. $f(x)$ is a quadratic polynomial (leading coefficient > 0) and if $f(x) = 0$ has two real roots α and β such that $-1 < \alpha < 1 < \beta < 3$ and $f(0) < f(2) < 0$, then show that $f(1) < f(2)$.

Answer keys

1. $a < 2$ **2.** 0 **3.** 3 **4.** II, III **5.** $m > \frac{5}{2}$ **6.** (5) **7.** $a > 11/9$

SOLVED PROBLEMS

PROBLEM 1

If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$ then find c.

SOLUTION

$x^3 + 3x^2 - 9x + c = 0$ has roots α, α, β

sum of roots $= 2\alpha + \beta = -3$ and $\alpha^2 + 2\alpha\beta = -9$

$$\Rightarrow \quad \alpha^2 + 2\alpha(-3 - 2\alpha) = -9 \Rightarrow 3\alpha^2 + 6\alpha - 9 = 0$$

$$\Rightarrow \quad 3(\alpha - 1)(\alpha + 3) = 0 \Rightarrow \alpha = 1 \text{ or } \alpha = -3$$

as α is a root of $x^3 + 3x^2 - 9x + c = 0$
If $\alpha = 1$, then $1 + 3 - 9 + c = 0 \Rightarrow c = 5$
If $\alpha = -3$, then $-27 + 27 + 27 + c = 0 \Rightarrow c = -27$.
So final values of $c = -27, 5$

Alternate

If $f(x) = x^3 + 3x^2 - 9x + c = (x - \alpha)^2 (x - \beta)$, then $(x - \alpha)$ is a factor of order 2. So, $x - \alpha$ is a factor of order one of $f'(x)$ i.e. $3x^2 + 6x - 9$.
Now, $f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$

$$\therefore \quad f'(x) = 0 \Rightarrow x = 1 \text{ or } x = -3.$$

This shows that either $\alpha = 1$ or $\alpha = -3$, as α is a root of $x^3 + 3x^2 - 9x + c = 0$
If $\alpha = 1$, then $1 + 3 - 9 + c = 0 \Rightarrow c = 5$
If $\alpha = -3$, then $-27 + 27 + 27 + c = 0 \Rightarrow c = -27$.

PROBLEM 2

If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then find $\alpha^4 + \beta^4 + \gamma^4$.

SOLUTION

We have,

$$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$$

$$\Rightarrow \quad 4 = 6 + 2(\beta\gamma + \gamma\alpha + \alpha\beta) \Rightarrow \beta\gamma + \gamma\alpha + \alpha\beta = -1$$

Also, $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta)$$

$$\Rightarrow \quad 8 - 3\alpha\beta\gamma = 2(6 + 1)$$

$$\Rightarrow \quad 3\alpha\beta\gamma = 8 - 14 = -6 \text{ or } \alpha\beta\gamma = -2$$

Now $(\alpha^2 + \beta^2 + \gamma^2)^2 = \sum \alpha^4 + 2\sum \alpha^2\beta^2$

$$\Rightarrow \quad 6^2 = \sum \alpha^4 + 2\left[\left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma(\Sigma\alpha)\right]$$

$$\Rightarrow \quad \Sigma\alpha^4 = 36 - 2[(-1)^2 - 2(-2)(2)] = 18$$

PROBLEM 3

If a, b, c, d are roots of the equation $x^4 - 2\pi x - \sqrt{2007} = 0$, then find the product of $\frac{a+b+c}{abc}$, $\frac{b+c+d}{bcd}$, $\frac{c+d+a}{cda}$, $\frac{d+a+b}{abd}$.

SOLUTION

As sum of roots $a + b + c + d = 0$
required product is

$$\frac{a+b+c}{abc} \cdot \frac{b+c+d}{bcd} \cdot \frac{c+d+a}{cda} \cdot \frac{d+a+b}{abd}$$

$$= \left(\frac{-d}{abc}\right)\left(\frac{-a}{bcd}\right)\left(\frac{-b}{cda}\right)\left(\frac{-c}{abd}\right)$$

$$= \frac{1}{(abcd)^2} = \frac{1}{2007}$$

PROBLEM 4

Let $f(x) = x^3 + x + 1$ suppose 'g' is a cubic polynomial such that $g(0) = -1$ and the roots of $g(x) = 0$ are the squares of the roots of $f(x) = 0$ then find $g(9)$.

SOLUTION

Let a, b, c are the roots of $f(x) = 0$

$$\Rightarrow \quad x^3 + x + 1 = (x - a)(x - b)(x - c)$$

$$\therefore \quad f(x) = x^3 + x + 1 = (x - a)(x - b)(x - c)$$

$$g(x) = (x - a^2)(x - b^2)(x - c^2)$$

$$\therefore \quad g(9) = (9 - a^2)(9 - b^2)(9 - c^2)$$

$$= (3 - a)(3 - b)(3 - c)(3 + a)(3 + b)(3 + c)$$

But $(3 + a)(3 + b)(3 + c) = -(-27 - 3 + 1) = -f(-3)$

$$= 29 \text{ and } (3 - a)(3 - b)(3 - c) = 31 = f(3)$$

$$\therefore \quad g(9) = 31 \times 29 = 899$$

PROBLEM 5

If the roots of the equation $\left(1 - q + \frac{p^2}{2}\right)x^2 + p(1 + q)x + q(q - 1) + \frac{p^2}{2} = 0$ are equal then show that $p^2 = 4q$.

SOLUTION

As roots are equal then $D = B^2 - 4AC = 0$

$$\Rightarrow \quad B^2 = 4AC$$

$$\Rightarrow \quad p^2(1+q)^2 = 4\left(1-q+\frac{p^2}{2}\right)\left\{q(q-1)+\frac{p^2}{2}\right\}$$

$$\Rightarrow \quad p^2(1+q)^2 = \{4(1-q)+2p^2\}\left\{q(q-1)+\frac{p^2}{2}\right\}$$

$$\Rightarrow \quad p^2(1+q)^2 = [-4q(1-q)^2 + 2p^2q(q-1) + 2p^2(1-q) + p^4]$$

$$\Rightarrow \quad p^2(1+q)^2 = -4q(1-q)^2 + p^2[2q^2 - 2q + 2 - 2q + p^2]$$

$$\Rightarrow \quad p^2[(1+q)^2 - 2q^2 + 4q - 2 - p^2] = -4q(1-q)^2$$

$$\Rightarrow \quad p^2[1 + 2q + q^2 - 2q^2 + 4q - 2 - p^2] = -4q(1-q)^2$$

$$\Rightarrow \quad p^2[-q^2 + 6q - 1 - p^2] = -4q(1-q)^2$$

$$\Rightarrow \quad p^2[-(1-q)^2 + 4q - p^2] = -4q(1-q)^2$$

$$\Rightarrow \quad -p^2(1-q)^2 + p^2(4q - p^2) = -4q(1-q)^2$$

$$\Rightarrow \quad (1-q)^2(-p^2 + 4q) + p^2(4q - p^2) = 0$$

$$\Rightarrow \quad (-p^2 + 4q)\{p^2 + (1-q)^2\} = 0$$

As $p^2 + (1-q)^2$ can not be zero, as it a a quadratic

So $-p^2 + 4q = 0 \Rightarrow p^2 = 4q$ Hence proved

PROBLEM 6

If $a < b < c < d$ then show that $(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots.

SOLUTION

Let $f(x) = (x - a)(x - c) + \lambda(x - b)(x - d)$ it is a variable quadratic equation in x with parameter λ.

Now $f(b) = (b - a)(b - c) < 0$

$$f(d) = (d - a)(d - c) > 0$$

$f(b)$ and $f(d)$ are of opposite sign that means it has at least one real root in (b, d).

Since one root is real , both roots must be real as when coefficient are real imaginary roots occurs in conjugate pairs.

PROBLEM 7

Let a, b, c be real numbers and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies $\alpha < \gamma < \beta$.

SOLUTION

$$a^2\alpha^2 + b\alpha + c = 0 \quad \ldots(1)$$

$$a^2\beta^2 - b\beta - c = 0 \quad \ldots(2)$$

Let $f(x) = a^2x^2 + 2bx + 2c$

$$f(\alpha) = a^2\alpha^2 + 2(b\alpha + c) \quad \text{using relation (1)}$$

$$= a^2\alpha^2 - 2a^2\alpha^2 = -a^2\alpha^2 < 0$$

$$f(\beta) = a^2\beta^2 + 2(b\beta + c) \quad \text{using relation (2)}$$

$$= a^2\beta^2 + 2a^2\beta^2 = 3a^2\beta^2 > 0$$

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs we know from theory of equations that a root γ of the equation $f(x) = 0$ lies between α and β

$$\therefore \quad \alpha < \gamma < \beta$$

PROBLEM 8

Find the maximum number of real roots of $x^6 - 3x^5 + 4x^3 + 3x^2 + 4 = 0$.

SOLUTION

Let $f(x) = x^6 - 3x^5 + 4x^3 + 3x^2 + 4$

Here $f(x)$ has two changes in sign, so $f(x) = 0$ has at most two positive real roots.

$f(-x) = x^6 + 3x^5 - 4x^3 + 3x^2 + 4$, has two changes in sign so it has almost two negative real roots. Hence maximum number of real roots = 4.

PROBLEM 9

Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$ such that $\alpha, \beta, \gamma, \delta$ are in G.P and $p \geq 2$. If $a, b, c \in \{1, 2, 3, 4, 5\}$, let the number of equation of the form $ax^2 + bx + c = 0$ which have real roots be r, then find the minimum value of $\frac{pqr}{1536}$.

SOLUTION

$$(\alpha + \beta) = 1, \alpha\beta = p, \gamma + \delta = 4, \gamma\delta = q$$

Since $\alpha, \beta, \gamma, \delta$ are in G.P

$$\therefore \quad \frac{\beta}{\alpha} = \frac{\delta}{\gamma} \Rightarrow \frac{\beta + \alpha}{\beta - \alpha} = \frac{\delta + \gamma}{\delta - \gamma}$$

$$\Rightarrow \quad \frac{(\beta + \alpha)^2}{(\beta + \alpha)^2 - 4\alpha\beta} = \frac{(\delta + \gamma)^2}{(\delta + \gamma)^2 - 4\delta\gamma}$$

$$\Rightarrow \quad \frac{1}{1 - 4p} = \frac{16}{16 - 4q} = \frac{4}{4 - q}$$

$$\Rightarrow \quad 4 - q = 4 - 16p \Rightarrow q = 16p$$

Now, $p \geq 2 \quad \therefore q \geq 32$ …(1)

For the given equation $ax^2 + bx + c = 0$ to have real roots $b^2 - 4ac \geq 0$

$$\therefore \quad ac \leq \frac{b^2}{4}$$

1	$\frac{b^2}{4}$	Possible values of ac such that $ac \le \frac{b^2}{4}$	No. of possible pairs (a, c)	Value of ac	Possible pairs (a, c)
				1	(1, 1)
2	1	1	1	2	(1, 2), (2, 1)
3	2.25	1, 2	3	3	(1, 3). (3, 1)
4	4	1, 2, 3, 4	8	4	(1, 4), (4, 1), (2, 3)
5	6.25	1, 2, 3, 4, 5, 6	12	5	(1, 5), (5, 1)
			Total 24	6	(2, 3), (3, 2)

Hence number of quadratic equation with real roots, $r = 24$ …(2)

Now from (1) and (2) the minimum value of $pqr = 2.32.24 = 1536$

PROBLEM 10

If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$ where 't' is a parameter has exactly one real solution of the form (x, y). Then find the sum of $(x + y)$.

SOLUTION

$$2x^2 + 4x(y - 3) + 7y^2 - 2y + t = 0$$

$D = 0$ (for one solution)

$\Rightarrow \quad 16(y - 3)^2 - 8(7y^2 - 2y + t) = 0$

$\Rightarrow \quad 2(y - 3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow \quad 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0$

$\Rightarrow \quad -5y^2 - 10y + 18 - t = 0$

$\Rightarrow \quad 5y^2 + 10y + t - 18 = 0$

Again $D = 0$ (for one solution)

$\Rightarrow \quad 100 - 20(t - 18) = 0$

$\Rightarrow \quad 5 - t + 18 = 0$

$\therefore \quad t = 23$

for $t = 23$; $5y^2 + 10y + 5 = 0$

$$(y + 1)^2 = 0 \Rightarrow y = -1$$

for $y = -1$; $2x^2 - 16x + 32 = 0$

$\Rightarrow \quad x^2 - 8x + 16 = 0 \quad \Rightarrow \quad x = 4$

we get $x + y = 3$

PROBLEM 11

Find the number of solutions of the inequality $\sqrt{3x^2 + 6x + 7} + \sqrt{5x^2 + 10x + 14} \le 4 - 2x - x^2$.

SOLUTION

We have

$$\sqrt{3x^2 + 6x + 7} = \sqrt{3(x + 1)^2 + 4} \ge 2$$

and $\sqrt{5x^2 + 10x + 14} = \sqrt{5(x + 1)^2 + 9} \ge 3$

$\therefore \quad$ L.H.S ≥ 5

R.H.S $= 4 - 2x - x^2 = 5 - (x + 1)^2 \le 5$

$\therefore$ the given inequation holds only when L.H.S = R.H.S = 5 when $x = -1$.

Therefore, $x = -1$ is only one solution to the given in equation.

PROBLEM 12

If $[x]$ denotes the greatest integer less than or equal to x then find the number of integer values of x which satisfies the equation $\left[x^2 - 2\right]^2 - 9\left[x^2 - 2\right] + 14 = 0.$

SOLUTION

$$t^2 - 9t + 14 = 0 \Rightarrow (t - 2)\,(t - 7)$$

$\Rightarrow \quad \left[x^2 - 2\right] = 2$ or $\left[x^2 - 2\right] = 7$

$\Rightarrow \quad 2 \le x^2 - 2 < 3$ or $7 \le x^2 - 2 < 8$

$\Rightarrow \quad 4 \le x^2 < 5$ or $9 \le x^2 < 10$

$\Rightarrow \quad 2 \le |x| < \sqrt{5}$ or $3 \le |x| < \sqrt{10}$

$\Rightarrow \quad x \in \left(-\sqrt{5}, -2\right] \cup \left[2, \sqrt{5}\right)$

or $\quad x \in \left(-\sqrt{10}, -3\right] \cup \left[3, \sqrt{10}\right)$

$\Rightarrow$ The integral values of x are $-2, 2, -3, 3$

PROBLEM 13

If $f : R - \{-4, 1\} \rightarrow R$ is defined as $f(x) = \dfrac{ax^2 + bx + c}{x^2 + 3x - 4}$, where $f(x) < 0 \ \forall x \in R - \{-4, 1\}$ If $|f(2)| = 2$, then find the number of solutions of the equation $\left|ax^2 + bx + c\right| = \left|x^2 + 3x - 4\right|$.

SOLUTION

$$f(x) = \frac{ax^2+bx+c}{x^2+3x-4} = \frac{ax^2+bx+c}{(x+4)(x-1)}$$

$$f(x) < 0, \forall\, x \in R - \{-4, 1\}$$

only possible when $ax^2 + bx + c = a(x+4)\,(x-1)$

also $|f(2)| = 2$, so $a = -2$

$$2\,|(x+4)(x-1)| = |(x+4)\,(x-1)|$$

$\Rightarrow \quad x = -4$ and 1. (Two solutions)

PROBLEM 14

Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$ which have two distinct real roots in the open interval $0 < x < 1$. Find the least positive integer a for which such a quadratic equation exists.

SOLUTION

Let α, β be two distinct real roots of

$$f(x) = ax^2 - bx + c = 0 \text{ lying in } (0,1).$$

Then $f(x) = a\,(x-\alpha)\,(x-\beta)$

Now $f(0)\,f(1) = a^2\,\alpha(1-\alpha)\,\beta(1-\beta)$

As $0 < \alpha < 1$, $0 < \alpha\,(1-\alpha) \le 1/4$, with equality holding for $\alpha = 1/2$.

Since $0 < \alpha, \beta < 1$ and $\alpha \ne \beta$

$$0 < \alpha(1-\alpha)\,\beta(1-\beta) < 1/16$$

$$0 < f(0)\,f(1) < \frac{a^2}{16} \qquad \dots(1)$$

As a, b, c are positive integers and $c(a-b+c) > 0$

$$f(0)\,f(1) = c(a-b+c) \ge 1 \quad [f(0)\,f(1) > 0] \qquad \dots(2)$$

(1) and (2) imply $a^2/16 > 1$, that is $a \ge 5$. Since the roots of $f(x) = 0$ are real and distinct, its discriminant $= b^2 - 4ac > 0$.

$$b^2 > 4ac \ge 20 \qquad [\because c \ge 1]$$

Hence the minimum possible value of b is 5. Let us try the least values of a, b and c, that is, $a = 5$, $b = 5$ and $c = 1$. It is easy to check that $5x^2 - 5x + 1 = 0$ has two distinct real roots lying between 0 and 1. Thus, the least positive integral value of a is 5.

PROBLEM 15

If $a_1, a_2, a_3 \dots, a_n$ $(n \ge 2)$ are real and $(n-1)\,a_1^2 - 2na_2 < 0$, then prove that at least two roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0$ are imaginary.

SOLUTION

Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the given equation

then $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = -a_1$

and $\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 \dots + \alpha_{n-1}\,\alpha_n = a_2$

Now $(n-1)a_1^2 - 2na_2 = (n-1)(\Sigma\alpha_1)^2 - 2n\,\Sigma\alpha_1\alpha_2$

$$= \sum_{1 \le i < j \le n}\sum (\alpha_i - \alpha_j)^2$$

But given that $(n-1)\,a_1^2 - 2na_2 < 0$

$$\therefore \quad \sum_{1 \le i < j \le n}\sum (\alpha_i - \alpha_j)^2 < 0$$

This can not be true for real roots a_i, as sum of squares of real numbers cannot be negative.
Hence it is true only when at least two roots are imaginary.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If the vertex of quadratic expression $y = 2x^2 - 4x + 6$ is (m, n), then the quadratic equation whose vertex $\left(2m, \frac{n}{2}\right)$ is
(A) $y = x^2 - 4x + 8$
(B) $y = x^2 + 4x + 6$
(C) $y = -2x^2 + 8x - 6$
(D) $y = -2x^2 + 8x - 8$

2. If α, β are the roots of $x^2 - p(x+1) - c = 0, c \neq 1$, then the value of $(\alpha + 1)(\beta + 1)$ is equal to
(A) $1 - c$
(B) $p + c$
(C) $1 + c$
(D) $1 - 2p$

3. Let α and β are the roots of the equation $x^2 - (p - 2)x - p - 1 = 0$ $(p \in R)$. If $\alpha^2 + \beta^2$ is least, then p equals
(A) 0
(B) 1
(C) 2
(D) 5

4. If α, β, γ are roots of equation $x^3 - 2x^2 - 1 = 0$ and $T_n = \alpha^n + \beta^n + \gamma^n$, then value of $\frac{T_{11} - T_8}{T_{10}}$ is equal to
(A) 1
(B) 2
(C) –1
(D) 3

5. If the quadratic polynomial $P(x) = 4x^2 + kx + 8x + 9$ where $k \in I$, is positive for every real 'x' then the greatest absolute values of 'k' is
(A) 20
(B) 19
(C) 4
(D) 3

6. If the quadratic equations $3x^2 + ax + 1 = 0$ and $2x^2 + bx + 1 = 0$ have a common root then the value of expression $\left|2a^2 - 5ab + 3b^2\right|$ is, where $(2a \neq 3b)$.
(A) 0
(B) 1
(C) 2
(D) 3

7. If (α, β), (β, γ), (γ, α) are respectively the roots of $x^2 - 2px + 2 = 0$; $x^2 - 2qx + 3 = 0$; $x^2 - 2rx + 6 = 0$ where α, β, γ are all positive then the value of $p + q + r$ is
(A) 1
(B) 2
(C) 5
(D) 6

8. A value of b for which the equations $x^2 + bx - 1 = 0$; $x^2 + x + b = 0$, have one root in common is
(A) $-\sqrt{2}$
(B) $-i\sqrt{3}$
(C) $i\sqrt{5}$
(D) $\sqrt{2}$

9. Let α, β be the roots of the equation $(x-a)(x-b). = c, c \neq 0$. Then the roots of the equation $(x-\alpha)((x-\beta)) + c = 0$ are
(A) a, b
(B) b, c
(C) a, c
(D) $a + c, b + c$

10. The set of values of 'a' for which the equation $(a - 1)x^2 - (a + 1)x + (a - 1) \geq 0$ is true for all $x \geq 2$ is
(A) $\left(1, \frac{7}{3}\right]$
(B) $(-6, 1)$
(C) $\left[\frac{7}{3}, \infty\right)$
(D) $(-3, -2)$

11. If $a \in R$ and the equation $(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (which $[x]$ denotes G.IF) has no integral solution and has exactly one solution in (2, 3) then a lies in the internal
(A) (0, 1)
(B) (1, 2)
(C) (–1, 3)
(D) (–1, 0)

12. The sum of all the real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is
(A) 1
(B) 2
(C) 3
(D) 4

13. Let α be a root of $ax^2 + bx + c = 0$ and β be a root of $-ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. Then the equation $\frac{a}{2}x^2 + bx + c = 0$ has a root γ such that
(A) $\gamma < \min\{\alpha, \beta\}$
(B) $\gamma > \max\{\alpha, \beta\}$
(C) γ lies between α and β
(D) γ-lies between α and β

14. If a, b, c and d are distinct positive real numbers such that a and b are the roots $x^2 - 10cx - 11d = 0$ and c and d are the roots of $x^2 - 10ax - 11b = 0$, then the value of $a + b + c + d$ is
(A) 1110
(B) 1010
(C) 1101
(D) 1210

15. The number of integral value(s) of 'a' so that the graph of $y = 16x^2 + 8(a + 5)x - 7a - 5$ is always above the x-axis is
(A) 12
(B) 13
(C) 14
(D) 15

16. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$ is equal to
(A) $\frac{b}{ac}$
(B) $\frac{-b}{ac}$
(C) $\frac{ac}{b}$
(D) $\frac{-ac}{b}$

17. Let α, β, γ and δ be the roots (real or non-real) of equation $x^4 - 3x + 1 = 0$. The value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$ is equal to
(A) 6
(B) 9
(C) 12
(D) 15

18. If $\tan^2 \frac{3\pi}{8}$ is a root of the equation $2x^2 - 3ax + 4b = 0$, where $a, b \in Q$, then the value of $a + 4b$ can be equal to

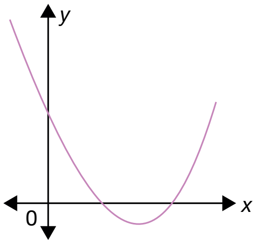
(A) 5 (B) 8
(C) 6 (D) 4

19. Let $P(x) = x^2 - (2 - p)x + p - 2$. If $P(x)$ assumes both positive and negative values $\forall x \in R$, then the range of p is
(A) $(-\infty, 2) \cup (6, \infty)$ (B) $(2, 6)$
(C) $(-\infty, 2)$ (D) $(6, \infty)$

20. If α, β, γ are roots of equation $111x^3 - 11x - 1 = 0$, then $(\alpha\beta)^{-2} + (\beta\gamma)^{-2} + (\gamma\alpha)^{-2}$ equals
(A) 2332 (B) 1331
(C) 1210 (D) 2442

21. If roots of the equation, $a(x - \ell)(x - m) + n = 0$ are α, β then roots of the equation, $a\left(\frac{x}{x-k} - \alpha\right)\left(\frac{x}{x-k} - \beta\right) - n = 0$ are
(A) $\frac{\ell}{\ell+1}, \frac{m}{m+1}$ (B) $\frac{k(\ell+1)}{\ell}, \frac{k(m+1)}{m}$
(C) $\frac{k\ell}{\ell-1}, \frac{km}{m-1}$ (D) $\frac{\ell-1}{\ell}, \frac{m-1}{m}$

22. The graph of $y = ax^2 + bx + c$ is as shown. Which one of the following is not correct?



(A) $a > 0$ (B) $b^2 - 4ac > 0$
(C) $b < 0$ (D) $abc > 0$

23. If the equation $x^3 + 2x^2 - 4x + 5 = 0$ has roots α, β and γ, then the value of $\frac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\beta\gamma}$ is equal to
(A) 5 (B) 8
(C) 12 (D) 15

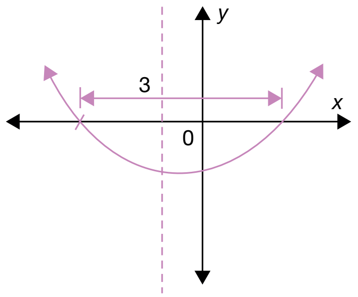
24. If $c^2 = 4d$ and the two equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one common roots, then the value of $2(b + d)$ is equal to
(A) $\frac{a}{c}$ (B) ac
(C) $2ac$ (D) $a + c$

25. This values of α and β for which the quadratic equation $x^2 + 2x + 2 + e^{-2\alpha} - \cos\beta = 0$ has a real solution, is
(A) $\alpha, \beta \in R$
(B) $\alpha \in (0, 1), \beta \in \left(\frac{\pi}{2}, 2\pi\right)$
(C) $\alpha \in (0, \infty), \beta \in \left(\frac{\pi}{2}, 2\pi\right)$
(D) no real value of α, β possible

26. If one of the roots of $ax^2 + ax + a + 1 = 0$ is less than 1 and the other is more than 1, then the complete set of values of a is given by
(A) $a \in \left(\frac{-1}{2}, 0\right)$ (B) $\frac{a^{11}+2^{11}}{a+2}$
(C) $\left(\frac{-1}{3}, 0\right)$ (D) $\left(0, \frac{1}{3}\right)$

27. If a, b, c, d are non-zero real number such that c and d are the roots of $x^2 + ax + b = 0$ and a and b are the roots of $x^2 + cx + d = 0$, then the absolute value of $(a + 2b + 3c + 4d)$ is
(A) 2 (B) 4
(C) 6 (D) 8

28. If graph of $f(x) = x^2 + bx + c$ is drawn in adjacent diagram, where $b, c \in I$, then number of such quadratic equation $f(x) = 0$ is



(A) 1 (B) 2
(C) 3 (D) 4

29. If α, β, γ be the roots of the equation $x^3 + ax + a = 0$ (where $a \in R, a \neq 0$) satisfy $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\gamma} + \frac{\gamma^2}{\alpha} = -8$, then a is
(A) -8 (B) -2
(C) 2 (D) 8

30. If $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$ are zeroes of $f(x) = x^2 - 5x - a$ where $\alpha, \beta \in (0, \infty)$ $\forall x \in R$, then set of values of a is
(A) $\left[-6, \frac{25}{4}\right]$ (B) $\left[\frac{25}{4}, 6\right]$
(C) $\left[-\frac{25}{4}, 6\right]$ (D) $\left(-\frac{25}{4}, -6\right)$

31. If α and β are the roots of the equation $x^2 - x + 11 = 0$, then the value of $3\alpha^3 - 3\alpha^2 - 2\beta^3 - 2\beta^2 + 11\alpha$ is
(A) 33 (B) -33
(C) 22 (D) -22

32. If α and β are the roots of the equation $x^2 - 2x - 5 = 0$, then the equation whose roots are $\frac{\alpha}{\alpha-1}$ and $\frac{\beta}{\beta-1}$ is
(A) $6x^2 - 12x + 5 = 0$
(B) $2x^2 + 8x - 5 = 0$
(C) $-2x^2 + 8x - 5 = 0$
(D) $6x^2 + 12x - 5 = 0$

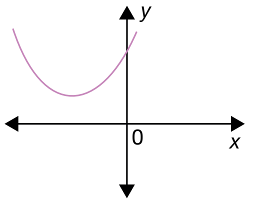
33. Given $f(x) = x^2 + \alpha x + \beta$ (where $\alpha, \beta \in R$). If the equation $f(f(x)) = 0$ has two roots 1 and 2, then $f(0)$ is equal to

(A) $\frac{-3}{2}$ (B) $\frac{-1}{2}$

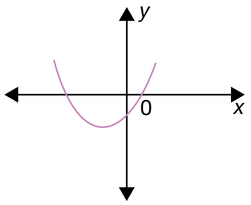
(C) $\frac{3}{2}$ (D) $\frac{1}{2}$

34. Which of the following graph is suitable for $y = \left(a + \frac{1}{a}\right)x^2 + 2x + 1$, where $a > 0$

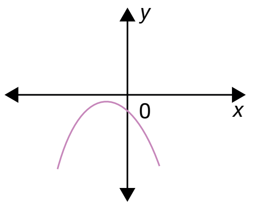
(A)



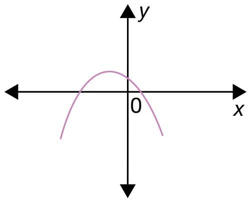
(B)



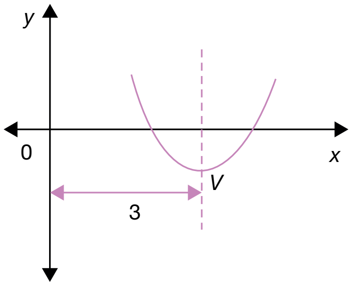
(C)



(D)



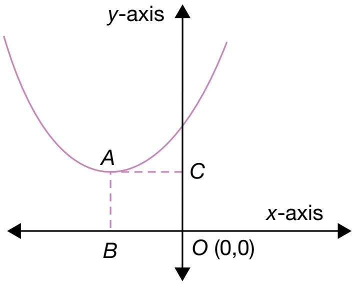
35. If $f(x) = x^2 + bx + c$ is drawn in adjacent diagram and $\ell(OV) = 5$ then zeros of $f(x)$ can be



(A) 2, 4 (B) $\frac{5}{2}, \frac{7}{2}$

(C) $3-\sqrt{2}, 3+\sqrt{2}$ (D) 1, 5

36. Consider the graph of $f(x) = ax^2 + bx + c$ in the given figure such that $\ell(AB) = 1$, $\ell(AC) = 4$ and $b^2 - 4ac = -4$, then value of $(a + b + c)$ is equal to



(A) 1 (B) 9

(C) 10 (D) 26

37. If α, β and γ are roots of $4x^3 + 8x^2 - x - 2 = 0$, then the value of $\frac{4(\alpha+1)(\beta+1)(\gamma+1)}{\alpha\beta\gamma}$ is equal to

(A) $\frac{3}{2}$ (B) 6

(C) $\frac{-3}{2}$ (D) –6

38. If roots of the equation $\ell x^2 + mx + n = 0$ are (2α) and $(\alpha+\beta)$, roots of the equation $px^2 + qx + r = 0$ are (2β) and $(\alpha+\beta)$ and D_1 and D_2 are the respective discriminants, then $\frac{D_1}{D_2}$ is equal to (where $\alpha \neq \beta$)

(A) $\frac{\ell mn}{pqr}$ (B) $\left(\frac{\ell}{p}\right)^2$

(C) $\left(\frac{m}{q}\right)^2$ (D) $\left(\frac{n}{r}\right)^2$

39. The range of values of 'a' for which all the roots of the equation $(a-1)(1+x+x^2)^2 = (a+1)(1+x^2+x^4)$ are imaginary is

(A) $(-\infty, -2]$ (B) $(2, \infty)$

(C) $(-2, 2)$ (D) $[2, \infty)$

40. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ then $aS_{n+1} + bS_n + cS_{n-1} = (n \geq 2)$

(A) 0 (B) $a + b + c$

(C) $(a + b + c)n$ (D) $n^2 abc$

41. A group of students decided to buy a Alarm Clock priced between ₹170 to ₹195. But at the last moment, two students backed out of the decision so that the remaining students had to pay 1 Rupee more than they had planned. If the students paid equal shares, the price of the Alarm Clock is

(A) 190 (B) 196

(C) 180 (D) 171

42. Let a, b and c be real numbers such that $a + 2b + c = 4$ then the maximum value of $ab + bc + ca$ is

(A) 1 (B) 2

(C) 3 (D) 4

43. If the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have a common root, then the value of k is

(A) – 2 (B) – 3

(C) $\frac{27}{4}$ (D) $-\frac{1}{4}$

44. If $P(Q - r)x^2 + Q(r - P)x + r(P - Q) = 0$ has equal roots then $\frac{2}{Q} =$ (where $P, Q, r \in R$)

(A) $\frac{1}{P} + \frac{1}{r}$ (B) $\frac{1}{P} - \frac{1}{r}$

(C) $P + r$ (D) Pr

45. If $x^2 + 4y^2 - 8x + 12 = 0$ is satisfied by real values of x and y then 'y' $\in$

(A) [2, 6] (B) [2, 5]

(C) [–1, 1] (D) [–2, –1]

46. The number of points (p, q) such that $p, q \in \{1, 2, 3, 4\}$ and the equation $px^2 + qx + 1 = 0$ has real roots is

(A) 7 (B) 8

(C) 9 (D) None of these

47. The value of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$ are

(A) $b = 2, c = 1$ (B) $b = 4, c = -1$
(C) $b = -1, c = 4$ (D) $b = -1, c = 1$

48. If the equations $x^2 + 2\lambda x + \lambda^2 + 1 = 0, \lambda \in R$ and $ax^2 + bx + c = 0$ where a, b, c are lengths of sides of triangle have a common root, then the possible range of values of λ is

(A) $(0, 2)$ (B) $\left(\sqrt{3}, 3\right)$
(C) $\left(2\sqrt{2}, 3\sqrt{2}\right)$ (D) $(0, \infty)$

49. The number of real or complex solutions of $x^2 - 6|x| + 8 = 0$ is

(A) 6 (B) 7
(C) 8 (D) 9

50. Sum of the roots of the equation is $4^x - 3(2^{x+3}) + 128 = 0$

(A) 5 (B) 6
(C) 7 (D) 8

51. The number of solutions of $\sqrt{3x^2 + x + 5} = x - 3$ is

(A) 0 (B) 1
(C) 2 (D) 4

52. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = x^2 - 2cx + b^2$ are such that min $f(x) >$ max $g(x)$, then relation between b and c, is

(A) no relation (B) $0 < c < b/2$
(C) $|c| < \dfrac{|b|}{\sqrt{2}}$ (D) $|c| > \sqrt{2}\,|b|$

53. The number of solutions of the equation $x^2 + [x] - 4x + 3 = 0$ is, where [ ] denotes G.I.F.

(A) 0 (B) 1
(C) 2 (D) 3

54. If the quadratic equations $3x^2 + ax + 1 = 0$ and $2x^2 + bx + 1 = 0$ have a common roots, then the value of the expression $5ab - 2a^2 - 3b^2$ is

(A) 0 (B) 1
(C) –1 (D) 2

55. If both roots of the quadratic equation $x^2 - 2kx + k^2 + 2k - 5 = 0$ are less than 4, then exhaustive set of values of k is

(A) $k \in R$ (B) $k < 4$
(C) $k \le \dfrac{5}{2}$ (D) $k > 0$

56. If the roots of the quadratic polynomial $f(x) = 2x^2 - 3x + k^2 - 3k + 4$ are of opposite sign, then the set of the values of k is

(A) ϕ (B) $(-4, 1)$
(C) $(4, -1)$ (D) $(-\infty, -4) \cup (1, \infty)$

57. If one of the roots of $ax^2 + bx + c = 0$ is greater than 2 and other is less than –1 and if the roots of $cx^2 + bx + a = 0$ are α and β, then

(A) $\alpha < \dfrac{1}{2}$ and $\beta > -1$ (B) $\alpha < \dfrac{1}{2}$ and $\beta < -1$
(C) $\alpha > \dfrac{1}{2}$ and $\beta > -1$ (D) $\alpha < 2$ and $\beta > -1$

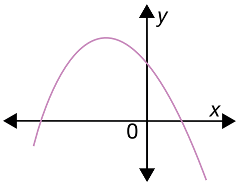
58. If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y then

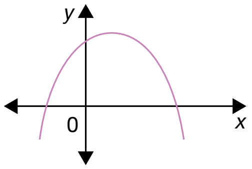
(A) $x \in [1, 3], y \in [1, 3]$ (B) $x \in [1, 3], y \in \left[\dfrac{-1}{3}, \dfrac{1}{3}\right]$
(C) $x \in \left[\dfrac{-1}{3}, \dfrac{1}{3}\right], y \in [1, 3]$ (D) $x \in \left[\dfrac{-1}{3}, \dfrac{1}{3}\right], y \in \left[\dfrac{-1}{3}, \dfrac{1}{3}\right]$

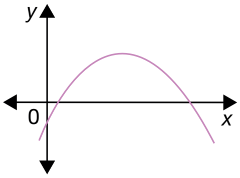
59. If the equation $x^2 + 4x \sin\theta + \tan\theta = 0$ $\left(0 < \theta < \dfrac{\pi}{2}\right)$ has repeated roots, then θ equals

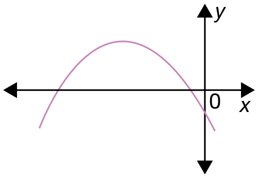
(A) $\dfrac{\pi}{2}$ (B) $\dfrac{\pi}{6}$
(C) $\dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$ (D) $\dfrac{\pi}{6}$ or $\dfrac{\pi}{12}$

60. Which one of the following can best represent the graph of quadratic polynomial $y = ax^2 + bx + c$ (where $a < 0, b < 0$ and $c > 0$)?

(A)


(B)


(C)


(D)


61. If $f(x)$ is a polynomial of least degree such that $f(r) = \dfrac{1}{r}, r = 1, 2, 3, \_\_\_9$, then $f(10) = \_\_\_$

(A) 1 (B) $\dfrac{1}{2}$
(C) $\dfrac{1}{10}$ (D) $\dfrac{1}{5}$

62. The values of 'a' for which the quadratic expression $ax^2 + (a-2)x - 2$ is negative for exactly two integral values of x, belongs to

(A) $[-1, 1]$ (B) $[1, 2)$
(C) $[3, 4]$ (D) $\left[-2,-1\right)$ $[-2, -1)$

63. If $f(x) = \prod_{i=1}^{i=3}(x - a_i) + \sum_{i=1}^{3} a_i - 3x$, where $a_i < a_{i+1}$, then $f(x) = 0$ has

(A) only one real root
(B) three real roots of which two of them are equal
(C) three distinct real roots
(D) three equal roots

64. If a, b and c are numbers for which the equation $\dfrac{x^2 + 10x - 36}{x(x-3)^2} = \dfrac{a}{x} + \dfrac{b}{x-3} + \dfrac{c}{(x-3)^2}$ is an identity, then $a + b + c$ equals

(A) 2 (B) 3
(C) 10 (D) 8

65. If one root of equation $x^2 - 4ax + a + f(a) = 0$ is three times of the other then minimum value of $f(a)$ is

(A) $\frac{-1}{6}$ (B) $\frac{-1}{10}$

(C) $\frac{-1}{5}$ (D) $\frac{-1}{12}$

66. Number of all integral values of x, so that $x^2 + 19x + 89$ is a perfect square is

(A) 0 (B) 1

(C) 2 (D) 3

67. If $ax^2 + bx + c; a, b, c \in R$ has no real zeroes, and if $c < 0$, then

(A) $a < 0$ (B) $a + b + c > 0$

(C) $4a + 2b + c > 0$ (D) $a - b + c > 0$

68. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then one of the roots of the equation $a^3x^2 + abcx + c^3 = 0$ in terms of α, β are

(A) $\frac{\alpha^2}{\beta}$ (B) α^3

(C) β^3 (D) $\alpha\beta^2$

69. If α, β are the roots of $x^2 - 3x + \lambda = 0 (\lambda \in R)$ and $\alpha < 1 < \beta$, then the true set of values of λ equals

(A) $\lambda \in \left(2, \frac{9}{4}\right]$ (B) $\lambda \in \left(-\infty, \frac{9}{4}\right]$

(C) $\lambda \in (2, \infty)$ (D) $\lambda \in (-\infty, 2)$

70. If $ax^2 + bx + c = 0$ and $5x^2 + 6x + 12 = 0$ have a common root where a, b and c are sides of a triangle ABC, then

(A) $\Delta\, ABC$ is obtuse angled (B) $\Delta\, ABC$ is acute angled

(C) $\Delta\, ABC$ is right angled (D) none of these

71. If $0 < a < 5, 0 < b < 5$ and $\frac{x^2+5}{2} = x - 2\cos(a + bx)$ is satisfied for atleast one real x, then value of $a + b$ may be equal to

(A) π (B) $\frac{\pi}{2}$

(C) 2π (D) 4π

72. The value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is

(A) $\frac{2}{3}$ (B) $-\frac{2}{3}$

(C) $\frac{1}{3}$ (D) $-\frac{1}{3}$

73. The polynomial $x^3 - 3x^2 - 9x + c$ can be written in the form $(x - \alpha)^2 (x - \beta)$ if value of c is

(A) 5 (B) –7

(C) 25 (D) 27

74. If the equations $2x^2 - 7x + 1 = 0$ and $ax^2 + bx + 2 = 0$ have a common root, then

(A) $a = 2, b = -7$ (B) $a = \frac{-7}{2}, b = 1$

(C) $a = 4, b = -14$ (D) $a = -4, b = 1$

75. If $p, q, r \in R$ and the quadratic equation $px^2 + qx + r = 0$ has no real root, then

(A) $p(p + q + r) > 0$ (B) $p(p + q + r) < 0$

(C) $r(p + q + r) > 0$ (D) $q(p + q + r) < 0$

76. For $x^2 - (\alpha + 2)|x| + 9 = 0$ to have real solutions, the range of 'α' is

(A) $(-\infty, 4]$ (B) $[4, \infty)$

(C) $(-\infty, 7] \cup [11, \infty)$ (D) $[-4, \infty)$

77. If α and β, α and γ, α and δ are the roots of the equations $ax^2 + 2bx + c = 0$, $2bx^2 + cx + a = 0$ and $cx^2 + ax + 2b = 0$ respectively where a, b, c are positive real numbers, then $\alpha^2 + \alpha =$

(A) –1 (B) 1

(C) 0 (D) abc

78. If $f(x) = (x^2 + 3x + 2)(x^2 - 7x + a)$ and $g(x) = (x^2 - x - 12)(x^2 + 5x + b)$ then the values of a and b, If $(x + 1)(x - 4)$ is HCF of $f(x)$ and $g(x)$

(A) $a = 10 : b = 6$ (B) $a = 4 : b = 12$

(C) $a = 12 : b = 4$ (D) $a = 6 : b = 10$

79. The equation $(x^2 + 3x + 4)^2 + 3(x^2 + 3x + 4) + 4 = x$ has

(A) all its solutions real but not all positive

(B) only two of its solutions real

(C) two of its solutions positive and two negative

(D) none of solutions real.

80. The values of a for which the roots of the equation $(a + 1)x^2 - 3ax + 4a = 0 (a \neq -1)$ are real and greater than 1

(A) $\left[-\frac{10}{7}, 1\right]$ (B) $\left[-\frac{12}{7}, 0\right]$

(C) $\left[-\frac{16}{7}, -1\right)$ (D) $\left(-\frac{16}{7}, 0\right)$

81. If both the roots of the equation $x^2 + 2bx + \log_3(b^2 - 4b + 4) = 0$ are of opposite sign then 'b' belongs to

(A) $(1, 3)$ (B) $(-\infty, 1) \cup (3, \infty)$

(C) $[1, 3]$ (D) $(1, 2) \cup (2, 3)$

82. If α, β are roots of $3x^2 + 2bx + c = 0$ whose discriminant is Δ_1; $\alpha + \delta, \beta + \delta$ are roots of $9x^2 + 2Bx + C = 0$ whose discriminant is Δ_2 then $\frac{\Delta_1}{\Delta_2}$ is

(A) $\frac{1}{9}$ (B) 9

(C) 3 (D) $\frac{1}{3}$

83. If both the roots of $(2a - 4)9^x - (2a - 3)3^x + 1 = 0$ are non-negative, then

(A) $0 < a < 2$ (B) $2 < a < \frac{5}{2}$

(C) $a < \frac{5}{4}$ (D) $a > 3$

84. Let $f(x)$ be a real valued function satisfying $af(x) + bf(-x) = px^2 + qx + r$, $\forall\, x \in R$. Where $p, q, r \in R - \{0\}$ and $a, b \in R$ such that $|a| \neq |b|$. Then the condition that $f(x) = 0$ will have real roots is

(A) $\left(\frac{a+b}{a-b}\right)^2 \leq \frac{q^2}{4pr}$ (B) $\left(\frac{a+b}{a-b}\right)^2 \leq \frac{4pr}{q^2}$

(C) $\left(\frac{a+b}{a-b}\right)^2 \leq \frac{q^2}{4pr}$ (D) $\left(\frac{a+b}{a-b}\right)^2 \geq \frac{4pr}{q^2}$

85. The values of 'a' for which the equation $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root
(A) 2
(B) –2
(C) 0
(D) 1

86. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{\alpha}{\alpha - 1}$ and $\frac{\alpha + 1}{\alpha}$ then value of $(a + b + c)^2$ is
(A) $2b^2 - ac$
(B) $b^2 - 2ac$
(C) $b^2 - 4ac$
(D) $4b^2 - 2ac$

87. The value of a, for which one root of the equation $(a - 5)x^2 - 2ax + (a - 4) = 0$ is smaller than 1 and the other greater than 2 is ____________
(A) $a \in (5, 24)$
(B) $a \in \left(\frac{20}{3}, \infty\right)$
(C) $a \in (5, \infty)$
(D) $a \in (-\infty, \infty)$

88. If the equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ have only positive roots then
(A) $a > b > c$
(B) $a < b < c$
(C) $a = b = c$
(D) $a > b; b < c$

89. If α is a root of $ax^2 + bx + c = 0$; β is a root of $-ax^2 + bx + c = 0$ and γ is a root of $ax^2 + 2bx + 2c = 0$ then
(A) $\gamma < \alpha < \beta$
(B) $\alpha < \beta < \gamma$
(C) $\alpha < \gamma < \beta$
(D) $\frac{\alpha}{\beta} < \gamma < \frac{\beta}{\alpha}$

90. If for any real x, we have $-1 \le \frac{x^2 + nx - 2}{x^2 - 3x + 4} \le 2$ then the value of n is
(A) $n \in \left[-1, \sqrt{40} - 6\right]$
(B) $n \in [-1, 3)$
(C) $n \in \left[-\sqrt{40} - 6, -1\right]$
(D) $n \in \left[1, \sqrt{40} + 6\right]$

91. If $\frac{mx^2 - 3x + 4}{x^2 + 3x + 4} \le 7, \forall x \in R$, then the value of m is
(A) $m \le 7$
(B) $m \ge 1$
(C) $m \in [1, 7)$
(D) $m \in (-\infty, 1]$

92. If α and β are roots of $a\left(x^2 - 1\right) + 2bx = 0$ and the quadratic equation whose roots are $2\alpha - \frac{1}{\beta}$ and $2\beta - \frac{1}{\alpha}$ is $px^2 + qx + r = 0$, then $p + q + r$ is equal to
(A) $2b$
(B) $6a - 8b$
(C) $6b - 8a$
(D) 0

93. If the equation $x^2 - 3x - a_i = 0$ has integral roots where $a_i \in N$ and $a_i \le 300$ then the value of $\sum a_i$ is equal to
(A) 360
(B) 1240
(C) 1600
(D) 780

94. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, and $b^3 + a^2c + ac^2 = kabc$, then the value of 'k' is
(A) 1
(B) 2
(C) 3
(D) 4

95. Let $ax^2 + bx + c = 0$, $a, b, c \in R$, be a quadratic equation. If $|b| > |a + c|$ then the roots of the equation will be such that
(A) exactly one root lies between –1 and 1
(B) both roots lie between –1 and 1
(C) one root less than –1 and other greater than 1
(D) one root less than 1 and other greater than 1.

96. The set of real values of a for which the equation $\frac{2a^2 + x^2}{a^3 - x^3} - \frac{2x}{ax + a^2 + x^2} + \frac{1}{x - a} = 0$ has a unique solution is
(A) $(-\infty, 1)$
(B) $(-1, \infty)$
(C) $(-1, 1)$
(D) R–{0}

97. Let $f(x) = x^3 + x + 1$. Suppose $P(x)$ is a cubic polynomial such that $P(0) = -1$ and the zeros of $P(x)$ are the squares of the roots of $f(x) = 0$. Then the value of $P(4)$ is
(A) 100
(B) –99
(C) 99
(D) –100

98. All possible quadratic equations $ax^2 + bx + 1 = 0$ taking $a \in \{1, 2, 3, ...n\}, b \in \{-1, -2, -3, ... -n\}$ and if (α_i, β_i) $i = 1, 2, 3, ...$ n represent solution sets of these equations then $\Sigma \frac{1}{\alpha_i} + \Sigma \frac{1}{\beta_i}$ equals
(A) $\frac{n(n+1)}{2}$
(B) $\frac{n^2(n+1)}{2}$
(C) $\frac{n^3(n+1)}{2}$
(D) n^2

99. Let $f(x) = x^2 - bx + c$, b is a odd positive integer, $f(x) = 0$ have two prime numbers as roots and $b + c = 35$. Then the minimum value of $f(x)$ is
(A) $-\frac{183}{4}$
(B) $\frac{173}{16}$
(C) $-\frac{81}{4}$
(D) data not sufficient

100. Given $\ell x^2 - mx + 5 = 0$ does not have two distinct real roots, then the minimum value of $5\ell + m$ is
(A) 5
(B) –5
(C) 1
(D) –1

101. The value of a for which exactly one root of the equation $e^a x^2 - e^{2a} x + e^a - 1 = 0$ lies between 1 and 2 are given by
(A) $\left(\log_e \frac{4 - \sqrt{13}}{4}, \log_e \frac{4 + \sqrt{13}}{2}\right)$
(B) $\left(0, \log_e \frac{4 + \sqrt{13}}{2}\right)$
(C) $\left(\log_e \frac{5 - \sqrt{17}}{4}, \log_e \frac{5 + \sqrt{17}}{4}\right)$
(D) $\left(\log_e \frac{6 - \sqrt{17}}{2}, \log_e \frac{6 + \sqrt{17}}{2}\right)$

102. Number of positive integral solution (x, y) of the equation $2xy - 4x^2 + 12x - 5y = 11$ is
(A) 0
(B) 1
(C) 2
(D) infinite many

103. The equation $x^2 + bx + c = 0$ has distinct roots. If 2 is subtracted from each root the result are the reciprocal of the original roots, then $b^2 + c^2$ is
(A) 2 (B) 3
(C) 4 (D) 5

104. The set of values of 'a' for which $ax^2 - (4 - 2a)\,x - 8 < 0$ for exactly three integral value of x is
(A) $2 \le a < 4$ (B) $1 \le a < 2$
(C) $2 < a < 4$ (D) $1 < a < 2$

105. The number of values of k for which the expression $\left(2x^2 - 4(k-1)x + 3\right)\left(x^2 + 2kx + k + 1\right)$ is a perfect square is
(A) 0 (B) 1
(C) 2 (D) 4

106. If α, β, γ are the roots of the equation $x^3 = x^2 + 1$. The equation whose roots are $\alpha^2 + \beta^3 + \gamma^4, \beta^2 + \gamma^3 + \alpha^4, \gamma^2 + \alpha^3 + \beta^4$ is
(A) $y^3 - 10y^2 - 33y = 37 = 0$ (B) $y^3 - 10y^2 + 33y - 37 = 0$
(C) $y^3 + 10y^2 + 33y + 37 = 0$ (D) $y^3 - 10y^2 - 33y - 37 = 0$

107. Let p and q be real numbers such that $p \ne O$, $p^3 \ne q$ and $p^3 \ne -q$. If α and β are non zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$ then a quadratic equation having $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ as its root is
(A) $\left(p^3 + q\right)x^2 - \left(p^3 + 2q\right)x + \left(p^3 + q\right) = 0$
(B) $\left(p^3 + q\right)x^2 - \left(p^3 - 2q\right)x + \left(p^3 + q\right) = 0$
(C) $\left(p^3 - q\right)x^2 - \left(5p^3 - 2q\right)x + \left(p^3 - q\right) = 0$
(D) $\left(p^3 - q\right)x^2 - \left(5p^3 + 2q\right)x + \left(p^3 - q\right) = 0$

108. If α and β are the roots of $ax^2 + bx + c = 0$ then the equation $ax^2 - bx(x-1) + C(x-1)^2 = 0$ has roots
(A) $\dfrac{\alpha}{1-\alpha}, \dfrac{\beta}{1-\beta}$ (B) $\dfrac{1-\alpha}{\alpha}, \dfrac{1-\beta}{\beta}$
(C) $\dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$ (D) $\dfrac{\alpha+1}{\alpha}, \dfrac{\beta+1}{\beta}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $f(x) = x^2 - 4x + 6$ for all $x \in R$, then which of the following statement(s) is/are correct?
(A) $f(x) > 6$, for all $x < 4$ (B) $f(x) > 6$, for all $x < 0$
(C) $f(x) > 0$, for all $x \in R$ (D) $f(x) > 2$, for all $x \in R$

2. If a and b are two numbers such that $a^2 + b^2 = 7$ and $a^3 + b^3 = 10$, then
(A) greatest value of $|a + b|$ is 5
(B) greatest value of $a + b$ is 4
(C) least value of $a + b$ is 1
(D) least value of $|a + b|$ is 1

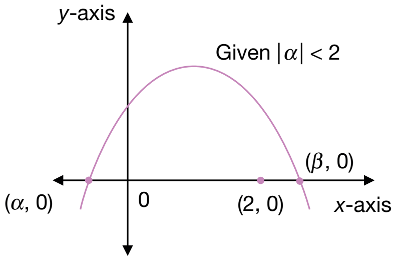
3. If 'a' and 'b' are distinct positive integers and the quadratic equations $(a-1)x^2 - \left(a^2 + 2\right)x + \left(a^2 + 2a\right) = 0$ and $(b-1)x^2 - \left(b^2 + 2\right)x + \left(b^2 + 2b\right) = 0$ have a common root. Then which of the following can be
(A) $a^2 + b^2 = 45$ (B) $a = 2b$
(C) $b = 2a$ (D) $ab = 18$

4. If the solution set for $f(x) < 3$ is $(0, \infty)$ and the solution set for $f(x) > -2$ is $(-\infty, 5)$. The interval in which x lies, for the equation $\left(f(x)\right)^2 \ge f(x) + 6$ is/are
(A) $(-\infty, \infty)$ (B) $(-\infty, 5]$
(C) $[5, \infty)$ (D) $(-\infty, 0]$

5. For $m \in R$, let $f(x) = x^3 + 3mx^2 - 3x - 3m + 2$ has 3 real roots x_1, x_2 and x_3, Also k denote the value of m for which $\left(x_1^2 + x_2^2 + x_3^2\right)$ is minimum. Which of the following statement(s) is/are correct?
(A) $f(x)$ has all 3 roots real and distinct corresponding to m for which is $\left(x_1^2 + x_2^2 + x_3^2\right)$ is minimum.
(B) The value of k lies in $[-1, 1]$
(C) The minimum value of $\sum_{i=1}^{3} x_i^2$ equals 6.
(D) Number of roots of $\sin x = k$ in $[0, 2\pi]$ equals 3.

6. The graph of the quadratic trinomial $y = ax^2 + bx + c$ has its vertex at $(4, -5)$ and two x-intercepts one positive and one negative. Which of the following hold good?
(A) $a > 0$ (B) $b < 0$
(C) $c < 0$ (D) $8a + b = 0$

7. The graph of $y = ax^2 + bx + c$ is shown in the figure, then which of the following is(are) correct?



(A) $ab^2c^3 < 0$ (B) $ab < 0$
(C) $bc(4a + 2b + c) > 0$ (D) $ab(4a - 2b + c) > 0$

8. The integral value(s) of 'a' for which the roots of $ax(x + 2) + a - (x + 2) = 0$ are rational is given by
(A) 2 (B) 5
(C) 12 (D) 20

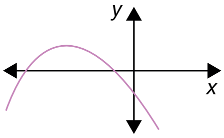
9. Let $f(x) = x^2 - 6kx + k^2 + 6k$, $k \in [-5, 5]$ and $x \in R$. If both roots of the equation $f(x) = 0$ are greater than unity and minimum integral value of $f(1)$ is β, then
(A) $\beta = 1$
(B) $\beta = 2$
(C) number of integral values of k is 4
(D) number of integral values of k is 5

10. If $1 + \log_5(x^2 + 1) \geq \log_5(ax^2 + 4x + a)$, $\forall\, x \in R$ then 'a' can be equal to
(A) 3
(B) $\frac{5}{2}$
(C) 2
(D) $\frac{3}{2}$

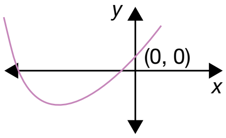
11. The value of 'k' for which both roots of the equation $4x^2 - 2x + k = 0$ are completely in $(-1, 1)$, then k may be equal to
(A) -1
(B) 0
(C) 2
(D) -3

12. For which of the following graphs of quadratic expression $y = ax^2 + bx + c$, then product abc is positive?

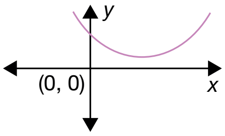
(A)



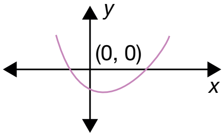
(B)



(C)



(D)



13. If $f(x) = 2x^2 + kx + 50$, $k \geq 20$ and magnitude of difference of the roots of the equation $f(x) = 0$ is 8, then which of the following can be true:
(A) minimum value of $f(x)$ is -32
(B) minimum value of $f(x)$ is -64
(C) graph of $f(x)$ cuts positive y-axis
(D) graph of $f(x)$ cuts negative y-axis

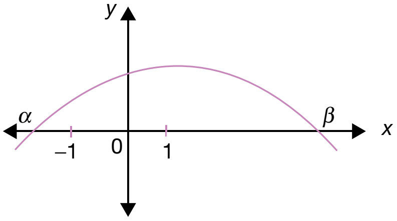
14. Let $f(x)$ be an even quadratic polynomial such that $-4 \leq f(1) \leq -1$ and $-1 \leq f(2) \leq 5$. Which of the following statements is true?
(A) Leading coefficient of $f(x)$ is $\frac{f(2) - f(1)}{3}$
(B) $f(0) = \frac{f(2) - 4f(1)}{3}$
(C) Maximum value of $f(3)$ is 20
(D) Maximum value of $f(3)$ is 12

15. The equation $x^3 + 4x^2 + px + q = 0$ and $x^3 + 6x^2 + px + r = 0$ have two roots common (where $p, q, r \in R$). If their uncommon roots are the roots of the equation $x^2 + 2ax + 8c = 0$, then:
(A) $a + c = 8$
(B) $3q = 2r$
(C) $a + c = 2$
(D) $3r = 2q$

16. If $(x - p)(x - q) + 5(x - r)(x - s) = 0$, where $p < r < q < s$, then
(A) one root lies in (p, r) and other root lies in (q, s)
(B) one root lies in (r, s) and other root lies in (s, ∞)
(C) one root lies in (p, q) and other root lies in (q, s)
(D) one root lies in $(-\infty, p)$ and other root lies in (s, ∞)

17. For the quadratic equation $x^2 + 2(a + 1)x + 12a - 8 = 0$ which of the following is/are true
(A) If $\frac{\pi}{2} \leq a \leq \frac{5\pi}{2}$, then roots are unreal
(B) If $a \leq \frac{\pi}{6}$, then roots are of opposite sign
(C) If $a > \pi^2$, then both roots are negative
(D) If $a > \pi^2$, then both roots are positive

18. The graph of quadratic polynomial $f(x) = ax^2 + bx + c$ is shown below, Which of the following are correct?



(A) $\frac{c}{a} < -1$
(B) $|\beta - \alpha| > 2$
(C) $f(x) > 0\ \forall\, x \in (0, \beta)$
(D) $abc < 0$

19. If $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots\infty}}}$, then which of the following is/are correct?
(A) $x^2 + \frac{9}{x^2} = 7$
(B) $x^3 - x^2 - x - 1 = \sqrt{13}$
(C) $x^2 + \frac{9}{x^2} = 14$
(D) $x^3 - x^2 - x - 1 = \sqrt{13} + 1$

20. If the equation $x^2 + (a - b)x - a - b + 1 = 0$ where $a, b \in R$ has unequal real roots for every real b then
(A) $a > 0$
(B) $a > 1$
(C) $a > -1$
(D) $a > 2$

21. The inequality $\left|\frac{x^2 + kx + 1}{x^2 + x + 1}\right| < 3$ is satisfied for all real values of x then
(A) $k \in (-1, 5)$
(B) The number of integral values of k is 5
(C) $k \in (-5, 1)$
(D) The equation $x^2 - \left(k^2 - 4k - 5\right)x - 1 = 0$ has one real root on $(0, 1)$

22. If α, β, γ are the roots of the equation $9x^3 - 7x + 6 = 0$ then the equation $x^3 + Ax^2 + Bx + C = 0$ has roots $3\alpha + 2, 3\beta + 2, 3\gamma + 2$, where
(A) $A = 6$
(B) $A = -5$
(C) $C = 24$
(D) $A + B + C = 23$

23. Let $f(x) = x^2 - ax + b$, a be odd positive integer and the roots of the equation $f(x) = 0$ are two distinct prime numbers. If $a + b = 35$, then
(A) sum of the roots = 17
(B) product of the roots = 22
(C) $\sum_{i=1}^{10} f(i) = -110$
(D) difference of the roots = 9

24. If both the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ in the variable x are less than 3 then 'a' can be

(A) 2

(B) 5/2

(C) $\sqrt{3}$

(D) –7

25. If a, b and c are integers, then the discriminant of $ax^2 + bx + c$ is of the form (where k is an integer)

(A) $4k$

(B) $4k + 1$

(C) $4k + 2$

(D) $4k + 3$

26. $P(x)$ is a cubic polynomial such that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$ and $P(4) = 16$, then

(A) $P(0) = -12$

(B) $P(5) = 53$

(C) $P(0) = 12$

(D) $P(-1) = -49$

27. If $\alpha, \beta, \gamma, \delta$ are roots of $x^3 + 2x^2 - 3x + 1 = 0$, then value of $\frac{\alpha\beta}{\alpha+\beta} + \frac{\alpha\gamma}{\alpha+\gamma} + \frac{\beta\gamma}{\beta+\gamma}$ is less than

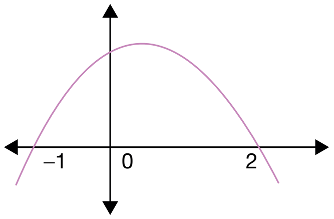
(A) 2

(B) 3

(C) 4

(D) 5

28. Consider the graph of $y = ax^2 + bx + c$ as shown in the figure, then



(A) $c > 0$

(B) $9a + 3b + c < 0$

(C) $a + b + c > 0$

(D) $4a + 2b + c < 0$

29. If α, β are roots of $p_1x^2 + p_1x + r_1 = 0$, β, γ are roots of $p_2x^2 + q_2x + r_2 = 0$ and α, γ are roots of $p_3x^2 + q_3x + r_3 = 0$, then

(A) $\alpha + \beta + \gamma = \frac{-1}{2}\left[\frac{q_1p_2p_3 + p_1q_2p_3 + p_1p_2q_3}{p_1p_2p_3}\right]$

(B) $\alpha + \beta + \gamma = \frac{-1}{2}\left[\frac{p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3}{p_1p_2p_3}\right]$

(C) $\prod_{i=1}^{3}(p_i + r_i - q_i) = p_1p_2p_3\left(1 + \sum\alpha + \sum\alpha\beta + \alpha\beta\gamma\right)^2$

(D) $\prod_{i=1}^{3}(p_i + r_i - q_i) = p_1p_2p_3\left(1 - \sum\alpha + \sum\alpha\beta - \alpha\beta\gamma\right)^2$

30. If $x^2 + x(a - 2) + (a - 2) < 0$ for atleast one real x, then 'a' can be

(A) 0

(B) 2

(C) 5

(D) 8

31. If all the three equations $x^2 + ax + 10 = 0$, $x^2 + bx + 8 = 0$ and $x^2 + (2a + 3b)x + 60 = 0$, where $a, b \in R$, have a common root, then value of $(a - b)$ can be

(A) –1

(B) 0

(C) 1

(D) 2

32. If a, b, c are positive integers such that $a > b > c$ are the quadratic equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in $(-1, 0)$ then

(A) $b + c > a$

(B) $c + a < 2b$

(C) Both roots of the given equation are rational

(D) The equation $ax^2 + 2bx + c = 0$ has both negative real roots

33. If the real numbers a, b, c are distinct, then the roots of the equation $\left(c^2 - ab\right)x^2 - 2\left(a^2 - bc\right)x + \left(b^2 - ac\right) = 0$ will be equal, if

(A) $a = 0$

(B) $a + b + c = 0$

(C) $a^3 + b^3 + c^3 - 3abc = 0$

(D) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

34. Let $a, b, c \in Q^+$ satisfying $a > b > c$. Which of the following statements (s) hold true for the quadratic polynomial $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$?

(A) The mouth of the parabola $y = f(x)$ opens upwards

(B) Both roots of the equation $f(x) = 0$ are rational

(C) x-coordinate of vertex of the graph is positive

(D) x-coordinate of vertex of the graph is negative

35. Let a, b and c be real numbers and $a \neq 0$. Let α and β be the roots of $ax^2 + bx + c = 0$. If α' and β' are roots of the equation $a^3x^2 + (abc)x + c^3 = 0$, then

(A) $\alpha' = \alpha^3\beta^2$

(B) $\beta' = \beta^3\alpha^2$

(C) $\alpha' = \alpha^2\beta$

(D) $\beta' = \alpha\beta^2$

36. If $c \in R$ and the negative of one of the solutions of $x^2 - 3x + c = 0$ is a solution of $x^2 + 3x - c = 0$, then the solutions of $x^2 - 3x + c = 0$ are

(A) –1

(B) 1

(C) 0

(D) 3

37. The equation $\left(ay - bx\right)^2 + 4xy = 0$ has rational solutions x, y for

(A) $a = \frac{1}{2}, b = 2$

(B) $a = 4, b = \frac{1}{8}$

(C) $a = 1, b = \frac{3}{4}$

(D) $a = 2, b = 1$

38. If one root of $kx^2 + x + 1 = 0$ is real and other is imaginary $(k \neq 0)$ then k^2 can not be equal to

(A) –4

(B) – 1

(C) 1

(D) 4

39. Let a, b, c be three distinct non-zero real numbers satisfying the system of equation: $\frac{1}{a} + \frac{1}{a-1} + \frac{1}{a-2} = 1$; $\frac{1}{b} + \frac{1}{b-1} + \frac{1}{b-2} = 1$; $\frac{1}{c} + \frac{1}{c-1} + \frac{1}{c-2} = 1$ then

(A) $a + b + c = 6$

(B) $abc = 2$

(C) $(1 - a)(1 - b)(1 - c) = 1$

(D) $(a - 2)(b - 2)(c - 2) = 2$

40. The real values of 'a' for which the equation $x^4 - 2x^2a - x^2 + 6x + a^2 - 3a = 0$ has all its roots real

(A) $a > 4$

(B) $a \geq 2$

(C) $a \geq 3/4$

(D) $a \geq -1$

41. Let 'm' be a real number, and suppose that two of the three solutions of the cubic equation $x^3+3x^2-34x=m$ differ by 1. Then possible values of 'm' is /are
(A) 120
(B) 80
(C) –48
(D) –32

42. If the equation $ax^2+bx+c=0$ and $bx^2+cx+a=0$ (a, b, c are unequal non zero real) have a common root then $f(x)=bx^3+cx^2+ax-5$ always passes through fixed point
(A) (1, –5)
(B) (0, –5)
(C) (–1, –5)
(D) (0, 5)

43. If $a, b \in R$ and $ax^2+bx+6=0$, $a \neq 0$ does not have two distinct real roots, then
(A) minimum possible value of $3a+b$ is 2
(B) minimum possible value of $3a+b$ is –2
(C) minimum possible value of $6a+b$ is –1
(D) minimum possible value of $6a+b$ is 1

44. If $\left|x^2+2x-8\right|+x-2=0$ then
(A) Number of roots are 3
(B) Sum of roots is –6
(C) Product of roots is 30
(D) Number of roots are 4

45. Let $f(x)=A.x^2+B.x+C$ when $A, B, C \in R$ If x is an integer then $f(x)$ is an integer then
(A) C is an integer
(B) $A+B$ is an integer
(C) B is an integer
(D) $2A$ is an integer

46. If $c \neq 0$ and the equation $\frac{p}{2x}=\frac{a}{x+c}+\frac{b}{x-c}$ has two equal roots, then p can be
(A) $\left(\sqrt{a}-\sqrt{b}\right)^2$
(B) $\left(\sqrt{a}+\sqrt{b}\right)^2$
(C) $a+b$
(D) $a-b$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Let graph of quadratic expression ax^2+bx+c, $a>0$ is symmetric about $x=\frac{-1}{2}$ and $4a+2b+c<0$.

1. Sum of roots of the equation $ax^2+bx+c=0$ is
(A) 1
(B) $\frac{1}{2}$
(C) $-\frac{1}{2}$
(D) –1

2. The value of $(a+c)$ is
(A) positive
(B) negative
(C) zero
(D) can't say

Comprehension 2

Let $3\alpha^3-\alpha^2=k\alpha-9$, $3\beta^3-\beta^2=k\beta-9$ and $3\gamma^3-\gamma^2=k\gamma-9$, where $\alpha>\beta>\gamma$, $\alpha+\gamma=0$.

3. The value of k is equal to
(A) 27
(B) 0
(C) – 9
(D) 3

4. The minimum value of $x^2+\alpha x+\beta$ is equal to
(A) $\frac{-23}{3}$
(B) $\frac{23}{12}$
(C) $\frac{-23}{12}$
(D) $\frac{23}{3}$

Comprehension 3

Let α, β, γ are roots of the equation $x^3+2x^2+3x+3=0$.

5. The value of $\frac{\alpha}{\alpha+1}+\frac{\beta}{\beta+1}+\frac{\gamma}{\gamma+1}$ is equal to
(A) 5
(B) 6
(C) – 5
(D) – 6

6. The value of $\left(\frac{\alpha}{\alpha+1}\right)^3+\left(\frac{\beta}{\beta+1}\right)^3+\left(\frac{\gamma}{\gamma+1}\right)^3$ is equal to
(A) 14
(B) 44
(C) 45
(D) 15

Comprehension 4

Let x_1, x_2, x_3, x_4 be the roots (real or complex) of the equation $x^4+ax^3+bx^2+cx+d=0$

If $x_1+x_2=x_3+x_4$ and $a,b,c,d \in R$, then

7. If $a=2$, then the value of $b-c$ is
(A) –1
(B) 1
(C) –2
(D) 2

8. If $b<0$ then how many different real values of 'a' we may have?
(A) 3
(B) 2
(C) 1
(D) 0

9. If $b+c=1$ and $a \neq -2$, then for real values of 'a' the value of $c \in$
(A) $\left(-\infty, \frac{1}{4}\right)$
(B) $(-\infty, 3)$
(C) $(-\infty, 1)$
(D) $(-\infty, 4)$

Comprehension 5

Let $f(x)=x^4+ax^3+bx^2+ax+1$ be a polynomial where a and b are real numbers, then

10. If $f(x)=0$ has two different pairs of equal roots, then the value of $a+b$ may be
(A) 0
(B) –4
(C) –2
(D) 4

11. If $f(x)=0$ has two different negative roots and two equal positive roots, then the least integral value of a is

(A) 1 (B) 2
(C) 3 (D) 4

Comprehension 6

Consider the inequation $9^x - a.3^x - a + 3 \le 0$ where 'a' is a real parameter. The given inequation has

12. At least one negative solution if
(A) $a \in (2, 3)$ (B) $a \in (2, \infty)$
(C) $a \in (-\infty, 2)$ (D) $a \in (-\infty, 3)$

13. At least one positive solutions if
(A) $a \in (-\infty, 2)$ (B) $a \in (0, 2)$
(C) $a \in (-\infty, -6)$ (D) $a \in (2, \infty)$

14. At least one solution in (1, 2) if
(A) $a \in (3, \infty)$ (B) $a \in \left(3, \frac{84}{10}\right)$
(C) $a \in \left(\frac{84}{10}, \infty\right)$ (D) $a \in R$

Comprehension 7

Consider the quadratic equation $(1 + k)x^2 - 2(1 + 2k)x + (3 + k) = 0$, where $k \in R - \{-1\}$.

15. The number of integral values of k such that the given quadratic equation has imaginary roots is
(A) 0 (B) 1
(C) 2 (D) 3

16. The set of values of k such that the given quadratic equation has both the roots positive is
(A) $k \in R$ (B) $k\ (-\infty, -3) \cup \left[\sqrt{\frac{2}{3}}, \infty\right)$
(C) $k \in (-\infty, -3) \cup (-1, \infty)$ (D) $k \in \left(-\infty, -\sqrt{\frac{2}{3}}\right] \cup \left[\frac{2}{3}, \infty\right)$

Comprehension 8

Let $f(x)$ and $g(x)$ be two polynomials of degree 2 defined as $f(x) = x^2 - 2(k - 1)x + 1$ and $g(x) = kax^2 + bx + c$. Given that $a^2 + b^2 + c^2 - 2a + 6b - 4c + 14 = 0$, $a, b, c \in R$ then

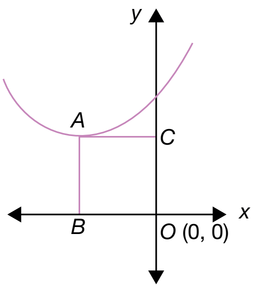
17. The value of $f(0) + g(0)$ is
(A) 1 (B) 2
(C) 3 (D) 4

18. The graph of $f(x)$ and $g(x)$ touched each other then sum of all the real values of k is
(A) 0 (B) 4
(C) 6 (D) 8

19. The number of values of k for which $f(x) = 0$ and $g(x) = 0$ has 3 real and distinct roots provided that there is no common solution among the 2 equation is/are
(A) 0 (B) 1
(C) 2 (D) 3

Comprehension 9

The graph of $f(x) = ax^2 + bx + c$ is given, for which $\ell(AB) = 4$, $\ell(AC) = 4$ and $b^2 - 4ac = -8$.



20. The value of $f(0) + 2f(1)$ is equal to
(A) 45 (B) 26
(C) 24 (D) 20

21. If α be one of the root of $f(x) = 0$ then value of $(\alpha^3 + 10\alpha^2 + 40\alpha + 39)$ is
(A) 0 (B) 9
(C) 10 (D) –9

22. Range of $h(x) = \left(\frac{3}{2} + a\right)x^2 + (b - 1)x + (c - 6)$ when $x \in [-2, 0]$ is
(A) $\left[\frac{39}{8}, 6\right]$ (B) $\left[\frac{39}{8}, 8\right]$
(C) $[6, 8]$ (D) $\left[\frac{39}{8}, \infty\right]$

Comprehension 10

Consider a rational $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 3x + 4}$ and a quadratic function $g(x) = x^2 - (b + 1)x + b - 1$, where b is a parameter.

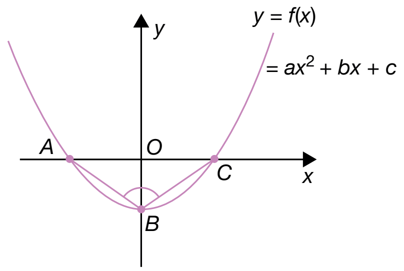
23. The sum of integers in the range of $f(x)$, is
(A) –5 (B) –6
(C) –9 (D) –10

24. If both roots of the equation $g(x) = 0$ are greater than –1, then b lies in the interval
(A) $(-\infty, -2)$ (B) $\left(-\infty, \frac{-1}{4}\right)$
(C) $(-2, \infty)$ (D) $\left(\frac{-1}{2}, \infty\right)$

25. The largest natural number b satisfying $g(x) > -2\ \forall\ x \in R$, is
(A) 1 (B) 2
(C) 3 (D) 4

Comprehension 11

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units.



26. $y = f(x)$ is given by

(A) $y = x^2 - 2\sqrt{2}$ (B) $y = x^2 - 12$

(C) $y = \frac{x^2}{2} - 2$ (D) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

27. The minimum value of $y = f(x)$ is

(A) –4 (B) –2

(C) $-2\sqrt{2}$ (D) –12

28. No. of integral values of '*K*' for which one root of $f(x) = 0$ is more than '*K*' and other less than '*K*'

(A) 4 (B) 5

(C) 6 (D) 7

Comprehension 12

Consider the quadratic equation $ax^2 + bx + c = 0$. The conditions for both roots to be positive are

(1) The roots must be real

(2) Sum of the roots > 0

(3) Product of the roots > 0

Similarly one can analyze the conditions for both roots to be negative, or one root to be positive and the other to be negative

29. The set of values of $k \in R$ for which the quadratic equation $x^2 - (k-3)x + k = 0$ has both roots positive is

(A) $[9, \infty)$ (B) $(0, 9]$

(C) $(-\infty, 1] \cup (9, \infty)$ (D) $(0, \infty)$

30. The least positive integral value of '*k*' for which the quadratic equation $x^2 + 4x + k = 0$ has both roots negative is

(A) 2 (B) 4

(C) 3 (D) 1

31. The values of λ *and* μ for which the quadratic equation $(\lambda - 3)x^2 + 4\mu x + (\lambda + 3) = 0$ has one root negative and the other positive is

(A) $\lambda \in (-3, 3), \mu \in R$

(B) $\lambda \in (-\infty, -3) \cup (3, \infty), \mu > 0$

(C) $\lambda \in (-\infty, -3) \cup (3, \infty), \mu \in R$

(D) $\lambda \in (-3, 3), \mu < 0$

Comprehension 13

If α is real parameter and $x^4 + (\alpha - 1)x^3 + (\alpha - 1)x + 1 = 0$.

32. The given equation has four real roots if

(A) $\alpha \le -\frac{1}{2}$ (B) $\alpha \ge \frac{5}{2}$

(C) $-\frac{5}{2} \le \alpha \le -\frac{1}{2}$ (D) none of these

33. The given equation has two distinct negative roots if

(A) $\alpha \le -\frac{1}{2}$ (B) $\alpha \le -\frac{5}{2}$

(C) $\alpha \ge \frac{5}{2}$ (D) none of these

34. If a non-zero complex number β is the solution of the given equation then all the value of h for which $\beta + \frac{1}{\beta}$ is real lie in the interval

(A) $(-\infty, 0) \cup (0, \infty)$ (B) $(0, \infty)$

(C) $(-\infty, -2]$ (D) R

Comprehension 14

Let $f(x) = x^2 + b_1 x + c_1,\ g(x) = x^2 + b_2 x + c_2$ Let the real roots of $f(x) = 0$ be α, β and real roots of $g(x) = 0$ be $\alpha + h, \beta + h$. The least value of $f(x)$ is $-1/4$ and least value of $g(x)$ occurs at $x = 7/2$ then

35. The least value of $g(x)$ is

(A) $-\frac{1}{4}$ (B) –1

(C) $-\frac{1}{3}$ (D) $-\frac{1}{2}$

36. The value of b_2 is

(A) –5 (B) 9

(C) –8 (D) –7

37. The roots of $f(x) = 0$ are

(A) 3, –4 (B) –3, 4

(C) 3, 4 (D) –3, –4

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1.

COLUMN-I	COLUMN-II
(A) If $x^2+x-a=0$ has integral roots and $a\in \mathrm{N}$ then a can be equal to	(p) 2
(B) If the equation $ax^2+2bx+4c=16$ has no real roots and $a+c>b+4$ then integral value of c may be equal to	(q) 12
(C) If $x^2+2bx+9b-14=0$, has only negative roots, then integral values of b may be	(r) 3
(D) If A is number of solutions of $\lvert x^2-x-6\rvert=x+2$ then A is	(s) 20

2. Let $4x^2-4(\alpha-2)x+\alpha-2=0$, $(a\in R)$ be a quadratic equation. The set of values of α of which

COLUMN-I	COLUMN-II
(A) Both roots are greater than $\frac{1}{2}$	(p) $\{2\}$
(B) Exactly one root lies in $\left(0,\frac{1}{2}\right)$	(q) ϕ
(C) Both roots are opposite in sign	(r) $R-[2,3]$
(D) Both roots are real but neither positive nor negative	(s) $(-\infty,2)$

3. Match the following for the equation $x^2+a\lvert x\rvert+1=0$ where a is a parameter

COLUMN-I	COLUMN-II
(A) No real root	(p) $a<-2$
(B) Two real roots	(q) $a=-2$
(C) Three real roots	(r) $a\in\phi$
(D) Four real roots	(s) $a\ge 0$

4. Consider an equation $(k-1)(k+1)x^2+(2k+7)(k-1)x+(2k+7)(x-2)=0$, $k\in R$. Find k, when

COLUMN-I	COLUMN-II
(A) Exactly one root zero and other is finite	(p) -1
(B) Both roots zero	(q) $\frac{-5}{2}$
(C) Exactly one root infinity	(r) 2
(D) Both roots infinity	(s) $\frac{-7}{2}$
	(t) 1

5. If $x^4-6x^3+8x^2+4ax-4a^2=0, a\in R$, then match the following

COLUMN-I	COLUMN-II
(A) Equation will have 4 real and distinct roots for $a\in$	(p) $(0,1)$
(B) Equation will have 2 distinct real roots for $a\in$	(q) $(3,4)$
(C) Equation will have at least one negative root for $a\in$	(r) $(-2,-1)$
(D) Equation will have 2 equal and 2 distinct real roots for $a\in$	(s) $\{2\}$

(*Continued*)

6. Let $P(x)=2x^2-12x+c\ \forall\ x\in R$ where c is a real constant, then

COLUMN-I	COLUMN-II
(A) If greatest value of $p(x)$ for $x\in[1,2]$ is 1, then c equals	(p) 8
(B) If smallest value of $P(x)$ for $x\in[1,5]$ is -1, then c equals	(q) 11
(C) If the greatest value of $P(x)$ for $x\in[1,4]$ is 2, then c equals	(r) 12
	(s) 17

7. Consider the quadratic trinomial

$$P(x)=3x^2-7kx+(5k-3)$$

where k is the parameter. Find the range of k in the following conditions given in column-I.

COLUMN-I	COLUMN-II
(A) One root is greater and other root is less than 2	(p) $(1,\infty)$
(B) On root lies in $(-2,0)$ and other root lies in $(0,1)$	(q) $(-\infty,0)$
(C) One root is less than -1 and other root is greater than 1	(r) ϕ
	(s) $\left(\frac{-9}{19},0\right)$

8. The expression $y=ax^2+bx+c(a,b,c\in R$ and $a\neq 0)$ represents a parabola which cuts the x-axis at the points which are roots of the equation $ax^2+bx+c=0$. Column-II contains values which correspond to the nature of roots mentioned in column-I.

COLUMN-I	COLUMN-II
(A) For $a=1, c=4$, if both roots are greater than 2 then b can be equal to	(p) 4
(B) For $a=-1, b=5$ if roots lie on either side of -1 then c can be equal to	(q) 8
(C) For $b=6, c=1$, if one root is less than -1 and the other root greater than $\frac{-1}{2}$ then a can be equal to	(r) 10
	(s) no real value

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If a, b, c, d are non - zero numbers such that c, d are the roots of the equation $x^2+ax+b=0$ and a, b are roots of the equation $x^2+cx+d=0$, then the value of $|a+b+c+d|$ is

2. Let x_1, x_2, x_3, x_4 be roots of $x^4 + ax^3 + bx^2 + cx + d = 0$. If $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$, then Find $b - c$ (Given that $a = 2$)

3. If $x^2 - 3ax + a^2 - 9 = 0$ has roots of opposite sign then $a \in (p, q)$ where $q - p =$

4. Let 't' be any root of $x^n + x^{n-1} + x^{n-2} + \ldots + x + 1 = 0$ then $\dfrac{\left(t^{2n+2}+3\right)\left(4-t^{3n+3}\right)}{4} =$

5. A polynomial of 6th degree $f(x)$, satisfies $f(x)=f(2-x)$ $\forall x \in R$. If $f(x) = 0$ has 4 distinct roots and two equal roots, then sum of roots of $f(x) = 0$ is

6. Let a, b and c be distinct real numbers such that $a^2-b= b^2-c=c^2-a$, then $(a+b)(b+c)(c+a)$ is equal to____

7. Let $P(x)=x^6-x^5-x^3-x^2-x$ and $Q(x)=x^4-x^3-x^2-1$. If α, β, γ and δ are the roots of $Q(x)=0$, then the value of $P(\alpha)+P(\beta)+P(\gamma)+P(\delta)$ is equal to

8. If both the roots of the equation $x^2-2ax+\left(a^2+a-3\right)=0$ are less than 3 then maximum value of $[a]$ is (where $[x]$ denotes the greatest integer function)

9. Let $f(x)=x^2-3x+[x]$ and n_1 is number of real solutions, n_2, n_3 are the integral solutions of $f(x) = 0$ then $(n_1+n_2+n_3)$ is equal to: ([.] denotes G.I.F)

10. Let $f(x)=ax^2+bx+c$ (where $a \neq 0, a, b, c \in R$) and $(a+c)^2<b^2$, then the number of distinct values of x in $(-1,1)$ satisfying the equation $f(x)=0$ will be

11. The total number of integral values of a so that $x^2-(a+1)x+a-1=0$ has integral roots is equal to ______

12. The number of real roots of the equation $2x^2-6x-5\sqrt{x^2-3x-6}=0$ is_______

13. All the values of parameter m, for which $x^2 + 2(m - 1)x + m + 5$ is positive for $x > 1$, lie in the interval $(-k, \infty)$ then k is

14. The number of values of a for which cubic equation $x^3 + 3x^2 + 3 - a = 0$ has all the roots as integers is ___________

15. The number of roots of the equation $x^2 - 2x - \log_2 |1 - x| = 3$ is _______________

16. If roots x_1 and x_2 of the equation $x^2 - 4ax + 1 = 0$ satisfy $x_1 \geq a$ and $x_2 \geq 0$, then $a \in \left[\dfrac{1}{k}, \infty\right)$, k must be _______________

17. If set of all real values of x satisfying $|x^2 - 3x - 1| < |3x^2 + 2x + 1| + |2x^2 + 5x + 2|$, $x^2 - 3x - 1 \neq 0$ is $(-\infty, -a) \cup (-b, \infty)$, then find the value of $a + \log ab$.

18. The number of integral solutions of equation $\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{b}\right)=4$ is ___ where $a, b, \in I$:

19. The number of real solutions of the system of equations $x=\dfrac{2z^2}{1+z^2}, y=\dfrac{2x^2}{1+x^2}, z=\dfrac{2y^2}{1+y^2}$ is__________

20. The number of roots of the equation $9\sqrt{\dfrac{x}{2-x}}-8\sqrt{\dfrac{2-x}{x}}+1=0$ is ________

21. The set of real parameter 'a' for which the equation $x^4-2ax^2+x+a^2-a=0$ has all real solutions, is given by $\left[\dfrac{m}{n}, \infty\right)$ where m and n are relatively prime positive integers, then the value of $(m+n)$ is __________

22. Number of real roots of $2x^{99}+3x^{98}+2x^{97}+3x^{96}+\ldots\ldots +2x+3=0$ is____

23. Let $p(x)=x^5+x^2+1$ have roots x_1, x_2, x_3, x_4 and x_5, $g(x)=x^2-2$, then the value of $g(x_1)g(x_2)g(x_3)$ $g(x_4)g(x_5)-30g(x_1x_2x_3x_4x_5)$, is

24. If roots x_1 and x_2 of $x^2+1=x/a$ satisfy $\left|x_1^2-x_2^2\right|>\dfrac{1}{a}$, then $a \in \left(-\dfrac{1}{2}, 0\right) \cup \left(0, \dfrac{1}{\sqrt{k}}\right)$ the numerical quantity k must be equal to____

25. If the equation $ax^2 - bx + 12 = 0$ where a and b are positive integers not exceeding 10, has roots both greater than 2 then the number of ordered pair (a, b) is _____.

26. The sum of squares of all integral values of $a > -5$ for which the inequality $x^2-ax+6a<0$ is satisfied for all $x \in (-1,1)$ must be equal to $6k$ then k is ___________

27. The solution set of the inequality $\left(\dfrac{1}{3}\right)^{\log_{\left(\frac{1}{9}\right)}\left(x^2-\frac{10}{3}x+1\right)} \leq 1$ is written as $x \in \left[0, \dfrac{1}{a}\right) \cup \left(a, \dfrac{10}{a}\right]$ then find a.

28. If $f(x)=\dfrac{ax+1}{x^2-1}$ gives all real values, then find sum of square of all integral values of 'a' given that $-2 \leq a \leq -1$

29. Let $f(x) = px^2 + px - 2$, $g(x) = x^2 + (2p + 1)x + (p + 2)$ $\forall x \in R$, where p is a real constant. Find the number of integral values of p so that $f(x) > g(x)$ $\forall x \in R$

30. Let **A** denotes the number of integers satisfying the inequality $\dfrac{3}{x^2-x+1} > 1$. **B** denotes the value of k for which the equation $(2k^2 - 5k - 3)x^2 + (k^2 - 5k + 6)x + (k^2 - 9) = 0$ has more than 2 roots and X denotes the value of the expression

$2x^4 - 30x^2 + 8x + 10$ where $x = \tan\left(\frac{5\pi}{12}\right)$. Find the value of $(X \div A + B)$.

31. The roots α and β of a quadratic equation are the square of two consecutive natural numbers. The geometric means of the two roots is 1 greater than the positive difference of the roots. If exactly one root of $x^2 - kx + 32 = 0$ lies between α and β then find the number of integral value(s) of k.

32. If exactly one root of quadratic equation $x^2 + (2a - 1)x + (a + 3) = 0$ lie in the interval $[0, 2]$ then find the square of the least integral value of a.

33. Find the number of integral value(s) of $a \in (8, \infty)$ for which the expression $\log_{10}(4x^2 - ax + 3)$ is always non-negative for all $x \in [1, 3]$.

34. If $x^4 + 2kx^3 + x^2 + 2kx + 1 = 0$ (where $k \in R$ and $k \in [-4, 4]$) has exactly two distinct positive and two distinct negative roots, then find the number of integral values of k?

35. Find the largest natural number 'a' for which the maximum value of $f(x) = a - 1 + 2x - x^2$ is smaller than the minimum value of $g(x) = x^2 - 2ax + 10 - 2a$.

36. If sum of maximum and minimum value of $y = \log_2(x^4 + x^2 + 1) - \log_2(x^4 + x^3 + 2x^2 + x + 1)$ can be expressed in form $((\log_2 m) - n)$, where m and n are coprime then compute $(m + n)$.

37. If α, β are roots of the equation $2x^2 + 6x + b = 0$ where $b < 0$, then find the least integral value of $\left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right)$.

38. If α, β, γ are roots of $x^3 + 2x^2 - x + 3 = 0$, then value of $\frac{(\alpha^3 + \alpha^2 + 3)(\beta^3 + \beta^2 + 3)(\gamma^3 + \gamma^2 + 3)}{\alpha\beta\gamma}$ is ________

39. If α, β are roots of $x^2 - (k + 4)x + k^2 - 12 = 0$ such that $4 \in (\alpha, \beta)$, then number of integral values of 'k' is ________

40. Number of integral values of x satisfying the inequality $(x^2 + 6)9^x - 21x \le 7x(3)^{2x} - 3x^2 - 18$ is

41. Number of integral solution(s) of x satisfying the inequality $\frac{\log_3(x^2 + 4x - 3) + \log_5(x^2 + 6x - 5)}{\sqrt{8x - 7 - x^2}} \ge 0$ is

42. If α, β, γ are roots of equation $f(x) = 0$ where $f(x) = x^3 - cx^2 + 4x - 2$ and $f\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = 2$ then value of 'c' is

43. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min(f(x)) < \max(g(x))$ then minimum integral value of $\frac{b^2}{c^2}$ $(c \ne 0)$ is ________

44. If the system of inequalities $6x^2 + 5x - 1 < 0$ and $(k^2 - 3k + 2)x - 2 > 0$ possesses solution, then find the least natural number k.

45. If α, β are roots of equation $3x^2 + x - 1 = 0$, then the value of $18\left(\sum_{r=1}^{\infty}(\alpha^r + \beta^r)\right)$.

46. If the expression $(m^2 + m + 1)a^2 - 2(m^2 + 1)a + m^2 - m + 1 - b$ vanishes for every $m \in R$, then find the value of $(a^2 + b^2)$.

47. The number of integral values of 'm' for which the equation $3x + my = m$ and $2x - 5y = 6$ have a solution satisfying the condition $x > 0$ and $y < 0$ is________

ARCHIVE: JEE MAIN

1. **[JEE Main 2020]**
Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2}\lambda \tan x = (1 - k)$, where $k(\ne -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is:
(A) 5 (B) 10
(C) $5\sqrt{2}$ (D) $10\sqrt{2}$

2. **[JEE Main 2020]**
Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If $P_k = (\alpha)^k + (\beta)^k$, $k \ge 1$, then which one of the following statements is not true?
(A) $p_5 = p_2 \cdot p_3$ (B) $p_3 = p_5 - p_4$
(C) $p_5 = 11$ (D) $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

3. **[JEE Main 2020]**
The number of real roots of the equation, $e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$ is:
(A) 4 (B) 3
(C) 1 (D) 2

4. **[JEE Main 2020]**
Let $a, b \in R$, $a \ne 0$ be such that the equation, $ax^2 - 2bx + 5 = 0$ has a repeated root α, which is also a root of the equation, $x^2 - 2bx - 10 = 0$. If β is the other root of this equation, then $a^2 + b^2$ is equal to
(A) 26 (B) 25
(C) 28 (D) 24

5. **[JEE Main 2020]**
Let $f(x)$ be a quadratic polynomial such that $f(-1) + f(2) = 0$. If one of the roots of $f(x) = 0$ is 3, then its other root lies in:
(A) $(-3, -1)$ (B) $(1, 3)$
(C) $(-1, 0)$ (D) $(0, 1)$

6. **[JEE Main 2020]**
Let α and β be the roots of the equation $5x^2 + 6x - 2 = 0$. If $S_n = \alpha^n + \beta^n$, $n = 1, 2, 3 \ldots$. then
(A) $5S_6 + 6S_5 = 2S_4$ (B) $5S_6 + 6S_5 + 2S_4 = 0$
(C) $6S_6 + 5S_5 + 2S_4 = 0$ (D) $6S_6 + 5S_5 = 2S_4$

7. **[JEE Main 2020]**
Consider the two sets:
$A = \{m \in R :$ both the roots of $x^2 - (m+1)x + m + 4 = 0$ are real$\}$
and $B = [-3, 5)$.
Which of the following is not true ?
(A) $A - B = (-\infty, -3) \cup (5, \infty)$
(B) $A \cap B = \{-3\}$
(C) $B - A = (-3, 5)$
(D) $A \cup B = R$

8. **[JEE Main 2020]**
If α and β are the roots of the equation $x^2 + px + 2 = 0$ and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of the equation $2x^2 + 2qx + 1 = 0$, then $\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$ is equal to:
(A) $\frac{9}{4}(9 + p^2)$
(B) $\frac{9}{4}(9 - q^2)$
(C) $\frac{9}{4}(9 - p^2)$
(D) $\frac{9}{4}(9 + q^2)$

9. **[JEE Main 2020]**
Let $\lambda \neq 0$ be in R. If α and β are the roots of the equation $x^2 - x + 2\lambda = 0$ and α and γ are the roots of the equation, $3x^2 - 10x + 27\lambda = 0$, then $\frac{\beta\gamma}{\lambda}$ is equal to:
(A) 36
(B) 27
(C) 9
(D) 18

10. **[JEE Main 2020]**
The set of all real values of λ for which the quadratic equations, $(\lambda^2 + 1)x^2 - 4\lambda x + 2 = 0$ always have exactly one root in the interval $(0, 1)$ is:
(A) $(-3, -1)$
(B) $(1, 3]$
(C) $(0, 2)$
(D) $(2, 4]$

11. **[JEE Main 2020]**
If α and β are the roots of the equation, $7x^2 - 3x - 2 = 0$, then the value of $\frac{\alpha}{1-\alpha^2} + \frac{\beta}{1-\beta^2}$ is equal to:
(A) $\frac{27}{16}$
(B) $\frac{1}{24}$
(C) $\frac{27}{32}$
(D) $\frac{3}{8}$

12. **[JEE Main 2020]**
The product of the roots of the equation
$9x^2 - 18|x| + 5 = 0$, is
(A) $\frac{25}{9}$
(B) $\frac{25}{81}$
(C) $\frac{5}{27}$
(D) $\frac{5}{9}$

13. **[JEE Main 2020]**
If α and β are the roots of the equation $2x(2x + 1) = 1$, then β is equal to:
(A) $2\alpha^2$
(B) $2\alpha(\alpha + 1)$
(C) $-2\alpha(\alpha + 1)$
(D) $2\alpha(\alpha - 1)$

14. **[JEE Main 2020]**
If α and β be two roots of the equation $x^2 - 64x + 256 = 0$ then the value of $\left(\frac{\alpha^3}{\beta^5}\right)^{\frac{1}{8}} + \left(\frac{\beta^3}{\alpha^5}\right)^{\frac{1}{8}}$ is
(A) 1
(B) 3
(C) 4
(D) 2

15. **[JEE Main 2020]**
The least positive value of 'a' for which the equation, $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is _______.

16. **[JEE Main 2019]**
The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is:
(A) 4
(B) 2
(C) 5
(D) 3

17. **[JEE Main 2019]**
If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval $[1, 5]$, then m lies in the interval:
(A) $(4, 5]$
(B) $(-5, -4)$
(C) $(3, 4)$
(D) $(5, 6)$

18. **[JEE Main 2019]**
Consider the quadratic $(c - 5)x^2 - 2cx + (c - 4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval $(0, 2)$ and its other root lies in the interval $(2, 3)$. Then the number of elements in S is -
(A) 12
(B) 18
(C) 10
(D) 11

19. **[JEE Main 2019]**
The value of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is -
(A) 1
(B) 2
(C) $\frac{15}{8}$
(D) $\frac{4}{9}$

20. **[JEE Main 2019]**
If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is:
(A) -81
(B) -300
(C) 100
(D) 144

21. **[JEE Main 2019]**
If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m - 4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is
(A) $-2 + \sqrt{2}$
(B) $4 - 3\sqrt{2}$
(C) $2 - \sqrt{3}$
(D) $4 - 2\sqrt{3}$

22. **[JEE Main 2019]**
The number of integral values of m for which the quadratic expression, $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$, is always positive, is:

(A) 7 (B) 8
(C) 3 (D) 6

23. **[JEE Main 2019]**
The sum of the solution of the equation $|\sqrt{x}-2|+\sqrt{x}(\sqrt{x}-4)+2=0$, $(x>0)$ is equal to -
(A) 4 (B) 10
(C) 9 (D) 12

24. **[JEE Main 2019]**
The number of integral value of m for which the equation $(1+m^2)x^2-2(1+3m)x+(1+8m)=0$ has no real root is -
(A) 1 (B) infinitely many
(C) 3 (D) 2

25. **[JEE Main 2019]**
Let $p, q \in Q$. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2+px+q=0$, then:
(A) $p^2-4q+12=0$ (B) $p^2-4p-16=0$
(C) $q^2-4p+14=0$ (D) $p^2-4q-12=0$

26. **[JEE Main 2019]**
If m is chosen in the quadratic equation $(m^2+1)x^2-3x+(m^2+1)=0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is:
(A) $8\sqrt{3}$ (B) $4\sqrt{3}$
(C) $10\sqrt{5}$ (D) $8\sqrt{5}$

27. **[JEE Main 2019]**
If α and β are the roots of the quadratic equation, $x^2+x\sin\theta-2\sin\theta=0$, $\theta\in\left(0,\dfrac{\pi}{2}\right)$, then $\dfrac{\alpha^{12}+\beta^{12}}{\left(\alpha^{-12}+\beta^{-12}\right)\cdot(\alpha-\beta)^{24}}$ is equal to:
(A) $\dfrac{2^{12}}{(\sin\theta-8)^6}$ (B) $\dfrac{2^6}{(\sin\theta+8)^{12}}$
(C) $\dfrac{2^{12}}{(\sin\theta+8)^{12}}$ (D) $\dfrac{2^{12}}{(\sin\theta+4)^{12}}$

28. **[JEE Main 2018]**
Let $S = \{x \in R.\ x \geq 0$ and $2\left|\sqrt{x}-3\right|+\sqrt{x}(\sqrt{x}-6)+6=0$. Then S
(A) Contains exactly one element.
(B) Contains exactly two elements.
(C) Contains exactly four elements.
(D) is an empty set.

29. **[JEE Main 2017]**
If, for a positive integer n, the quadratic equation, $x(x+1)+(x+1)(x+2)+\ldots\ldots+(x+n-1)(x+n)=10n$ has two consecutive integral solutions, then n is equal to
(A) 11 (B) 12
(C) 9 (D) 10

30. **[JEE Main 2016]**
The sum of all real values of x satisfying the equation $\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is:
(A) 5 (B) 3
(C) –4 (D) 6

31. **[IIT JEE 2011 and JEE Main 2015]**
Let α and β be the roots of equation $x^2-6x-2=0$. If $a_n=\alpha^n-\beta^n$, for $n \geq 1$, then the value of $\dfrac{a_{10}-2a_8}{2a_9}$ is equal to:
(A) 3 (B) –3
(C) 6 (D) –6

32. **[JEE Main 2014]**
Let α and β the roots of equation $px^2+qx+r=0$, $p\neq 0$. If p, q, r are in A.P. and $\dfrac{1}{\alpha}+\dfrac{1}{\beta}=4$, then the value of $|\alpha-\beta|$ is
(A) $\dfrac{\sqrt{61}}{9}$ (B) $\dfrac{2\sqrt{17}}{9}$
(C) $\dfrac{\sqrt{34}}{9}$ (D) $\dfrac{2\sqrt{13}}{9}$

33. **[JEE Main 2013]**
If the equations $x^2+2x+3=0$ and $ax^2+bx+c=0$, $a, b, c \in R$, have a common root, then $a:b:c$ is -
(A) 1 : 2 : 3 (B) 3 : 2 : 1
(C) 1 : 3 : 2 (D) 3 : 1 : 2

34. **[AIEEE-2012]**
The equation $e^{\sin x}-e^{-\sin x}-4=0$ has:
(A) exactly four real roots.
(B) infinite number of real roots.
(C) no real roots.
(D) exactly one real root.

35. **[AIEEE-2011]**
Let for $a \neq a_1 \neq 0$, $f(x)=ax^2+bx+c$, $g(x)=a_1x^2+b_1x+c_1$ and $p(x)=f(x)-g(x)$. If $p(x)=0$ only for $x=-1$ and $p(-2)=2$, then the value of $p(2)$ is:
(A) 18 (B) 3
(C) 9 (D) 6

36. **[AIEEE-2011]**
Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are:
(A) –4, –3 (B) 6, 1
(C) 4, 3 (D) –6, –1

37. **[AIEEE-2009]**
If the roots of the equation $bx^2+cx+a=0$ be imaginary, then for all real values of x, the expression $3b^2x^2+6bcx+2c^2$ is:
(A) Greater than $-4ab$
(B) Less than $-4ab$
(C) Greater than $4ab$
(D) Less than $4ab$

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Single Correct Choice Type Problem

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. **[JEE Advanced 2020]**
Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$ is
(A) 0 (B) 8000
(C) 8080 (D) 16000

2. **[JEE Advanced 2016]**
Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x \sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x \tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$ then $\alpha_1 + \beta_2$ equals
(A) $2(\sec\theta - \tan\theta)$ (B) $2\sec\theta$
(C) $-2\tan\theta$ (D) 0

3. **[JEE Advanced 2014]**
The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has
(A) only purely imaginary roots
(B) all real roots
(C) two real and two purely imaginary roots
(D) neither real nor purely imaginary roots

4. **[IIT JEE 2011]**
A value of b for which the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have one root in common is –
(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$
(C) $i\sqrt{5}$ (D) $\sqrt{2}$

5. **[IIT JEE 2010]**
Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex number satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$. then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is:
(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Multiple Correct Choice Type Problem

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[JEE Advanced 2015]**
Let S be the set of all non-zero numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals. is(are) a subset(s) of S?
(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$
(C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$ where $\alpha \neq \beta$. For $n = 0, 1, 2, \ldots$, let $a_n = p\alpha^n + q\beta^n$.

Fact: If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

1. **[JEE Advanced 2017]**
$a_{12} =$
(A) $a_{11} - a_{10}$ (B) $a_{11} + a_{10}$
(C) $2a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

2. **[JEE Advanced 2017]**
If $a_4 = 28$, then $p + 2q =$
(A) 21 (B) 14
(C) 7 (D) 12

Integer/Numerical Answer Type Question

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[IIT JEE 2009]**
The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is:

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** A	**3.** B	**4.** B	**5.** B	**6.** B	**7.** D	**8.** B	**9.** A	**10.** C
11. D	**12.** D	**13.** C	**14.** D	**15.** A	**16.** A	**17.** B	**18.** C	**19.** A	**20.** D
21. C	**22.** D	**23.** B	**24.** B	**25.** D	**26.** C	**27.** D	**28.** A	**29.** A	**30.** C
31. D	**32.** A	**33.** A	**34.** A	**35.** D	**36.** D	**37.** D	**38.** B	**39.** C	**40.** A
41. C	**42.** D	**43.** B	**44.** A	**45.** C	**46.** A	**47.** B	**48.** A	**49.** A	**50.** C
51. A	**52.** D	**53.** A	**54.** B	**55.** C	**56.** A	**57.** A	**58.** B	**59.** C	**60.** A
61. D	**62.** B	**63.** C	**64.** A	**65.** D	**66.** C	**67.** A	**68.** D	**69.** D	**70.** D
71. A	**72.** A	**73.** D	**74.** C	**75.** A	**76.** B	**77.** A	**78.** C	**79.** D	**80.** C
81. D	**82.** A	**83.** B	**84.** D	**85.** B	**86.** C	**87.** A	**88.** C	**89.** C	**90.** A
91. D	**92.** C	**93.** C	**94.** C	**95.** A	**96.** D	**97.** C	**98.** B	**99.** C	**100.** D
101. C	**102.** C	**103.** D	**104.** A	**105.** B	**106.** B	**107.** B	**108.** C		

Multiple Correct Choice Type Questions

1. B, C	**2.** A, B, D	**3.** B, C	**4.** C, D	**5.** B, C, D
6. A, B, C, D	**7.** A, B, C, D	**8.** A, C, D	**9.** B, D	**10.** A, B
11. A, B	**12.** B, D	**13.** A, C	**14.** A, C	**15.** A, B
16. A, C	**17.** A, B, C	**18.** A, B, C, D	**19.** A, B	**20.** A, B, C
21. A, B, D	**22.** C, D	**23.** B, C, D	**24.** C, D	**25.** A, B
26. A, B, D	**27.** A, B, C, D	**28.** A, B, C	**29.** A, C	**30.** A, D
31. A, C	**32.** B, C, D	**33.** A, B, C	**34.** A, B, C	**35.** C, D
36. C, D	**37.** A, C	**38.** A, B, C, D	**39.** A, B, C, D	**40.** A, B
41. A, C	**42.** A, B	**43.** B, C	**44.** A, B, C	**45.** A, B, D
46. A, B				

Linked Comprehension Type Questions

1. D	**2.** B	**3.** A	**4.** C	**5.** A	**6.** B	**7.** B	**8.** C	**9.** A	**10.** D
11. A	**12.** A	**13.** D	**14.** B	**15.** B	**16.** B	**17.** C	**18.** C	**19.** A	**20.** A
21. D	**22.** B	**23.** B	**24.** D	**25.** B	**26.** D	**27.** C	**28.** B	**29.** A	**30.** D
31. A	**32.** D	**33.** D	**34.** C	**35.** A	**36.** D	**37.** C			

Matrix Match/Column Match Type Questions

1. A → (p, q, s)	B → (q, s)	C → (p, q, s)	D → (r)
2. A → (q)	B → (r, s)	C → (s)	D → (p)
3. A → (s)	B → (q)	C → (r)	D → (p)
4. A → (r)	B → (s)	C → (p)	D → (t)
5. A → (p)	B → (q, r)	C → (p, q, r, s)	D → (s)
6. A → (q)	B → (s)	C → (r)	
7. A → (p)	B → (s)	C → (r)	
8. A → (s)	B → (q, r)	C → (p)	

Integer/Numerical Answer Type Questions

1. 2	**2.** 1	**3.** 6	**4.** 3	**5.** 6	**6.** 1	**7.** 6	**8.** 1	**9.** 5	**10.** 1
11. 1	**12.** 0	**13.** 1	**14.** 2	**15.** 4	**16.** 2	**17.** 2	**18.** 1	**19.** 2	**20.** 1
21. 7	**22.** 1	**23.** 7	**24.** 5	**25.** 1	**26.** 5	**27.** 3	**28.** 5	**29.** 0	**30.** 8
31. 1	**32.** 9	**33.** 0	**34.** 8	**35.** 1	**36.** 5	**37.** 10	**38.** 5	**39.** 7	**40.** 6
41. 5	**42.** 3	**43.** 1	**44.** 6	**45.** 6	**46.** 1	**47.** 8			

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1. B	**2.** A	**3.** C	**4.** B	**5.** C	**6.** A	**7.** A	**8.** C	**9.** D	**10.** B
11. A	**12.** B	**13.** C	**14.** D	**15.** 8.00	**16.** D	**17.** A	**18.** D	**19.** B	**20.** B
21. B	**22.** A	**23.** B	**24.** B	**25.** D	**26.** D	**27.** C	**28.** D	**29.** A	**30.** B
31. A	**32.** D	**33.** A	**34.** C	**35.** A	**36.** B	**37.** A			

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Single Correct Choice Type Questions

1. D	**2.** C	**3.** D	**4.** B	**5.** B

Multiple Correct Choice Type Questions

1. A, D

Linked Comprehension Type Questions

1. B	**2.** D

Integer/Numerical Answer Type Questions

1. 2

From the Author's Desk

“ Dear learners,

Progression is a scoring chapter from Algebra, consisting of numerous concepts. It has substantial weightage in JEE Mains and Advanced. A.P./G.P./H.P./A.G.P, problems based on Means, Miscellaneous series, Method of Difference problems are key concepts to focus on. Progression chapter can be mixed up with many other chapters, such as Determinant & Matrices, Quadratic equation, Complex number, P & C and Probability.

It is a conceptual chapter, so practice is the key to handle it. Students are advised to solve adequate questions covering various concepts in prescribed time limit. Do not forget to attempt PYQs from JEE Mains & Advanced.

Good Luck!! ”

CHAPTER

3 Progression and Series

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Arithmetic progression
(b) Geometric progression
(c) Harmonic progression
(d) Arithmetic geometric progression
(e) Problems based on means
(f) Miscellaneous series
(g) Method of difference

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

A sequence is a function whose domain is the set of natural numbers. If the range of a sequence is a subset of a real number (complex number), then it is called a real sequence (complex sequence).
If $t_n = f(n)$, then the sequence is written as $\{t_1, t_2, t_3, \ldots\}$.

REAL SEQUENCE

A sequence whose range is a subset of R is called a real sequence. They can follow some logic, rule or pattern.
On the basis of the number of terms there are two types of sequence.

- **(i) Finite sequences:** A sequence is said to be finite if it has finite number of terms.
- **(ii) Infinite sequences:** A sequence is said to be infinite if it has infinitely many terms.

EXAMPLES:

(i) 2, 5, 8, 11, 14, 17, 20, 23, 26 (finite sequence)
(ii) 4, 1, –2, –5, … (infinite sequence)

SERIES

By adding or subtracting the terms of a sequence, we get an expression which is called a series.
If $a_1, a_2, a_3, \ldots a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n$ is a series.

EXAMPLES:

(i) $1 + 2 + 3 + 4 + \ldots + n$
(ii) $2 + 4 + 8 + 16 + \ldots$
(iii) $-1 + 3 - 9 + 27 - \ldots$

PROGRESSION

The word progression collectively refers to sequence or series of finite or infinite type

ILLUSTRATION 1

For Sequence

$$(1), (2, 3), (4, 5, 6, 7), (8, 9, 10, \ldots 15), \ldots$$

write n^{th} term.

SOLUTION

Rewriting given terms,

$$T_1 = (2^0),$$

$$T_2 = (2, 3) = (2, 2^2 - 1)$$

$$T_3 = (4, 5, 6, \ldots, 7) = (2^2, 2^2 + 1, 2^2 + 2, \ldots, 2^3 - 1)$$

$$T_4 = (8, \ldots, 15) = (2^3, 2^3 + 1, 2^3 + 2, \ldots, 2^4 - 1)$$

$$T_5 = (16, \ldots, 31) = (2^4, 2^4 + 1, 2^4 + 2, \ldots, 2^5 - 1)$$

Clearly we can write n^{th} term as

$$T_n = (2^{n-1}, 2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1)$$

ILLUSTRATION 2

For sequence (1), (2, 3, 4), (5, 6, 7, 8, 9), (10, 11, 12, …, 15, 16), write n^{th} term.

SOLUTION

$$T_1 = (1),$$

$$T_2 = (2, 3, 4) = (1^2 + 1, 1^2 + 2, 2^2)$$

$$T_3 = (5, 6, 7, 8, 9) = (2^2 + 1, 2^2 + 2, \ldots, 3^2)$$

$$T_4 = (10, 11, 12, \ldots, 15, 16) = (3^2 + 1, 3^2 + 2, \ldots, 4^2)$$

Clearly we can write n^{th} term as

$$T_n = ((n-1)^2 + 1, (n-1)^2 + 2, \ldots, n^2)$$

ILLUSTRATION 3

Write down the sequence whose n^{th} term is $\dfrac{(-2)^n}{(-1)^n + 2}$.

SOLUTION

Let $t_n = \dfrac{(-2)^n}{(-1)^n + 2}$

put $n = 1, 2, 3, 4, \ldots$

we get $t_1 = -2,\ t_2 = \dfrac{4}{3},\ t_3 = -8,\ t_4 = \dfrac{16}{3}$

so the sequence is $-2, \dfrac{4}{3}, -8, \dfrac{16}{3}, \ldots$

ARITHMETIC PROGRESSION (A.P.)

A.P. is a sequence whose successive terms are obtained by adding a fixed number 'd' to the preceding terms.
This fixed number 'd' is called the common difference
If a is the first term and d the common difference, then A.P. can be written as $a, a + d, a + 2d, \ldots, a + (n - 1)\,d, \ldots$

EXAMPLE:

$-4, -1, 2, 5 \ldots$

Let 'a' be the first term and 'd' be the common difference of an A.P., then

General n^{th} Term from starting $T_n = a + (n-1)\,d$, where $d = T_n - T_{n-1}$

$$d = T_2 - T_1 = T_3 - T_2 = T_4 - T_3$$

$$= T_5 - T_4 = \ldots = T_{n-1} - T_{n-2} = T_n - T_{n-1}$$

The Sum of First *n* Terms of an A.P.

$$S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$= \frac{n}{2}[2(\text{first term}) + (n-1)(\text{Common difference})]$$

$$= \frac{n}{2}[a + \ell] = \frac{n}{2}[\text{first term} + \text{Last Term}]$$

ILLUSTRATION 4

If in an A.P., 3rd term is 18 and 7th term is 30, then find sum of its first 17 terms.

SOLUTION

Let a be the first term and d be the common difference

$$T_3 = a + 2d = 18$$

$$T_7 = a + 6d = 30$$

We get $d = 3, a = 12$

sum of its first 17 terms $S_{17} = \dfrac{17}{2}[2 \times 12 + 16 \times 3] = 612$

ILLUSTRATION 5

Find the sum of all odd numbers between 1 and 1000 which are divisible by 3.

SOLUTION

Odd numbers between 1 and 1000 are

3, 5, 7, 9, 11, 13, … 993, 995, 997, 999.

Those numbers which are divisible by 3 are

3, 9, 15, 21, … 993, 999

They form an A.P. of which $a = 3, d = 6, \ell = 999$

$\therefore \quad n = 167$

$$S = \frac{n}{2}[a + \ell] = 83667$$

ILLUSTRATION 6

The ratio between the sum of n term of two A.P.'s is $3n + 8 : 7n + 15$. Then find the ratio between their 12th term.

SOLUTION

$$\frac{S_n}{S_n'} = \frac{(n/2)[2a + (n-1)d]}{(n/2)[2a' + (n-1)d']} = \frac{3n+8}{7n+15}$$

or $\quad \dfrac{a + \{(n-1)/2\}d}{a' + \{(n-1)/2\}d'} = \dfrac{3n+8}{7n+15} \qquad \ldots(1)$

we have to find $\dfrac{T_{12}}{T_{12}'} = \dfrac{a + 11\,d}{a' + 11d'}$

choosing $(n-1)/2 = 11$ or $n = 23$ in (1)

we get $\dfrac{T_{12}}{T_{12}'} = \dfrac{a + 11d}{a' + 11d'} = \dfrac{3(23) + 8}{(23) \times 7 + 15} = \dfrac{77}{176} = \dfrac{7}{16}$

ILLUSTRATION 7

The numbers $t(t^2 + 1)$, $-t^2/2$ and 6 are three consecutive terms of an A.P. If t be real,then find the next two term of A.P.

SOLUTION

if a, b, c are in AP then

$$2b = a + c \quad \Rightarrow \quad -t^2 = t^3 + t + 6$$

or $\quad t^3 + t^2 + t + 6 = 0$

or $\quad (t+2)\,(t^2 - t + 3) = 0$

$\therefore \quad t^2 - t + 3 \neq 0 \quad \Rightarrow \quad t = -2$

the given numbers are $-10, -2, 6$
which are in an A.P. with $d = 8$. The next two numbers are 14, 22.

ILLUSTRATION 8

If 1, $\log_y x$, $\log_z y$, $-15 \log_x z$ are in A.P., then prove that $x = y^{-1} = z^3$.

SOLUTION

Let d be the common difference. then

$$\log_y x = 1 + d \quad \Rightarrow \quad x = y^{1+d}$$

$$\log_z y = 1 + 2d \quad \Rightarrow \quad y = z^{1+2d}$$

and $\quad -15 \log_x z = 1 + 3d \quad \Rightarrow \quad z = x^{-(1+3d)/15}$

$\therefore \quad x = y^{\,1+d} = z^{(1+2d)(1+d)}$

$$= x^{-(1+d)(1+2d)(1+3d)/15}$$

$\Rightarrow \quad (1+d)\,(1+2d)\,(1+3d) = -15$

$\Rightarrow \quad 6d^3 + 11d^2 + 6d + 16 = 0$

$\Rightarrow \quad (d+2)\,(6d^2 - d + 8) = 0$

$\Rightarrow \quad d = -2$

[Note that $6d^2 - d + 8 = 0$ has complex roots]

$\therefore \quad x = y^{1+d} = y^{-1},\ y = z^{1-4} = z^{-3}$

$\therefore \quad x = (z^{-3})^{-1} = z^3.$

Also $x = y^{-1} = z^3$

ILLUSTRATION 9

Let $\langle a_n \rangle$ be an arithmetic sequence. If $\sum_{r=1}^{10^{99}} a_{2r} = 10^{100}$ and $\sum_{r=1}^{10^{99}} a_{2r-1} = 10^{99}$, then find the common difference of arithmetic sequence.

SOLUTION

let d be the common difference of the A.P., then

$$a_{2r} = a_{2r-1} + d$$

$\Rightarrow \quad \displaystyle\sum_{r=1}^{10^{99}} a_{2r} = \sum_{r=1}^{10^{99}} (a_{2r-1} + d)$

$\Rightarrow \quad \displaystyle\sum_{r=1}^{10^{99}} a_{2r} = \left(\sum_{r=1}^{10^{99}} a_{2r-1}\right) + 10^{99}\, d$

$\Rightarrow \quad 10^{100} = 10^{99} + 10^{99} d$

$\Rightarrow \quad d = \dfrac{10^{100} - 10^{99}}{10^{99}} = \dfrac{10^{99}(10-1)}{10^{99}} = 9$

ILLUSTRATION 10

If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms then prove that

$$(m+n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m+p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

SOLUTION

Let the A.P. be $a, a+d, a+2d, \ldots$

Given $T_1 + T_2 + \ldots + T_m = T_{m+1} + \ldots + T_{m+n}$ $\quad$ …(1)

Adding $T_1 + T_2 + \ldots + T_m$ on both sides in (1), we get

$$2(T_1 + T_2 + \ldots + T_m)$$

$$= T_1 + T_2 + \ldots + T_m + T_{m+1} + \ldots + \ldots + T_{m+n}$$

$\Rightarrow \quad 2S_m = S_{m+n}$

$\therefore \quad 2 \,.\, \dfrac{m}{2}\{2a + (m-1)\,d\} = \dfrac{m+n}{2}\,\{2a + (m+n-1)\,d\}$

Let $2a + (m-1)\,d = x$

$\Rightarrow \quad mx = \dfrac{m+n}{2}\,\{x + nd\}$

$\Rightarrow \quad (m-n)\,x = (m+n)\,nd$ $\quad$ …(2)

Again $T_1 + T_2 + \ldots + T_m = T_{m+2} + \ldots + T_{m+p}$

Similarly $(m-p)\,x = (m+p)\,pd$ $\quad$ …(3)

Dividing (2) by (3), we get

$$\frac{m-n}{m-p} = \frac{(m+n)\,n}{(m+p)\,p}$$

$\Rightarrow \quad (m-n)\,(m+p)\,p = (m-p)\,(m+n)\,n$

Dividing both sides by mnp, we have

$$(m+p)\left(\frac{1}{n} - \frac{1}{m}\right) = (m+n)\left(\frac{1}{p} - \frac{1}{m}\right)$$

Hence, $(m+n)\left(\dfrac{1}{m} - \dfrac{1}{p}\right) = (m+p)\left(\dfrac{1}{m} - \dfrac{1}{n}\right)$

ILLUSTRATION 11

The first, second and middle terms of an AP are a, b, c respectively then find their sum.

SOLUTION

We have first term = a, second term = b

$$d = \text{common difference} = b - a$$

It is given that the middle term is c. This means that there are an odd number of terms in the AP. Let there be $(2n + 1)$ terms in the AP. Then $(n + 1)^{\text{th}}$ term is the middle term.

$$\therefore \quad \text{middle term} = c \quad \Rightarrow \quad a + nd = c$$

$$\Rightarrow \quad a + n\,(b - a) = c \quad \Rightarrow \quad n = \frac{c-a}{b-a}$$

$$\text{Sum} = \frac{2n+1}{2}\,[2a + (2n + 1 - 1)d]$$

$$= \frac{1}{2}\left\{2\left(\frac{c-a}{b-a}\right)+1\right\}\left[2a+2\left(\frac{c-a}{b-a}\right)(b-a)\right]$$

$$= \frac{1}{2}\left\{2\left(\frac{c-a}{b-a}\right)+1\right\}\{2c\} = \frac{2c(c-a)}{b-a}+c$$

ILLUSTRATION 12

If $a_1, a_2, a_3 \ldots a_n$ are in AP where $a_i > 0 \;\forall\; i$ then show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}.$$

SOLUTION

Let d be the common difference of the A.P.

$$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1}$$

Now required term

$$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\ldots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$$

$$= -\left(\frac{\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\ldots+\left(\sqrt{a_{n-1}}-\sqrt{a_n}\right)}{d}\right)$$

$$= -\frac{\left(\sqrt{a_1}-\sqrt{a_n}\right)}{d} = \frac{1}{d}\,\frac{\left(a_n-a_1\right)}{\sqrt{a_n}+\sqrt{a_1}}$$

$$= \frac{(n-1)d}{d[\sqrt{a_n}+\sqrt{a_1}]} \qquad [\because\, a_n = a_1 + (n-1)d]$$

$$= \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

ILLUSTRATION 13

The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in A.P. then find magnitude of the difference between the largest and smallest roots.

SOLUTION

Let the roots are $a - d$, a, $a + d$

$$\therefore \quad 3a = \frac{144}{64} = \frac{9}{4} \quad \Rightarrow \quad a = \frac{3}{4}$$

$$\text{Also } a(a^2 - d^2) = \frac{15}{64} = \frac{3}{4}\left(\frac{9}{16}-d^2\right)$$

$$= \frac{15}{64};\ \frac{9}{16}-d^2=\frac{5}{16}$$

$$d^2 = \frac{1}{4} \quad \Rightarrow \quad d = \frac{1}{2} \text{ or } -\frac{1}{2}$$

if $a = \frac{3}{4}$ and $d = \frac{1}{2}$ the numbers are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}$;

if $a = \frac{3}{4}$ and $d = -\frac{1}{2}$ the numbers are $\frac{5}{4}, \frac{3}{4}, \frac{1}{4}$

$\therefore$ Difference between largest and smallest root is equal to $\frac{5}{4} - \frac{1}{4} = 1$

ILLUSTRATION 14

Find number of terms common to the two sequences 17, 21, 25, …, 417 and 16, 21, …, 466.

SOLUTION

$$417 = 17 + (n - 1)4 \quad \Rightarrow \quad 400 = 4(n - 1)$$

$$\Rightarrow \quad n = 101$$

Similarly $466 = 16 + (m - 1)5$

$$\Rightarrow \quad 450 = 5(m - 1) \quad \Rightarrow \quad m = 91$$

Let T_n is common to both for some n for which m is an integer.

$$17 + (n - 1)4 = 16 + (m - 1)5$$

$$\Rightarrow \quad 1 + 4n - 4 = 5m - 5$$

$$\Rightarrow \quad 5m = 4n + 2$$

Hence $n = 2, 7, 12, \ldots, 97$ (20 common terms).

ILLUSTRATION 15

Let a_n, $n \in N$ is an A.P. with common difference 'd' and all whose terms are non-zero. If n approaches infinity, then find the sum $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_na_{n+1}}$.

SOLUTION

$$d = a_2 - a_1 = a_3 - a_2 = \ldots = a_{n+1} - a_n$$

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_na_{n+1}}$$

$$= \frac{1}{d}\left[\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\ldots+\frac{a_{n+1}-a_n}{a_n\cdot a_{n+1}}\right]$$

$$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \ldots + \frac{1}{a_n} - \frac{1}{a_{n+1}}\right]$$

$$= \frac{1}{d}\left[\frac{1}{a_1} - \frac{1}{a_{n+1}}\right]$$

$$= \frac{1}{d}\left[\frac{a_{n+1} - a_1}{(a_1)(a_{n+1})}\right]$$

$$= \frac{1}{d}\left[\frac{a_1 + nd - a}{(a_1)(a_{n+1})}\right]$$

$$= \frac{n}{a_1[a_1 + nd]} = \frac{1}{a_1\left[\frac{a_1}{n} + d\right]}$$

$\Rightarrow$ as $n \to \infty$ then $S = \dfrac{1}{a_1 d}$

PROPERTIES OF ARITHMETIC PROGRESSION

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow$ $2b = a + c$

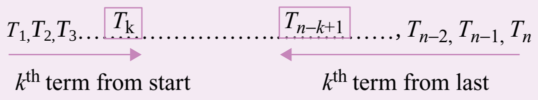
if a, b, c, d are in A.P. $\Rightarrow$ $a + d = b + c$.

(iii)

Number of Consecutive Terms in AP	Terms We Can Assume	Common Difference of AP
Three	$a - d, a, a + d$	d
Four	$a - 3d, a - d, a + d, a + 3d$	$2d$
Five	$a - 2d, a - d, a, a + d, a + 2d$	d
Six	$a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$	$2d$
Seven	$a - 3d, a - 2d, a - d, a, a + d, a + 2d, a + 3d$	d

(iv) The sum of the terms of an A.P. equidistant from the beginning and end is constant and equal to the sum of first and last term.

$$T_k + T_{n-k+1} = \text{First Term} + \text{Last Term}$$


$T_1, T_2, T_3 \ldots$ T_k $\ldots\ldots$ T_{n-k+1} $\ldots\ldots, T_{n-2}, T_{n-1}, T_n$
kth term from start
kth term from last

Proof:

Let there are total n terms are present of a certain AP

kth term from start $T_k = a + (k - 1)d$

kth term from last T_{n-k+1}

$= (n - k + 1)^{\text{th}}$ term from start $= a + (n - k)d$

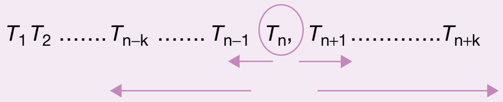
$T_k + T_{n-k+1} = a + (k - 1)d + a + (n - k)d$

$= a + (a + (n - 1)d) =$ First Term + Last Term

$T_k + T_{n-k+1} = T_1 + T_n = T_2 + T_{n-1} = T_3 + T_{n-2}$

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it.

$$T_n = \frac{T_{n-k} + T_{n+k}}{2}$$


$T_1\, T_2 \ldots\ldots T_{n-k} \ldots\ldots T_{n-1}$ $T_n,$ $T_{n+1} \ldots\ldots\ldots\ldots T_{n+k}$

Proof:

For an AP we can write $T_{n-k} = a + (n - k - 1)d$ and $T_{n+k} = a + (n + k - 1)d$

Adding both we get

$$T_{n-k} + T_{n+k} = a + (n - k - 1)d + a + (n + k - 1)d$$

$$= 2(a + (n - 1)d) = 2T_n$$

$$T_n = \frac{T_{n-k} + T_{n+k}}{2} = \frac{T_{n-1} + T_{n+1}}{2}$$

$$= \frac{T_{n-2} + T_{n+2}}{2} = \frac{T_{n-3} + T_{n+3}}{2}$$

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number k, then the resulting sequence is also an AP.

Let $T_1, T_2, T_3, T_4, T_5, T_6, \ldots T_{n-2}, T_{n-1}, T_n$ are in AP with common difference d

Changes Applied to Each Term of an AP	Resultant AP	New Common Difference of Resultant AP
Addition of k	$T_1 + k, T_2 + k, T_3 + k, T_4 + k, \ldots T_{n-2} + k, T_{n-1} + k, T_n + k$	d
Subtraction of k	$T_1 - k, T_2 - k, T_3 - k, T_4 - k, \ldots T_{n-2} - k, T_{n-1} - k, T_n - k$	d
Multiplication of k	$kT_1, kT_2, kT_3, kT_4, \ldots kT_{n-2}, kT_{n-1}, kT_n$	kd
Division of k	$\frac{T_1}{k}, \frac{T_2}{k}, \frac{T_3}{k}, \ldots \frac{T_{n-2}}{k}, \frac{T_{n-1}}{k}, \frac{T_n}{k}$	d/k

(vii) The sum and difference of two AP's is also an AP.

ARITHMETIC MEAN (MEAN OR AVERAGE) (A.M.)

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a and c.
A.M. for any n numbers $a_1, a_2, \ldots, a_n$ is;

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}.$$

n-ARITHMETIC MEANS BETWEEN TWO NUMBERS

If a, b are any two given numbers and $a, A_1, A_2, \ldots, A_n, b$ are in A.P., then $A_1, A_2, \ldots A_n$ are the n A.M.'s between a and b.
Let $A_1, A_2, A_3, \ldots, A_n$ be the AM's. then $a, A_1, A_2, \ldots, A_n, b$ will be in A.P.
As we can observe that b is $(n+2)^{\text{th}}$ term of AP

Now $T_{n+2} = b = a + (n+1)\,d \quad \Rightarrow \quad d = \dfrac{b-a}{n+1}$
(common difference)

$$\therefore \quad T_2 = A_1 = a + d = a + \left(\frac{b-a}{n+1}\right)$$

$$T_3 = A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right)$$

$$T_4 = A_3 = a + 3d = a + 3\left(\frac{b-a}{n+1}\right)$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\therefore \quad T_{k+1} = A_n = a + n\left(\frac{b-a}{n+1}\right) \ (n^{\text{th}} \text{ term of the A.M.})$$

We get $A_1 = a + \dfrac{b-a}{n+1}, A_2 = a + \dfrac{2(b-a)}{n+1}, \ldots,$

$$A_n = a + \frac{n(b-a)}{n+1}$$

Note

Sum of n A.M.'s inserted between a and b is equal to n times the single A.M. between a and b

i.e. $\displaystyle\sum_{r=1}^{n} A_r = A_1 + A_2 + A_3 + \ldots + A_n = nA,$

where A is the single A.M. between a and b

i.e. $A = \dfrac{a+b}{2}$

ILLUSTRATION 16

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the sum of the squares of the other three numbers. Find the numbers.

SOLUTION

Let the numbers be $a-d, a, a+d, a+2d$ where, $a, d \in I, d > 0$ according the hypothesis

$$(a-d)^2 + a^2 + (a+d)^2 = a + 2d$$

i.e., $\quad 2d^2 - 2d + 3a^2 - a = 0$

$$\therefore \quad d = \frac{1}{2}\left[1 \pm \sqrt{\left(1 + 2a - 6a^2\right)}\right]$$

Since, d is positive integer,

$$1 + 2a - 6a^2 > 0 \quad \Rightarrow \quad a^2 - \frac{a}{3} - \frac{1}{6} < 0$$

$$\Rightarrow \quad \left(a - \frac{1-\sqrt{7}}{6}\right)\left(a - \frac{1+\sqrt{7}}{6}\right) < 0$$

$$\therefore \quad \left(\frac{1-\sqrt{7}}{6}\right) < a < \left(\frac{1+\sqrt{7}}{6}\right)$$

Since, a is an integer, $a = 0$

then, $d = \dfrac{1}{2}[1 \pm 1] = 1$ or 0. Since, $d > 0$

$\therefore \quad d = 1$

Hence, the numbers are $-1, 0, 1, 2$

ILLUSTRATION 17

If $a(b+c), b(c+a), c(a+b)$ are in A.P.,

prove that $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are also in A.P.

SOLUTION

$\because \quad a(b+c), b(c+a), c(a+b)$ are in A.P.

subtract $ab + bc + ca$ from each

$\quad -bc, -ca, -ab$ are in A.P.

divide by $-abc$

$\quad \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

ILLUSTRATION 18

If $\dfrac{a+b}{1-ab}, b, \dfrac{b+c}{1-bc}$ are in A.P. then

prove that $\dfrac{1}{a}, b, \dfrac{1}{c}$ are in A.P.

SOLUTION

$\therefore \quad \dfrac{a+b}{1-ab}, b, \dfrac{b+c}{1-bc}$ are in A.P.

$$\Rightarrow \quad b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b$$

$$\Rightarrow \quad \frac{-a\left(b^2+1\right)}{1-ab} = \frac{c\left(1+b^2\right)}{1-bc}$$

$\Rightarrow \quad -a + abc = c - abc$

$\Rightarrow \quad a + c = 2abc$

divide by ac

$$\frac{1}{c}+\frac{1}{a}=2b \quad \Rightarrow \quad \frac{1}{a}, b, \frac{1}{c} \text{ are in A.P.}$$

ILLUSTRATION 19

Insert 10 A.M. between 3 and 80.

SOLUTION

$3, A_1, A_2, \ldots, A_9, A_{10}, 80$ are in A.P.

Here 3 is the first term and 80 is the 12th term of A.P.

$$T_{12} = 80 = 3 + (11)d \quad \Rightarrow \quad d = 7$$

so the series is 3, 10, 17, 24, …, 73, 80

$\therefore$ Required means are 10, 17, 24, …, 73.

ILLUSTRATION 20

If the equation $x^4 - (12k+5)x^2 + 16k^2 = 0$ $(k > 0)$ has four real solutions which are in A.P. then the value of $k = \frac{a}{b}$ where a and b are co-prime, then find the value of $(a - 2b)$.

SOLUTION

$x^4 + 0 \cdot x^3 - (12k+5)x^2 + 0 \cdot x + 16\,k^2 = 0$

Let the roots are $A - 3d, A - d, A + d, A + 3d$

Sum of roots $= 0$ $\qquad$ ($\because$ Coefficient of $x^3 = 0$)

$A - 3d + A - d + A + d + A + 3d \quad \Rightarrow \quad 4A = 0 \quad \Rightarrow \quad A = 0$

Product of roots $= (-3d)(-d)(d)(3d) \quad \Rightarrow \quad 9d^4 = 16k^2$

$$\Rightarrow \quad d^2 = \frac{4k}{3} \qquad \ldots(1)$$

Sum of roots taken two at a time i.e.,

$$\Sigma x_1 x_2 = -10d^2 = -(12k+5)$$

$$10 \cdot \left(\frac{4k}{3}\right) = 12k + 5$$

$$40k = 36k + 15$$

$$k = \frac{15}{4} = \frac{a}{b}$$

Hence, $(a - 2b) = 7$

ILLUSTRATION 21

If in a finite A.P. the number of terms be odd, then prove that its middle term is the A.M. between the first and last term and its sum is equal to the product of middle term and no. of terms.

SOLUTION

Let there are $2n + 1$ terms and first term as 'a' and common difference is 'd'

Middle term $T_{n+1} = a + (n + 1 - 1)\, d = a + nd$

Last Term $= T_{2n+1} = a + (2n + 1 - 1)\, d = a + 2nd$

Clearly we can see

$$2T_{n+1} = T_{2n+1} + T_1 = 2(a + nd)$$

Middle term is arithmetic mean of first and last term

$$S_{2n+1} = \frac{2n+1}{2}(T_1 + T_{2n+1}) = (2n+1)(\text{Middle Term})$$

$$= (\text{No. of terms})(\text{Middle Term})$$

ILLUSTRATION 22

If S_1, S_2, S_3 be the sum of first n terms of 3A.P's the first terms of each being 1 and respective common differences are 1, 2 and 3, then show that $S_1 + S_3 = 2S_2$.

SOLUTION

$$S_1 = \frac{n}{2}(2.1 + (n-1)1) = \frac{n}{2}(n+1)$$

$$S_2 = \frac{n}{2}(2.1 + (n-1)2) = n^2$$

$$S_3 = \frac{n}{2}(2.1 + (n-1)3) = \frac{n}{2}(3n-1)$$

$$S_1 + S_3 = \frac{n}{2}(n + 1 + 3n - 1) = \frac{4n^2}{2} = 2S_2$$

ILLUSTRATION 23

If $x_1, x_2, x_3, \ldots$ upto $2n$ terms are in AP, then the find value of $x_1^2 - x_2^2 + x_3^2 - x_4^2 + \ldots$ up to $2n$ terms.

SOLUTION

Let d is common difference of A.P.

$$d = x_2 - x_1 = x_3 - x_2 = \ldots = x_{2n} - x_{2n-1}$$

$$\left(x_1^2 - x_2^2\right) + \left(x_3^2 - x_4^2\right) + \ldots + \left(x_{2n-1}^2 - x_{2n}^2\right)$$

$$= (x_1 - x_2)(x_1 + x_2) + (x_3 - x_4)(x_3 + x_4) + \ldots$$

$$+ (x_{2n-1} - x_{2n})(x_{2n-1} + x_{2n})$$

$$= -d(x_1 + x_2 + x_3 + \ldots + x_{2n})$$

$$= -d.n.(x_1 + x_{2n}),$$

$$\left(\text{using } x_{2n} = x_1 + (2n-1)d \Rightarrow d = \frac{x_{2n} - x_1}{2n-1}\right)$$

$$= -\frac{(x_{2n} - x_1)(x_{2n} + x_1)}{2n-1} \,.\, n = \frac{n\left(x_1^2 - x_{2n}^2\right)}{2n-1}$$

ILLUSTRATION 24

If the arithmetic mean of the roots of a quadratic equation is 7 and the arithmetic mean of their reciprocals is $\frac{7}{2}$, then find the quadratic equation.

SOLUTION

$$\frac{\alpha+\beta}{2}=7 \quad \Rightarrow \quad \alpha+\beta=14$$

$$\frac{\frac{1}{\alpha}+\frac{1}{\beta}}{2}=\frac{7}{2} \quad \Rightarrow \quad \frac{\alpha+\beta}{\alpha\beta}=7 \quad \Rightarrow \frac{14}{\alpha\beta}=7$$

$\Rightarrow \quad \alpha\beta=2.$

Hence, Quadratic equation is $x^2-(\alpha+\beta)x+\alpha\beta=0$

or $\quad x^2-14x+2=0$

ILLUSTRATION 25

Let 'p' be the greatest integer for which $5m^2-16m$, $2pm$, p^2 are distinct consecutive terms of an A.P. where $m \in R$. If common difference of the A.P. is $\left(\frac{A}{B}\right)$, n $A, B \in N$ then find the least value of $(26B-A)$.

SOLUTION

$$5m^2-16+p^2=4pm$$

$$5m^2-4pm+p^2-16=0$$

It is quadratic in m, where $m \in R$,
For real roots $D \geq 0$

$$16p^2-4(5)(p^2-16) \geq 0$$

$\Rightarrow \quad 4(4p^2-5p^2+80) \geq 0$

$\Rightarrow \quad p^2-80 \leq 0$

$$p \in [-\sqrt{80}, \sqrt{80}]$$

$\therefore$ Greatest integral value of 'p' is 8.

Now, $5m^2-32m+48=0 \quad \Rightarrow \quad (5m-12)(m-4)=0$

$\Rightarrow \quad m=12/5$ and $m=4$(rejected)

$$5m^2-16, 2pm, p^2 \quad \Rightarrow \quad \frac{64}{5}, \frac{192}{5}, \frac{320}{5}$$

$\therefore \quad$ Common difference $=\dfrac{128}{5}$

ILLUSTRATION 26

If 19 Arithmetic means $A_1, A_2, A_3, \ldots, A_{19}$ are inserted between 102 and 2 such that their sum is S, then find the value of $\dfrac{S}{(A_{17}+2)A_{10}}$.

SOLUTION

$$S=A_1+A_2+\ldots+A_{19}=19\left(\frac{a+b}{2}\right)$$

$\Rightarrow \quad S=19\times\left(\dfrac{2+102}{2}\right)=19\times 52$

$102, A_1, A_2, \ldots, A_{19}, A_{21}$ are in A.P.

$$2=102+(21-1)d \quad \Rightarrow \quad d=\frac{2-102}{20}=-5$$

$$A_{17}=102+17\times(-5)=17$$

$$A_{10}=102+10\times(-5)=52$$

Now $\dfrac{S}{(A_{17}+2)A_{10}}=\dfrac{19.52}{(17+2)52}=1$

Test Your Concepts-I

(Solutions on page H.52)

1. Find the expression $\dfrac{1+3+5+\ldots+99}{2+4+6+\ldots+100}$ when simplified.
2. If the ratio of the sum to 'n' terms of two A.P's is $(5n+3):(3n+4)$, then find the ratio of their 17th terms.
3. Between the numbers 2 and 20, 8 arithmetic means are inserted, then find sum of these means.
4. Find the number of terms common between the two series $2+5+8+\ldots$ up to 50 terms and the series $3+5+7+9+\ldots$ upto 60 terms.
5. Find sum of first hundred numbers common to the two A.P.'s 12, 15, 18, ... and 17, 21, 25 ...
6. If 11 A.M. s are inserted between 28 and 10, then find number of integral Arithmetic Means.
7. If $\log 2$, $\log(2^x-1)$ and $\log(2^x+3)$ are in A.P., then find x.
8. If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3=2$.
9. If a, b, c, d, e, f are Arithmetic Means between 2 and 12, then find $a+b+c+d+e+f$.
10. $a_1, a_2, a_3, \ldots a_n$ are in A.P. such that $a_n=100$, $a_{50}-a_{49}=\frac{3}{5}$, then find 15th term of A.P. from end.
11. If the sum of the first 11 terms of an arithmetical progression equals that of the first 19 terms, then find the sum of its first 30 terms.
12. Let $a_1, a_2, a_3, \ldots, a_{49}, a_{50}$ are in arithmetic progression. If $a_1=4$ and $a_{50}=144$, then find the value of $\dfrac{1}{\sqrt{a_1}+\sqrt{a_2}}+\dfrac{1}{\sqrt{a_2}+\sqrt{a_3}}+\dfrac{1}{\sqrt{a_3}+\sqrt{a_4}}\ldots+\dfrac{1}{\sqrt{a_{49}}+\sqrt{a_{50}}}$.
13. If 100 times the 100th term of an A.P. with non-zero common difference equals the 50 times its 50th term, then find the 150th term of this A.P.

14. Two consecutive numbers k, $k + 1$ are removed from natural numbers 1, 2, 3, ... n such that arithmetic mean reduces by 1, then find $(n - k)$.

15. If the first term of an A.P. is an integer with common difference 2, then find the number of possible values of k for which the sum of first k terms is 153 is $(k > 1)$.

Answer keys

1. $\frac{50}{51}$ **2.** 168:103 **3.** 88 **4.** 20 **5.** 61500 **6.** 5 **7.** $\log_2 5$ **8.** 10 **9.** 42

10. $\frac{458}{5}$ **11.** 0 **12.** $\frac{49}{14}$ **13.** 0 **14.** 1 **15.** 5

GEOMETRIC PROGRESSION (G.P.)

G.P. is a sequence whose each of the succeeding term is equal to the preceding terms multiplied by a constant, this constant number is called the common ratio of the GP.

For non-zero G.P., the ratio of successive term to preceding term is common ratio (r) Therefore a, ar, ar^2, ar^3, ar^4, … is a G.P. with 'a' as the first term and 'r' as common ratio.

Common ratio $(r) = T_2/T_1 = T_3/T_2 = T_4/T_3 = T_5/T_4$

$= \dots T_{n-1}/T_{n-2} = T_n/T_{n-1}$

General term (n^{th} term) from start of a G.P.

$T_n = ar^{n-1} = (\text{First Term})\,(\text{Common Ratio})^{n-1}$

Sum of first 'n' term of a finite G.P.

$$S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r} & \text{where } r \neq 1 \\ na & \text{where } r = 1 \end{cases}$$

When a G.P. is defined for infinite terms then It converges to a finite value only when its common ratio r as $-1 < r < 1$

Sum of an infinite G.P. is

$$S_\infty = \frac{a}{1-r} \quad (-1 < r < 1 \text{ or } |r| < 1)$$

PROPERTIES OF GEOMETRIC PROGRESSION

(i) If a, b, c are in G.P. then $b^2 = ac$
(using $r = b/a = c/b$)
If a, b, c, d are in G.P. then $ad = bc$
(using $r = b/a = d/c$)

(ii)

Number of Consecutive Terms in G.P.	Terms We Can Assume	Common Ratio of G.P.
Three	$\frac{a}{r}, a, ar$	r
Four	$\frac{a}{r^3}, \frac{a}{r}, a, ar^3$	r^2
Five	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r
Six	$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$	r^2
Seven	$\frac{a}{r^3}, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, ar^3$	r

(Continued)

(iii) If $a_1, a_2, a_3, \dots$ are in G.P. where each $a_i > 0$, then $\log a_1$, $\log a_2$, $\log a_3$, ... are in A.P. and its converse is also true.

(iv) The product of the terms of a G.P. equidistant from the beginning and end is constant and equal to the product of first and last term.

$T_k \cdot T_{n-k+1} = (\text{First Term})\,(\text{Last Term})$

$T_k \cdot T_{n-k+1} = T_1 \cdot T_n = T_2 \cdot T_{n-1} = T_3 \cdot T_{n-2}$

Proof:

Let there are total n terms are present of a certain G.P.

k^{th} term from start $T_k = a\,r^{k-1}$

k^{th} term from last $T_{n-k+1} = (n - k + 1)^{\text{th}}$ term from start $= ar^{n-k}$

$T_k \cdot T_{n-k+1} = ar^{k-1} \cdot ar^{n-k} = a \cdot (ar^{n-1})$

$= (\text{First Term})(\text{Last Term})$

$T_k \cdot T_{n-k+1} = T_1 \cdot T_n = T_2 \cdot T_{n-1} = T_3 \cdot T_{n-2}$

(v) Square of any term of an G.P. (except the first and last) is equal to product of terms which are equidistant from it.

$T_n^{\,2} = T_{n-k} \cdot T_{n+k}$

$$T_n^{\,2} = T_{n-1} \cdot T_{n+1} = T_{n-2} \cdot T_{n+2} = T_{n-3} \cdot T_{n+3}$$
$$= T_{n-4} \cdot T_{n+4} = T_{n-5} \cdot T_{n+5}$$

Proof:

For a G.P. we can write

$T_{n-k} = ar^{n-k-1}$ and $T_{n+k} = ar^{n+k-1}$

Multiplying both we get

$$T_{n-k} \cdot T_{n+k} = ar^{n-k-1} \cdot ar^{n+k-1} = (ar^{n-1})^2 = T_n^{\,2}$$

$$T_n^{\,2} = T_{n-1} \cdot T_{n+1} = T_{n-2} \cdot T_{n+2} = T_{n-3} \cdot T_{n+3}$$
$$= T_{n-4} \cdot T_{n+4} = T_{n-5} \cdot T_{n+5}$$

(vi) Let $T_1, T_2, T_3, T_4, T_5, T_6, \ldots, T_{n-2}, T_{n-1}, T_n$ are in G.P. with common ratio r

Changes Applied to Each Term of an G.P.	**Resultant G.P.**	**New Common Ratio of Resultant G.P.**
Multiplication of k	$kT_1, kT_2, kT_3, kT_4, \ldots,$ kT_{n-2}, kT_{n-1}, kT_n	r
Division of k	$\frac{T_1}{k}, \frac{T_2}{k}, \frac{T_3}{k}, \ldots,$ $\frac{T_{n-2}}{k}, \frac{T_{n-1}}{k}, \frac{T_n}{k}$	r
Raised to the power k	$(T_1)^k, (T_2)^k, (T_3)^k, (T_4)^k, \ldots, (T_{n-1})^k, (T_n)^k$	r^k

(vii) If $a_1, a_2, \ldots, a_n$ are in G.P. with common ratio r_1 and $b_1, b_2, \ldots, b_n$ are in G.P. with common ratio r_2 then

(a) product of two G.P.s as $a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n$ is also a G.P. (New common ratio $r_1.r_2$)

(b) division of two G.P.s as $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$ is also a G.P. (New common ratio r_1/r_2)

GEOMETRIC MEAN (G.M.)

(i) If a, b, c are in G.P., b is called as the G.M. of a and c. and we have $b^2 = ac$

(ii) Geometric Mean of a and

$$c = \begin{cases} \sqrt{ac} \text{ where } a > 0, c > 0 \\ -\sqrt{ac} \text{ where } a < 0, c < 0 \end{cases}$$

For opposite sign of a and c, no real geometric Mean is defined

INSERTING *n*-GEOMETRIC MEANS BETWEEN *a* AND *b*

If a, b are two given numbers and $G_1, G_2, G_3, \ldots, G_n$ are n Geometric Means are inserted between a and b.

Now $a, G_1, G_2, \ldots, G_n, b$ are in G.P.

As we can observe that b is $(n+2)^{\text{th}}$ term of GP

Now $T_{n+2} = b = ar^{n+1} \quad \Rightarrow \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$T_2 = G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$T_3 = G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$T_4 = G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}}$$

$$\vdots \qquad\qquad \vdots$$

$$T_{k+1} = G_k = ar^k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$$

Note

The product of n Geometric Means between a and b is equal to the n^{th} power of the single Geometric Mean between a and b

i.e. $\displaystyle\prod_{r=1}^{n} G_r = G_1G_2G_3 \ldots G_{n-1}G_n = \left(\sqrt{ab}\right)^n = G^n,$

where G is the single G.M. between a and b.

ILLUSTRATION 27

The n^{th} term of the series $3, \sqrt{3}, 1, \ldots$ is $\frac{1}{243}$, then find n

SOLUTION

$$T_n = ar^{n-1} = 3 \cdot \left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243}$$

$$3^{-\left(\frac{n-1}{2}\right)} = \frac{1}{729} = 3^{-6} \quad \Rightarrow \quad n = 13$$

ILLUSTRATION 28

If $\log_x a, a^{x/2}$ and $\log_b x$ are in G.P., then find x.

SOLUTION

As $\log_x a, a^{x/2}, \log_b x$ are in G.P.,

$$(a^{x/2})^2 = \log_x a \,.\, \log_b x$$

$$\Rightarrow \quad a^x = \frac{\log a}{\log x} \cdot \frac{\log x}{\log b} = \frac{\log a}{\log b} = \log_b a$$

$$\Rightarrow \quad x = \log_a(\log_b a)$$

ILLUSTRATION 29

If the m^{th}, n^{th}, and p^{th} terms of an A.P. and G.P. be equal and be respectively x, y, z, then prove that

$$x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = 1 \quad \text{or} \quad x^y\, y^z\, z^x = x^z \cdot y^x \cdot z^y$$

SOLUTION

Let 'a' be the first term and d be the common difference of the A.P.

$$\Rightarrow \quad x = a + (m-1)\,d,\ y = a + (n-1)\,d,\ z = a + (p-1)\,d.$$

Let A be the first term and R be the common ratio of the G.P.

$$\Rightarrow \quad x = AR^{m-1},\ y = AR^{n-1},\ z = AR^{p-1}$$

$$\Rightarrow \quad x^{y-z} \cdot y^{z-x} \cdot z^{x-y}$$

$$= (AR^{m-1})^{y-z} \cdot (AR^{n-1})^{z-x} \cdot (AR^{p-1})^{x-y}$$

$$= A^{\circ} \cdot (R^{m-1})^{(n-p)d}\,(R^{n-1})^{(p-m)d}\,(R^{p-1})^{(m-n)d}$$

$$= R^{\circ} = 1$$

$$\Rightarrow \quad x^y\,y^z\,z^x = x^z \cdot y^x \cdot z^y$$

ILLUSTRATION 30

Let $a_1, a_2, a_3, \ldots, a_n$ be in geometric progression. If $a_1 + a_3 + a_5 = 455$ and $a_2 + a_4 + a_6 = 1365$, then find the ratio between each consecutive term.

SOLUTION

Let given G.P. is $a, ar, ar^2, ar^3, \ldots$

$$a_1 + a_3 + a_5 = a + ar^2 + ar^4 = 455$$

$$a_2 + a_4 + a_6 = ar + ar^3 + ar^5 = 1365$$

$$\Rightarrow \quad \frac{ar(1+r^2+r^4)}{a(1+r^2+r^4)} = \frac{1365}{455}$$

$$\Rightarrow \quad r = 3.$$

ILLUSTRATION 31

Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. Find the numbers.

SOLUTION

Let the three numbers in G.P. be $\frac{a}{r}, a, ar$

Given $\frac{a}{r} + a + ar = 70$ …(1)

and $\frac{4a}{r}, 5a, 4ar$ are in A.P.

$$\therefore \quad 10a = \frac{4a}{r} + 4ar \quad \Rightarrow \quad \frac{10a}{4} = \frac{a}{r} + ar$$

or $\frac{5a}{2} = 70 - a \quad \Rightarrow \quad 5a = 140 - 2a$ {from (1)}

or $7a = 140 \quad \therefore \quad a = 20$

from (1), we get

$$\frac{20}{r} + 20 + 20r = 70 \quad \Rightarrow \quad \frac{20}{r} + 20r = 50$$

$$\Rightarrow \quad 2 + 2r^2 = 5r \quad \Rightarrow \quad 2r^2 - 5r + 2 = 0$$

$$\Rightarrow \quad (r-2)\,(2r-1) = 0$$

$$\therefore \quad r = 2 \text{ or } \frac{1}{2}$$

Hence, the three numbers are 10, 20, 40 or 40, 20, 10

ILLUSTRATION 32

Express $0.4\overline{23}$ in the form of $\frac{p}{q}$, (where $p, q \in I, q \neq 0$).

SOLUTION

$$S = 0.4\overline{23}$$

$$= 0.42323232323\ldots = 0.4 + 0.023 + 0.00023 + 0.0000023 + \ldots$$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots \infty = \frac{4}{10} + \frac{a}{1-r} = \frac{4}{10} + \frac{23}{990}$$

$$= \frac{419}{990}$$

ILLUSTRATION 33

Evaluate $6 + 66 + 666 + \ldots$ upto n terms.

SOLUTION

Let $S = 6 + 66 + 666 + \ldots$ upto n terms.

$$= \frac{6}{9}\,[9 + 99 + 999 + \ldots]$$

$$= \frac{2}{3}\,[(10-1) + (10^2-1) + (10^3-1) + \ldots + \text{upto } n \text{ terms}]$$

$$= \frac{2}{3}\,[10 + 10^2 + 10^3 + \ldots + 10^n - n]$$

$$= \frac{2}{3}\left(\frac{10(10^n-1)}{9} - n\right)$$

ILLUSTRATION 34

In a G.P. $T_2 + T_5 = 216$ and $T_4 : T_6 = 1 : 4$ and all terms are integers, then find its first term.

SOLUTION

Let G.P. is $a, ar, ar^2, ar^3, \ldots$

$$T_2 + T_5 = ar(1+r^3) = 216 \text{ and } \frac{T_4}{T_6} = \frac{ar^3}{ar^5} = \frac{1}{4}$$

$$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = \pm 2$$

when $r = -2$, then $-2a\,(1-8) = 216$

$$\therefore \quad a = \frac{216}{14} = \frac{108}{7}, \text{ which is not an integer.}$$

when $r = 2$ then $2a(9) = 216 \quad \Rightarrow \quad a = 12$

ILLUSTRATION 35

The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of this infinite series is 24. Then find its first term and common ratio.

SOLUTION

Let a be the first term and r be the common ratio of G.P.

$$a + ar + ar^2 + ar^3 + \ldots$$

$$= \frac{a}{1-r} = 2 \quad \Rightarrow \quad a = 2(1-r) \text{ where } -1 < r < 1$$

$$a^3 + a^3r^3 + a^3r^6 + \ldots$$

$$= \frac{a^3}{1-r^3} = 24 \quad \Rightarrow \quad a^3 = 24(1-r^3)$$

$$\Rightarrow \quad 8(1-r)^3 = 24(1-r^3)$$

$$\Rightarrow \quad (1-r)^3 = 3(1-r^3)$$

$$\Rightarrow \quad (1-r)(1 + r^2 - 2r - 3(1 + r + r^2)) = 0$$

$$\Rightarrow \quad (1-r)(2r^2 + 5r + 2) = 0$$

$$\Rightarrow \quad (1-r)(2r+1)(r+2) = 0$$

$$r \neq 1, -2 \text{ as } -1 < r < 1$$

$$r = -\frac{1}{2} \text{ acceptable}$$

Solving we get $a = 3, r = -\dfrac{1}{2}$

ILLUSTRATION 36

Between 4 and 2916 odd number $(2n + 1)$ G.M's are inserted. Then find the $(n+1)^{\text{th}}$ G.M.

SOLUTION

$$4, G_1.G_2, \ldots G_{n+1}, \ldots G_{2n}, G_{2n+1}, 2916$$

G_{n+1} will be the middle mean of $(2n + 1)$ odd means and it will be equidistant from Ist and last term

$\therefore \quad 4, G_{n+1}, 2916$ will also be in G.P.

$$\therefore \quad G_{n+1}^2 = 4 \times 2916 = 4 \times 9 \times 324 = 4 \times 9 \times 4 \times 81$$

$$G_{n+1} = 2 \times 3 \times 2 \times 9 = 108$$

ILLUSTRATION 37

If the roots of the cubic equation $x^3 - 5x^2 + kx - 27 = 0$ are in G.P. then find the value of k.

SOLUTION

$$x^3 - 5x^2 + kx - 27 = 0$$

Let roots be $\dfrac{a}{r}, a, ar$

Sum of the roots $= \dfrac{a}{r} + a + ar = 5$ …(1)

Sum of product of roots taken two at a time

$$= \frac{a^2}{r} + a^2 + a^2r = k \qquad \ldots(2)$$

Product of roots $= \dfrac{a}{r} \cdot a \cdot ar = a^3 = 27 \quad \Rightarrow \quad a = 3$ …(3)

Using $a = 3$ in (1) and (2) we get

$$\frac{1}{r} + 1 + r = \frac{5}{3}$$

$$k = a^2\left(\frac{1}{r} + 1 + r\right) = 9\left(\frac{5}{3}\right) = 15$$

ILLUSTRATION 38

Let $a_1a_2a_3a_4$ be a four digit natural number such that a_1, a_2, a_3 are in arithmetic progression and a_2, a_3, a_4 are in geometric progression. If $a_4 = a_1 - 5$ and $d + 4r = 0$ then find $\displaystyle\sum_{i=1}^{4} a_i$.

SOLUTION

Given, a_2, a_3, a_4 are in geometric progression

Let $a_2 = a, a_3 = ar, a_4 = ar^2$

Also, given $a_1 = a_4 + 5$

$$\therefore \quad a_1 = ar^2 + 5$$

Now, $a_1a_2\,a_3 \to$ A.P.

$$\therefore \quad d = a_2 - a_1 = a_3 - a_2$$

$$d = a - ar^2 - 5 = ar - a$$

$$\therefore \quad a(r^2 + r - 2) = -5 \qquad \ldots(1)$$

Also, $d + 4r = 0$

$$\therefore \quad ar - a + 4r = 0 \quad \Rightarrow \quad a(1-r) = 4r \qquad \ldots(2)$$

From (1) and (2)

$$\frac{r^2 + r - 2}{1-r} = \frac{-5}{4r}$$

or $\quad 4r^3 + 4r^2 - 13r + 5 = 0$

or $\quad (2r-1)(2r^2 + 3r - 5) = 0$

or $\quad (2r-1)(2r+5)(r-1) = 0$

or $\quad r \neq 1, -\dfrac{5}{2}$

$$\therefore \quad r = \frac{1}{2} \quad \Rightarrow \quad a = 4$$

Hence, $a_1 = 6, a_2 = 4, a_3 = 2, a_4 = 1$v

$$\sum_{i=1}^{4} a_i = 6 + 4 + 2 + 1 = 13$$

ILLUSTRATION 39

If the sum of three numbers of an infinite G.P. is 21 and the sum of their squares is 189 then find the common ratio of G.P.

SOLUTION

Let the three numbers in G.P. be a, ar, ar^2

$$\therefore \quad a(1+r+r^2) = 21 \qquad \dots(1)$$

$$\text{and} \quad a^2(1+r^2+r^4) = 189 \qquad \dots(2)$$

$$\Rightarrow \quad \frac{a^2(1+r^2+r^4)}{a^2(1+r+r^2)^2} = \frac{189}{(21)^2}$$

$$\Rightarrow \quad \frac{(r^2+r+1)(r^2-r+1)}{(r^2+r+1)^2} = \frac{3}{7}$$

$$\Rightarrow \quad \frac{r^2-r+1}{r^2+r+1} = \frac{3}{7} \quad \Rightarrow \quad 4r^2-10r+4=0$$

$$\Rightarrow \quad 2r^2-4r-r+2=0 \quad \Rightarrow \quad 2r(r-2)-1(r-2)=0$$

$$\therefore \quad r = 2 \text{ or } r = \frac{1}{2} \text{ (but } r = 2 \text{ rejected)}$$

ILLUSTRATION 40

If the sum of the series

$$0.9\left(99+999+9999+\dots+\underbrace{999\dots9}_{\substack{\text{string of 101 digits}\\ \text{all equal to 9}}}\right)$$

can be expressed in the form of $(10^a - b)$ where $a, b \in N$, then find $(a + b)$.

SOLUTION

We have,

$$\text{Given sum} = 0.9[(10^2-1)+(10^3-1)+\dots+(10^{101}-1)]$$

$$= 0.9\left[10^2\left(\frac{10^{100}-1}{10-1}\right)-100\right]$$

$$= 10\left(10^{100}-1\right)-90$$

$$= 10^{101}-100$$

$$= (10^a - b) \quad \text{(Given).}$$

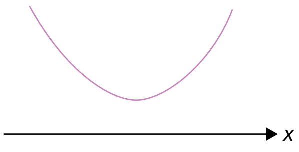
So $\quad a = 101$ and $b = 100$

Hence, $(a + b) = 101 + 100 = 201$

ILLUSTRATION 41

If a, b and c are three consecutive positive terms of a G.P. then show that the graph of $y = ax^2 + bx + c$ lies entirely above the x-axis.

SOLUTION



Given $b^2 = ac$ as a, b, c are in GP

Discriminant of quadratic expression

$$D = b^2 - 4ac = b^2 - 4b^2 = -3b^2 < 0$$

Both roots are imaginary and $a > 0$

It means graph of $y = ax^2 + bx + c$ lies entirely above the x-axis

ILLUSTRATION 42

Let $a_1, a_2, a_3, \dots, a_{10}$ are in G.P. with $a_{51} = 25$ and $\sum_{i=1}^{101} a_i = 125$, then find the value of $\sum_{i=1}^{101}\left(\frac{1}{a_i}\right)$.

SOLUTION

Let 1st term be 'a' and common ratio be 'r'

$$\sum_{i=1}^{101} a_i = \frac{a(1-r^{101})}{1-r} = 125 \qquad \dots(1)$$

$$\sum_{r=1}^{101} \frac{1}{a_i} = \left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_{101}}\right)$$

$$= \frac{\frac{1}{a}\left(\left(\frac{1}{r}\right)^{101}-1\right)}{\frac{1}{r}-1} = \frac{1}{a}\left(\frac{1-r^{101}}{1-r}\right)\cdot\frac{1}{r^{100}}$$

$$= \frac{125}{(ar^{50})^2} = \frac{125}{625} = \frac{1}{5}, \text{ using (1)}$$

ILLUSTRATION 43

Consider the ten numbers $ar, ar^2, \dots ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6 then find the product of these ten numbers.

SOLUTION

$$\text{Given } ar + ar^2 + \dots + ar^{10} = \frac{ar(r^{10}-1)}{r-1} = 18 \qquad \dots(1)$$

$$\text{Also } \frac{1}{ar}+\frac{1}{ar^2}+\dots+\frac{1}{ar^{10}} = \frac{\frac{1}{ar}\left(1-\frac{1}{r^{10}}\right)}{1-\frac{1}{r}} = 6$$

$$\Rightarrow \quad \frac{r}{ar^{11}}\cdot\frac{(r^{10}-1)}{r-1} = 6$$

$$\Rightarrow \quad \frac{1}{a^2r^{11}}\cdot\frac{ar(r^{10}-1)}{r-1} = 6 \qquad \dots(2)$$

From (1) and (2)

$$\frac{1}{a^2r^{11}}\times 18=6 \quad\Rightarrow\quad a^2r^{11}=3$$

Now Product of 10 terms = $ar \cdot ar^2 \cdot ar^3 \ldots ar^{10} = a^{10}r^{55}$ $= (a^2r^{11})^5 = 3^5 = 243$.

ILLUSTRATION 44

Let a_n denotes the n^{th} term of a G.P. given that $S=\sum\limits_{n=1}^{10} a_n$ and $T=\sum\limits_{n=1}^{10}\frac{1}{a_n}$, then find the value of $\prod\limits_{n=1}^{10} a_n$ in terms of S and T.

SOLUTION

Let $a_1 = a$

$$S = a + ar + ar^2 + \ldots + ar^9 = \frac{a(1-r^{10})}{1-r} \qquad \ldots(1)$$

$$T=\frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}+\ldots\frac{1}{ar^9}$$

$$=\frac{1}{a}\frac{\left(\frac{1}{ar^{10}}-1\right)}{\left(\frac{1}{r}-1\right)}=\frac{1}{a}\frac{(1-r^{10})\cdot r}{r^{10}(1-r)}$$

$$T=\frac{1}{a}\frac{(1-r^{10})}{r^9(1-r)} \qquad \ldots(2)$$

Now from (1) and (2)

$$\frac{S}{T}=\frac{a(1-r^{10})}{1-r}\cdot\frac{ar^9(1-r)}{(1-r^{10})}$$

$$\therefore \quad \frac{S}{T}=a^2r^9$$

Also, $\prod\limits_{n=1}^{10} a_n = a\cdot ar \cdot ar^2 \ldots ar^9 = a^{10}\times r^{45} = (a^2r^9)^5 = \left(\frac{S}{T}\right)^5$

ILLUSTRATION 45

Let $T_n = 1+2+3+\ldots+n$. If $\sum\limits_{i=3}^{\infty}\sum\limits_{k=1}^{\infty}\left(\frac{3}{T_i}\right)^k=\frac{p}{q}$ where p and q are co-prime, then find the value of $(3q-p-8)$.

SOLUTION

$$T_n=\frac{n(n+1)}{2}$$

$$\therefore \quad T_i=\frac{i(i+1)}{2}$$

Now, $\sum\limits_{k=1}^{\infty}\left(\frac{3}{T_i}\right)^k=\left(\frac{3}{T_i}\right)+\left(\frac{3}{T_i}\right)^2+\left(\frac{3}{T_i}\right)^3+\ldots\infty$

$$=\frac{\frac{3}{T_i}}{1-\frac{3}{T_i}}=\frac{3}{T_i-3}=\frac{3}{\frac{i(i+1)}{2}-3}=\frac{6}{i^2+i-6}$$

$$\therefore \sum_{i=3}^{\infty}\sum_{k=1}^{\infty}\left(\frac{3}{T_i}\right)^k=\sum_{i=3}^{\infty}\left(\frac{6}{i^2+i-6}\right)=\sum_{i=3}^{\infty}\left(\frac{6}{(i+3)(i-2)}\right)$$

$$=\frac{1}{5}\sum_{i=3}^{\infty}\left(\frac{6}{i-2}-\frac{6}{(i+3)}\right)=\frac{6}{5}\sum_{i=3}^{\infty}\left(\frac{1}{i-2}-\frac{1}{i+3}\right)$$

$$=\frac{6}{5}\left[\left(1-\frac{1}{6}\right)+\left(\frac{1}{2}-\frac{1}{7}\right)+\left(\frac{1}{3}-\frac{1}{8}\right)+\left(\frac{1}{4}-\frac{1}{9}\right)\right.$$

$$\left.+\left(\frac{1}{5}-\frac{1}{10}\right)+\left(\frac{1}{6}-\frac{1}{11}\right)+\ldots\right]$$

$$=\frac{6}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=\frac{137}{50}\equiv\frac{p}{q}$$

now $(3q-p-8) = 150 - 137 - 8 = 5$

ILLUSTRATION 46

In a geometric progression consisting of positive terms if $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, (where $n \in N$ and a_n denotes n^{th} term of G.P.) and so on, then find the common ratio of this progression.

SOLUTION

Let G.P. is $a, ar, ar^2, \ldots (r > 0)$

$a_n = a_{n-1} + a_{n-2}$

Put $n = 3$, $a_3 = a_2 + a_1 \quad\Rightarrow\quad ar^2 = ar + a$

$$r^2-r-1=0 \quad\Rightarrow\quad r=\frac{1+\sqrt{5}}{2}\text{(accepted)}$$

$$r=\frac{1-\sqrt{5}}{2}\text{ (rejected)}$$

ILLUSTRATION 47

Let 9 GM's are inserted between 100 and 3200. Then find number of rational GMs.

SOLUTION

$100, G_1, G_2, \ldots, G_9, 3200$ are in G.P.

$$3200 = 100\,r^{10} \quad\Rightarrow\quad r=\pm\sqrt{2}$$

Consider $r=\sqrt{2}$ acceptable, as GM lies between the numbers.

$\Rightarrow \quad G_1 = 100r = 100\sqrt{2}$

$\Rightarrow \quad G_2, G_4, G_6, G_8$ are rational terms.

Four GMs are rational

ILLUSTRATION 48

If $S_1, S_2, S_3, \ldots S_n$ are the sums of infinite geometric series whose first terms are 2, 4, 6 … $2n$ and whose common ratios are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7} \ldots, \frac{1}{2n+1}$ and $T_n = \frac{1}{S_1S_2S_3} + \frac{1}{S_2S_3S_4} + \ldots + \frac{1}{S_nS_{n+1}S_{n+2}}$ then find value of T_n when $n \to \infty$.

SOLUTION

$$S_1 = \frac{2}{1-\frac{1}{3}} = 3,\ S_2 = \frac{4}{1-\frac{1}{5}} = 5,$$

$$S_3 = \frac{6}{1-\frac{1}{7}} = 7, \ldots, S_n = (2n+1)$$

$$T_n = \sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)}$$

$$T_n = \frac{1}{4}\sum_{r=1}^{n}\left[\frac{1}{(2r+1)(2r+3)} - \frac{1}{(2r+3)(2r+5)}\right]$$

$$T_n = \frac{1}{4}\left[\left(\frac{1}{3.5}-\frac{1}{5.7}\right)+\left(\frac{1}{5.7}-\frac{1}{7.9}\right)+\ldots + \frac{1}{(2n+1)(2n+3)} - \frac{1}{(2n+3)(2n+5)}\right]$$

$$T_n = \frac{1}{4}\sum_{r=1}^{n}\left[\frac{1}{15} - \frac{1}{(2n+3)(2n+5)}\right]$$

$$\lim_{n\to\infty} T_n = \frac{1}{4.15} = \frac{1}{60}$$

ILLUSTRATION 49

Find general solution of the equation

cosec $x = 1 + \sin x + \sin^2 x + \sin^3 x + \ldots$ upto ∞ terms.

SOLUTION

cosec $x = 1 + \sin x + \sin^2 x + \sin^3 x + \ldots$ upto ∞ terms

$\Rightarrow \quad \text{cosec}\, x = \frac{1}{1-\sin x}; 1 - \sin x = \sin x$

$\Rightarrow \quad \sin x = \frac{1}{2}$ we get $x = n\pi + (-1)^n \frac{\pi}{6}, n \in \text{I}$

ILLUSTRATION 50

If x, y and z are three distinct real numbers in geometric progression and $ay = x + y + z$ then find 'a'.

SOLUTION

Let $x = a_1, y = a_1 r, z = a_1 r^2$

$ay = x + y + z \Rightarrow a(a_1 r) = a_1 + a_1 r + a_1 r^2 \Rightarrow r^2 + (1-a)r + 1 = 0$

For real roots

$D \geq 0 \quad \Rightarrow \quad (1-a)^2 \geq 4 \quad \Rightarrow \quad (a-1-2)(a-1+2) \geq 0$

$\Rightarrow \quad (a-3)(a+1) \geq 0 \quad \Rightarrow \quad a \in (-\infty, -1] \cup [3, \infty)$

If $a = 3$ then $r = 1$

all numbers will be same if $a = -1$ then $r = -1$ two numbers will be same

$\therefore \quad$ for distinct x, y, z, we get $a \in (-\infty, -1) \cup (3, \infty)$

Test Your Concepts-II

(Solutions on page H.53)

1. If S, P and R are the sum, product and sum of the reciprocals of n terms of an increasing G.P. and $S^n = R^n \cdot P^k$, then find k.

2. If a, b, c are in G.P. then prove that $\frac{1}{b-a}, \frac{1}{2b}, \frac{1}{b-c}$ are in A.P.

3. If x, y, z are in G.P and $a^x = b^y = c^z$, then state whether $log_b a = \log_c b$ is true/false.

4. If the p^{th}, q^{th}, r^{th} terms of an A.P are in G.P, then find common ratio of G.P.

5. Find the sum to n terms of the series, 5 + 55 + 555 + ...

6. The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

7. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. find the common ratio of G.P.

8. The geometric series $a + ar^2 + ar^3 + \ldots \infty$ has a sum of 7 and the terms involving odd powers of r has a sum of 3 then find value of $(a + r)$.

9. If $x = 30°$, then find the sum of the series $1 + \cos(2x) + \cos^2(2x) + \cos^3(2x) + \ldots \infty$.

10. The first, third and eighth terms of an arithmetic sequence with common difference 9 (taken in that order) are in geometric progression. Find the seventh term of arithmetic sequence.

11. For $x > 0$, the find the sum of the series $\frac{1}{1+x} - \frac{(1-x)}{(1+x)^2} + \frac{(1-x)^2}{(1+x)^3} - \ldots \infty$.

12. A geometric sequence has four positive terms a_1, a_2, a_3, a_4. If $\frac{a_3}{a_1} = 9$ and $a_1 + a_2 = \frac{4}{3}$, then find a_4.

13. Let α, β, γ be the roots of the cubic $2x^3 + 9x^2 - 27x - 54 = 0$. If α, β, γ are in G.P., then find the value of $\frac{2}{3}(|\alpha| + |\beta| + |\gamma|)$.

Answer keys

1. 2 **3.** True **4.** $\frac{q-r}{p-q}$ **5.** $\frac{50}{81}\left(10^n - 1\right) - \frac{5}{9}n$ **6.** $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$ **7.** $2 + \sqrt{3}$

8. 5/2 **9.** 2 **10.** 66 **11.** 1/2 **12.** 9 **13.** 7

HARMONIC PROGRESSION (H.P.)

A sequence is said to be harmonic progression (H.P.) if the reciprocal of its terms are in Arithmetic progression (A.P.)

(i) No term of H.P. can be zero

(ii) If a, b, c are in H.P. then $1/a, 1/b, 1/c$ are in A.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac}$$

or $b = \frac{2ac}{a+c}$

(iii) If $a_1, a_2, a_3, \ldots, a_n$ are in H.P. then

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n} \text{ are in A.P.}$$

(iv) Let n^{th} term of this H.P. from start is T_n and n^{th} term of corresponding A.P. is T'_n

$$T_{n(\text{H.P.})} = \frac{1}{T'_{n(\text{A.P.})}}$$

$$T'_n = \frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)$$

$$\therefore\ n^{\text{th}} \text{ term of this H.P. } T_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)}$$

(using $T_n = \frac{1}{T'_n}$)

(v) $S_{n(\text{H.P.})} \neq \frac{1}{S'_{n(\text{A.P.})}}$

There is no direct formula for the sum of a H.P.

HARMONIC MEAN

(i) If a, b, c are in H.P., b is called as the Harmonic Mean between a and c, then $b = \frac{2ac}{a+c}$

(ii) H.M. of a and b is $H = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a+b}$

(iii) If $a_1, a_2, \ldots a_n$ are 'n' non-zero numbers then Harmonic Mean 'H' of these numbers is given by

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

(iv) n H.Ms. inserted between a and b are given by

Let $H_1, H_2, H_3, \ldots, H_n$ are the HMs, then

$a, H_1, H_2, H_3, \ldots, H_n, b$ are in H.P.

$\therefore$ $1/a, 1/H_1, 1/H_2, \ldots, 1/H_n, 1/b$ are in A.P.

$1/b$ is the T_{n+2} term of this A.P.

$$T_{n+2} = \frac{1}{b} = \frac{1}{a} + (n+2-1)\,d$$

$$\Rightarrow d = \frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$$

$$T_2 = \frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$$

$$T_3 = \frac{1}{H_2} = \frac{1}{a} + 2d = \frac{1}{a} + 2\frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$$

$$T_4 = \frac{1}{H_3} = \frac{1}{a} + 3d = \frac{1}{a} + 3\frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$$

....

....

$$\therefore T_{m+1} = \frac{1}{H_m} = \frac{1}{a} + (m)\frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$$

ILLUSTRATION 51

The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

SOLUTION

Let a be the first term and d be the common difference of corresponding A.P.

$$T_7 = a + 6d = 10$$

$$T_{12} = a + 11d = 25$$

By difference $5d = 15$ or $d = 3, a = -8$

$$T_{20} = a + 19d = -8 + 19 \times 3 = 49$$

20th term of H.P. $= \dfrac{1}{T_{20}} = \dfrac{1}{49}$

ILLUSTRATION 52

Find the largest positive term of the H.P., whose first two term are $\dfrac{2}{5}$ and $\dfrac{12}{23}$.

SOLUTION

The corresponding A.P. is

$$\frac{5}{2}, \frac{23}{12} \ldots \text{ or } \frac{30}{12}, \frac{23}{12}, \frac{16}{12}, \frac{9}{12}, \frac{2}{12}, \frac{-5}{12} \ldots$$

The H.P. is $\dfrac{12}{30}, \dfrac{12}{23}, \dfrac{12}{16}, \dfrac{12}{9}, \dfrac{12}{2}, -\dfrac{12}{5}, \ldots$

Largest positive term of H.P. $= \dfrac{12}{2} = 6$

ILLUSTRATION 53

Insert 4 Harmonic mean between $\dfrac{3}{4}$ and $\dfrac{3}{19}$.

SOLUTION

$$\frac{3}{4}\, H_1, H_2, H_3, H_4, \frac{3}{19} \text{ are in H.P.}$$

$$\frac{4}{3}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{19}{3} \text{ are in A.P.}$$

Let 'd' be the common difference of corresponding A.P.

$$T_6 = \frac{4}{3} + 5d = \frac{19}{3} \quad \text{or} \quad d = \frac{\frac{19}{3} - \frac{4}{3}}{5} = 1.$$

$$\therefore \quad \frac{1}{H_1} = \frac{4}{3} + 1 = \frac{7}{3} \quad \text{or} \quad H_1 = \frac{3}{7}$$

$$\frac{1}{H_2} = \frac{4}{3} + 2 = \frac{10}{3} \quad \text{or} \quad H_2 = \frac{3}{10}$$

$$\frac{1}{H_3} = \frac{4}{3} + 3 = \frac{13}{3} \quad \text{or} \quad H_3 = \frac{3}{13}$$

$$\frac{1}{H_4} = \frac{4}{3} + 4 = \frac{16}{3} \quad \text{or} \quad H_4 = \frac{3}{16}.$$

ILLUSTRATION 54

If a, x, y, z, b are in A.P., then $x + y + z = 15$ and if a, x, y, z, b are in HP, then $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{5}{3}$ then find positive numbers a and b.

SOLUTION

By property of A.P. $x + z = a + b$ and

$$y = \frac{a+b}{2}$$

$$\Rightarrow \quad x + y + z = \frac{3}{2}(a+b) \qquad (\because x + y + z = 15)$$

$$\Rightarrow \quad a + b = 10 \qquad \ldots(1)$$

Also $\dfrac{1}{a}, \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}, \dfrac{1}{b}$ are in A.P., so as above

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \qquad \left(\because \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}\right)$$

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{b} = \frac{10}{9} \quad \Rightarrow \quad ab = 9 \qquad \ldots(2)$$

From (1) and (2) a, b are 9, 1 or 1, 9

ILLUSTRATION 55

If $H_1, H_2, H_3, \ldots, H_n$ are n-harmonic means lying between a and b then show that $\dfrac{H_1 + a}{H_1 - a} + \dfrac{H_n + b}{H_n - b} = 2n$.

SOLUTION

Given that $a, H_1, H_2, H_3, \ldots, H_n, b$ are in H.P.

Then $\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \dfrac{1}{H_3}, \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ … are in A.P.

Let d be its common difference.

$$\therefore \quad T_{n+2} = \frac{1}{b} = \frac{1}{a} + (n + 2 - 1)\, d \qquad \left\{\because \frac{1}{b} \text{ is } (n+2)^{\text{th}} \text{ term}\right\}$$

thus $d = \dfrac{a-b}{ab\,(n+1)}$

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab\,(n+1)}$$

$$\text{or,} \quad \frac{1}{H_1} = \frac{bn+a}{ab(n+1)} \quad \Rightarrow \quad \frac{a}{H_1} = \frac{bn+a}{b(n+1)}$$

Using componendo and dividendo rule, we get

$$\frac{H_1 + a}{H_1 - a} = \frac{1 + \dfrac{bn+a}{b(n+1)}}{1 - \left(\dfrac{bn+a}{b(n+1)}\right)} = \frac{2bn + (a+b)}{b-a} \qquad \ldots(1)$$

$$\text{Again,} \quad \frac{1}{H_n} = \frac{1}{b} - d = \frac{1}{b} - \frac{a-b}{ab(n+1)}$$

or, $\dfrac{1}{H_n} = \dfrac{an+b}{ab(n+1)} \Rightarrow \dfrac{b}{H_n} = \dfrac{an+b}{a(n+1)}$

or, $$\frac{H_n+b}{H_n-b} = \frac{1+\dfrac{an+b}{a(n+1)}}{1-\left(\dfrac{an+b}{a(n+1)}\right)} = \frac{2an+(a+b)}{a-b} \qquad \ldots(2)$$

From (1) and (2), we get

$$\frac{H_1+a}{H_1-a} + \frac{H_n+b}{H_n-b} = \frac{2(b-a)\cdot n}{(b-a)} = 2n$$

Hence proved.

ILLUSTRATION 56

If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then prove that $a_1a_2 + a_2a_3 + \ldots + a_{n-1}\,a_n = (n-1)\,a_1a_n$

SOLUTION

Let d be common difference of the corresponding A.P.

So $\dfrac{1}{a_2} - \dfrac{1}{a_1} = \dfrac{1}{a_3} - \dfrac{1}{a_2} =, \ldots, \dfrac{1}{a_n} - \dfrac{1}{a_{n-1}} = d$

$\Rightarrow \quad a_1 - a_2 = d(a_1a_2),\ a_2 - a_3 = d(a_2a_3), \ldots, (a_{n-1} - a_n)$

$\qquad = d\,(a_{n-1}\,a_n)$

Adding these relations, we get

$$a_1 - a_n = d(a_1a_2 + a_2a_3 + \ldots + a_{n-1}\,a_n) \qquad \ldots(1)$$

Also $\dfrac{1}{a_n} = T_n = \dfrac{1}{a_1} + (n-1)d$

$\Rightarrow \quad \dfrac{1}{a_n} - \dfrac{1}{a_1} = (n-1)d$

$\Rightarrow \quad a_1 - a_n = (n-1)d(a_1a_n) \qquad \ldots(2)$

From (1) and (2), we have

$$(n-1)\,(a_1\,a_n) = a_1a_2 + a_2a_3 + \ldots + a_{n-1}\,a_n$$

Hence proved.

ILLUSTRATION 57

If a, b, c are in harmonic progression, then show that $\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right) = \dfrac{2}{bc} - \dfrac{1}{b^2} = \dfrac{1}{4}\left(\dfrac{3}{c^2}+\dfrac{2}{ca}-\dfrac{1}{a^2}\right)$

SOLUTION

As a, b, c are in H.P. then $1/a, 1/b, 1/c$ are in A.P.

$\Rightarrow \quad \dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$ or $\dfrac{2}{b} = \dfrac{1}{a} + \dfrac{1}{c} \qquad \ldots(1)$

Therefore,

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right) = \left(\frac{1}{b}+\frac{1}{c}+\frac{1}{c}-\frac{2}{b}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{c}\right)$$

[using (1)]

$$= \left(\frac{2}{c}-\frac{1}{b}\right)\left(\frac{1}{b}\right) = \frac{2}{bc} - \frac{1}{b^2}$$

Also $\left(\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}\right)\left(\dfrac{1}{c}+\dfrac{1}{a}-\dfrac{1}{b}\right)$

$$= \left(\frac{1}{2a}+\frac{1}{2c}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{2a}-\frac{1}{2c}\right) \qquad \text{[using (1)]}$$

$$= \left(\frac{3}{2c}-\frac{1}{2a}\right)\left(\frac{1}{2c}+\frac{1}{2a}\right) = \frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$$

Hence proved.

ILLUSTRATION 58

If three unequal positive real numbers a, b, c are in G.P. and $b-c, c-a, a-b$ are in H.P. then find the value of $\dfrac{a+b+c}{b}$.

SOLUTION

As a, b, c are in G.P., $b^2 = ac$ and $b-c, c-a, a-b$ are in H.P.

$$\frac{2}{c-a} = \frac{1}{b-c} + \frac{1}{a-b} = \frac{a-c}{(b-c)(a-b)}$$

$\Rightarrow \quad 2(b-c)\,(a-b) = -(a-c)^2$

$\Rightarrow \quad 2(ab - ac - b^2 + bc)$

$\qquad = -(\sqrt{a}-\sqrt{c})^2(\sqrt{a}+\sqrt{c})^2$

$\Rightarrow \quad 2(ab - 2b^2 + bc)$

$\qquad = -(\sqrt{a}-\sqrt{c})^2(\sqrt{a}+\sqrt{c})^2$

$\Rightarrow \quad 2b\,(\sqrt{a}-\sqrt{c})^2 = -(\sqrt{a}-\sqrt{c})^2(\sqrt{a}+\sqrt{c})^2 \quad [\because \sqrt{a}-\sqrt{c} \neq 0]$

$\Rightarrow \quad 2b = -(a+c+2\sqrt{ac}) = -(a+c+2b)$

$\Rightarrow \quad a+b+c = -3b \quad \Rightarrow \quad \dfrac{a+b+c}{b} = -3$

ILLUSTRATION 59

Consider three sequences an A.P., a G.P. and a H.P. as follows

$a, A_1, A_2, A_3, \ldots, A_9, b$ (A.P.)

$a, G_1, G_2, G_3, \ldots, G_9, b$ (G.P.)

$a, H_1, H_2, H_3, \ldots, H_9, b$ (H.P.)

If a, b are the roots of the equation $x^2 - (p+1)x + 16 = 0$, $p \in r$, then find the value of $\displaystyle\prod_{r=1}^{4} A_rH_{10-r}G_{10-2r}$.

SOLUTION

We need terms

$$\underbrace{A_1H_9G_8}_{}\ \underbrace{A_2H_8G_6}_{}\ \underbrace{A_3H_7G_4}_{}\ \underbrace{A_4H_6G_2}_{}$$

$$A_r = a + \left(\frac{b-a}{10}\right)r$$

$$\frac{1}{H_r} = \frac{1}{a} + \frac{\left(\frac{1}{b}-\frac{1}{a}\right)}{10}r = \frac{10b+(a-b)r}{10ab}$$

$$H_{10-r} = \frac{10ab}{10b+(a-b)(10-r)}$$

$$A_r H_{10-r} = \left(\frac{10a+(b-a)r}{10}\right)\left(\frac{10ab}{10b+(a-b)(10-r)}\right) = ab$$

Now, $A_1H_9 = A_2H_8 = A_3H_7 = A_4H_6 = ab = 16$

Now, $G_2G_8 = G_4G_6 = ab = 16$

$\therefore$ Expression $\prod_{r=1}^{4} A_r H_{10-r} G_{10-2r} = 16^4 \times G_2G_4G_6G_8$

$$= 16^4 \times 16^2 = 4^{12}$$

ILLUSTRATION 60

Find the harmonic mean of the roots of the equation $\left(5+\sqrt{2}\right)x^2 - \left(4+\sqrt{5}\right)x + 8 + 2\sqrt{5} = 0.$

SOLUTION

$\left(5+\sqrt{2}\right)x^2 - \left(4+\sqrt{5}\right)x + 8 + 2\sqrt{5} = 0$ has roots α and β.

H.M. of roots is $\frac{2\alpha\beta}{\alpha+\beta}$

$$\alpha\beta = \frac{8+2\sqrt{5}}{5+\sqrt{2}};\ \alpha+\beta = \frac{4+\sqrt{5}}{5+\sqrt{2}}$$

$\therefore$ H.M of $\alpha, \beta = \frac{2\times 2(4+\sqrt{5})}{4+\sqrt{5}} = 4$

ILLUSTRATION 61

If p, q, r in H.P. and p and r be different having same sign then find nature of the roots of the equation $px^2 + qx + r = 0$.

SOLUTION

As, p, q, r are in H.P. then $q = \frac{2pr}{p+r}$

Discriminant $D = q^2 - 4pr$ also $q = \frac{2pr}{p+r}$

$$= \left(\frac{2pr}{p+r}\right)^2 - 4pr = -4pr\left[1 - \frac{pr}{(p+r)^2}\right]$$

$$D = -4pr\left[\frac{p^2+r^2+pr}{(p+r)^2}\right]$$

p and r are of same sign then $pr > 0$

$\therefore \quad D < 0$

Hence roots of given quadratic equation are imaginary

ILLUSTRATION 62

Let $\alpha, \beta, \gamma, \delta$ are zeroes of $P(x) = 5x^4 + px^3 + qx^2 + rx + s$ ($p, q, r, s \in R$) and α, γ, δ are zeroes of $Q(x) = x^3 - 9x^2 + ax - 24$ ($\alpha < \beta < \gamma < \delta$). If α, γ, δ (taken in that order) are in arithmetic progression and $\alpha, \beta, \gamma, \delta$ (taken in that order) are in harmonic progression, then find the value of $\left|\frac{P(1)}{Q(1)}\right|$.

SOLUTION

$$2\gamma = \alpha + \delta \text{ and } \alpha + \gamma + \delta = 9 \quad \Rightarrow \quad \gamma = 3$$

$$\alpha\gamma\delta = 24 \quad \Rightarrow \quad \gamma = 3,\ \delta = 4,\ \alpha = 2$$

$\because \quad \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ are in A.P. $\quad \Rightarrow \quad \beta = \frac{12}{5}$

$$\left|\frac{P(x)}{Q(x)}\right| = \left|\frac{5(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}{(x-\alpha)(x-\gamma)(x-\delta)}\right|$$

$$= |5(x-\beta)|$$

Now, $\left|\frac{P(1)}{Q(1)}\right| = \left|5\left(1-\frac{12}{5}\right)\right| = 7$

ILLUSTRATION 63

Let $a_1, a_2, \ldots$ be in A.P., $g_1, g_2, \ldots$ be in G.P. and $h_1, h_2, \ldots$ be in H.P. If $a_1 = g_1 = h_1 = 1$, $a_{10} = g_{10} = 2$ and $h_{10} = \frac{1}{5}$, then find the value of $a_{19}g_{10}h_{28} + a_{28}g_{19}h_{19}$.

SOLUTION

Given $a_1 = g_1 = h_1 = 1$

$$a_{10} = a_1 + 9d = 1 + 9d = 2 \quad \Rightarrow \quad d = \frac{1}{9}$$

$$g_{10} = g_1 \cdot r^9 = 2$$

$\Rightarrow \quad r = 2^{1/9};\ g_{19} = 1(2^{1/9})^{18} = 4.$

Given $h_1, h_2, \ldots h_{10}$ are in H.P.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_1+d'}, \ldots$ are in H.P.

$$h_{10} = \frac{1}{5} = \frac{1}{a_1+9d'}$$

$\Rightarrow \quad a_1 + 9d' = 5 \quad \Rightarrow \quad d' = \frac{4}{9}$

$\therefore \quad a_{19}g_{10}h_{28} + a_{28}g_{19}h_{19}$

$$= \left(1+18\times\frac{1}{9}\right)2\left(\frac{1}{1+27\times\frac{4}{9}}\right) + \left(1+27\times\frac{1}{9}\right)2^2\left(\frac{1}{1+18\times\frac{4}{9}}\right)$$

$$= 3\times 2\times\frac{1}{13} + 4\times 4\times\frac{1}{9} = \frac{262}{117}$$

ARITHMETIC-GEOMETRIC PROGRESSION

A series is said to be A.G.P. if its terms are formed by multiplying corresponding terms of an A.P. and a G.P.

$S = 1 + 2x + 3x^2 + 4x^3 + \dots$ is Arithmetic Geometric Progression which is consisting of AP as 1, 2, 3, 4, … and G.P. as $1, x, x^2, x^3, x^4, \dots$

$S = 3 + 5x + 7x^2 + 9x^3 + \dots$ is also an Arithmetic Geometric Progression which is consisting of AP 3, 5, 7, 9, … and G.P. $1, x, x^2, x^3, x^4, \dots$

(i) $T_{n(\text{AGP})} = T_{n(\text{AP})} T_{n(\text{GP})}$

(ii) $S_{n(\text{AGP})} \neq S_{n(\text{AP})} S_{n(\text{GP})}$

Sum of *n* terms of A.G.P. (Arithmetic Geometric Progression)

$$S_n = a + (a+d)r + (a+2d)r^2 + \dots + (a+(n-1)d)r^{n-1} \quad \dots(1)$$

$$\therefore \quad rS_n = 0 + ar + (a+d)r^2 + \dots + (a+(n-2)d)r^{n-1} + (a+(n-1)d)r^n \quad \dots(2)$$

Subtracting (2) from (1),

$$(1-r)S_n = a + dr + dr^2 + \dots + dr^{n-1} - (a+(n-1)d)r^n$$

$$= a + dr\frac{(1-r^{n-1})}{(1-r)} - (a+(n-1)d)r^n$$

$$\therefore \quad S_n = \frac{a}{(1-r)} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{(1-r)}$$

For infinite terms AG.P. ($n \to \infty$)and with $-1 < r < 1$

We know $r^n \to 0$

$$\text{then } S_\infty = \frac{a}{1-r} + \frac{dr(1-0)}{(1-r)^2} - 0$$

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

ILLUSTRATION 64

Find $1 + 2.2 + 3.2^2 + 4.2^3 + \dots + 100.2^{99}$.

SOLUTION

We can see here 1, 2, 3, 4, … are in AP

and $1, 2, 2^2, 2^3, 2^4 \dots$ are in GP

Let

$$S = 1 + 2.2 + 3.2^2 + 4.2^3 + \dots + 100.2^{99} \quad \dots(1)$$

$$\Rightarrow \quad 2S = 0 + 2 + 2.2^2 + 3.2^3 + \dots + 99.2^{99} + 100.2^{100} \quad \dots(2)$$

Subtracting (2) from (1), we get

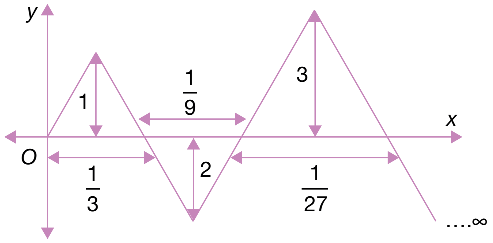
$$-S = (1 + 2 + 2^2 + 2^3 + \dots + 2^{99}) - 100.2^{100}$$

$$\Rightarrow \quad S = 100.2^{100} - \frac{2^{100}-1}{2-1}$$

$$= 100.2^{100} - 2^{100} + 1 = 1 + 99.2^{100}$$

ILLUSTRATION 65

Infinite number of triangles are formed as shown in figure. Find total area of these triangles.



SOLUTION

We have

$$A = \left(\frac{1}{2}\times\frac{1}{3}\times 1\right) + \left(\frac{1}{2}\times\frac{1}{9}\times 2\right) + \left(\frac{1}{2}\times\frac{1}{27}\times 3\right) + \dots \infty$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \dots\infty\right) = \frac{S}{2}$$

Let $S = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots \infty$, then

$$\frac{S}{3} = 0 + \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \dots \infty$$

(Subtracting)

$$\Rightarrow \quad \frac{2S}{3} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \infty = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$\Rightarrow \quad S = \frac{3}{4}$$

$$\therefore \quad \text{Now required area} = \frac{S}{2} = \frac{3}{8}$$

ILLUSTRATION 66

If r^{th} term of a series is $(2r+1)2^{-r}$, then find the sum of its infinite terms.

SOLUTION

Here $T_r = (2r+1)2^{-r}$

$$S = \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \dots \quad \dots(1)$$

$$\frac{S}{2} = 0 + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \dots \quad \dots(2)$$

Subtracting (2) from (1), we get

$$\frac{S}{2} = \frac{3}{2} + \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4} + \dots$$

Or $$\frac{S}{2}=\frac{3}{2}+\frac{2}{2^2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\right)$$

$$=\frac{3}{2}+\frac{2}{2^2}\left(\frac{1}{1-\frac{1}{2}}\right)=\frac{5}{2}$$

We have $S_\infty = 5$

ILLUSTRATION 67

Find the sum to n terms of the series

$$1+5\left(\frac{4n+1}{4n-3}\right)+9\left(\frac{4n+1}{4n-3}\right)^2+13\left(\frac{4n+1}{4n-3}\right)^3+\ldots$$

SOLUTION

Let $x=\frac{4n+1}{4n-3}$,

$$S = 1 + 5x + 9x^2 + \ldots + (4n-3)x^{n-1}$$

$$Sx = 0 + x + 5x^2 + \ldots +(4n-3)x^n$$

$$(1-x)S = 1 + 4x + 4x^2 + \ldots + 4\,x^{n-1} - (4n-3)x^n.$$

$$S(1-x) = 1 + \frac{4x}{1-x}\,[1-x^{n-1}] - (4n-3)x^n$$

$$S=\frac{1}{1-x}\left[1+\frac{4x}{1-x}-\frac{4x^n}{1-x}-(4n-3)x^n\right]$$

Put $x=\frac{4n+1}{4n-3}$

$$=-\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n]$$

$$= n\,(4n-3).$$

ILLUSTRATION 68

Find sum to infinite terms of the series

$$1+2\left(\frac{10}{11}\right)+3\left(\frac{10}{11}\right)^2+4\left(\frac{10}{11}\right)^3+\ldots$$

SOLUTION

Let $x=\left(\frac{10}{11}\right)$

let $S = 1 + 2x + 3x^2 + 4x^3 +$ …(1)

$xS = 0 + x + 2x^2 + 3x^3 +$ …(2)

take difference $(1-x)\,S = 1 + x + x^2 + x^3 + \ldots$

or $$S=\frac{1}{(1-x)^2}=\frac{1}{\left(1-\left(\frac{10}{11}\right)\right)^2}=121$$

ILLUSTRATION 69

Evaluate the sum to infinity of the series

$$1+\frac{3}{4}+\frac{6}{4^2}+\frac{10}{4^3}+\frac{15}{4^4}+\ldots$$

SOLUTION

Let $S=1+\frac{3}{4}+\frac{6}{4^2}+\frac{10}{4^3}+\frac{15}{4^4}+\ldots$ (1)

$\frac{S}{4}=0+\frac{1}{4}+\frac{3}{4^2}+\frac{6}{4^3}+\frac{10}{4^4}+\ldots$ (2)

Now (1) – (2)

$\Rightarrow \frac{3S}{4}=1+\frac{2}{4}+\frac{3}{4^2}+\frac{4}{4^3}+\frac{5}{4^4}+\ldots$ (3)

$\Rightarrow \frac{3S}{16}=0+\frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{4}{4^4}+\ldots$ (4)

Now (3) – (4)

$$\Rightarrow \frac{9S}{16}=1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots$$

$$\Rightarrow \frac{9S}{16}=\frac{1}{1-\frac{1}{4}} \quad\Rightarrow\quad \frac{9S}{16}=\frac{4}{3}$$

$$\Rightarrow \quad S=\frac{64}{27}$$

ILLUSTRATION 70

Find the sum of the series $1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\,\infty$, $|x| < 1$.

SOLUTION

Here the numbers $1^2, 2^2, 3^2, 4^2, \ldots$ i.e., 1, 4, 9, 16, … are not in A.P.

but $1, 4-1=3, 9-4=5, 16-9=7$, are in A.P.

Let $S_\infty = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots\,\infty$

$S_\infty = 1 + 4x + 9x^2 + 16x^3 + \ldots\,\infty$ …(1)

$\therefore\quad xS_\infty = 0 + x + 4x^2 + 9x^3 + \ldots\,\infty$ …(2)

Subtracting (2) from (1), we get

$(1-x)\,S_\infty = 1 + 3x + 5x^2 + 7x^3 + \ldots\,\infty$ …(3)

$x\,(1-x)\,S_\infty = 0 + x + 3x^2 + 5x^3 + \ldots\,\infty$ …(4)

Subtracting (4) from (3), we get

$$(1-x)\,(1-x)\,S_\infty = 1 + 2x + 2x^2 + 2x^3 + \ldots\,\infty$$

$$(1-x)^2S_\infty = 1+\frac{2x}{1-x}=\frac{1+x}{1-x}$$

Hence, $S_\infty=\frac{(1+x)}{(1-x)^3}$

RELATION AMONG ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

For Two Positive Numbers

Let A, G and H be arithmetic, Geometric and Harmonic means of any two numbers 'a' and 'b' where $a, b \in R^+$.

and $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

then $G^2 = AH$ and $A \geq G \geq H$ (Equality holds when all numbers are equal)

(i) $G^2 = AH$

Proof:

$$G^2 = AH$$

$$\Rightarrow \quad \left(\sqrt{ab}\right)^2 = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) \quad \Rightarrow \quad ab = ab$$

(ii) $A \geq G \geq H$

Proof:

$$A \geq G$$

$$\Rightarrow \quad \frac{a+b}{2} \geq \sqrt{ab} \quad \Rightarrow \quad a+b \geq 2\sqrt{ab}$$

$$\Rightarrow \quad a+b-2\sqrt{ab} \geq 0$$

$$\Rightarrow \quad (\sqrt{a}-\sqrt{b})^2 \geq 0 \qquad \text{Always true}$$

$\because$ square of a real number is always non negative

$$G \geq H \quad \Rightarrow \quad \sqrt{ab} \geq \frac{2ab}{a+b} \quad \Rightarrow \quad (a+b)\sqrt{ab} \geq 2ab$$

$$\Rightarrow \quad a+b \geq 2\sqrt{ab} \quad \Rightarrow \quad (a+b-2\sqrt{ab}) \geq 0$$

$$\Rightarrow \quad (\sqrt{a}-\sqrt{b})^2 \geq 0 \qquad \text{Always True}$$

$\because$ square of a real number is always non-negative

$$A \geq G \geq H$$

Equality holds when all numbers are equal

For More Than 2 Positive Numbers

(i) Arithmetic Mean of n positive numbers

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

(ii) Geometric Mean of n positive numbers

$$G = (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

(iii) Harmonic Mean of n positive numbers

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}$$

$$A \geq G \geq H$$

Equality holds when all numbers are equal

ILLUSTRATION 71

If the AM and GM between two numbers are in the ratio $m : n$ then show that the number are in the ratio as $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.

SOLUTION

Let two numbers are a and b

$$A = \frac{a+b}{2}, \; G = \sqrt{ab}$$

Now $\frac{A}{G} = \frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$

$$\Rightarrow \quad \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{m+n}{m-n}$$

$$\Rightarrow \quad \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right) = \frac{\sqrt{m+n}}{\sqrt{m-n}}$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}$$

ILLUSTRATION 72

For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is $(a \neq b)$

(i) A.M. between a and b
(ii) G.M. between a and b
(iii) H.M. between a and b

SOLUTION

(i) A.M. between a and b is $\frac{a+b}{2}$

we have $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$

$$\Rightarrow 2a^{n+1} + 2b^{n+1} = (a+b)(a^n+b^n)$$

$$\Rightarrow 2a^{n+1} + 2b^{n+1} = a^{n+1} + ab^n + ba^n + b^{n+1}$$

$$\Rightarrow a^{n+1} - ab^n + b^{n+1} - ba^n = 0$$

$$\Rightarrow a^n(a-b) - b^n(a-b) = 0$$

$$\Rightarrow (a-b)(a^n - b^n) = 0$$

$$a - b \neq 0$$

$$\therefore \; a^n - b^n = 0$$

$$\Rightarrow \left(\frac{a}{b}\right)^n = 1$$

$$\Rightarrow \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$$

Hence, $n = 0$

(ii) $\because$ G.M. between a and b is $\sqrt{ab}$

we have $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$

$$\Rightarrow a^{n+1}+b^{n+1}=\sqrt{ab}\,(a^n+b^n)$$

$$\Rightarrow a^{n+1}+b^{n+1}=a^{n+1/2}\cdot b^{1/2}+a^{1/2}\cdot b^{n+1/2}$$

$$\Rightarrow a^{n+1}-a^{n+1/2}\cdot b^{1/2}-a^{1/2}\cdot b^{n+1/2}+b^{n+1}=0$$

$$\Rightarrow a^{n+1/2}\,(a^{1/2}-b^{1/2})-b^{n+1/2}\,(a^{1/2}-b^{1/2})=0$$

$$\Rightarrow a^{1/2}-b^{1/2}\neq 0$$

$$\therefore\ a^{n+1/2}-b^{n+1/2}=0$$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+\frac{1}{2}}=1=\left(\frac{a}{b}\right)^{0}$$

$$\therefore\ n+\frac{1}{2}=0$$

Hence, $n=-\dfrac{1}{2}$

(iii) Because H.M. between a and b is $\dfrac{2ab}{a+b}$

We have $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\dfrac{2ab}{a+b}$

$$\Rightarrow (a^{n+1}+b^{n+1})\,(a+b)=2ab\,(a^n+b^n)$$

$$\Rightarrow a^{n+2}+a^{n+1}b+b^{n+1}\,a+b^{n+2}=2a^{n+1}\,b+2ab^{n+1}$$

$$\Rightarrow a^{n+2}-a^{n+1}\,b-b^{n+1}\,(a-b)=0$$

$$\Rightarrow a^{n+1}\,(a-b)-b^{n+1}\,(a-b)=0$$

$$\Rightarrow (a-b)\,(a^{n+1}-b^{n+1})=0$$

$$a-b\neq 0$$

$$\therefore\ a^{n+1}-b^{n+1}=0\Rightarrow a^{n+1}=b^{n+1}$$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+1}=1=\left(\frac{a}{b}\right)^{0}$$

Hence, $n+1=0$, therefore, $n=-1$

ILLUSTRATION 73

The A.M. of two numbers exceeds the G.M. by 2 and the G.M. exceeds the H.M. by $\frac{8}{5}$; find the numbers.

SOLUTION

Let the numbers be a and b, now using the relation

$$G^2=\text{AH}=(G+2)\left(G-\frac{8}{5}\right)\quad\Rightarrow\quad G=8\text{ and }A=10$$

i.e. $G=8$ or $ab=64$

also $A=10$ or $a+b=20$

Hence the two numbers are 4 and 16.

ILLUSTRATION 74

If A_1, A_2 be the A.M.'s, G_1, G_2 be the G.M.'s and H_1, H_2 be the H.M.'s between any two numbers then show that

$$\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$$

SOLUTION

Let the two numbers be a and b.

Then

$$A_1+A_2=\text{sum of A.M.'s}=\left(\frac{a+b}{2}\right)2=a+b \qquad \ldots(1)$$

and $$G_1G_2=\text{Product of G.M.'s}=\left(\sqrt{ab}\right)^2=ab \qquad \ldots(2)$$

and sum of reciprocals of H.M.'s $=\dfrac{1}{H_1}+\dfrac{1}{H_2}$

$$=2\left(\frac{1}{\left(\dfrac{2ab}{a+b}\right)}\right)=\frac{a+b}{ab}=\frac{A_1+A_2}{G_1G_2} \qquad \{\text{from (1) and (2)}\}$$

$$\therefore\quad \frac{1}{H_1}+\frac{1}{H_2}=\frac{A_1+A_2}{G_1G_2}\quad\Rightarrow\quad \frac{H_1+H_2}{H_1H_2}=\frac{A_1+A_2}{G_1G_2}$$

$$\Rightarrow\quad \frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$$

Hence proved.

ILLUSTRATION 75

If the roots of cubic $x^3-6x^2+kx-8=0$ are positive real numbers, then find the value of k.

SOLUTION

$x^3-6x^2+kx-8=0$ has roots α, β and γ

Arithmetic mean of roots $=\dfrac{\alpha+\beta+\gamma}{3}=\dfrac{6}{3}=2.$

Geometric mean of roots $=(\alpha\beta\gamma)^{1/3}=(8)^{1/3}=2.$

$\text{AM}=\text{GM}=2$

$\therefore$ Numbers α, β, γ are equal.

$\Rightarrow$ $\alpha=\beta=\gamma=2$

So, $k=\alpha\beta+\beta\gamma+\gamma\alpha$

$=4+4+4=12$

ILLUSTRATION 76

If a, b, c are non-zero real numbers then find the minimum value of the expression

$$\left(\frac{(a^8+4a^4+1)(b^4+3b^2+1)(c^2+2c+2)}{a^4b^2}\right)$$

SOLUTION

$$E=\left(a^4+4+\frac{1}{a^4}\right)\left(b^2+3+\frac{1}{b^2}\right)((c+1)^2+1),$$

using AM $\ge$ GM

$$a^4+\frac{1}{a^4}\ge 2, b^2+\frac{1}{b^2}\ge 2 \text{ and } (c+1)^2\ge 0$$

Minimum values exists for $a=1, b=1, c=-1$

$$E_{\min}=(6)\,(5)\,(1)=30$$

ILLUSTRATION 77

If a, b, c are positive real numbers such that $a+b+c=3$, then find the minimum value of

$$\frac{a}{6-a}+\frac{b}{6-b}+\frac{c}{6-c}$$

SOLUTION

Let $6-a=x, 6-b=y, 6-c=z$

adding $6-a+6-b+6-c=x+y+z$

$\Rightarrow\quad 18-(a+b+c)=x+y+z$

$\Rightarrow\quad x+y+z=15$

$$\therefore\quad \frac{a}{6-a}+\frac{b}{6-b}+\frac{c}{6-c}=\frac{6-x}{x}+\frac{6-y}{y}+\frac{6-z}{z}$$

$$=6\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-3$$

Here x, y, z are real and positive

Using AM $\ge$ HM

$$\Rightarrow\quad \frac{x+y+z}{3}\ge\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\frac{9}{x+y+z}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\frac{9}{15}=\frac{3}{5}$$

From equation (1) and (2), we get

$$\frac{a}{6-a}+\frac{b}{6-b}+\frac{c}{6-c}=6\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-3\ge 6\left(\frac{3}{5}\right)-3=\frac{3}{5}.$$

ILLUSTRATION 78

Let p, q, r and s be four positive numbers such that $pqrs=1$ unity then find the least value of

$$(1+p)\,(1+q)\,(1+r)\,(1+s)$$

SOLUTION

A.M. $\ge$ G.M. (for positive numbers)

$1+p\ge 2\sqrt{p}, 1+q\ge 2\sqrt{q}, 1+r\ge 2\sqrt{r}, 1+s=2\sqrt{s}$

Product of all above four inequality

We get $(1+p)\,(1+q)\,(1+r)\,(1+s)\ge 2^4\sqrt{pqrs}$

$$(1+p)\,(1+q)\,(1+r)\,(1+s)\ge 16$$

Required least value $=16$

ILLUSTRATION 79

If A.M., G.M. and H.M. of the first and the last terms of the series 100, 101, 102, …, $n-2$, $n-1$, n are the terms of the series itself, then find the value of $n(100<n\le 500)$.

SOLUTION

We have A.M. $=\dfrac{100+n}{2}$,

G.M. $=10\sqrt{n}$ and H.M. $=\dfrac{200n}{100+n}$

For A.M. to be an integer, n must be even.
For G.M. to be an integer, n must be perfect square.

$\Rightarrow\quad n=4k^2$

$$\therefore\quad \text{H.M.}=\frac{200k^2}{25+k^2}$$

Since numerator is multiple of 25, denominator must be a multiple of 25 as H.M. is an integer.
Also $100<n\le 500$

$\Rightarrow\quad 25<k^2\le 125$

$\Rightarrow\quad k^2=50, 75, 100$ and 125

$\therefore$ Satisfying these values, we get $k^2=100$ as the only solutions.
Hence $n=4k^2=4(100)=400$

ILLUSTRATION 80

If a, b, c (all distinct positive) are in H.P., then find nature of roots of equation

$$x^2-kx+2b^{101}-a^{101}-c^{101}=0\ (k\in R)$$

SOLUTION

(G.M. > H.M.)

$$\sqrt{ac}>\frac{2ac}{a+c}=b\Rightarrow\sqrt{ac}>b$$

A.M. > G.M. $\Rightarrow \dfrac{a^{101}+c^{101}}{2}>(a^{101}.c^{101})^{1/2}=(\sqrt{ac})^{101}>(b)^{101}$

$\Rightarrow\quad a^{101}+c^{101}>2b^{101}$

Product of roots of equation is negative
Hence roots of given quadratic equation are real and distinct roots.

ILLUSTRATION 81

If $x, y, z \in R^+$ such that $x+y+z=4$, then find the maximum possible value of xyz^2.

SOLUTION

We are looking for maximum value of xyz^2

It means we need 4 terms and maximum values occurs when all numbers are equal

As $x+y+z=4$, we can take four positive numbers as x, y, $z/2$, $z/2$

A.M. ≥ G.M.

$$\frac{x+y+\frac{z}{2}+\frac{z}{2}}{4} \geq \left(x, y . \frac{z}{2} . \frac{z}{2}\right)^{1/4}$$

$$\Rightarrow \quad 1 \geq \left(\frac{xyz^2}{4}\right)^{1/4} \quad \Rightarrow \quad xyz^2 \leq 4$$

ILLUSTRATION 82

Let $2p + 3q + 4r = 15$ and if the expression $p^3q^5r^7$ takes the maximum value, then find $p : q : r$ where $p, q, r \in R^+$.

SOLUTION

We are looking for maximum value of $p^3q^5r^7$

It means we need 15 terms and maximum values occurs when all numbers are equal

As $2p + 3q + 4r = 15$, we can take 15 positive numbers as $\frac{2p}{3}, \frac{2p}{3}, \frac{2p}{3}, \frac{3q}{5}, \frac{3q}{5}, \frac{3q}{5}, \frac{3q}{5}, \frac{3q}{5}, \frac{4r}{7}, \frac{4r}{7}, \frac{4r}{7}, \frac{4r}{7}, \frac{4r}{7}, \frac{4r}{7}, \frac{4r}{7}$

For maximum value $\frac{2p}{3} = \frac{3q}{5} = \frac{4r}{7}$

$$\Rightarrow \quad p : q : r = \frac{3}{2} : \frac{5}{3} : \frac{7}{4} = 18 : 20 : 21$$

ILLUSTRATION 83

If a, b, c are three positive roots of the equation $x^3 - px^2 + qx - 48 = 0$ $(p, q > 0)$ then find the minimum value of $2\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)$.

SOLUTION

Consider $\frac{1}{a}, \frac{2}{b}, \frac{3}{c}$ and use A.M. ≥ G.M.,

we get $\frac{1}{3}\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right) \geq \left(\frac{1}{a}.\frac{2}{b}.\frac{3}{c}\right)^{1/3} = \left(\frac{6}{abc}\right)^{1/3}$

but $abc = 48$

$$\therefore \quad \frac{1}{3}\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right) \geq \left(\frac{6}{48}\right)^{1/3} = \frac{1}{2};$$

$$\therefore \quad \left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)_{\min} = \frac{3}{2}$$

$$\Rightarrow \quad 2\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)_{\min} = 2 \times \frac{3}{2} = 3$$

Test Your Concepts-III

(Solutions on page H.54)

1. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = ka_1a_5$ find k.
2. If $\log_{ax} x$, $\log_{bx} x \log_{cx} x$ are in H.P. where a, b, c, x belong to $(1, \infty)$, then show that a, b, c are in G.P.
3. If $b+c, c+a, a+b$ are in H.P then show that a^2, b^2, c^2 are in A.P.
4. If a, b, c are in HP, then find $\frac{1}{b-a}+\frac{1}{b-c}$ in terms of b only.
5. Find the limiting sum of the infinite series, $\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3}$ ... whose n^{th} term is $\frac{n}{10^3}$.
6. Find the sum to infinity of the series $1 - 3x + 5x^2 - 7x^3 + \ldots, \infty$ when $|x| < 1$.
7. If a be one A.M. and G_1 and G_2 be two geometric means between b and c, prove that $G_1^3 + G_2^3 = 2abc$.
8. If the positive numbers a, b, c are in G.P., then show that $\log_a 2$, $\log_b 2$, $\log_c 2$ are in HP.
9. Find the minimum value of $\left(16x + \frac{4}{x} + 1\right)$, $x > 0$.
10. If the relation between x and y such that the 20th arithmetic mean between x and $2y$ is same as the 20th arithmetic mean between $2x$ and y, (99 means being inserted in each case) if $y = kx$, then find k.
11. If $(a+b+c+d)^4 \leq 256$ abcd $(a, b, c, d > 0)$, then find the value of $\frac{a+b+c}{d}$.
12. If $\sqrt{3}$, H, $\sqrt{2}$ in H.P., then find the value of $\frac{H+\sqrt{2}}{H-\sqrt{2}} + \frac{H+\sqrt{3}}{H-\sqrt{3}}$.
13. If $x + y + z = 24$, then maximum value of $(x + 1)(y + 2)(z - 3)$ is (where $x > -1$, $y > -2$, $z > 3$).
14. If x, y, z are arbitrary positive real numbers satisfying the equation $4xy + 6yz + 8zx = 9$ and maximum possible value of (xyz) is M, then find (2016M).

15. If p, q, r are positive numbers satisfying $pqr = 3^{-5/2}$, then find the minimum value of $p^2 + 9q^2 + 6pq + 2r^2$.

16. If a, b, c are positive numbers in A.P., such that their product is 64, then find the minimum value of b.

Answer keys

1. 4	**4.** $\frac{2}{b}$	**5.** $\frac{10}{81}$	**6.** $\frac{1-x}{(1+x)^2}$	**9.** 17	**10.** 4
11. 3	**12.** 2	**13.** 512	**14.** 756	**15.** 2	**16.** 4

MISCELLANEOUS PROGRESSION

$T_r = r^{\text{th}}$ term of a progression

Sum of first n terms $S_n = \sum_{r=1}^{n} T_r$

$$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$

$$S_n = S_{n-1} + T_n$$

$\Rightarrow$ **$S_n - S_{n-1} = T_n$ Valid for all sequence**

(i) $$\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r.$$

(ii) $$\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r.$$

(iii) $$\sum_{r=1}^{n} \lambda f(r) \pm \mu g(r) = \lambda \sum_{r=1}^{n} f(r) \pm \mu \sum_{r=1}^{n} g(r)$$

(iv) $$\sum_{r=1}^{n} f(r)\cdot q(r) \neq \sum_{r=1}^{n} f(r) \cdot \sum_{r=1}^{n} q(r)$$

(v) $$\sum_{r=1}^{n} \frac{1}{f(r)} \neq \frac{1}{\sum_{r=1}^{r} f(r)}$$

(vi) $\sum_{r=1}^{n} k = k + k + k + \ldots\ n$ times $= nk$; where k is a constant.

(vii) $$\sum_{r=1}^{n} r = 1 + 2 + 3 \ldots + n = \frac{n(n+1)}{2}$$

Proof:

$1 + 2 + 3 + \ldots + n = \frac{n}{2}.(\text{First Term} + \text{Last Term})$ using the formula for summation of A.P.

$$= \frac{n(n+1)}{2}$$

Alternate

$$r^2 - (r-1)^2 = 2r - 1$$

$r = 1$ $\quad 1^2 - 0^2 = 2 - 1$

$r = 2$ $\quad 2^2 - 1^2 = 4 - 1$

$r = 3$ $\quad 3^2 - 2^2 = 6 - 1$

$r = 4$ $\quad 4^2 - 3^2 = 8 - 1$

$\vdots \qquad \vdots \quad \vdots$

$r = n$ $\quad n^2 - (n-1)^2 = 2n - 1$

adding all of them

$$n^2 - 0^2 = 2 + 4 + 6 + 8 + \ldots + 2n - (1 + 1 + 1 + \ldots + 1)$$

$$n^2 = 2(1 + 2 + 3 + 4 + \ldots + n) - n$$

$$n^2 + n = 2(1 + 2 + 3 + \ldots + n)$$

$$1 + 2 + 3 + \ldots + n = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

(viii) $$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof:

As we know $(r-1)^3 = r^3 - 3r^2 + 3r - 1$

or $\quad r^3 - (r-1)^3 = 3r^2 - 3r + 1$

$r = 1$ $\quad 1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$

$r = 2$ $\quad 2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$

$r = 3$ $\quad 3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$

$\vdots \qquad \vdots \quad \vdots$

$r = n$ $\quad n^3 - (n-1)^3 = 3 \cdot n^2 - 3n + 1$

adding all terms,

we get $n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + (1 + 1 + \ldots + 1)$

$$3(1^2 + 2^2 + 3^2 + \ldots + n^2) = n^3 + 3\,\frac{n(n+1)}{2} - n$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{3}\left[n^2 + \frac{3n+3}{2} - 1\right]$$

$$= \frac{n}{3}\left[\frac{2n^2 + 3n + 1}{2}\right]$$

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

(ix) $$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

$$\Rightarrow \sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$$

Proof:

We use the expression $r^4-(r-1)^4$ for getting required summation

$$r^4-(r-1)^4=(r^2-(r-1)^2)\,(r^2+(r-1)^2)$$

$$=(2r-1)(2r^2-2r+1)$$

$$r^4-(r-1)^4=4r^3-6r^2+4r-1$$

$r=1$ $\quad 1^4-0^4=4.1^3-6.1^2+4.1-1$

$r=2$ $\quad 2^4-1^4=4.2^3-6.2^2+4.2-1$

$r=3$ $\quad 3^4-2^4=4.3^3-6.3^2+4.3-1$

$r=4$ $\quad 4^4-3^4=4.4^3-6.4^2+4.4-1$

$\quad\vdots \qquad \vdots \quad \vdots$

$r=n$ $\quad n^4-(n-1)^4=4n^3-6n^2+4n-1$

adding all terms

we get $n^4-0=4(1^3+2^3+3^3+\ldots+n^3)-6(1^2+2^2+3^2+\ldots+n^2)+4(1+2+3+\ldots+n)-(1+1+1+\ldots+1)$

$$n^4=4(1^3+2^3+3^3+\ldots+n^3)-6.\frac{n(n+1)(2n+1)}{6}+4\frac{n(n+1)}{2}-n$$

on simplification we get

$$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

$$=\left(\sum_{r=1}^{n} r\right)^2$$

ILLUSTRATION 84

If sum of n terms of a sequence is given by $S_n=3n^2-4n$, find its 50th term.

SOLUTION

Let T_n is nth term of the sequence

so $\quad T_n=S_n-S_{n-1}=3n^2-4n-3(n-1)^2+4(n-1)=6n-7$

so $\quad T_{50}=293.$

ILLUSTRATION 85

Find the sum of the series to n terms whose nth term is $3n+2$.

SOLUTION

$$S_n=\sum_{n=1}^{n} T_n=\sum_{n=1}^{n}(3n+2)=3\sum_{n=1}^{n} n+\sum_{n=1}^{n} 2$$

$$=\frac{3(n+1)\;n}{2}+2n=\frac{n}{2}(3n+7)$$

ILLUSTRATION 86

If for a sequence, general term is $T_k=k^3+3^k$, then find $\sum_{k=1}^{n} T_k$.

SOLUTION

$$\sum_{k=1}^{n} T_k=\sum_{k=1}^{n} k^3+\sum_{k=1}^{n} 3^k=\left(\frac{n(n+1)}{2}\right)^2+\frac{3(3^n-1)}{3-1}$$

$$=\left(\frac{n(n+1)}{2}\right)^2+\frac{3}{2}(3^n-1)$$

ILLUSTRATION 87

If $\sum_{r=1}^{n} r(r+1)(2r+3)=an^4+bn^3+cn^2+dn+e$, then find $2a+3b+2c+3d+e$.

SOLUTION

We have

$$\sum_{r=1}^{n} r(r+1)(2r+3)=\sum_{r=1}^{n}(2r^3+5r^2+3r)$$

$$=2\sum_{r=1}^{n} r^3+5\sum_{r=1}^{n} r^2+3\sum_{r=1}^{n} r$$

$$=2\left[\frac{1}{4}n^2(n+1)^2\right]+5\left[\frac{1}{6}n(n+1)(2n+1)\right]+3\left[\frac{1}{2}n(n+1)\right]$$

$$=\frac{1}{6}n\,(n+1)\,[3n\,(n+1)+5\,(2n+1)+9]$$

$$=\frac{1}{6}n\,(n+1)\,(3n^2+13n+14)$$

$$=\frac{1}{6}[3n^4+16n^3+27n^2+14n]$$

$$=\frac{1}{2}n^4+\frac{8}{3}n^3+\frac{9}{2}n^2+\frac{7}{3}n$$

$$= an^4 + bn^3 + cn^2 + dn + e$$

$$2a + 3b + 2c + 3d + e = 1 + 8 + 9 + 7 + 0 = 25$$

ILLUSTRATION 88

Find the sum to n terms of the series $1^2 + (1^2 + 3^2) + (1^2 + 3^2 + 5^2) + \ldots$

SOLUTION

We have $t_r = 1^2 + 3^2 + 5^2 + \ldots + (2r - 1)^2$

$$= \sum_{k=1}^{r} (2k-1)^2 = \sum_{k=1}^{r} \left(4k^2 - 4k + 1\right)$$

$$= 4\left\{\frac{1}{6} r(r+1)(2r+1)\right\} - \frac{4}{2} r(r+1) + r$$

$$= \frac{1}{6} r\,[4(r+1)(2r+1) - 12(r+1) + 6]$$

$$= \frac{1}{6} r\,[8r^2 - 2] = \frac{1}{3}(4r^3 - r)$$

Thus, sum of n terms of the given series

$$\frac{4}{3}\sum_{r=1}^{n} r^3 - \frac{1}{3}\sum_{r=1}^{n} r = \frac{4}{3}\cdot\frac{1}{4} n^2 (n+1)^2 - \frac{1}{3}\cdot\frac{1}{2} n(n+1)$$

$$= \frac{1}{6} n(n+1)\,[2n(n+1) - 1]$$

$$= \frac{1}{6} n(n+1)(2n^2 + 2n - 1)$$

ILLUSTRATION 89

Find the sum of the series

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots + n \text{ terms}$$

SOLUTION

Here n^{th} term $T_n = \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + (2n-1)}$

$$= \frac{\sum_{r=1}^{n} n^3}{\frac{n}{2}(1 + 2n - 1)} = \frac{\left[n\left(\frac{n+1}{2}\right)\right]^2}{n^2}$$

$$= \frac{1}{4}(n+1)^2 = \frac{1}{4}(n^2 + 2n + 1) = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$$

$\therefore$ sum of n terms

$$S_n = \sum_{r=1}^{n} T_n = \frac{1}{4}\sum_{r=1}^{n} n^2 + \frac{1}{2}\sum_{r=1}^{n} n + \frac{1}{4}\sum_{r=1}^{n} 1$$

$$= \frac{1}{4}\,\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\,\frac{n(n+1)}{2} + \frac{1}{4} n$$

$$= \frac{1}{24} n(n+1)(2n+1) + \frac{1}{4} n(n+1) + \frac{1}{4} n$$

$$= \frac{n}{24}\{2n^2 + 3n + 1 + 6n + 6 + 6\}$$

Hence, $S_n = \dfrac{n}{24}(2n^2 + 9n + 13)$

ILLUSTRATION 90

If $x, y \in R$ satisfy the equation $3x^2 + 2y^2 - 6y - 3x + \dfrac{21}{4} = 0$, and the value of $\sum_{r=0}^{10}(2y + rx) = k$ then find the value of k.

SOLUTION

$$3x^2 + 2y^2 - 6y - 3x + \frac{21}{4} = 0,$$

$$\Rightarrow \quad 3\left(x - \frac{1}{2}\right)^2 + 2\left(y - \frac{3}{2}\right)^2 = 0$$

For solution to exist.

$\therefore \quad x = \dfrac{1}{2}$ and $y = \dfrac{3}{2}$

Hence, $\sum_{r=0}^{10}(2y + rx) = \sum_{r=0}^{10}\left(3 + \dfrac{r}{2}\right)$

$$= \sum_{r=0}^{10} 3 + \frac{1}{2}\sum_{r=0}^{10} r = 3 \times 11 + \frac{1}{2}\cdot\frac{10\cdot 11}{2} = \frac{121}{2} = 60.5$$

ILLUSTRATION 91

If the n^{th} term of a certain series is of the form $a + bn + c.\,2^n$ where a, b, c are constants. The first three terms are 2, –1, and –3, then show that a, c, b are in A.P. Also find the sum of first n terms of the series.

SOLUTION

$$t_n = a + bn + c.2^n$$

$$t_1 = a + b + 2c = 2 \qquad \ldots(1)$$

$$t_2 = a + 2b + 4c = -1 \qquad \ldots(2)$$

$$t_3 = a + 3b + 8c = -3 \qquad \ldots(3)$$

On solving (1), (2), (3), we get

$$a = 5,\ b = -4,\ c = 1/2$$

Clearly $a + b = 2c$

$\therefore \quad a, c, b$ are in A.P.

Now, $t_n = 5 - 4n + 2^{n-1}$

$$S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} 5 - 4\sum_{n=1}^{n} n + \sum_{n=1}^{n} 2^{n-1}$$

$$= 5n - 4.\frac{n(n+1)}{2} + \frac{2^n - 1}{2-1}$$

$$= 5n - 2\left(n^2 + n\right) + 2^n - 1$$

$$S = 2^n - 2n^2 + 3n - 1.$$

ILLUSTRATION 92

If $S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, then show that $\sum_{k=1}^{n}(2k+1)S(k)$ $= (n+1)^2 S(n) - \frac{n(n+1)}{2}$

SOLUTION

$$S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \qquad \ldots(1)$$

$$\sum_{k=1}^{n}(2k+1)S(k)$$

$$= 3S(1) + 5S(2) + 7S(3) + \ldots + (2n+1)\,S(n)$$

$$= 3 + 5\left(1+\frac{1}{2}\right) + 7\left(1+\frac{1}{2}+\frac{1}{3}\right) + \ldots +$$

$$(2n+1)\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$

$$= 1(3 + 5 + 7 + \ldots (2n+1)) +$$

$$\frac{1}{2}(5 + 7 + \ldots (2n+1)) + \frac{1}{3}(7 + 9 + \ldots +$$

$$(2n+1)) + \ldots + \frac{(2n+1)}{n}$$

$$= \left((n+1)^2 - 1\right) + \frac{1}{2}\left((n+1)^2 - 2^2\right) +$$

$$\frac{1}{3}\left((n+1)^2 - 3^2\right) + \ldots + \frac{1}{n}\left((n+1)^2 - n^2\right)$$

$$= (n+1)^2\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) - (1 + 2 + 3 + \ldots n)$$

$$= (n+1)^2\,S(n) - \frac{n(n+1)}{2}$$

Hence proved.

ILLUSTRATION 93

If sum $S = \sum_{r=1}^{2015}\frac{(-1)^{r+1}}{r}$, then show that $S = \sum_{r=1008}^{2015}\frac{1}{r}$

SOLUTION

$$S = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2014} + \frac{1}{2015}$$

$$S = \sum_{r=1}^{2015}\frac{1}{r} - 2\sum_{r=1}^{1007}\frac{1}{2r} = \sum_{r=1}^{2015}\frac{1}{r} - \sum_{r=1}^{1007}\frac{1}{r}$$

$$\Rightarrow \quad S = \sum_{r=1008}^{2015}\frac{1}{r} \text{ Hence proved.}$$

METHOD OF DIFFERENCE APPROACH FOR FINDING n^{th} TERM

Given sequence $T_1, T_2, T_3, T_4, T_5, T_6$

1st difference

$$T_1' = T_2 - T_1,\ T_2' = T_3 - T_2,\ T_3' = T_4 - T_3,\ T_4' = T_5 - T_4$$

2nd difference

$$T_1'' = T_2'' - T_1',\ T_2'' = T_3' - T_2',\ T_3'' = T_4' - T_3',\ T_4'' = T_5' - T_4'$$

3rd difference

$$T_1''' = T_2'' - T_1'',\ T_2''' = T_3'' - T_2'',\ T_3'''$$

$$= T_4'' - T_3'',\ T_4''' = T_5'' - T_4''$$

$$\vdots$$

kth difference

$$T_1^{\,k} = T_2^{\,k-1} - T_1^{\,k-1},\ T_2^{\,k} = T_3^{\,k-1} - T_2^{\,k-1},\ T_3^{\,k}$$

$$= T_4^{\,k-1} - T_3^{\,k-1},\ T_4^{\,k} = T_5^{\,k-1} - T_4^{\,k-1}$$

This process is continued untill the kth differences when **first time** we observe that the kth differences are all equal or they form a G.P. with common ratio different from 1.

CASE-1: After kth difference all terms are equal.
T_n = polynomial of degree k

$T_n = a_0 n^k + a_1 n^{k-1} + \ldots + a_k$, where $a_0, a_1, \ldots, a_k$ are calculated by using terms of the given sequence.

CASE-2: The kth differences are in G.P. with common ratio $r\ (r \neq 1)$

$T_n = \lambda r^{n-1}$ + polynomial of degree $(k-1)$

$T_n = \lambda r^{n-1} + a_0\, n^{k-1} + a_1\, n^{k-2} + \ldots + a_{k-1}$ where $a_0, a_1, \ldots, a_k$ and λ are calculated by using terms of the given sequence.

ILLUSTRATION 94

Find the sum of the series 3 + 7 + 14 + 24 + 37 + … upto 10 terms.

SOLUTION

Here the given series is not A.P., G.P., or H.P.

Given sequence is 3,7, 14, 24, 37, …

The sequence of first order difference between successive term 4, 7, 10, 13, … clearly it is an A.P.

The sequence of the second order difference is 3, 3, 3, 3, … all equal terms

$$T_n = an^2 + bn + c$$

$$T_1 = a + b + c = 3,\ T_2 = 4a + 2b + c = 7,$$

$$T_3 = 9a + 3b + c = 14$$

On solving we get $a = 3/2,\ b = -1/2,\ c = 2$

n^{th} term of given sequence $T_n = \frac{1}{2}[3n^2 - n + 4]$

Alternate

Let $\quad S = 3 + 7 + 14 + 24 + 37 + \ldots + T_n$

$$S = 0 + 3 + 7 + 14 + 24 + \ldots + T_{n-1} + T_n$$

after subtracting

$$0 = 3 + \underbrace{4 + 7 + 10 + 13 + \ldots (n-1)\text{term}}_{\text{A.P.}} - T_n$$

$$T_n = 3 + \frac{(n-1)}{2}[2(4) + (n-2)\,3]$$

$$= \frac{\left(3n^2 - n + 4\right)}{2}$$

$$S_n = \sum_{n=1}^{n} T_n = \frac{\left[3\sum_{n=1}^{n} n^2 - \sum_{n=1}^{n} n + 4n\right]}{2}$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

Putting $n = 10$

$$S_{10} = \frac{1}{2}\left[\frac{10 \times 11 \times 21}{2} - \frac{10 \times 11}{2} + 40\right]$$

$$= \frac{[1155 - 55 + 40]}{2} = \frac{1140}{2} = 570$$

ILLUSTRATION 95

Find n^{th} term and sum of n terms of the series,

$$1 + 3 + 7 + 15 + 31 + \ldots$$

SOLUTION

$$1, 3, 7, 15, 31 \ldots$$

Take first difference we get $2, 4, 8, 16 \ldots$ a G.P. with common ratio 2

then n^{th} term $T_n = \lambda 2^{n-1} + a$

$$T_1 = \lambda 2^0 + a = 1$$

$$T_2 = \lambda 2^1 + a = 3$$

We get $\lambda = 2,\ a = -1,\ T_n = 2^n - 1$

Alternate:

Let n^{th} term and sum of n terms of the series be denoted by T_n and S_n respectively.

$$\therefore \quad S_n = 1 + 3 + 7 + 15 + 31 + \ldots + T_{n-1} + T_n \qquad \ldots(1)$$

$$S_n = 0 + 1 + 3 + 7 + 15 + 31 + \ldots + T_{n-1} + T_n \qquad \ldots(2)$$

Subtracting (1) from (2), we get

$$0 = 1 + 2 + 4 + 8 + 16 + \ldots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = 1 + 2 + 4 + 8 + 16 + \ldots\ n \text{ terms}$$

$$= 1.\frac{\left(2^n - 1\right)}{2 - 1}$$

Hence, $T_n = 2^n - 1$

$\therefore$ sum of n term $S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n}\left(2^n - 1\right)$

$$= \sum_{n=1}^{n} 2^n - \sum_{n=1}^{n} 1 = (2 + 2^2 + 2^3 + \ldots + 2^n) - n$$

Hence, $S_n = 2.\frac{\left(2^n - 1\right)}{2 - 1} - n = 2(2^n - 1) - n$

ILLUSTRATION 96

Find the n^{th} term of the series 1, 3, 8, 16, 27, 41, …

SOLUTION

Given sequence is 1, 3, 8, 16, 27, 41, …
The sequence of first order difference between successive term 2, 5, 8, 11, 14, … clearly it is an A.P.
The sequence of the second order difference is 3, 3, 3, 3, … all equal terms

$$T_n = an^2 + bn + c$$

$$T_1 = a + b + c = 1,\ T_2 = 4a + 2b + c = 3,$$

$$T_3 = 9a + 3b + c = 8$$

On solving we get $a = 3/2,\ b = -5/2,\ c = 2$

n^{th} term of given sequence $T_n = \frac{1}{2}[3n^2 - 5n + 4]$

Alternate

$$s = 1 + 3 + 8 + 16 + 27 + 41 + \ldots + T_n \qquad \ldots(1)$$

$$s = 0 + 1 + 3 + 8 + 16 + 27 \ldots + T_{n-1} + T_n \qquad \ldots(2)$$

Take Difference

$$T_n = 1 + 2 + 5 + 8 + 11 + \ldots + (T_n - T_{n-1})$$

$$= 1 + (2 + 5 + 8 + 11 + \ldots \text{ upto } (n-1) \text{ terms})$$

$$T_n = 1 + \left(\frac{n-1}{2}\right)[2 \times 2 + (n-2)3] = \frac{1}{2}[3n^2 - 5n + 4]$$

ILLUSTRATION 97

Find the general term and sum of n terms of the series

$$1 + 5 + 19 + 49 + 101 + 181 + 295 + \ldots$$

SOLUTION

The sequence of first order difference between successive term 4, 14, 30, 52, 80 …

The sequence of the second order difference is 10, 16, 22, 28, … clearly it is an A.P.

We get third order difference as 6,6,6,6,… all equal terms so let n^{th} term is of degree 3 polynomial

$$T_n = an^3 + bn^2 + cn + d$$

$$a + b + c + d = 1 \qquad \ldots(1)$$

$$8a + 4b + 2c + d = 5 \qquad \ldots(2)$$

$$27a + 9b + 3c + d = 19 \qquad \ldots(3)$$

$$64a + 16b + 4c + d = 49 \qquad \ldots(4)$$

From (1), (2), (3) and (4)

$$a = 1, b = -1, c = 0, d = 1 \quad \Rightarrow \quad T_n = n^3 - n^2 + 1$$

$$S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^3 - r^2 + 1)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n$$

$$= \frac{n\left(n^2-1\right)(3n+2)}{12} + n$$

METHOD OF DIFFERENCE FOR SUMMATION (SPLIT METHOD/$V_n - V_{n-1}$ METHOD)

Given sequence $T_1, T_2, T_3, T_4, T_5, T_6, \ldots$ and we are looking to find out their summation

If possible we can try to express r^{th} term as difference of two terms as $T_r = V(r) - V(r-1)$

$$T_1 = V(1) - V(0),$$

$$T_2 = V(2) - V(1),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T_{n-1} = V(n-1) - V(n-2)$$

$$T_n = V(n) - V(n-1)$$

Adding all terms we get

$$S_n = T_1 + T_2 + T_3 + T_4 + T_5 + \ldots + T_{n-1} + T_n$$

$$= V(n) - V(0)$$

This can be practiced with the help of examples given below.

ILLUSTRATION 98

Find sum of n terms of series $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots$

SOLUTION

$$\text{Given series} = \sum_{r=1}^{n} r \cdot (r+1) = \sum_{r=1}^{n} (r^2 + r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}[2n + 1 + 3] = \frac{n(n+1)(n+2)}{3}$$

Alternate

$$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} \frac{r(r+1)}{3}[(r+2)-(r-1)]$$

$$= \frac{1}{3}\sum_{r=1}^{n}[r(r+1)(r+2)-(r-1)r(r+1)]$$

$$= \frac{1}{3}[1 \cdot 2 \cdot 3 - 0 \cdot 1 \cdot 2] + \frac{1}{3}[2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3] +$$

$$\frac{1}{3}[3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4] + \ldots +$$

$$\frac{1}{3}[n(n+1)(n+2) - (n-1)n(n+1)]$$

$$= \frac{1}{3}[n(n+1)(n+2) - 0]$$

$$= \frac{n(n+1)(n+2)}{3}$$

ILLUSTRATION 99

Evaluate $\frac{3}{2} + \frac{3}{6} + \frac{3}{12} + \frac{3}{20} + \ldots \infty$

SOLUTION

$$S = 3\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots\right)$$

$$S = 3\left(\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots\right)$$

$$\therefore \quad S = 3$$

ILLUSTRATION 100

Find the sum $\sum_{r=1}^{n} \frac{1}{(ar+b)[a(r+1)+b]}$

SOLUTION

Here we observe that the factor in the denominator of the r^{th} term differ by a constant term. Hence T_r can be expressed as–

$$T_r = \frac{1}{a}\left\{\frac{a(r+1)+b]-(ar+b)}{(ar+b)\{a(r+1)+b\}}\right\}$$

$$= \frac{1}{a}\left(\frac{1}{ar+b} - \frac{1}{a(r+1)+b}\right)$$

$$S_n = \sum_{r=1}^{n} T_r = \frac{1}{a}\sum_{r=1}^{n}(V_{r-1} - V_r) \text{ where } V_r = \frac{1}{a(r+1)b}$$

$$S_n = \frac{1}{a}(V_0 - V_n)$$

Hence $S_n = \frac{1}{a}\left(\frac{1}{a+b} - \frac{1}{a(n+1)+b}\right) = \frac{n}{(a+b)[a(n+1)+b]}$

ILLUSTRATION 101

Sum to n terms of the series

$$\frac{1}{(1+x)(1+3x)} + \frac{1}{(1+3x)(1+5x)} + \frac{1}{(1+5x)(1+7x)} + \ldots$$

SOLUTION

Let T_r be the general term of the series

$$T_r = \frac{1}{[1+(2r-1)x][1+(2r+1)x]}$$

So $\quad T_r = \frac{1}{2x}\left[\frac{(1+(2r+1)x)-(1+(2r-1)x)}{(1+(2r-1)x)(1+(2r+1)x)}\right]$

$$= \frac{1}{2x}\left[\frac{1}{(1+(2r-1)x)} - \frac{1}{(1+(2r+1)x)}\right]$$

$$= \frac{1}{2x}[V(2r-1) - V(2r+1)]$$

$\therefore \quad S_n = \Sigma T_r = T_1 + T_2 + T_3 + \ldots + T_n$

$$= \frac{1}{2x}\left[\begin{array}{l}(V(1)-V(3)) + (V(3)-V(5)) \\ + \ldots + (V(2n-1)-V(2n+1))\end{array}\right]$$

$$= \frac{1}{2x}[V(1) - V(2n+1)]$$

$$= \frac{1}{2x}\left[\frac{1}{1+x} - \frac{1}{(1+(2n+1)x)}\right]$$

$$= \frac{n}{(1+x)[1+(2n+1)x]}$$

ILLUSTRATION 102

Find the sum of n terms of series

$$\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + \ldots$$

SOLUTION

$$T_r = \frac{1+2+3+\ldots+r}{1^3+2^3+3^3+\ldots+r^3}$$

$$= \frac{\frac{r(r+1)}{2}}{\left(\frac{r(r+1)}{2}\right)^2} = \frac{1}{\frac{r(r+1)}{2}}$$

$$T_r = \frac{2}{r(r+1)} = 2\left[\frac{r+1-r}{r(r+1)}\right] = 2\left[\frac{1}{r} - \frac{1}{r+1}\right]$$

$T_1 + T_2 + T_3 + \ldots + T_n$

$$= 2\left(\left[\frac{1}{1} - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \ldots + \left[\frac{1}{n} - \frac{1}{n+1}\right]\right)$$

$$S_n = 2\left[1 - \frac{1}{n+1}\right] = 2\left[\frac{n+1-1}{n+1}\right] = \frac{2n}{n+1}$$

ILLUSTRATION 103

Find the sum of n terms of the series $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \ldots$

SOLUTION

$$T_r = \frac{3+(r-1)\cdot 2}{1^2+2^2+3^2+\ldots+r^2} = \frac{2r+1}{r(r+1)(2r+1)/6}$$

$$= \frac{6}{r(r+1)} = 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\therefore \quad \sum_{r=1}^{n} T_r = 6\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= 6\left(\frac{1}{1} - \frac{1}{n+1}\right) = \frac{6n}{n+1}$$

ILLUSTRATION 104

Sum to n terms of the series

$$\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + \ldots$$

SOLUTION

$$T_n = \frac{1}{(3n-2)(3n+1)(3n+4)}$$

$$= \frac{1}{6}\left[\frac{(3n+4)-(3n-2)}{(3n-2)(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$S_n = \sum_{r=1}^{n} T_r$$

$$= \frac{1}{6}\left[\left(\frac{1}{1.4} - \frac{1}{4.7}\right) + \left(\frac{1}{4.7} - \frac{1}{7.10}\right) + \ldots + \left(\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right)\right]$$

$$= \frac{1}{6}\left[\frac{1}{4} - \frac{1}{(3n+1)(3n+4)}\right]$$

ILLUSTRATION 105

If $S_n = \sum_{r=1}^{n} \frac{2r+1}{r^4+2r^3+r^2}$, then find S_{10}.

SOLUTION

$$S_n = \sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \sum_{r=1}^{n} \left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$\therefore \quad S_{10} = \left(\frac{1}{1} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \ldots + \left(\frac{1}{10^2} - \frac{1}{11^2}\right)$$

$$S_{10} = 1 - \frac{1}{121} = \frac{120}{121}$$

ILLUSTRATION 106

Find sum upto n terms

$$\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)}$$

SOLUTION

Given expression

$$= \frac{1}{4}\left[\frac{5-1}{1\cdot3\cdot5} + \frac{7-3}{3\cdot5\cdot7} + \frac{9-5}{5\cdot7\cdot9} + \ldots + \frac{(2n+3)-(2n-1)}{(2n-1)(2n+1)(2n+3)}\right]$$

$$= \frac{1}{4}\left[\left(\frac{1}{1\cdot3} - \frac{1}{3\cdot5}\right) + \left(\frac{1}{3\cdot5} - \frac{1}{5\cdot7}\right) + \left(\frac{1}{5\cdot7} - \frac{1}{7\cdot9}\right) + \ldots + \left(\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$$

$$= \frac{1}{4}\left[\frac{(2n+1)(2n+3)-3}{3(2n+1)(2n+3)}\right]$$

$$= \frac{1}{4}\left(\frac{4n^2+2n+6n+3-3}{3(2n+1)(2n+3)}\right) = \frac{(n^2+2n)}{3(2n+1)(2n+3)}$$

$$= \frac{n(n+2)}{3(2n+1)(2n+3)}$$

ILLUSTRATION 107

Find sum upto n terms

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$$

SOLUTION

$$T_r = \frac{r}{1+r^2+r^4} = \frac{r}{r^4+2r^2+1-r^2}$$

$$T_r = \frac{r}{(r^2+1)^2-r^2} = \frac{r}{(r^2+1+r)(r^2+1-r)}$$

$$T_r = \frac{1}{2}\left[\frac{(r^2+r+1)-(r^2-r+1)}{(r^2-r+1)(r^2-r+1)}\right]$$

$$T_r = \frac{1}{2}\left[\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}\right]$$

$$T_1 = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3}\right]$$

$$T_2 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{7}\right]$$

$$T_3 = \frac{1}{2}\left[\frac{1}{7} - \frac{1}{13}\right]$$

$$\vdots \qquad \vdots \quad \vdots$$

$$T_n = \frac{1}{2}\left[\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right]$$

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^2+n+1}\right] = \frac{1}{2}\left(\frac{n^2+n+1-1}{n^2+n+1}\right)$$

$$= \frac{n(n+1)}{2(n^2+n+1)}$$

ILLUSTRATION 108

Find the sum

$$\frac{n}{1\cdot2\cdot3} + \frac{n-1}{2\cdot3\cdot4} + \frac{n-2}{3\cdot4\cdot5} + \ldots + \frac{1}{n(n+1)(n+2)}$$

SOLUTION

$$T_r = \frac{n-(r-1)}{r(r+1)(r+2)} = \frac{n-r+1}{r(r+1)(r+2)}$$

$$T_r = \frac{n+1-r}{r(r+1)(r+2)} = \frac{n+1}{r(r+1)(r+2)} - \frac{1}{(r+1)(r+2)}$$

$$T_r = \frac{n+1}{2}\left[\frac{(r+2)-r}{r(r+1)(r+2)}\right] - \left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right]$$

$$T_r = \frac{n+1}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right] - \left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$r=1,\ T_1 = \frac{n+1}{2}\left[\frac{1}{1\cdot2} - \frac{1}{2\cdot3}\right] - \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$r=2,\ T_2 = \frac{n+1}{2}\left[\frac{1}{2\cdot3} - \frac{1}{3\cdot4}\right] - \left[\frac{1}{3} - \frac{1}{4}\right]$$

$$r=3,\ T_3 = \frac{n+1}{2}\left[\frac{1}{3\cdot4} - \frac{1}{4\cdot5}\right] - \left[\frac{1}{4} - \frac{1}{5}\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad \vdots$$

$$r=n,\ T_n = \frac{n+1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right] - \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

Adding all n terms, we get

$$S_n = \frac{n+1}{2}\left[\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right] - \left[\frac{1}{2} - \frac{1}{n+2}\right]$$

$$S_n = \frac{n+1}{4} - \frac{1}{2(n+2)} - \frac{1}{2} + \frac{1}{n+2}$$

$$= \frac{n+1-2}{4} + \frac{1}{2(n+2)} = \frac{(n-1)(n+2)+2}{4(n+2)}$$

$$= \frac{n^2+n}{4(n+2)}$$

ILLUSTRATION 109

If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$ then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

SOLUTION

$$S_n = \sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$$

$$\because \quad T_n = S_n - S_{n-1}$$

$$= \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8}$$

$$= \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$= \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} \qquad \ldots(1)$$

$$\sum_{r=1}^{n} \frac{1}{T_r} = \sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \sum_{r=1}^{n} \frac{r+2-r}{r(r+1)(r+2)}$$

$$= \sum_{r=1}^{n}\left(\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right)$$

$$= \left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{2.3} - \frac{1}{3.4}\right) + \left(\frac{1}{3.4} - \frac{1}{4.5}\right) + \ldots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) = \frac{(n+1)(n+2)-2}{2(n+1)(n+2)}$$

$$= \frac{n^2+3n}{2(n+1)(n+2)}$$

Test Your Concepts-IV

(Solutions on page H.55)

1. Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 + ...
2. Find the sum to n terms of the series 5, 7, 13, 31, 85, ...
3. Find sum upto n terms $\frac{4}{1\cdot 2\cdot 3} + \frac{5}{2\cdot 3\cdot 4} + \frac{6}{3\cdot 4\cdot 5} + \ldots$
4. Find $\sum_{r=2}^{\infty}\left(\frac{1}{r^2-1}\right)$
5. If $t_n = \frac{n^3-n}{2}$, then find $\sum_{n=2}^{\infty}\frac{1}{t_n}$
6. The sum $\sum_{k=1}^{100}\frac{k}{k^4+k^2+1}$ is equal to $\frac{N}{10101}$ their find the value of N.
7. If $S_\infty = \frac{1}{1.3.5.7.9} + \frac{1}{3.5.7.9.11} + \frac{1}{5.7.9.11.13} + \ldots$, then find the value of $\frac{1}{120S_\infty}$.
8. Let $V_{n+1} = 3V_n - 2V_{n-1}\ \forall\ n \in N$ and $V_0 = 2$, $V_1 = 3$, then find value of $\left(\frac{V_6}{13}\right)$.
9. If S_n denotes the sum of n term of the series 1.2 + 2.3 + 3.4 + ... and σ_n denotes the sum of n terms of $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \ldots$, then find the value of $18S_n\sigma_{n-1} - S_n + 2$.
10. If S denotes the sum of infinite series $\frac{1}{1.3.5} + \frac{1}{2.4.6} + \frac{1}{3.5.7} + \frac{1}{4.6.8} + \ldots$, then find the value of $\frac{22}{S}$.
11. Find the n^{th} term and the sum of n terms of the series 2, 5, 12, 31, 86, ...
12. If $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6}n\left(2n^2+9n+13\right)$, then find $\sum_{r=1}^{n}\sqrt{t_r}$

Answer keys

1. $3n(n+1)^2-2n$

2. $\frac{1}{2}[3^n+8n-1]$

3. $\left(\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}\right)$

4. 3/4

5. $\frac{1}{2}$

6. 5050

7. 7

8. 5

9. 0

10. 192

11. $t_n=3^{n-1}+n-1,\ S_n=\frac{3^n-1+n^2-n}{2}$

12. $\frac{1}{2}n(n+3)$

SERIES EXPANSION OF SOME FUNCTIONS

(i) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\frac{x^n}{n!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

(ii) $\ell n\,(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots;-1<x<1$

(iii) $a^x=1+(x\,\ell n\,a)+\frac{(x\,\ell n\,a)^2}{2!}+\frac{(x\,\ell n\,a)^3}{3!}+\ldots\frac{(x\,\ell n\,a)^n}{n!}+\ldots$

ILLUSTRATION 110

Evaluate $\sum_{n=1}^{\infty}\frac{2n^2+n+1}{n!}$

SOLUTION

$$\sum_{n=1}^{\infty}\frac{2n(n-1)+3n+1}{n!}=2\sum_{n=1}^{\infty}\frac{n(n-1)}{n(n-1)(n-2)!}+3\sum_{n=1}^{\infty}\frac{n}{n(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$=2\sum_{n=2}^{\infty}\frac{1}{(n-2)!}+3\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+(e-1)$$

$$=2e+3e+e-1=6e-1$$

ILLUSTRATION 111

Find the sum of the series $1+\frac{1}{4.2!}+\frac{1}{16.4!}+\frac{1}{64.6!}+\ldots$

SOLUTION

$$1+\frac{1}{4.2!}+\frac{1}{16.4!}+\ldots=\frac{e^{1/2}+e^{-1/2}}{2}=\frac{e+1}{2\sqrt{e}}$$

ILLUSTRATION 112

Find $\frac{1^2}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\ldots..\ \infty$

SOLUTION

$$\sum_{n=1}^{\infty}\frac{1^2+2^2+3^2+\ldots.+n^2}{n!}=\sum_{n=1}^{\infty}\frac{n(n+1)(2n+1)}{6n!}$$

$$=\frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{2n^3}{n!}+\sum_{n=1}^{\infty}\frac{3n^2}{n!}+\sum_{n=1}^{\infty}\frac{n}{n!}\right)$$

$$=\frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{2(n^2-1)+2}{(n-1)!}+\sum_{n=1}^{\infty}\frac{3n}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\right)$$

$$=\frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{2(n-1)(n+1)}{(n-1)!}+\sum_{n=1}^{\infty}\frac{2}{(n-1)!}+\sum_{n=1}^{\infty}\frac{3(n-1+1)}{(n-1)!}+e\right)$$

$$=\frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{2(n-2+3)}{(n-2)!}+\sum_{n=1}^{\infty}\frac{2}{(n-1)!}+\sum_{n=1}^{\infty}\frac{3}{(n-2)!}+\sum_{n=1}^{\infty}\frac{3}{(n-1)!}+e\right)$$

$$=\frac{1}{6}\left(\sum_{n=1}^{\infty}\frac{2}{(n-3)!}+\sum_{n=1}^{\infty}\frac{6}{(n-2)!}+2e+3e+3e+e\right)$$

$$=\frac{1}{6}(2e+6e+9e)=\frac{17e}{6}$$

ILLUSTRATION 113

If $|x|<1$ and $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ find x in terms of y.

SOLUTION

$$y=x-x^2+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.$$

$$y=\log_e(1+x)\ \Rightarrow\ 1+x=e^y$$

$$\Rightarrow\quad x=y+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots$$

SOLVED PROBLEMS

PROBLEM 1

If following equations have equal roots then specify that a, b, c follows which series among A.P./G.P./H.P.

(i) $(b - c)\,x^2 + (c - a)x + (a - b) = 0$

(ii) $a(b - c)\,x^2 + b(c - a)x + c(a - b) = 0$

(iii) $(a^2 + b^2)\,x^2 - 2b(a + c)x + (b^2 + c^2) = 0$

SOLUTION

(i) $(b - c)\,x^2 + (c - a)x + (a - b) = 0$ has equal rootthen

$D = (c - a)^2 - 4\,(b - c)\,(a - b) = 0$

$\Rightarrow c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow (c + a)^2 - 4b(a + c) + 4b^2 = 0$

$\Rightarrow (c + a - 2b)^2 = 0 \quad \Rightarrow \quad c + a - 2b = 0$

$\Rightarrow 2b = c + a$

So a, b, c in A.P.

(ii) $a(b - c)\,x^2 + b(c - a)x + c(a - b) = 0$ has equal roots

$D = (b(c - a))^2 - 4a\,(b - c)\,c(a - b)$

$= (b^2c^2 + a^2b^2 - 2ab^2c) - 4ac\,(ab - ac + bc - b^2)$

$= (b^2c^2 + a^2b^2 + 2ab^2c) + 4a^2c^2 - 4ac(ab + bc)$

$= (bc + ab)^2 + 4a^2c^2 - 4ac(ab + bc) = 0$

$\Rightarrow (ab + bc - 2ac)^2 = 0 \quad \Rightarrow \quad 2ac = ab + bc$

$b = 2ac/(a + c)$

so a, b, c in H.P.

(iii) given equation

$(a^2 + b^2)\,x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ has equal roots

$D = 4b^2\,(a + c)^2 - 4\,(a^2 + b^2)\,(b^2 + c^2)$

$= 4(b^2a^2 + b^2c^2 + 2ab^2c - a^2b^2 - a^2c^2 - b^4 - b^2c^2)$

$= 4(2ab^2c - a^2c^2 - b^4) = -4\,(b^2 - ac)^2 = 0$

We get $b^2 = ac$ $\therefore$ a, b, c in G.P.

PROBLEM 2

If $\dfrac{a_1}{a_2}, \dfrac{b_1}{b_2}, \dfrac{c_1}{c_2}$ are in A.P. and the equations $a_1x^2 + 2b_1x + c_1 = 0$ and $a_2x^2 + 2b_2\,x + c_2 = 0$ have a common root, prove that a_2, b_2, c_2 are in G.P.

SOLUTION

It is given that the equation $a_1x^2 + 2b_1\,x + c_1 = 0$ and $a_2x^2 + 2b_2\,x + c_2 = 0$ have a common root. So, let α be the common root.

$$\therefore \quad a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \text{ and}$$

$$a_2\alpha^2 + 2b_2\alpha + c_2 = 0$$

$$\Rightarrow \quad \frac{\alpha^2}{2\,(b_1c_2 - b_2c_1)} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{2(a_1b_2 - a_2b_1)}$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } \alpha = \frac{c_1a_2 - c_2a_1}{2(a_1b_2 - a_2b_1)}$$

Now, $\alpha^2 = (\alpha)^2$

$$\Rightarrow \quad \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left\{\frac{c_1a_2 - c_2a_1}{2(a_1b_2 - a_2b_1)}\right\}^2$$

$$\Rightarrow \quad 4\,(b_1c_2 - b_2c_1)\,(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$$

$$\Rightarrow 4b_2c_2\left(\frac{b_1}{b_2} - \frac{c_1}{c_2}\right).a_2b_2\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right) = (c_2a_2)^2\left(\frac{c_1}{c_2} - \frac{a_1}{a_2}\right)^2$$

$$\Rightarrow \quad 4\,a_2b_2^2c_2\left(\frac{b_1}{b_2} - \frac{c_1}{c_2}\right)\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right) = a_2^2\,c_2^2\left(\frac{c_1}{c_2} - \frac{a_1}{a_2}\right)^2 \qquad \ldots(1)$$

It is given that $\dfrac{a_1}{a_2}, \dfrac{b_1}{b_2}, \dfrac{c_1}{c_2}$ are in A.P. Let D be the common difference of this A.P. Then,

$$\therefore \quad \frac{b_1}{b_2} - \frac{a_1}{a_2} = \frac{c_1}{c_2} - \frac{b_1}{b_2} = D \text{ and } \frac{c_1}{c_2} - \frac{a_1}{a_2} = 2\,D.$$

Putting these values in (1), we get

$$4\,a_2b_2^{\,2}\,c_2 \times (-D) \times (-D) = a_2^{\,2}\,c_2^{\,2}\,(2D)^2$$

$$\Rightarrow \quad b_2^{\,2} = a_2c_2 \quad \Rightarrow \quad a_2, b_2, c_2 \text{ are in GP.}$$

PROBLEM 3

Find largest term in the expansion of $\dfrac{1}{201}, \dfrac{4}{208}, \dfrac{9}{227}, \dfrac{16}{264}, \ldots$

SOLUTION

$$T_n = \frac{x^2}{x^3 + 200} = f(x) \text{ where } x \in N$$

$$f'(x) = \frac{\left(x^3 + 200\right)2x - x^2\left(3x^2\right)}{\left(x^3 + 200\right)^2} = \frac{x\left(400 - x^3\right)}{\left(x^3 + 200\right)^2}$$

$f(x)$ increasing in $0 < x < 400^{\frac{1}{3}}$ and decreasing in $x > 400^{\frac{1}{3}}$ largest term is either T_7 or T_8

$$T_7 = \frac{49}{543},\ T_8 = \frac{8}{89}.$$

T_7 is largest term

PROBLEM 4

If $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \ldots \infty = \dfrac{\pi^4}{90}$

Find $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots \infty$

SOLUTION

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \ldots$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} \ldots\right)$$

$$\frac{\pi^4}{90} = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) + \frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right]$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{90} - \frac{1}{2^4}\left[\frac{\pi^4}{90}\right]$$

$$= \frac{\pi^4}{90}\left(1 - \frac{1}{16}\right) = \frac{\pi^4}{90} \times \frac{15}{16} = \frac{\pi^4}{96}$$

PROBLEM 5

Let $<a_n>$ be an arithmetic sequence such that arithmetic mean of $a_1, a_3, a_5, \ldots a_{99}$ is 1. Find the value of

$$\left|\sum_{r=1}^{50} (-1)^{\frac{r(r+1)}{2}} \cdot a_{2r-1}\right|$$

SOLUTION

$$a_1 + a_3 + \ldots + a_{99} = 50$$

$$\Rightarrow \quad a + (a + 2d) + (a + 4d) + \ldots + (a + 98d) = 50$$

$$\Rightarrow \quad 50a + 2d(1 + 2 + \ldots + 49) = 50$$

$$\Rightarrow \quad 50a + \frac{2d(50)(49)}{2} = 50 \quad \Rightarrow \quad a + 49d = 1;$$

$$\left|\sum_{r=1}^{50} (-1)^{\frac{r(r+1)}{2}} \cdot a_{2r-1}\right|$$

$$= |-a_1 - a_3 + a_5 + a_7 - a_9 - a_{11} + \ldots + a_{93} + a_{95} - a_{97} - a_{99}|$$

$$= \left|\begin{matrix} -a_1 + (-a_3 + a_5) + (a_7 - a_9) + (-a_{11} + a_{13}) \\ + (a_{15} - a_{17}) + \ldots + (-a_{91} + a_{93}) + (a_{95} - a_{97}) - a_{99} \end{matrix}\right|$$

$$= |-a_1 + 2d - 2d + 2d - 2d + \ldots + 2d - a_{99}|$$

$$\Rightarrow \quad |-a_1 - a_{99}| = |-2a - 98d|$$

$$= |-2(a + 49d)| = |-2| = 2$$

PROBLEM 6

If s_1, s_2, s_3 denote the sum of n terms of 3 arithmetic series whose first terms are unity and their common difference are in H.P., Prove that $n = \dfrac{2s_3s_1 - s_1s_2 - s_2s_3}{s_1 - 2s_2 + s_3}$

SOLUTION

Let the common differences of three A.P.'s are d_1, d_2 and d_3. Given d_1, d_2, d_3 are in H.P.

$\therefore \quad \dfrac{1}{d_1}, \dfrac{1}{d_2}, \dfrac{1}{d_3}$ are in A.P.

$$\therefore \quad \frac{1}{d_2} - \frac{1}{d_1} = \frac{1}{d_3} - \frac{1}{d_2} \qquad \ldots(1)$$

Now $s_1 = \dfrac{n}{2}\{2.1 + (n-1)d_1\} \quad \Rightarrow \quad s_1 - n = \dfrac{(n-1)n}{2} d_1$

Similarly $s_2 - n = \dfrac{(n-1)nd_2}{2}$

and $s_3 - n = \dfrac{(n-1)nd_3}{2}$

Substituting d_1, d_2 and d_3 in (1), then

$$\frac{n(n-1)}{2(s_2 - n)} - \frac{n(n-1)}{2(s_1 - n)} = \frac{n(n-1)}{2(s_3 - n)} - \frac{n(n-1)}{2(s_2 - n)}$$

$$\Rightarrow \quad \frac{1}{s_2 - n} - \frac{1}{s_1 - n} = \frac{1}{s_3 - n} - \frac{1}{s_2 - n}$$

$$\Rightarrow \quad \frac{s_1 - s_2}{(s_2 - n)(s_1 - n)} = \frac{s_2 - s_3}{(s_3 - n)(s_2 - n)}$$

$$\Rightarrow \quad \frac{s_1 - s_2}{s_1 - n} = \frac{s_2 - s_3}{s_3 - n}$$

$$\Rightarrow \quad s_1 s_3 - ns_1 - s_2 s_3 + ns_2 = s_1 s_2 - s_1 s_3 - ns_2 + ns_3$$

$$\Rightarrow \quad n(s_1 - 2s_2 + s_3) = 2s_3s_1 - s_1 s_2 - s_2 s_3$$

$$\therefore \quad n = \frac{2s_3s_1 - s_1s_2 - s_2s_3}{s_1 - 2s_2 + s_3}$$

PROBLEM 7

Let S_n, $n = 1, 2, 3, \ldots$ be the sum of infinite geometric series whose first term is n and the common ratio is $\dfrac{1}{n+1}$.

Evaluate $\displaystyle\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \ldots + S_nS_1}{S_1^2 + S_2^2 + \ldots + S_n^2}$.

SOLUTION

$$S_n = \frac{A}{1 - R} = \frac{n}{1 - \dfrac{1}{n+1}}$$

$$S_n = n + 1$$

$$\therefore \quad S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \ldots + S_n S_1$$

$$= 2 \cdot (n+1) + 3 \cdot n + 4 \cdot (n-1) + \ldots + (n+1) \cdot 2$$

$$= \sum_{r=1}^{n} (r+1)(n-r+2)$$

$$= \sum_{r=1}^{n} (nr - r^2 + 2r + n - r + 2)$$

$$= \sum_{r=1}^{n} \{(n+1)r - r^2 + (n+2)\}$$

$$= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + (n+2)\sum_{r=1}^{n} 1$$

$$= \frac{(n+1)n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + n\,(n+2)$$

$$= \frac{n}{6}\{3(n^2+2n+1) - (2n^2+3n+1) + 6n + 12\}$$

$$= \frac{n}{6}\{n^2+9n+14\} \qquad \ldots(1)$$

and $S_1^2 + S_2^2 + \ldots + S_n^2 = 2^2 + 3^2 + \ldots + (n+1)^2$

$$= \frac{(n+1)(n+2)(2n+3)}{6} - 1 = \frac{n(2n^2+9n+13)}{6} \qquad \ldots(2)$$

From (1) and (2), we get

$$\lim_{n\to\infty} \frac{S_1S_n + S_2S_{n-1} + S_3S_{n-2} + \ldots + S_nS_1}{S_1^2 + S_2^2 + \ldots + S_n^2}$$

$$= \lim_{n\to\infty} \frac{n^2+9n+14}{2n^2+9n+13} = \lim_{n\to\infty} \frac{1+\frac{9}{n}+\frac{14}{n^2}}{2+\frac{9}{n}+\frac{13}{n^2}}$$

$$= \frac{1+0+0}{2+0+0} = \frac{1}{2}$$

PROBLEM 8

The sum of the first 10 terms of the series $\frac{7}{2^2 5^2} + \frac{13}{5^2 8^2} + \frac{19}{8^2 11^2} + \ldots$ is $\frac{m}{n}$ then find the value of $(n - 12m)$?

SOLUTION

$$S = \frac{7}{2^2 5^2} + \frac{13}{5^2 8^2} + \frac{19}{8^2 11^2} + \ldots$$

$$\Rightarrow \quad 3S = \frac{21}{2^2 5^2} + \frac{39}{5^2 8^2} + \frac{57}{8^2 11^2} + \ldots$$

$$\Rightarrow \quad 3S = \sum_{r=1}^{10} \frac{(3r+2)^2 - (3r-1)^2}{(3r-1)^2(3r+2)^2}$$

$$\Rightarrow \quad 3S = \sum_{r=1}^{10} \left(\frac{1}{(3r-1)} - \frac{1}{(3r+2)^2} \right)$$

$$= \left(\frac{1}{2^2} - \frac{1}{5^2}\right) + \left(\frac{1}{5^2} - \frac{1}{8^2}\right) + \ldots + \left(\frac{1}{29^2} - \frac{1}{32^2}\right)$$

$$\Rightarrow \quad 3S = \frac{1}{2^2} - \frac{1}{2^{10}}$$

$$\Rightarrow \quad 3S = \frac{2^8-1}{2^{10}} = \frac{255}{1024}$$

$$\Rightarrow \quad S = \frac{85}{1024} = \frac{m}{n}$$

Hence, $(n - 12m) = 1024 - 85 \times 12 = 4$

PROBLEM 9

Let r_1, r_2, r_3 be the three (not necessarily distinct) solution to the equation $x^3 + 4x^2 - ax + 1 = 0$. If a can be any real number, then find the minimum value of $\left(r_1 + \frac{1}{r_1}\right)^2 + \left(r_2 + \frac{1}{r_2}\right)^2 + \left(r_3 + \frac{1}{r_3}\right)^2$.

SOLUTION

We have $x^3 + 4x^2 - ax + 1 = 0$ has roots r_1, r_2 and r_3.

$\therefore \quad \Sigma r_1 = -4$, $\Sigma r_1 r_2 = -a$ and $r_1 r_2 r_3 = -1$

$$\text{Now let } E = \left(r_1 + \frac{1}{r_1}\right)^2 + \left(r_2 + \frac{1}{r_2}\right)^2 + \left(r_3 + \frac{1}{r_3}\right)^2$$

$$= (\Sigma r_1)^2 - 2\Sigma r_1 r_2 + \left[\frac{(r_2r_3)^2 + (r_3r_1)^2 + (r_1r_2)^2}{(r_1r_2r_3)^2}\right] + 6$$

$$= 22 + 2a + \left[\left(\sum r_1r_2\right)^2 - 2\left(r_1^2 r_2 r_3 + r_2^2 r_3 r_1 + r_3^2 r_2 r_1\right)\right]$$

$$= 22 + 2a + [a^2 + 2(-4)]$$

$$= 22 - 8 + a^2 + 2a$$

$$\therefore \quad E = (a+1)^2 + 13$$

Hence minimum value of $E = 13$ (when $a = -1$)

PROBLEM 10

If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$ then prove that

$$x^{y+z}\, y^{z+x}\, z^{x+y} = 1$$

Also prove that $x^{y+z} + y^{z+x} + z^{x+y} \geq 3$.

SOLUTION

For log$x$, log$y$, log$z$ to defined $x > 0$, $y > 0$, $z > 0$

Now $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y} = k$ (let)

$$\Rightarrow \quad x = 10^{k(y-z)},\ y = 10^{k(z-x)},\ z = 10^{k(x-y)}$$

$$\Rightarrow \quad (x)^{y+z} \cdot (y)^{z+x} \cdot (z)^{x+y}$$

$$= \left(10^{k(y-z)}\right)^{(y+z)} \left(10^{k(z-x)}\right)^{(z+x)} \left(10^{k(x-y)}\right)^{(x+y)}$$

$$= 10^{k(y^2+z^2+x^2-x^2-y^2-z^2)}$$

$$= 10^0 = 1$$

i.e. $\quad x^{y+z} \cdot y^{z+x} \cdot z^{x+y} = 1 \qquad \ldots(1)$

Again take the positive number x^{y+z}, y^{z+x} and z^{x+y}

A.M. ≥ G.M.

$$\frac{x^{y+z} + y^{z+x} + z^{x+y}}{3} \geq \sqrt[3]{x^{y+z} y^{z+x} z^{x+y}} = 1$$

$$\therefore \quad x^{y+z} + y^{z+x} + z^{x+y} \geq 3$$

Hence proved.

PROBLEM 11

A Set of consecutive natural numbers from 1 to n are written and one number is removed from it. The arithmetic mean of the remaining numbers is found to be $\frac{71}{2}$, then the sum of all possible value(s) of the number removed from the set is λ then find the value of λ.

SOLUTION

$$\frac{\frac{n(n+1)}{2} - n}{n-1} \leq \frac{71}{2} \leq \frac{\frac{n(n+1)}{2} - 1}{n-1}$$

$$\Rightarrow \quad \frac{n^2 - n}{2(n-1)} \leq \frac{71}{2} \leq \frac{n^2 + n - 2}{2(n-1)}$$

$$\Rightarrow \quad \frac{n}{2} \leq \frac{71}{2} \leq \frac{n+2}{2}$$

$$\Rightarrow \quad n \leq 71 \leq n+2$$

Hence, n can be 69, 70 or 71.

For $n = 69$, $\frac{\frac{69(70)}{2} - x}{68} = \frac{71}{2} \quad \Rightarrow \quad x = 1$

For $n = 70$, $\frac{\frac{70(71)}{2} - x}{69} = \frac{71}{2}$

$\Rightarrow$ we do not get $x \in N$

For $n = 71$, $\frac{\frac{71 \cdot 72}{2} - x}{70} = \frac{71}{2}$ we get $x = 71$

Hence $\lambda = 1 + 71 = 72$

PROBLEM 12

The roots α and β of a quadratic equation are the square of two consecutive natural numbers. The geometric mean of the two roots is 1 greater than the difference of the roots. If α and β lies between the roots of $x^2 - cx + 1 = 0$, then find the minimum integral value of c.

SOLUTION

Suppose roots be a^2, $(a+1)^2$, where $a \in N$.

So the equation is $(x - a^2)\,(x - (a+1)^2) = 0$

Given, $\sqrt{a^2(a+1)^2} = 1 + |(a+1)^2 - a^2|$

$\Rightarrow \quad a(a+1) = 1 + (2a+1)$ (As $a \in N$)

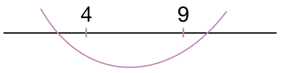
$\Rightarrow \quad a^2 + a = 1 + (2a+1)$

$\Rightarrow \quad a^2 - a - 2 = 0 \quad \Rightarrow \quad (a-2)\,(a+1) = 0$

$\Rightarrow \quad a = 2$ or -1 (Reject, $a = -1$ as $a \in N$)

Hence roots are 4 and 9.

Let $f(x) = x^2 - cx + 1$



so, $f(4) < 0 \quad \Rightarrow \quad 17 - 4c < 0 \quad \Rightarrow \quad c > \frac{17}{4}$ ...(1)

and $f(9) < 0 \quad \Rightarrow \quad 82 - 9c < 0 \quad \Rightarrow \quad c > \frac{82}{9}$ ...(2)

$\therefore$ (1) $\cap$ (2) gives

Hence, $c > \frac{82}{9}$

$\Rightarrow$ Minimum integral value of $c = 10$.

PROBLEM 13

Let the numbers tan x, cos x and sec x are the first three terms of an arithmetic progression. If the n^{th} term of the progression is cot x, then find the value of n.

SOLUTION

Given, $2 \cos x = \tan x + \sec x = \frac{1 + \sin x}{\cos x}$

$\Rightarrow \quad 2\cos^2 x = 1 + \sin x \quad \Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad 2\sin^2 x + 2\sin x - \sin x - 1 = 0$

$\Rightarrow \quad (\sin x + 1)\,(2 \sin x - 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}$ or -1 (reject)

Now, $T_n = \cot x = \tan x + (n-1)d$

$\therefore \quad (n-1) = \frac{\cot x - \tan x}{d}$ where $d = \sec x - \cos x$

$$\Rightarrow \quad n = 1 + \frac{\cos 2x}{(\cos x \cdot \sin x)(\sec - \cos x)}$$

$$\Rightarrow \quad n = 1 + \frac{\cos 2x}{\sin^3 x} = 1 + \frac{1 - 2\sin^2 x}{\sin^3 x} = 1 + \frac{1 - 2 \times \frac{1}{4}}{\frac{1}{8}}$$

$\Rightarrow \quad n = 1 + 4 = 5.$

PROBLEM 14

If $a_n = \sqrt{1 + \left(1 - \frac{1}{n}\right)^2} + \sqrt{1 + \left(1 + \frac{1}{n}\right)^2}$; $(n \geq 1)$, then find value of $\sum_{i=1}^{20} \frac{1}{a_i}$.

SOLUTION

$$a_n = \frac{\sqrt{n^2 + (n-1)^2} + \sqrt{n^2 + (n+1)^2}}{n}$$

rationalise

$$a_n = \frac{4n}{n\left[\sqrt{n^2+(n+1)^2} - \sqrt{n^2+(n-1)^2}\right]}$$

Now $\displaystyle\sum_{i=1}^{20} \frac{1}{a_i} = \sum_{i=1}^{20} \frac{\sqrt{i^2+(i+1)^2}-\sqrt{i^2+(i-1)^2}}{4}$

$$= \left[\frac{1}{4}\left[\begin{array}{l}\left(\sqrt{1^2+2^2}-\sqrt{1^2+0^2}\right)+\left(\sqrt{2^2+3^2}-\sqrt{2^2+1^2}\right)\\ +\left(\sqrt{3^2+4^2}-\sqrt{3^2+2^2}\right)+\ldots+\left(\sqrt{20^2+21^2}-\sqrt{20^2+19^2}\right)\end{array}\right]\right]$$

$$= \frac{1}{4}\left[\sqrt{20^2+21^2}-1\right] = \frac{1}{4}[29-1] = 7$$

PROBLEM 15

Find sum of first n terms of the series $6 + 15x + 28x^2 + 45x^3 + \ldots$

SOLUTION

As we can see 6, 15, 28, 45, …

On first difference we get 9, 13, 17,…

On second difference we get 4, 4, …

r^{th} term of coefficients is of the form quadratic $ar^2 + br + c$.

For determining a, b, c put $r = 1, 2, 3$

$$6 = a + b + c \qquad \ldots(1)$$

$$15 = 4a + 2b + c \qquad \ldots(2)$$

$$28 = 9a + 3b + c \qquad \ldots(3)$$

(2) – (1) gives

$$9 = 3a + b \qquad \ldots(4)$$

(3) – (2) gives

$$13 = 5a + b \qquad \ldots(5)$$

Subtracting (4) from (5)

$$4 = 2a \quad \Rightarrow \quad a = 2 \text{ and } b = 3, c = 1$$

Hence given series is $T_r = \displaystyle\sum_{r=1}^{n} (2r^2 + 3r + 1)x^{r-1}$

If the difference after two operations remains equal, the process is as followings:

Let $S_n = 6 + 15x + 28x^2 + 45x^3 + \ldots + (2n^2 + 3n + 1)x^{n-1}$

$\therefore\ xS_n = 0 + 6x + 15x^2 + 28x^3 + \ldots + (2n^2 + 3n + 1)x^n$

Subtracting $(1 - x)\, S_n = 9x + 13x^2 + 17x^3 + \ldots$ up to $(n - 1)$ terms $+ [6 - (2n^2 + 3n + 1)x^n]$

Let us find the sum of $(n - 1)$ terms of

$$S = 9x + 13x^2 + 17x^3 + \ldots + (4n + 5)x^{n-1}$$

$$xS = 0 + 9x^2 + 13x^3 + 17x^4 + \ldots + (4n + 5)x^n$$

$$S - xS = 4[x^2 + x^3 + \ldots (n - 2) \text{ terms}] + [9x - (4n + 5)x^n]$$

$$S(1 - x) = 9x - (4n + 5)x^n + 4x^2\left(\frac{1-x^{n-2}}{1-x}\right)$$

$$S = \frac{9x}{1-x} - \frac{(4n+5)x^n}{1-x} + 4x^2\left(\frac{1-x^{n-2}}{(1-x)^2}\right)$$

$$S_n = \frac{9x}{(1-x)^2} - \frac{(4n+5)x^n}{(1-x)^2} + 4x^2\frac{(1-x^{n-2})}{(1-x)^3} + \frac{[6-(2n^2+3n+1)x^n]}{1-x}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If fourth term of a G.P. is 3, the product of the first seven terms is

(A) 3^4 (B) 3^7

(C) 7^4 (D) 4^7

2. a, b, c, d are in increasing G.P. If the A.M. between a and b is 6 and the A.M. between c and d is 54., then the A.M. of a and d is

(A) 15 (B) 48

(C) 44 (D) 42

3. Let $a_1, a_2, a_3, \dots, a_{4001}$ are in A.P. such that $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\dots+\frac{1}{a_{4000}a_{4001}}=10$ and $a_2+a_{4000}=50$ then $|a_1-a_{4001}|$ is equal to

(A) 20 (B) 30

(C) 40 (D) none of these

4. Let $t_1=7, t_2=7^7, t_3=7^{7^7}$ and so on. The digit at the tens place of number t_{1000} is

(A) 8 (B) 0

(C) 6 (D) 4

5. If $\frac{p+q.5^x}{p-q.5^x}=\frac{q+r.5^x}{q-r.5^x}=\frac{r+s.5^x}{r-s.5^x}$ then p, q, r, s are in

(A) A.P. (B) G.P.

(C) H.P. (D) none of these

6. Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be arithmetic progression such that $a_1=25, b_1=75$ and $a_{100}+b_{100}=100$, then the sum of first hundred term of the progression $a_1+b_1, a_2+b_2, \dots$ is equal to

(A) 1000 (B) 100000

(C) 10000 (D) 24000

7. The value of $\sum_{n=1}^{2008}\frac{\sqrt{n^4+2n^3+3n^2+2n+1}}{n(n+1)}$

(A) $2008+\frac{2008}{2009}$ (B) $2007+\frac{2008}{2009}$

(C) $2008+\frac{2007}{2009}$ (D) $2008+\frac{2007}{2008}$

8. Consider the quadratic equation $ax^2-bx+c=0,\ a,b,c\in N$. If the given equation has two distinct real roots belonging to (1, 2) then

(A) $1<a<5$ (B) $a\geq 5$

(C) $a=4$ (D) $a=3$

9. The largest term of the sequence $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692} \dots$ is

(A) $\frac{16}{692}$ (B) $\frac{4}{524}$

(C) $\frac{49}{1529}$ (D) $\frac{1}{35}$

10. A sequence of numbers A_n, n = 1, 2, 3, … is defined as follows: $A_1=\frac{1}{2}$ and for each $n\geq 2, A_n=\left(\frac{2n-3}{2n}\right)A_{n-1}$ then

(A) $\sum_{k=1}^{n}A_k=1$ for all $n\geq 1$ (B) $\sum_{k=1}^{n}A_k>1$ for all $n\geq 1$

(C) $\sum_{k=1}^{n}A_k<1$ for all $n\geq 1$ (D) None of these

11. If $S_n=\left[\frac{1}{1.2.3}+\frac{1}{2.3.4}+\dots+\frac{1}{n(n+1)(n+2)}\right]$. value of $8S_\infty$.

(A) 2 (B) 4

(C) 6 (D) 8

12. The sequence $\{a_n\}$ is defined by $x_{k+1}=x_k^2+x_k$ and $x_1=\frac{1}{2}$. Then $\left[\frac{1}{x_1+1}+\frac{1}{x_2+1}+\dots+\frac{1}{x_{100}+1}\right]$ (where [.] denotes the greatest integer function) equals to

(A) 0 (B) 2

(C) 4 (D) None of these

13. If the sum to infinity of the series, $1+4x+7x^2+10x^3+\cdots$, is $\frac{35}{16}$, where $|x|<1$, then 'x' equals to

(A) 19/7 (B) 1/5

(C) 1/4 (D) None of these

14. If a, b, c and d are four positive real numbers such that $abcd=1$, the minimum value of $(1+a)$ $(1+b)$ $(1+c)$ $(1+d)$ is

(A) 4 (B) 1

(C) 16 (D) 18

15. If $t_r=\frac{1^2+2^2+3^2+\dots+r^2}{1^3+2^3+3^3+\dots+r^3}$, $S_n=\sum_{r=1}^{n}(-1)^r t_r$, then $\lim_{n\to\infty}S_n$ equals

(A) $-\frac{2}{9}$ (B) $-\frac{1}{3}$

(C) $\frac{2}{3}$ (D) none of these

16. A set of consecutive positive integers starting with 1 is written on a blackboard. One number is erased, and thus the A.M. of the remaining number is found to be $35\frac{7}{17}$. The erased number is

(A) 6 (B) 7
(C) 8 (D) 9

17. Let a_1, a_2, a_3, a_4, a_5 be five terms geometric sequence satisfies the condition $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 100$, where each term is an integer. Then the number of five terms in geometric progression are
(A) 6 (B) 7
(C) 8 (D) 10

18. If $S_n = \frac{1.2}{3!} + \frac{2.2^2}{4!} + \frac{3.2^3}{5!} + \ldots +$ up to n terms, then sum of infinite terms is
(A) $\frac{4}{\pi}$ (B) $\frac{3}{e}$
(C) $\frac{\pi}{3}$ (D) 1

19. The sum of first n terms $(n > 1)$ of an A.P. is 153 and the common difference is 2. If the first term is an integer, the number of possible values of n is
(A) 3 (B) 5
(C) 4 (D) 6

20. If a, b, c be three unequal positive quantities in A.P., then
(A) $a^{100} + c^{100} > 2b^{100}$ (B) $a^3 + c^3 > 2b^3$
(C) $a^5 + c^5 > 2b^5$ (D) All of the above

21. If the length of sides of a right triangle are in A.P., then the sines of the acute angle are
(A) $\frac{3}{5}, \frac{4}{5}$ (B) $\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}$
(C) $\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{\sqrt{5}+1}{2}}$ (D) $\sqrt{\frac{\sqrt{3}-1}{2}}, \sqrt{\frac{\sqrt{3}+1}{2}}$

22. If x, y, z are distinct positive numbers, then $x^{\ln y - \ln z} + y^{\ln z - \ln x} + z^{\ln x - \ln y} \in$
(A) $(0, \infty)$ (B) $(1, \infty)$
(C) $(3, \infty)$ (D) $(1, 3)$

23. Let $f(n), g(n)$ denotes the sum of n terms of the sequences 2, 5, 10, 17, 26, … and 2, 6, 12, 20, … respectively then the value $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)}$ is
(A) 0 (B) 1
(C) 3 (D) does not exist

24. If $a_1, a_2, \ldots. a_n$ n distinct odd natural numbers not divisible by any prime greater than 5, then $\frac{1}{a_1} + \frac{1}{a_2} + \ldots. \frac{1}{a_n}$ is less than
(A) $\frac{15}{8}$ (B) $\frac{16}{8}$
(C) $\frac{8}{15}$ (D) $\frac{15}{4}$

25. Consider the sequence of numbers $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_n$, where $\alpha_0 = 17.23$, $\alpha_1 = 33.23$ and $\alpha_{r+2} = \frac{\alpha_r + \alpha_{r+1}}{2}\ \forall r \geq 0$ the value of $|\alpha_{10} - \alpha_9|$ is equals
(A) $\frac{41}{50}$ (B) $\frac{28}{61}$
(C) $\frac{1}{32}$ (D) $\frac{33}{277}$

26. If $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}$ then the value of $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots$ is
(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{6}$
(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{36}$

27. If a, b, c, d are in G.P., then $\left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right)$ equals to
(A) $a\,b + b\,c + c\,d$ (B) $(a\,b + b\,c + c\,d)^2$
(C) $(a\,b + b\,c + c\,d)^4$ (D) none of these

28. If $a_i > 0$ for $1 \leq i \leq 5$ and $a_1 a_2 \ldots a_5 = 6$, then the least value of $a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5$ is
(A) $6(5!)^{1/5}$ (B) $15(6!)^{1/5}$
(C) $(6!)^{1/5}$ (D) $5(6!)^{1/5}$

29. If a, b and c are each positive, and $a + b + c = 6$ then the minimum value of $\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2$ is
(A) $\frac{75}{2}$ (B) $\frac{75}{4}$
(C) $\frac{65}{4}$ (D) $\frac{65}{2}$

30. If a,b,c are three positive numbers, then the minimum value of the expression $\frac{ab(a+b) + bc(b+c) + ca(c+a)}{bca}$
(A) 3 (B) 4
(C) 6 (D) 1

31. Sum of the series $\frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \cdots$ to ∞ is
(A) $\frac{38}{81}$ (B) $\frac{4}{19}$
(C) $\frac{36}{371}$ (D) $\frac{18}{19}$

32. If sum of n terms of A.P. is $cn(n-1)$ *when* $c \neq 0$.The sum of the squares of these terms is
(A) $c^2 n^2 (n+1)^2$ (B) $\frac{2}{3} c^2 n(n-1)(2n-1)$
(C) $\frac{2c^2}{3} n(n+1)(2n+1)$ (D) $\frac{2c^2(n-1)(n-2)(2n-3)}{3}$

33. If $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$ where a, b, c, d, e are non-negative real numbers and range of e is $\left[\ell, \frac{m}{n}\right]$ (where G.C.D of m, $n = 1$) then $\ell + m + n$ is equal to

(A) 21 (B) 26
(C) 20 (D) 31

34. If A, G, H be respectively the A.M., GM and the HM between two positive numbers if $x\text{A} = y\text{G} = z\text{H}$ where x, y, z are non zero positive quantities then x, y, z are in

(A) A.P (B) G.P
(C) H.P (D) A.G.P

35. If a, b, c are three distinct numbers such that a, b, c are in A.P. and $b - a$, $c - b$, a are in G.P., then $a : b : c$ are in

(A) $2:3:4$ (B) $3:4:5$
(C) $1:3:5$ (D) $1:2:3$

36. The 2008th term of the sequence 1, $\underbrace{2,2,2,}_{3}\ \underbrace{3,3,3,3,3,3,}_{6}\ \underbrace{4,4,4,4,4,4,4,4,4,4,}_{10}\ldots$ where n occurs $\dfrac{n(n+1)}{2}$ times in the sequence, equals

(A) 24 (B) 23
(C) 22 (D) 21

37. If $a_k = \dfrac{1}{k(k+1)}$, for $k = 1, 2, 3 \ldots n$, then $\left(\displaystyle\sum_{k=1}^{n} a_k\right)^2 =$

(A) $\dfrac{n}{n+1}$ (B) $\dfrac{n^2}{(n+1)^2}$
(C) $\dfrac{n^4}{(n+1)^4}$ (D) $\dfrac{n^6}{(n+1)^6}$

38. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}\left(\sum_{n=1}^{k} 2^{n-1}\right) =$

(A) e (B) $e^2 + e$
(C) e^2 (D) $e^2 - e$

39. If $H_1, H_2, \ldots, H_{20}$ be 20 harmonic means between 2 and 3, then $\dfrac{H_1+2}{H_1-2} + \dfrac{H_{20}+3}{H_{20}-3} =$

(A) 20 (B) 21
(C) 40 (D) 38

40. If $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$ to $\infty = \dfrac{\pi^2}{6}$, then $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots =$

(A) $\dfrac{\pi^2}{8}$ (B) $\dfrac{\pi^2}{12}$
(C) $\dfrac{\pi^2}{3}$ (D) $\dfrac{\pi^2}{2}$

41. If 1, $\log_{81}(3^x + 48)$, $\log_9\left(3^x - \dfrac{8}{3}\right)$ are in A.P., then the value of x equals

(A) 9 (B) 6
(C) 2 (D) 4

42. If a, b, c are in H.P., then $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in

(A) A.P. (B) H.P.
(C) G.P. (D) None of these

43. In a sequence of $(4n + 1)$ terms, the first $(2n + 1)$ terms are in A.P., whose common difference is 2, and the last $(2n + 1)$ ratio is 0.5 if the middle terms of the A.P and G.P are equal then the middle term of the sequence is

(A) $n2^{n-1}/(2^n - 1)$ (B) $n.2^{n+1}/(2^{2n} - 1)$
(C) $n.2^n$ (D) $n2^{n+1}/(2^n - 1)$

44. The sum of the series $\dfrac{x}{1-x^2} + \dfrac{x^2}{1-x^4} + \dfrac{x^4}{1-x^8} + \cdots$ to infinite terms, if $|x| < 1$ is

(A) $\dfrac{x}{1-x}$ (B) $\dfrac{1}{1-x}$
(C) $\dfrac{1+x}{1-x}$ (D) 1

45. If n arithmetic means are inserted between two sets of numbers a, $2b$ and $2a$, b, where $a, b \in R$. Suppose that m^{th} arithmetic mean between these two sets of numbers is same, then the ratio $a{:}b$ equals

(A) $n - m + 1 : m$ (B) $n - m + 1 : n$
(C) $m : n - m + 1$ (D) $n : n - m + 1$

46. The ratio of sum of first three terms of a G.P. to the sum of first six terms is $64:91$,the common ratio of G.P. is

(A) $\dfrac{1}{4}$ (B) $\dfrac{3}{4}$
(C) $\dfrac{5}{4}$ (D) $\dfrac{7}{4}$

47. Sum of the series $3 + 5 + 9 + 17 + 33 + \cdots$ to n terms is

(A) $2^{n+1} - n - 2$ (B) $2^{n+1} + n - 2$
(C) $2^n + n - 2$ (D) $2^{n+1} - n + 2$

48. If one A.M. A and two G.M.s p and q be inserted between two numbers a and b, then which of the following is hold good

(A) $a^3 + b^3 = 2Apq$ (B) $p^3 + q^3 = 2Apq$
(C) $a^3 + b^3 = 2Aab$ (D) None of these.

49. The number of divisors of 6912, 52480, 32000 are in

(A) A.P Only (B) G.P. Only
(C) A.P. , G.P. and H.P. (D) None of these

50. If a,b,c, are in A.P. and p, p' are respectively A.M. and G.M. between a and b while q, q' are respectively A.M.. And G.M. between b and c, then

(A) $p^2 + q^2 = p'^2 + q'^2$ (B) $pq = p'q'$
(C) $p^2 - q^2 = p'^2 - q'^2$ (D) $p^2 + p'^2 = q^2 + q'^2$

51. Through the centroid of an equilateral triangle a line parallel to the base is drawn. On this line, an arbitrary point p is taken inside the triangle. Let h denote the distance of p from the base of the triangle. Let h_1 and h_2 be the distance of p from the other two sides of the triangle, then

(A) h is the H.M. of h_1, h_2 (B) h is the G.M. of h_1, h_2
(C) h is the A.M, of h_1, h_2 (D) None of these

52. a, b, c are positive integers forming an increasing G.P. whose common ratio is a natural number, $b - a$ is cube of a natural number and $\log_6 a + \log_6 b + \log_6 c = 6$, then $a + b + c =$

(A) 100 (B) 111
(C) 122 (D) 189

53. Let a_n be the n^{th} term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is

(A) $\alpha - \beta$ (B) $\beta - \alpha$
(C) $\frac{\alpha - \beta}{2}$ (D) None of these

54. If the n^{th} term of the sequence defined by $t_n = \frac{\sqrt{n}}{n + 2008}$ is the greatest term then $n =$

(A) 2006 (B) 2007
(C) 2008 (D) 2009

55. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their G.M., then $a : b$ is

(A) $6 + \sqrt{7} : 6 - \sqrt{7}$ (B) $2 + \sqrt{3} : 2 - \sqrt{3}$
(C) $5 + \sqrt{6} : 5 - \sqrt{6}$ (D) None of these

56. Sum of first n terms of a sequence is given by $3\,S_n = T_n^{\,2} + 3T_n - 2, \left(T_n > 0\right)$ where T_n is the nth term of sequence, then the value of T_2^2 is

(A) $2 - \sqrt{2}$ (B) $2 + \sqrt{2}$
(C) $2 + 3\sqrt{2}$ (D) $3 + 2\sqrt{2}$

57. Sum of n terms of series $ab + (a+1)(b+1) + (a+2)(b+2) + \ldots + (a + (n-1))(b + (n-1))$ if $ab = \frac{1}{6}$ and $(a + b) = \frac{1}{3}$, is

(A) $\frac{n}{6}(1 - 2n)^2$ (B) $\frac{n}{6}(1 + n - 2n^2)$
(C) $\frac{n}{6}(1 - 2n + 2n^2)$ (D) none of these

58. If log $(a + c)$, $\log(a + b)$, $\log(b + c)$ are in A.P. and a, c, b are in H.P, then the value of $a + b$ is (given $a, b, c > 0$)

(A) $2c$ (B) $3c$
(C) $4c$ (D) $6c$

59. The positive integral values of n such that $1.2^1 + 2.2^2 + 3.2^3 + 4.2^4 + 5.2^5 + \ldots + n.2^n = 2^{(n+10)} + 2$ is

(A) 313 (B) 513
(C) 413 (D) 613

60. The arithmetic mean of the nine numbers in the given set {9, 99, 999, … 999999999} is a 9 digit number N, all whose digits are distinct. The number N does not contain the digit

(A) 0 (B) 2
(C) 5 (D) 9

61. The minimum value of the expression $\frac{9x^2 \sin^2 x + 4}{x \sin x}$ for $x \in (0, \pi)$ is

(A) $\frac{16}{3}$ (B) 6
(C) 12 (D) $\frac{8}{3}$

62. There is a certain sequence of positive real numbers. Beginning from the third term, each term of the sequence is the sum of all the previous terms. The seventh term is equal to 1000 and the first term is equal to 1. The second term of this sequence is equal to

(A) 246 (B) $\frac{123}{2}$
(C) $\frac{123}{4}$ (D) 124

63. The sum of the series $(1^2 + 1).1! + (2^2 + 1).2! + (3^2 + 1).\,3! + \ldots + (n^2 + 1).\,n!$ is :

(A) $(n + 1).\,(n + 2)!$ (B) $n.(n + 1)!$
(C) $(n + 1).\,(n + 1)!$ (D) None of these

64. Find the sum of the infinite series $\frac{1}{9} + \frac{1}{18} + \frac{1}{30} + \frac{1}{45} + \frac{1}{63} + \cdots$

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$
(C) $\frac{1}{5}$ (D) $\frac{2}{3}$

65. The 20th term of 2, 9, 20, 35, 54, … is

(A) 819 (B) 820
(C) 1009 (D) 1010

66. If $x > 1, y > 1, z > 1$ and x, y, z are in G.P. then $(\ln x^2)^{-1}$, $(\ln xy)^{-1}$, $(\ln xz)^{-1}$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) none of these

67. If $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ and $n > 2$, then S_n would always be

(A) more than $n(n+1)^{\frac{1}{n}} - n$
(B) less than $n(n + 1)^{1/n} - n$
(C) equal to $n(n+1)^{\frac{1}{n}} - n$
(D) greater than or equal to $\frac{n(n+1)^{\frac{1}{n}}}{(n+5)}$

68. If b is the arithmetic mean between a and x; b is the geometric mean between 'a' and y; 'b' is the harmonic mean between a and z, $(a, b, x, y, z > 0)$ then the value of xyz is

(A) a^3 (B) b^3
(C) $\frac{b^3(2a - b)}{2b - a}$ (D) $\frac{b^3(2b - a)}{2a - b}$

69. The first term of an infinite geometric series is 2 and its sum be denoted by S. If $|S - 2| < \frac{1}{10}$ then the true set of the range of common ratio of the series is

(A) $\left(\frac{1}{10}, \frac{1}{5}\right)$ (B) $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$
(C) $\left(-\frac{1}{19}, \frac{1}{20}\right) - \{0\}$ (D) $\left(-\frac{1}{19}, \frac{1}{21}\right) - \{0\}$

70. The sum of first 'n' terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$ is

(A) 2^{n-1} (B) $1-2^{-n}$

(C) $2^{-n}-n+1$ (D) $2^{-n}+n-1$

71. The sum to n terms of the series

$$\frac{1}{\lfloor\underline{1}}\left(\frac{1}{2}\right)+\frac{1.3}{\lfloor\underline{2}}\left(\frac{1}{2}\right)^2+\frac{1.3.5}{\lfloor\underline{3}}\left(\frac{1}{2}\right)^3+\cdots \text{ upto } n \text{ terms is}$$

(A) $\frac{1.3.5\ldots(2n-1)(2n+1)}{2^n\lfloor\underline{n}}-1$

(B) $1-\frac{1.3.5\ldots(2n-1)}{\lfloor\underline{n}\ \lfloor\underline{n}}$

(C) $1-\frac{1.3.5\ldots(2n-3)}{2^{n-1}\lfloor\underline{n-1}}$

(D) $\frac{1.3.5\ldots(2n-3)}{2^{n-1}\lfloor\underline{n-1}}$

72. The sequence $\{x_1, x_2, \ldots, x_{50}\}$ has the property that for each k, x_k is k less than the sum of other 49 numbers. The value of $96x_{20}$ is

(A) 300 (B) 315

(C) 1024 (D) 0

73. If the first and $(2n-1)^{th}$ terms of an A.P; a G.P. and H.P. are equal and their n^{th} terms are p, q and s respectively, then which of the following options is/are correct?

(A) $p \ge q \ge s$ (B) $p+s=q$

(C) $ps=q^2$ (D) $p=q=s$

74. The fourth and fifth term of a sequence $\{t_n\}_{n\ge1}$ are 4 and 5 respectively and the n^{th} term is given as $t_n = 2t_{n-1}-t_{n-2}$, $n \ge 3$ $(n \in N)$. Then the sum to first 2009 terms is

(A) 2019045 (B) 2013021

(C) 2017036 (D) 2018040

75. If a, b, c are in A.P. b, c, d are in G.P. and c, d, e are in H.P. then a, c, e are in

(A) A.P. (B) G.P.

(C) H.P. (D) None

76. If a_1 is the greatest value of $f(x)$; where $f(x)=\left(\frac{1}{2+[\sin x]}\right)$ (where [.] denotes greatest integer function) and $a_{n+1}=\frac{(-1)^{n+2}}{(n+1)}+a_n$, then $\lim\limits_{n\to\infty}(a_n)$ is

(A) 1 (B) e^2

(C) ln2 (D) ln3

77. The sum of the series $\frac{9}{5^2.2.1}+\frac{13}{5^3.3.2}+\frac{17}{5^4.4.3}+\cdots$ upto infinity

(A) 1 (B) $\frac{9}{5}$

(C) $\frac{1}{5}$ (D) $\frac{2}{5}$

78. If a, b, c, d are distinct integers in A.P. such that $d=a^2+b^2+c^2$ then $a+b+c+d$ is

(A) 0 (B) 1

(C) 2 (D) None

79. $\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\ldots\ldots=$

(A) $\frac{1}{4}$ (B) $\log_e\left(\frac{3}{4}\right)$

(C) $\log_e\left(\frac{3}{2}\right)$ (D) $\log_e\left(\frac{2}{3}\right)$

80. If α, β, γ are positive real roots of the equation $x^3 - mx^2 + 12x - 4n = 0$ $(m, n \in R)$ such that $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)=8\alpha\beta\gamma$, then the value of $(m+n)$ is equal to

(A) 12 (B) 16

(C) 0 (D) 8

81. Let α and β are the roots of $x^2-10x+2k^2+2k=0$ and α, γ are the roots of $x^2-(3k+2)x+k^3+3=0$ (where $k \in I$). If α, γ and β are in A.P., then the value of $\alpha+\beta^2+\gamma^3$ is

(A) 135 (B) 147

(C) 165 (D) 178

82. If the roots of the cubic, $x^3+ax^2+bx-\frac{1}{9}=0$ are in A.P. then the value of $(2a^3-9ab+1)$ is equal to

(A) 1 (B) 2

(C) 3 (D) 4

83. If α, β, γ are the 3 positive roots of cubic, $x^3 - ax^2 + bx - 16\sqrt{6}=0$, then the minimum value of $\left(\frac{4}{\alpha\beta}+\frac{3}{\beta\gamma}+\frac{2}{\gamma\alpha}\right)$ is

(A) 1 (B) $\frac{3}{2}$

(C) $\frac{3}{4}$ (D) $\frac{2}{3}$

84. If a, b, c (in that order) are non-zero real numbers in arithmetic progression, then the equation $3ax^2-4bx+c=0$ has

(A) two distinct roots (B) no real solution

(C) rational roots (D) equal real roots

85. a, b, c are all different and non-zero real numbers which are in arithmetic progression. If the roots of the quadratic equation $ax^2+bx+c=0$ are α and β such that $\frac{1}{\alpha}+\frac{1}{\beta}, \alpha+\beta, \alpha^2+\beta^2$ are in geometric progression, then the value of $\frac{a}{c}$ is

(A) 1 (B) 2

(C) 3 (D) 4

86. If 'a' is least of two prime roots of equation $x^2 - tx + n = 0$; $n \in N$ and t is odd and a, b, c are 1st, r^{th} and $(r^2)^{\text{th}}$ term of an A.P. with common difference 2, then roots of quadratic equation $ax^2+cx+b=0$ are always

(A) both negative

(B) both imaginary

(C) both positive

(D) one positive and one negative

87. If p, q, r be in H.P and p and r be different having same sign, then the root of the equation $px^2 + 2qx + r = 0$ will be –

(A) real (B) equal

(C) Imaginary (D) None of these

88. Consider a decreasing G.P. : $g_1, g_2, g_3, \ldots, g_n, \ldots$ such that $g_1 + g_2 + g_3 = 13$ and $g_1^2 + g_2^2 + g_3^2 = 91$ then which of the following does not hold?

(A) The greatest term of the G.P. is 9

(B) $3g_4 = g_3$

(C) $g_1 = 1$

(D) $g_2 = 3$

89. The sum of the infinite series

$$1 + \left(1+\frac{1}{5}\right)\left(\frac{1}{2}\right) + \left(1+\frac{1}{5}+\frac{1}{5^2}\right)\left(\frac{1}{2^2}\right) + \ldots \text{ is}$$

(A) $\frac{20}{9}$ (B) $\frac{10}{9}$

(C) $\frac{5}{9}$ (D) $\frac{5}{3}$

90. The value of x satisfying the equation $(2011)^{\log_{10} x} = 11^{\log_{10}(1+3+5+\ldots+4021)}$ lies in the interval

(A) (95, 100) (B) (100, 105)

(C) (110, 115) (D) (120, 125)

91. If 'b' be the p^{th} term of a G.P., whose $(p + q)^{\text{th}}$ and $(p - q)^{\text{th}}$ terms are $(a > 0)$ and c respectively and also $f(x) = ax^2 - 2bx + c$, then for all $x \in R$

(A) $af(x) \geq 0$ (B) $af(x) \leq 0$

(C) $af(x) > 0$ (D) $af(x) < 0$

92. In an arithmetic progression, if $S_n = n(5 + 3n)$ and $t_n = 32$, then the value of n is

[**Note:** S_n and t_n denote the sum of first n terms and n^{th} term of arithmetic progression respectively.]

(A) 4 (B) 5

(C) 6 (D) 7

93. Let a_1, a_2, a_3 ….. be an arithmetic progression with common difference 2. Let S_n be the sum of first n terms of the sequence. If $\frac{S_{3n}}{S_n}$ does not depend on n then the sum of the first 10 terms of the sequence is

(A) 100 (B) 125

(C) 150 (D) 225

94. The sum of a certain infinite geometric series is 20. When all the terms in the series are squared, the sum of the resulting series is 80. If the first term of the original series is expressed in lowest terms as $\frac{p}{q}$, $(p, q \in N)$ then the value of $(p + q)$ is

(A) 21 (B) 22

(C) 23 (D) 24

95. If the sum of first n terms of an arithmetic progression is denoted by S_n and $S_n = 6n\sec^2\theta + n(n - 1)(\sin^2\theta\,(4 + \tan^2\theta) + \cos^2\theta\,(4 + \cot^2\theta))$, where $\theta \in \left(0, \frac{\pi}{2}\right)$. Then the minimum value of common difference of the arithmetic progression is

(A) 10 (B) 12

(C) 14 (D) 16

96. the value of $\sum_{n=1}^{\infty} \frac{5^n - 2^n}{10^n}$ is equal to

(A) 1 (B) $\frac{1}{4}$

(C) $\frac{1}{3}$ (D) $\frac{3}{4}$

97. If the roots of the equation $72x^3 - 108x^2 + 46x - 5 = 0$ are in arithmetic progression then the difference between largest and smallest root lies in the interval

(A) $\left(0, \frac{1}{2}\right)$ (B) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

(C) $\left(\frac{1}{\sqrt{2}}, 1\right)$ (D) (1, 2)

98. If $\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{8x^4} + \frac{1}{16x^5} + \ldots. \infty = \frac{1}{64}$, then the value of x is

(A) $\frac{125}{2}$ (B) $\frac{127}{2}$

(C) $\frac{129}{2}$ (D) $\frac{131}{2}$

99. The first term of an A.P. is 5, the last is 45 and their sum is 400. If the number of terms is n and d is the common difference, then $\left(\frac{n}{d}\right)$ is equal to

(A) 9 (B) 8

(C) 7 (D) 6

100. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ $\forall\, n \geq 3$. If a_2 and a_5 are +ve integers and $a_5 \leq 512$, then number of possible value of a_5 are

(A) 1 (B) 2

(C) 3 (D) 4

101. If $S = 1.2(2^2 - 1^2) + 2.3(3^2 - 2^2) + 3.4(4^2 - 3^2) + \ldots$ upto 50 terms, then S is

(A) $50(52)^3$ (B) $50 \times (51)^2 \times 26$

(C) $49(50)^3$ (D) $50 \times (51)^2 \times 13$

102. Let g_n be the n^{th} term of a G.P. If $\prod_{r=1}^{100} g_{2r} = \alpha$ and $\prod_{r=1}^{100} g_{2r-1} = \beta$, then the common ratio of G.P. is

(A) $\left(\frac{\beta}{\alpha}\right)^{\frac{1}{100}}$ (B) $\left(\frac{\alpha}{\beta}\right)^{\frac{1}{100}}$

(C) $\left(\frac{\beta}{\alpha}\right)^{\frac{1}{200}}$ (D) $\frac{\beta}{\alpha}$

103. If $(1 + x)(1 + x^3)(1 + x^5) \ldots. (1 + x^{21})$ is a polynomial of 'n' degree, then value of 'n' is

(A) 121 (B) 231
(C) 85 (D) 21

104. The sum

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\dots+\frac{2008}{2006!+2007!+2008!}$$

is equal to

(A) $\frac{1}{2}-\frac{1}{2006!}$ (B) $\frac{1}{2}-\frac{1}{2008!}$

(C) $\frac{1}{2006!-2008!}$ (D) $\frac{1}{2007!}-\frac{1}{2008!}$

105. Let $a_0, a_1, a_2, \cdots, a_n \dots$ be a sequence of numbers satisfying $(3-a_{n+1}).(6+a_n)=18$ and $a_0=3$ then $\sum_{i=0}^{n}\frac{1}{a_i}=$

(A) $\frac{1}{3}\left(2^{n+2}+n-3\right)$ (B) $\frac{1}{3}\left(2^{n+2}-n+3\right)$

(C) $\frac{1}{3}\left(2^{n+2}-n-3\right)$ (D) $\frac{1}{3}\left(2^{n+2}+n+3\right)$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $S_1, S_2, S_3 \dots$ be squares such that for each $n > 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm then for which of the following values of 'n' is the area of S_n less than 1 cm^2?

(A) 7 (B) 8
(C) 9 (D) 10

2. Let $f_1, f_2, f_3 \dots$ be a sequence of integers satisfying $f_{n-1}+f_n=2n$ $\forall n \geq 2$ if $f_1=100$ then

(A) $f_{100}=2$ (B) $f_{200}=102$
(C) $f_{99}=198$ (D) $f_9=100$

3. If the sum of the first 100 terms of an AP is –1 and the sum of even terms lying in first 100 terms is 1. Then

(A) common difference of the sequence is $\frac{3}{50}$

(B) first term of the sequence is $\frac{-149}{50}$

(C) 100th term $=\frac{74}{25}$

(D) the sum of squares of the first hundred terms is $\frac{14999}{50}$

4. If the roots of the equation $x^3+bx^2+cx-1=0$ form an increasing G.P., then

(A) $b+c=0$
(B) $b \in (-\infty, -3)$
(C) one of the roots is 1
(D) one root is smaller than 1 and one root is more than 1.

5. Given that $x+y+z=15$ when a, x, y, z, b are in A. P. and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{3}$ when a, x, y, z, b are in H.P. Then

(A) G.M. of a and b is 3
(B) one possible value of $(a+2b)$ is 11
(C) A.M. of a and b is 5
(D) H.M. of a and b is $\frac{9}{5}$

6. If $a_1, a_2, a_3 \dots\dots a_n$ is sequence of positive numbers which are in A.P. with common difference 'd' and $a_1+a_4+a_7+\dots a_{16}=147$ then

(A) $a_1+a_6+a_{11}+a_{16}=98$

(B) $a_1+a_{16}=49$

(C) $a_1+a_4+a_7+\dots a_{16}=6a_1+45\,d$

(D) Maximum value of $a_1a_2\dots a_{16}$ is $\left(\frac{49}{2}\right)^{16}$

7. If a, b, c, d are distinct positive numbers in H.P., then

(A) $\sqrt{\frac{a+d}{b+c}}+\sqrt{\frac{b+c}{a+d}}=\sqrt{\frac{ad}{bc}}+\sqrt{\frac{bc}{ad}}$

(B) $abc+bcd=abd+acd$

(C) $\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{d}$

(D) $\left(\frac{1}{a}+1\right)\left(\frac{1}{d}+1\right)<\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right)$

8. Let $9+99+999+\dots$ upto n terms $=a$, then the correct option(s) is/are –

(A) The value of $7+77+777+\dots$ upto n terms $=\frac{7a}{9}$

(B) The value of $7+77+777+\dots$ upto n terms $=\frac{9a}{9}$

(C) The value of $2^2+22^2+222^2+\dots$ upto n terms $=\frac{4}{81}\left[\frac{100(10^{2n}-1)}{99}-(n+2a)\right]$

(D) The value of $2^2+22^2+222^2+\dots$ upto n terms $=\frac{4}{9}\left[\frac{100(10^{2n}-1)}{99}-(n+2a)\right]$

9. A sequence is defined as $a_1=a_2=1$ and $a_n=a_{n-1}-a_{n-2}\,\forall\, n \geq 3$ then

(A) $\frac{1}{a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_n a_{n+1}}$

(B) $\sum_{n=2}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = 1$

(C) $\sum_{n=2}^{\infty} \frac{a_n}{a_{n-1}a_{n+1}} = 2$

(D) $\{a_n\}$ is an increasing sequence

10. If positive distinct quantities a,b,c are in H.P, then

(A) $b > \frac{a+c}{2}$

(B) $\frac{1}{a-b} - \frac{1}{b-c} < 0$

(C) $ac > b^2$

(D) $bc(1-a).\ ac(1-b),\ ab\ (1-c)$ are in A.P.

11. If $a, b, c \in R^+$, among the following which is/are always true

(A) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{3\sqrt[3]{abc}} \geq 4$

(B) $\frac{(a+b)(b+c)(c+a)}{abc} \geq 8$

(C) $a^2+b^2+c^2 \leq ab+bc+ca$

(D) $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c$

12. a,b,c are three positive numbers and abc^2 has greatest value $\frac{1}{64}$, then

(A) $a=b=\frac{1}{2}, c=\frac{1}{4}$
(B) $a=b=\frac{1}{4}, c=\frac{1}{2}$

(C) $a=b=c=\frac{1}{3}$
(D) $a+b+c=1$

13. If for $a, b > 0$, $a.\ 2^x + b.\ 3^y + 1 - \sqrt{4^x+9^y+1}\sqrt{a^2+b^2+1} \geq 0$ holds true for some real x and y, then

(A) A.M. of a,b is $\frac{2^x+3^y}{2}$
(B) $x = 9^{2009}$, $y = (2009)^9$

(C) A.M. of a,b is $\frac{4^x+9^y}{2}$
(D) $x\log_a 2 + y\log_b 3 = 2$

14. Let $S_n\,(n \geq 1)$ be a sequence of sets defined by $S_1 = \{0\}$,

$S_2 = \left\{\frac{3}{2}, \frac{5}{2}\right\}, S_3 = \left\{\frac{8}{3}, \frac{11}{3}, \frac{14}{3}\right\}, S_4 = \left\{\frac{15}{4}, \frac{19}{4}, \frac{23}{4}, \frac{27}{4}\right\}, \ldots$

then

(A) third element in S_{20} *is* $\frac{439}{20}$

(B) third element in S_{20} *is* $\frac{431}{20}$

(C) sum of elements in S_{20} is 589

(D) sum of elements in S_{20} is 609

15. If a, b and c are three terms of an A.P. such that $a \neq b$, then $\frac{b-c}{a-b}$ may be equal to

(A) $\sqrt{2}$
(B) $\sqrt{3}$

(C) 1
(D) 3

16. $(1^2 - a_1) + (2^2 - a_2) + (3^2 - a_3) + \cdots (n^2 - a_n) = \frac{n}{3}(n^2-1)$ then

(A) $a_n = n$
(B) $a_1, a_2, a_3 \ldots.$ are in A.P.

(C) $a_1, a_2, a_3 \ldots.$are in G.P.
(D) $\sum_{i=1}^{n} a_n = \frac{n(n+1)}{2}$

17. If the first and $(2n-1)^{\text{th}}$ terms of an A.P; a G.P. and H.P. are equal and their n^{th} terms are p, q and s respectively, then which of the following options is/are correct?

(A) $p \geq q \geq s$
(B) $p + s = q$

(C) $ps = q^2$
(D) $p = q = \mathrm{s}$

18. The sequence $\{a_n\}; n \geq 1$ satisfy $a_1 = 0$, $a_{n+1} = 5a_n + \sqrt{24a_n^2+1}$. Then

(A) a_n is rational for all n

(B) a_n is rational for $n \leq 24$

(C) $a_{n+1} = 10a_n - a_{n-1}$

(D) a_n is +ve integer for all $n \geq 2$

19. Define $S_n = \frac{1}{\underline{|3}} + \frac{5}{\underline{|4}} + \frac{11}{\underline{|5}} + \ldots.. + \frac{n^2+n-1}{\underline{|n+2}}$ then

(A) $t_{100} = \frac{10099}{\underline{|102}}$
(B) $S_{2009} = \frac{1}{2} - \frac{1}{\underline{|2009}}.\frac{1}{2011}$

(C) $S_{2009} = \frac{1}{4} - \frac{1}{\underline{|2009}}.\frac{1}{2011}$
(D) $S_{\infty} = \frac{1}{2}$

20. The cubes of the natural numbers are grouped as

$(1^3), (2^3, 3^3), (4^3, 5^3, 6^3) \ldots.$

Let S_n denote the sum of the cubes n^{th} group, then $8.S_n$ is divisible by

(A) n^2
(B) n^3

(C) n^2+1
(D) n^2+3

21. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \frac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$, terms of the sequence being distinct. Given that a_2 and a_5 are positive integers and $a_5 \leq 162$ then the possible value(s) of a_5 can be

(A) 2
(B) 32

(C) 64
(D) 162

22. a, b, c are the first three terms of geometric series. If the H.M. of a and b is 12 and that of 'b' and 'c' is 36 then which of the following hold(s) good ?

(A) Sum of the first term and common ration of the G.P is 11

(B) Sum for the first five terms of the G.P is 948

(C) If the value of the first term and common ration of the given G.P is taken as the first term and common difference of an A.P then its 8th term is 29

(D) The number 648 is one of the term of the G.P

23. Let T_n and t_n be the n^{th} term of two AP's respectively and ratio of sum to 'n' terms of these AP's is $\frac{2n-3}{4n+1}$ $\forall\, n \in N$, then identify the correct option/s

(A) $\frac{T_{10}}{t_{11}} = \frac{7}{17}$ (B) $\frac{T_{10}}{t_{11}} = \frac{17}{7}$

(C) $\frac{t_6}{T_6} = \frac{45}{19}$ (D) $\frac{t_6}{T_6} = \frac{19}{45}$

24. If $a_1, a_2, a_3 \ldots$ are in A.P. with common difference d, $b_k = a_k + a_{k+1} \ldots + a_{k+n-1}$ $(k = 1, 2, 3 \ldots)$

(A) $\sum_{k=1}^{n} b_k = n^2 a_n$ (B) $\sum_{k=1}^{n} b_k = (n+1)^2 a_n$

(C) $b_k = \frac{n}{2}\left[a_n + a_1 + 2d(k-1)\right]$ (D) $\sum_{k=1}^{n} b_k = n(n+1)a_n$

25. If a, b, c, d are four unequal positive number which are in A.P then

(A) $\frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$ (B) $\frac{1}{a} + \frac{1}{d} < \frac{1}{b} + \frac{1}{c}$

(C) $\frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$ (D) $\frac{1}{b} + \frac{1}{c} > \frac{4}{a+d}$

26. If the first and $(2n-1)^{\text{th}}$ terms of an A.P., G.P. and H.P. of positive terms are equal and their n^{th} terms are a, b, c respectively, then

(A) $a = b = c$ (B) $a \geq b \geq c$

(C) $b^2 = ac$ (D) $a + c = 2b$

27. In a G.P., the product of first four terms is 4 and the second term is the reciprocal of the fourth term. The sum of infinite terms of the G.P. can be

(A) -8 (B) $-\frac{8}{3}$

(C) $\frac{8}{3}$ (D) 8

28. If 3 positive real number a, b, c are in A.P with $abc = 4$ then $[b]$ can be equal to (where [.] represents the integral part)

(A) 1 (B) 2

(C) 3 (D) 4

29. If a, b, c are first three terms of a G.P. if the harmonic mean of a and b is 12 and arithmetic mean of b and c is 3, then

(A) no term of this G.P. is square of an integer

(B) arithmetic mean of a, b, c is 3

(C) $b = \pm 6$

(D) common ratio of this G.P. is 2

30. For a positive integer n, let $S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots. + \frac{1}{2^n - 1}$. Then, so G.P. is 3, –6, 12, ….

(A) $S_n \leq n$ (B) $S_n > n$

(C) $S_{2n} \leq n$ (D) $S_{2n} > n$

31. $T_r = \frac{1}{r\sqrt{r+1} + (r+1)\sqrt{r}}$, then (here $r \in N$)

(A) $T_r > T_{r+1}$ (B) $T_r < T_{r+1}$

(C) $\sum_{r=1}^{99} T_r = \frac{9}{10}$ (D) $\sum_{r=1}^{n} T_r < 1$

32. If $S_{(n)} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, $(n \in N)$, then $S_{(1)} + S_{(2)} + \ldots + S_{(n-1)}$ is equal to

(A) $nS_{(n)} - n$ (B) $nS_{(n)} - 1$

(C) $(n-1)_{(n-1)} - n$ (D) $nS_{(n-1)} - n + 1$

33. If $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1 a_2 \ldots a_n = 1$, then $(2 + a_1)(2 + a_2)\ldots.(2 + a_n)$ is greater than

(A) $2^{n/2}$ (B) $2^{3n/2}$

(C) 2^{2n} (D) None of these

34. The p^{th} term T_p of HP is $q(p+q)$ and q^{th} term T_q is $p(p+q)$ when $p > 1, q > 1, (p \neq q)$ then

(A) $T_{p+q} = pq$ (B) $T_{pq} = p + q$

(C) $T_{p+q} > T_{pq}$ (D) $T_{pq} > T_{p+q}$

35. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their geometrical mean then $a : b$ is

(A) $2 + \sqrt{3} : 2 - \sqrt{3}$ (B) $7 + 4\sqrt{3} : 1$

(C) $1 : 7 - 4\sqrt{3}$ (D) $2 : \sqrt{3}$

36. If b_1, b_2, b_3 $(b_1 > 0)$ are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(A) $r > 3$ (B) $1 < r < \frac{3}{2}$

(C) $r < 1$ (D) None of these

37. If a, b, c are in H.P., then

(A) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{c+a-b}$ are in H.P.

(B) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$

(C) $a - \frac{2}{b}, \frac{2}{b}, c - \frac{2}{b}$ are in G.P.

(D) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

38. If in a ΔABC, a, b, c are in A.P. then it is necessary that

(A) $\frac{2}{3} < \frac{b}{c} < 2$ (B) $\frac{1}{3} < \frac{b}{c} < \frac{2}{3}$

(C) $\frac{2}{3} < \frac{b}{a} < 2$ (D) $\frac{1}{3} < \frac{b}{a} < \frac{2}{3}$

39. Let $S_1, S_2, \ldots, S_n$ be the sums of infinite geometric series. Whose 1^{st} terms are $1, 2, 3, \ldots, n$ and common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ respectively. Then

(A) $S_1 + S_2 + \ldots + S_n = \frac{n(n+3)}{2}$

(B) $S_1.S_2 \ldots S_n = \lfloor\underline{n+1}$

(C) $\frac{1}{S_1 S_2} + \frac{1}{S_2 S_3} + \ldots \frac{1}{S_{n-1} S_n} = \frac{n-1}{2(n+1)}$

(D) $S_1^2.S_2^3.S_3^4 \ldots S_n^{n+1} = \frac{1024}{3}$

40. Three numbers in A.P. with common difference 'd' are removed from first n natural numbers and average of remaining number is found to be $\frac{43}{4}$ then ordered pair (n, d) can be

(A) $(19, 5)$ (B) $(19, 2)$

(C) $(23, 5)$ (D) $(19, 8)$

41. For the series $S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \ldots$

(A) 7th term is 16

(B) 7th term is 18

(C) Sum of first 10 terms is $\frac{505}{4}$

(D) Sum of first 10 terms is $\frac{405}{4}$

42. If $\sum_{r=1}^{n} r(r+1) = \frac{(n+a)(n+b)(n+c)}{3}$, where $a < b < c$, then

(A) $2b = c$ (B) $a^3 - 8b^3 + c^3 = 8abc$

(C) c is prime number (D) $(a+b)^2 = 0$

43. The third term of a G.P. is 64. The product of first five terms is

(A) 4^{15} (B) $\frac{2^{33}}{8}$

(C) 2^{15} (D) $\frac{2^{17}}{4}$

44. Let $a_n = \underbrace{(111,\ldots 1)}_{n\ times}$, then

(A) a_{912} is not prime (B) a_{951} is not prime

(C) a_{480} is not prime (D) a_{91} is not prime

45. If a, b, c, d are in G.P. and $a - 2, b - 7, c - 9, d - 5$ are in A.P. then $|a + b + c + d|$ is divisible by

(A) 2 (B) 3

(C) 5 (D) 7

46. If a, b, c are in H.P., then

(A) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ will be in H.P.

(B) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(C) $\frac{1}{a}+\frac{1}{b+c}, \frac{1}{b}+\frac{1}{c+a}, \frac{1}{c}+\frac{1}{a+b}$ are in H.P.

(D) $\frac{\sqrt{bc}}{\sqrt{b}+\sqrt{c}}, \frac{\sqrt{ca}}{\sqrt{c}+\sqrt{a}}, \frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$ are in A.P.

47. If n^{th} term of a sequence is $\frac{n^3+n^2+2}{n(n+1)}$ and S_n is sum of first n terms then identify the correct option/s

(A) $5S_9 = 234$ (B) $9S_5 = 234$

(C) $3S_5 = 50$ (D) $5S_3 = 50$

48. Four A.M's A_1, A_2, A_3, A_4 are inserted between the numbers 5 and $\frac{45}{2}$, then which of the following is a G.P. -

(A) $\left(A_1 - \frac{5}{2}\right), A_2, \left(A_3 + \frac{5}{2}\right)$

(B) $A_2, \left(A_3 + \frac{5}{2}\right), (A_4 + 8)$

(C) $(A_4 - 1), A_2, \left(A_1 - \frac{1}{2}\right)$

(D) $\left(A_1 - \frac{3}{2}\right), \left(A_3 - \frac{3}{2}\right), (A_4 + 9)$

49. Let $a_1, a_2, a_3 \ldots a_n$ are terms of a sequence such that $\sum_{i=1}^{\pi} a_i = \frac{(n+1)[(n+2)(2n+3)-6]}{6}$ and $V_n = \sum_{i=1}^{\pi} \frac{1}{a_i}$, then

(A) a_r, a_{r+1}, a_{r+2} are in A.P., then number of possible values of r is 0

(B) a_r, a_{r+1}, a_{r+2} are in A.P., then number of possible values of r is 2.

(C) $V_s = \frac{23}{48}$

(D) $V_8 = \frac{29}{45}$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

If A, G and H are respectively A.M., G.M., H.M.'s between a and b both being unequal and positive, then $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b} \Rightarrow G^2 = AH$

So, we can say that a, b are roots of the equation $x^2 - 2Ax + G^2 = 0$

Now, quadratic equation $x^2 - Px + Q = 0$ and quadratic equation $a(b-c)\,x^2 + b(c-a)\,x + c(a-b) = 0$ have a root common and satisfy the relation $b = \frac{2ac}{(a+c)}$, where a, b, c are real numbers.

1. The value of $[2P - Q]$ is (where [.] denotes the greatest integer function)

(A) 2 (B) 3

(C) 5 (D) 6

2. The sum of the A.M. and G.M. of two positive number is equal to the difference between the numbers. the numbers are in the ratio

(A) $1:3$ (B) $1:6$

(C) $9:1$ (D) $1:12$

Comprehension 2

Let $a_m (m = 1, 2, \ldots, p)$ be the possible integral values of a for which the graphs of $f(x) = ax^2 + 2bx + b$ and $g(x) = 5x^2 - 3bx - a$ meets at some point for all real values of b.

$$\text{Let } t_r = \prod_{m=1}^{p}(r - a_m) \text{ and } S_n = \prod_{r=1}^{n} t_r,\ n \in N.$$

3. The minimum possible value of a is

(A) $\frac{1}{5}$ (B) $\frac{5}{26}$

(C) $\frac{3}{38}$ (D) $\frac{2}{43}$

4. The sum of values of n for which S_n vanishes is

(A) 8 (B) 9

(C) 10 (D) 15

5. The value of $\sum_{r=5}^{\infty}\frac{1}{t_r}$ is equal to

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$

(C) $\frac{1}{15}$ (D) $\frac{1}{18}$

Comprehension 3

Let $M_n = \{0.a_1a_2a_3 \ldots a_n \,|\, a_i = 0 \text{ or } 1\ \forall\ 1 \le i \le n-1 \text{ and } a_n = 1\}$ be a set of decimal fractions, T_n and S_n be the number of elements and sum of elements is M_n respectively then

6. $T_n =$

(A) 2^{n-1} (B) 2^n

(C) 2^{n+1} (D) $2n-1$

7. $S_n =$

(A) $\frac{2^{n-1}}{18}\left(-\frac{28}{10^n}+1\right)$ (B) $\frac{2^{n-1}}{18}\left(1+\frac{8}{10^n}\right)$

(C) $\frac{2^n}{18}\left(1+\frac{18}{10^n}\right)$ (D) $\frac{2^n}{18}\left(-\frac{18}{10^n}+1\right)$

8. $\lim_{n\to\infty}\frac{S_n}{T_n} =$

(A) $\frac{1}{10}$ (B) 18

(C) $-\frac{1}{10}$ (D) $\frac{1}{18}$

Comprehension 4

If r^{th} term of a series can be written as $a_r = f(r) - f(r-1)$, then $S_n = \sum_{r=1}^{n} a_r = f(n) - f(0)$ and $S_\infty = \lim_{n\to\infty} S_n$, then

9. Value of $\sum_{r=1}^{\infty}\frac{(4r-1)5^r}{r^2+r}$ is

(A) 5 (B) –5

(C) 10 (D) none of these

10. If $3(3!) + 4(4!) + 5(5!) + \ldots + 50$ terms $= a! - b$ then $a - b$ is equal to

(A) 45 (B) 46

(C) 47 (D) 48

Comprehension 5

Let a_1, a_2, a_3 be three numbers in arithmetic progression (A.P.) and g_1, g_2, g_3 be three numbers in geometric progression (G.P.). Also the values of $a_1 + g_1, a_2 + g_2$ and $a_3 + g_3$ are respectively 85, 76 and 84 and. $\sum_{i=1}^{3} a_i = 126$

11. The number of possible values that a_1 takes is

(A) 1 (B) 2

(C) 3 (D) 4

12. The sum of the common difference of the AP and the common ratio of the G.P. can be

(A) 27 (B) –24

(C) 31 (D) –33

13. The number of possible values that $\frac{a_2}{g_2}$ takes is

(A) 1 (B) 2

(C) 3 (D) 4

Comprehension 6

The sum of an infinitely decreasing geometric progression whose first term is a and common ratio r, is equal to least value of the quadratic trinomial $P(x) = 3x^2 - x + \frac{25}{12}$, in [0, 2]. Also the first term of the geometric progression is equal to the square of its common ratio.

14. The value of $(r^2 + 2r)$ is

(A) 1 (B) 2

(C) $\frac{3}{2}$ (D) $\frac{1}{2}$

15. If the minimum value of $P(x)$ lies between roots of equation $x^2 + (k+1)x + a + 2\sqrt{3} = 0$, then the maximum integral value of k is

(A) – 6 (B) – 1

(C) – 7 (D) – 10

16. If the range of function $y = \frac{P(x)}{P'(x)}$ is $(-\infty, p] \cup [q, \infty)$, then $(p + q)$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{6}$ (D) 0

Comprehension 7

Let equation $x^3 + px^2 + qx - q = 0$ where $p, q \in R - \{0\}$ has 3 real roots α, β, γ in H.P., then

17. $(9p + 2q)$ has the value of equal to

(A) 9 (B) –18
(C) – 27 (D) 1

18. Minimum value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ is (you may use the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$ for any $a, b, c \in R$)

(A) $\frac{1}{3}$ (B) 1
(C) $\frac{4}{3}$ (D) 3

19. $\frac{p}{q}$ has the minimum value equal to

(A) $\frac{-1}{2}$ (B) $\frac{-1}{3}$
(C) $\frac{-1}{4}$ (D) –1

Comprehension 8

Let $A_1, A_2, A_3, \ldots, A_m$ be arithmetic means between – 2 and 1027 and $G_1, G_2, G_3, \ldots, G_n$ be geometric means between 1 and 1024. Product of geometric means is 2^{45} and sum of arithmetic means is 1025×171.

20. The value of $n + m$ is

(A) 349 (B) 351
(C) 353 (D) 355

21. The value of $G_1 + G_2 + G_3 + \ldots G_n$ is

(A) 1022 (B) 2044
(C) 512 (D) none of these

22. The numbers $2A_{171}, G_5^2 + 1, 2A_{172}$ are in

(A) A.P. (B) G.P.
(C) H.P. (D) A.G.P.

Comprehension 9

Let S_r denote the sum of first r terms of an arithmetic progression whose first term is $(r - 1)$ and the common difference is $(2r - 3)$. Let $3T_r = S_{r+1} - S_r$ and $V_r = T_{r+2} - T_r + 1$, for $r = 1, 2, 3, 4, 5, \ldots$

23. The value of $\sum_{r=1}^{12} S_r$ equals

(A) 5814 (B) 5184
(C) 5148 (D) 5418

24. The value of $T_1 + T_2 + T_3 + \ldots + T_n$ equals

(A) $\frac{n(n+1)(2n+3)}{6}$ (B) $\frac{n(n+1)(2n+1)}{6}$
(C) $\frac{n(2n+1)(2n+3)}{6}$ (D) $\frac{n(2n+3)(2n+4)}{6}$

25. Which of the following is a correct statement?

(A) $V_1, V_2, V_3, \ldots$ are in A.P. with common difference 2.
(B) $V_1, V_2, V_3, \ldots$ are in A.P. with common difference 4.
(C) $V_1, V_2, V_3, \ldots$ are in A.P. with common difference 6.
(D) $V_1, V_2, V_3, \ldots$ are in A.P. with common difference 8.

Comprehension 10

Let $< a_n >$ and $< b_n >$ be the arithmetic sequences each with common difference 2 such that $a_1 < b_1$ and let $c_n = \sum_{k=1}^{n} a_k$, $d_n = \sum_{k=1}^{n} b_k$. Suppose that the points $A_n(a_n, c_n)$, $B_n(b_n, d_n)$ are all lying on the parabola $C : y = px^2 + qx + r$ where p, q, r are constants.

26. The value of p equals

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$
(C) $\frac{1}{2}$ (D) 2

27. The value of q equals

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$
(C) $\frac{1}{2}$ (D) 2

28. If $r = 0$ then the value of a_1 and b_1 are

(A) $\frac{1}{2}$ and 1 (B) 1 and $\frac{3}{2}$
(C) 0 and 2 (D) $\frac{1}{2}$ and 2

Comprehension 11

Let $A_1, A_2, A_3, \ldots A_n$ and $H_1, H_2, \ldots H_n$ are n arithmetic means and harmonic means between two positive number a and b.
On the basis of above information answer the following.

29. The value of second arithmetic means is:

(A) $\frac{b + a(n+2)}{(n+1)}$ (B) $\frac{2b - a(n-1)}{(n+1)}$
(C) $\frac{2b + a(n-1)}{(n+1)}$ (D) $\frac{b + a(n-2)}{(n+1)}$

30. The value of $\frac{A_2 H_2}{ab}$ is

(A) $\frac{2b + a(n+1)}{2a + b(n+1)}$ (B) $\frac{2b + a(n-1)}{2a + b(n-1)}$
(C) $\frac{2a + b(n+1)}{2b + a(n+1)}$ (D) $\frac{2a + b(n-1)}{2b + a(n-1)}$

Comprehension 12

Let $f(t) = \sum_{k=0}^{\infty} \frac{1}{(t+2)^k}$, $t > -1$.

31. Value of $\sum_{r=1}^{\infty} \frac{r}{(f(0))^{r-1}}$ is

(A) 1
(B) 2
(C) 3
(D) 4

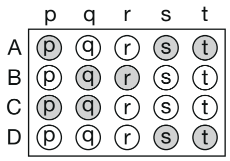
32. Value of $\sum_{k=1}^{\infty} \frac{f(k)}{k(k+2)}$ is

(A) $\frac{2}{3}$
(B) $\frac{1}{3}$
(C) $\frac{1}{2}$
(D) 1

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Column Matching:

COLUMN-I	COLUMN-II
(A) For an A.P. $a_1, a_2, a_3, \ldots, a_n, \ldots$ $a_1 = \frac{5}{2}; a_{10} = 16$. If $\sum_{i=1}^{n} a_i = 110$ then 'n' equals	(p) 7
(B) The interior angles of a convex non-equiangular polygon of 9 sides are in A.P. The least positive integer that limits the upper value of the common difference between the measures of the angles in degree is	(q) 10
(C) For an increasing G.P. $a_1, a_2, a_3, \ldots, a_n$, $a_6 = 4a_4; a_9 - a_7 = 192$. If $\sum_{i=4}^{n} a_i = 1016$ then 'n' equals	(r) 11
(D) Number of positive integral ordered pairs of (a, b) such that $6, a, b$ are in harmonical progression, is	(s) 12

2. Column Matching:

COLUMN-I	COLUMN-II
(A) Let S_n, S_{2n}, S_{3n} are the sums of $n, 2n, 3n$ terms of an arithmetic progression, then $\frac{S_{3n}}{S_{2n} - S_n}$ is equal to	(p) 3
(B) The value of $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = 220$, then the value of n equals	(q) 4
(C) Let g_n be the nth term of the geometric progression of positive numbers. If $\sum_{n=1}^{100} g_{2n} = \frac{10}{3}$ and $\sum_{n=1}^{100} g_{2n-1} = \frac{5}{9}$, then the common ratio of geometric progression, is	(r) 6
(D) A geometric progression consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupy the odd place then its common ratio is	(s) 10

(*Continued*)

3. Column Matching:

COLUMN-I	COLUMN-II
(A) The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is	(p) 8
(B) Let $a_1, a_2, \ldots, a_{10}$, be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$ then a_4h_7 is	(q) 3

(*Continued*)

COLUMN-I	COLUMN-II
(C) If the numbers, $(3^{1+x}+3^{1-x})$, $(a/2)$, (9^x+9^{-x}) form an A.P., then possible value(s) of 'a' is/are $(x \in R)$	(r) 4
(D) Between 2 and 5 six geometric means are inserted. If their product can be expressed as $(10)^n$ then the value of n equals	(s) 6

4.

COLUMN-I	COLUMN-II
(A) Suppose that $F(n+1)=\dfrac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$ and $F(1) = 2$. Then $F(101)$ equals	(p) 42
(B) If $a_1, a_2, a_3, \ldots a_{21}$ are in A.P. and $a_3+a_5+a_{11}+a_{17}+a_{19}=10$ then the value of $\sum_{i=1}^{21} a_i$ is	(q) 1620
(C) 10th term of the sequence $S = 1 + 5 + 13 + 29 + \ldots$, is	(r) 52
(D) The sum of all two digit numbers which are not divisible by 2 or 3 is	(s) 2045
	(t) $2 + 4 + 6 + \ldots. + 12$

5.

COLUMN-I	COLUMN-II
(A) If $p^{\text{th}}, q^{\text{th}}, r^{\text{th}}$ and S^{th} terms of an A.P are in G.P then $p-q, q-r, r-s$	(p) are all equal
(B) If $\ln x$, $\ln y$, $\ln z$ $(x, y, z > 1)$ are in G.P then $2x + \ln(\ln x)$, $3x+\ln(\ln y), 4x+\ln(\ln z)$	(q) are in A.P
(C) If $n!, 3\times n!$ and $(n+1)!$ are in G.P then $n!, 5\times n!$ and $(n+1)!$	(r) are in G.P
(D) If the arithmetic mean of $(b-c)^2, (c-a)^2$ and $(a-b)^2$ is same as that of $(b+c-2a)^2, (c+a-2b)^2,$ $(a+b-2c)^2$ then a, b, c	(s) are in H.P

6. Observe the following lists :

COLUMN-I	COLUMN-II
(A) If three unequal number a, b, c are in A.P. and $b-a, c-b$, a are in GP., then $\dfrac{a^3+b^3+c^3}{3abc}$ is equal to	(p) 4
(B) Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers, then $\dfrac{y^3+z^3}{xyz}$ is equal to	(q) 8
(C) If $a_1, a_2, a_3, \cdots, a_{50}$ are 50 distinct numbers in A.P. and $a_1^{\,2}-a_2^{\,2}+a_3^{\,2}, \cdots, a_{50}^{\,2}=\left(\dfrac{5}{7}\right)^n \left(a_1^{\,2}-a_{50}^{\,2}\right), (n \in N)$ then $n =$	(r) 2
(D) If a, b, c are positive real number such that $a + b + c = 1$, then minimum value of $\dfrac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)}$ is	(s) 3

7. Match the following:

COLUMN–I	COLUMN–II
(A) The arithmetic mean of two positive numbers is 6 and their geometric mean G and harmonic mean H satisfy the relation $G^2 + 3H = 48$, then product of the two number is	(p) $\dfrac{240}{77}$
(B) The sum of the series $\dfrac{5}{1^2 4^2}+\dfrac{11}{4^2 7^2}+\dfrac{17}{7^2.10^2}+\ldots$ is	(q) 32
(C) If the first two terms of a Harmonic Progression be $\dfrac{1}{2}$ and $\dfrac{1}{3}$, then the Harmonic Mean of the first four terms is	(r) $\dfrac{1}{3}$
(D) Geometric mean of – 4 and – 9	(s) $\dfrac{2}{7}$
	(t) –6

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Three numbers, the third of which is 4 from a decreasing G.P. If the last term is replaced by 3, the three numbers form an A.P, then the first number of the G.P. is

2. Find the sum to infinity of a decreasing G.P. with the common ratio x such that $|x| < 1; x \neq 0$. The ratio of the fourth term to the second term is $\frac{1}{16}$ and the ratio of third term to the square of the second term is $\frac{1}{9}$.

3. If $\sum_{\alpha=4}^{n+3} 4(\alpha-3) = An^2 + Bn + C$, then find the value of $A + B - C$.

4. If $(1-P)(1+3x+9x^2+27x^3+81x^4+243x^5) = 1-P^6,\ P \neq 1$, then find the value of $\frac{P}{x}$.

5. If $(1^2-a_1)+(2^2-a_2)+(3^2-a_3)+\ldots+(n^2-a_n) = \frac{1}{3}n(n^2-1)$, then find the value of a_7.

6. If $x_i > 0 \quad \forall i = 1,2,3$. Then least value of $\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} \frac{1}{x_i}\right)$ is ________.

7. The sum $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + \ldots + \sqrt{1+\frac{1}{999^2}+\frac{1}{1000^2}}$ is equal to $\frac{10^k-1}{1000}$, where $k =$ __.

8. If three positive real numbers a, b, c are in A.P such that $abc = 4$ then minimum value of b is $K^{l/m}$ where k, l, m are prime numbers, then $k + l - m$ is ________.

9. If $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots. + \frac{1}{2^{n-1}}$; the least value of '$n$' such that $2 - S_n < \frac{1}{100}$ is ________.

10. log $(a + c)$, log $(a + b)$, log $(b + c)$ are in A.P and a, c, b are in H.P where $a, b, c > 0$. If $a + b = k\,c/4$ then the value of 'k' is

11. If $x, y \in R^+$ satisfy $\log_8 x + \log_4 y^2 = 5$ and $\log_8 y + \log_4 x^2 = 7$ then the value of $\frac{(x^2+y^2)}{2080} =$ ________.

12. If $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P such that $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$ then a_4h_7 is equal to

13. Consider the sequence of integers 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, … then the units digit in the 150th term of the sequence is

14. If $x = 2$ then the sum to infinity of the series $\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \cdots$ is $\frac{\log\lambda}{2}$ then $\lambda =$

15. Number of natural numbers n satisfying the condition $[\sqrt{n+1}] \neq [\sqrt{n}]$ and $1 \leq n \leq 100$ is ________ (where [.] denotes the greatest integer function)

16. The number of values of $n \in Z$ for which $n^2 + n + 2$ is perfect square is ________

17. If the roots of $x^5 - 40x^4 + px^3 + qx^2 + rx + S = 0$ are in G.P. and sum of their reciprocals is 10, then the value of $\sqrt{|2S|}$ is ________

18. The first two terms of a Harmonic progression are $\frac{2}{5}$ and $\frac{12}{23}$. The value of the largest positive term of the H.P is

19. Number of integral values of m satisfying $17^2 + n^4 = m^2$ is ________ $(n \in I)$

20. If a, b, c are 3 positive real numbes, then minimum value of $\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b}$ is ________

21. Let T_n is n^{th} term of an A.P. If $\sum_{m=1}^{5^{99}} T_{2m} = 5^{100}$ and $\sum_{m=1}^{5^{99}} T_{2m-1} = 5^{99}$, then common difference of A.P. is ________

22. Let $a_1 + a_2 + \ldots + a_n = 2n^2(3 + a_{n+1} + a_{n+2} + \ldots + a_{50})$ holds for $n \in \{1, 2, \ldots, 50\}$. Then the lowest odd prime factor of a_1 is ________

23. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where $f(x) = 2^x$

24. A sequence is obtained by deleting all perfect squares from set of natural numbers. The remainder when the 2003rd term of new sequence is divided by 2048, is

25. If $(r)_n$ denotes the number $rrr \ldots (n$ digits) where $r = 1, 2, 3, \ldots, 9$ and $a = (6)_n, b = (8)_n, c = (4)_{2n}$ then $a^2 + b - c =$

26. If $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{kr}{1\times3\times5\times7\times9\times\ldots\times(2r+1)} = 1$ then $k^2 =$

27. If a, b, c are in H.P and positive, if $\left(\frac{a+b}{2a-b}\right) + \left(\frac{c+b}{2c-b}\right) \geq \sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\ldots\infty}}}$ then greatest value of λ.

28. Least value of $\left(\frac{a}{3}+\frac{b}{3}+\frac{c}{3}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \ldots$ where $a, b, c > 0$

29. Find $\left[\frac{1^2-2^2+3^2-4^2+\ldots+2009^2-2010^2}{1^2-2^2+3^2-4^2+\ldots+1003^2-1004^2}\right]$ where [.] is G.I.F.

30. If $S_n = 1.n + 2.(n-1) + 3.(n-2) + \ldots + n.1$ and $S_{25} = 325\lambda$ then λ is

31. Given a, b, c are positive integers forming an increasing G.P., $b - a$ is a prefect square of a natural number, and $\log_6 a + \log_6 b + \log_6 c = 6$. Find the value of $(a + b + c - 105)$.

32. Let a, b and c be distinct real numbers such that $a^2 - b = b^2 - c = c^2 - a$, then $(a+b)(b+c)(c+a)$ is equal to __________

33. If $a^2b^3c^4, a^3b^4c^5, a^4b^5c^6$ are in A.P. $\forall a,b,c > 0$ then least value of $a + b + c$ is

34. The value of $x.y.z = 55$ or $\frac{343}{55}$ according as the series a, x, y, z, b form an A.P or H.P respectively, where a and b are positive natural numbers. Find the sum $a + b$.

35. If the sum of the n terms of the series $1^3 + 3.2^2 + 3^3 + 3.4^2 + 5^3 + 3.6^2 + \ldots..$, where n is an even number, is given by $\frac{n}{k}\left(n^3 + an^2 + bn + c\right)$ then $b - a + c - k$ is

36. Let x, y, z, w be four non-zero real numbers such that x, y, z (in order) are in arithmetic progression and y, z, w (in order) are in geometric progression. If $x + w = 16$ and $y + z = 8$, then find the absolute value of $(x^2 - y^2 + 2z - w)$.

37. Let S_k be defined as $\sum_{n=1}^{\infty}\left(\frac{1}{k}\right)^n$ where k is a positive integer greater than one. If the maximum value of the expression $\frac{S_k \cdot S_{k+2}}{(S_{k+1})^2}$ can be expressed in the lowest rational as $\frac{p}{q}$, find the value of $(p + q)$.

38. Let $x = 111\ldots11$ (20 digits),
$y = 333\ldots33$ (10 digits)
and $z = 222\ldots22$ (10 digits), then find the value of $\frac{x - y^2}{z}$.

39. Let α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ where $\beta = 4\alpha$ $(\alpha > 0)$. If $3a = 2(c - b)$ and $S = \sum_{r=0}^{\infty}\beta(\alpha^r)$, then find the value of $18S$.

40. Let $a_1, a_2, a_3, \ldots, a_n$ be real numbers in arithmetic progression such that $a_1 = 15$ and a_2 is an integer. Given $\sum_{r=1}^{10}(a_r)^2 = 1185$. If $S(n) = \sum_{r=1}^{n} a_r$, then find maximum value of n for which $S(n) \geq S(n - 1)$.

41. If the roots of the equation $x^3 - ax^2 + 14x - 8 = 0$ are all real and positive, then the minimum value of $[a]$ (where $[a]$ is the greatest integer of a) is

42. If the roots of the equation $x^3 + px^2 + qx + r = 0$ are in G.P. such that geometric mean among the three roots satisfy the equation $px + k_1 q = 0$ and other two roots satisfy the equation $pqx^2 - k_2(q - p^2)qx + p^2 r = 0$ then the value of $k_1 + k_2$ is

43. If $p + q + r \leq 3\sqrt[3]{pqr}$ (where $p, q, r \in R^+$), then the value of $\frac{p + 2q + 3r}{p + q + r}$ is

44. If roots of the equation $x^3 + kx^2 + kx - 15 = 0$ are in A.P., then the value $2k^3 - 9k^2$ is

45. If the roots of equation $48x^3 - ax^2 + 12x - 1 = 0$ are in H.P., then the value of $\frac{a}{11}$ is

46. If $T_r = \frac{8r}{(16r^4 - 8r^2 + 1)}$, then the value of $100\left(\sum_{r=1}^{\infty} T_r\right)$ is equal to

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1. **[JEE Main 2020]**
The greatest positive integer k, for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + \ldots + 49^2 + 49 + 1$, is:
(A) 65 (B) 32
(C) 60 (D) 63

2. **[JEE Main 2020]**
Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is
(A) $\frac{21}{2}$ (B) 27
(C) 16 (D) 7

3. **[JEE Main 2020]**
If the sum of the first 40 terms of the series, $3 + 4 + 8 + 9 + 13 + 14 + 18 + 19 + \ldots$ is $(102)m$, then m is equal to :
(A) 20 (B) 5
(C) 10 (D) 25

4. **[JEE Main 2020]**
Let $a_1, a_2, a_3, \ldots$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to:
(A) 171 (B) -513
(C) $\frac{511}{3}$ (D) -171.

5. **[JEE Main 2020]**
Let $f: R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is:
(A) 0 (B) 3
(C) 4 (D) 2

6. **[JEE Main 2020]**
If the 10th term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 terms is:

(A) 50 (B) $100\frac{1}{2}$

(C) 100 (D) $50\frac{1}{4}$

7. **[JEE Main 2020]**

The product $2^{\frac{1}{4}} \cdot 4^{\frac{1}{16}} \cdot 8^{\frac{1}{48}} \cdot 16^{\frac{1}{128}} \cdot \ldots$ to ∞ is equal to:

(A) $2^{\frac{1}{4}}$ (B) $2^{\frac{1}{2}}$

(C) 2 (D) 1

8. **[JEE Main 2020]**

If $x = \sum_{n=0}^{\infty} (-1)^n \tan^{2n}\theta$ and $y = \sum_{n=0}^{\infty} \cos^{2n}\theta$, for $0 < \theta < \frac{\pi}{4}$, then:

(A) $x(1-y) = 1$ (B) $x(1+y) = 1$

(C) $y(1-x) = 1$ (D) $y(1+x) = 1$

9. **[JEE Main 2020]**

Let a_n be the n^{th} term of a G.P. of positive terms. If $\sum_{n=1}^{100} a_{2n+1} = 200$ and $\sum_{n=1}^{100} a_{2n} = 100$, then $\sum_{n=1}^{200} a_n$ is equal to

(A) 300 (B) 175

(C) 225 (D) 150

10. **[JEE Main 2020]**

If the sum of first 11 terms of an A.P. are $a_1, a_2, a_3 \ldots$ is 0 $(a_1 \neq 0)$, then the sum of the A.P. $a_1, a_3, a_5 \ldots. a_{23}$ is ka_1, where k is equal to:

(A) $\frac{121}{10}$ (B) $-\frac{72}{5}$

(C) $\frac{72}{5}$ (D) $-\frac{121}{10}$

11. **[JEE Main 2020]**

Let S be the sum of the first 9 terms of the series:

$\{x + ka\} + \{x^2 + (k+2)a\} + \{x^3 + (k+4)a\} + \{x^4 + (k+6)a\} + \ldots$

Where $a \neq 0$ and $x \neq 1$. If $S = \frac{x^{10} - x + 45a(x-1)}{x-1}$, then k is equal to:

(A) –5 (B) 1

(C) –3 (D) 3

12. **[JEE Main 2020]**

If $|x| < 1$, $|y| < 1$ and $x \neq y$, then the sum to infinity of the following series

$(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$

(A) $\frac{x+y-xy}{(1-x)(1-y)}$ (B) $\frac{x+y-xy}{(1+x)(1+y)}$

(C) $\frac{x+y+xy}{(4+x)(1+y)}$ (D) $\frac{x+y+xy}{(1-x)(1-y)}$

13. **[JEE Main 2020]**

The sum of the first three terms of a G.P. is S and their product is 27. Then all such S lie in:

(A) $[-3, \infty)$ (B) $(-\infty, 9]$

(C) $(-\infty, -9] \cup [3, \infty)$ (D) $(-\infty, -3] \cup [9, \infty)$

14. **[JEE Main 2020]**

If the first term of an A.P. is 3 and the sum of its first 25 terms is equal to the sum of its next 15 terms, then the common difference of this A.P. is:

(A) $\frac{1}{4}$ (B) $\frac{1}{5}$

(C) $\frac{1}{7}$ (D) $\frac{1}{6}$

15. **[JEE Main 2020]**

Let $a_1, a_2 \ldots.. a_n$ be a given A.P. whose common difference is an integer and $S_n = a_1 + a_2 + \ldots + a_n$. If $a_1 = 1$, $a_n = 300$ and $15 \leq n \leq 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to:

(A) (2480, 249) (B) (2490, 249)

(C) (2490, 248) (D) (2480, 248)

16. **[JEE Main 2020]**

The minimum value of $2^{\sin x} + 2^{\cos x}$ is:

(A) $2^{1-\frac{1}{\sqrt{2}}}$ (B) $2^{-1+\sqrt{2}}$

(C) $2^{1-\sqrt{2}}$ (D) $2^{-1+\frac{1}{\sqrt{2}}}$

17. **[JEE Main 2020]**

If the sum of the series $20 + 19\frac{3}{5} + 19\frac{1}{5} + 18\frac{4}{5} + \ldots$ upto n^{th} term is 488 and the n^{th} term is negative, then:

(A) n^{th} term is $-4\frac{2}{5}$ (B) $n = 41$

(C) n^{th} term is – 4 (D) $n = 60$

18. **[JEE Main 2020]**

Let α and β be the roots of $x^2 - 3x + p = 0$ and γ and δ be the roots of $x^2 - 6x + q = 0$. If $\alpha, \beta, \gamma, \delta$ form a geometric progression. Then ratio $(2q + p) : (2q - p)$ is :

(A) 3 : 1 (B) 33 : 31

(C) 9 : 7 (D) 5 : 3

19. **[JEE Main 2020]**

If $1 + (1 - 2^2.1) + (1 - 4^2.3) + (1 - 6^2.5) + \ldots + (1 - 20^2.19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to:

(A) (10, 97) (B) (11, 103)

(C) (10, 103) (D) (11, 97)

20. **[JEE Main 2020]**

If the sum of the first 20 terms of the series $\log_{(7^{1/2})} x + \log_{7^{1/3}} x + \log_{(7^{1/4})} x + \ldots.$ is 460, then x is equal to:

(A) $7^{46/21}$ (B) $7^{1/2}$

(C) e^2 (D) 7^2

21. **[JEE Main 2020]**

If the sum of the second, third and fourth terms of a positive term G.P. is 3 and the sum of its sixth, seventh and eighth terms is 243, then the sum of the first 50 terms of this G.P. is :

(A) $\frac{2}{13}(3^{50} - 1)$ (B) $\frac{1}{26}(3^{50} - 1)$

(C) $\frac{1}{13}(3^{50} - 1)$ (D) $\frac{1}{26}(3^{49} - 1)$

22. [JEE Main 2020]

If $3^{2\sin 2\alpha - 1}$, 14 and $3^{4-2\sin 2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P. is:

(A) 66
(B) 65
(C) 81
(D) 78

23. [JEE Main 2020]

If $2^{10} + 2^9 \times 3^1 + 2^8 \times 3^2 + \ldots + 2 \times 3^9 + 3^{10} = S - 2^{11}$, then S is equal to:

(A) $\dfrac{3^{11}}{2} + 2^{10}$
(B) $3^{11} - 2^{12}$
(C) 3^{11}
(D) 2×3^{11}

24. [JEE Main 2020]

The common difference of the A.P. $b_1, b_2, \ldots, b_m$ is 2 more than the common difference of A.P. $a_1, a_2, \ldots.. a_n$.

If $a_{40} = -159$, $a_{100} = -399$ and $b_{100} = a_{70}$. Then b_1 is equal to:

(A) -127
(B) -81
(C) 81
(D) 127

25. [JEE Main 2020]

Let a, b, c, d and p be any non - zero distinct real numbers such that $(a^2 + b^2 + c^2)\, p^2 + 2(ab + bc + cd)\, p + (b^2 + c^2 + d^2) = 0$. Then:

(A) a, c, p are in G.P.
(B) a, c, p are in A.P.
(C) a, b, c, d are in G.P.
(D) a, b, c, d are in A.P.

26. [JEE Main 2020]

The sum $\sum_{k=1}^{20}(1 + 2 + 3 + \ldots. + k)$ is ________.

27. [JEE Main 2020]

The sum, $\sum_{n=1}^{7} \dfrac{n(n+1)(2n+1)}{4}$ is equal to _________.

28. [JEE Main 2020]

The number of terms common to the two A.P.¢s 3, 7, 11, …, 407 and 2, 9, 16, …, 709 is _______.

29. [JEE Main 2020]

If m arithmetic means (A.Ms) and three geometric means (G.Ms) are inserted between 3 and 243 such that 4th A.M. is equal to 2nd G.M., then m is equal to ________.

30. [JEE Main 2019]

If a, b and c be three distinct real numbers in G.P. and $a + b + c = xb$, then x cannot be :

(A) -2
(B) 2
(C) -3
(D) 4

31. [JEE Main 2019]

Let $a_1, a_2, \ldots, a_{30}$ be an A.P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{(2i-1)}$.

If $a_5 = 27$, $S - 2T = 75$, then a_{10} is equal to:

(A) 47
(B) 42
(C) 52
(D) 57

32. [JEE Main 2019]

The sum of the following series

$$1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + \ldots + 5^2)}{11} + \ldots \text{ up to 15 terms, is:}$$

(A) 7830
(B) 7520
(C) 7820
(D) 7510

33. [JEE Main 2019]

Let a, b and c be the 7th, 11th and 13th terms respectively of a non-constant A.P. If these are also the three consecutive terms of a G.P. then $\dfrac{a}{c}$ is equal to:

(A) $\dfrac{1}{2}$
(B) 4
(C) 2
(D) $\dfrac{7}{13}$

34. [JEE Main 2019]

The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is:

(A) 1256
(B) 1465
(C) 1365
(D) 1356

35. [JEE Main 2019]

The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\dfrac{27}{19}$. Then the common ratio of this series is:

(A) $\dfrac{4}{9}$
(B) $\dfrac{1}{3}$
(C) $\dfrac{2}{3}$
(D) $\dfrac{2}{9}$

36. [JEE Main 2019]

Let $a_1, a_2, \ldots, a_{10}$ be a G.P. If $\dfrac{a_3}{a_1} = 25$, then $\dfrac{a_9}{a_5}$ equals:

(A) 5^4
(B) $4(5^2)$
(C) 5^3
(D) $2(5^2)$

37. [JEE Main 2019]

If 19th terms of non-zero A.P. is zero, then its (49th term) : (29th term) is :

(A) 4 : 1
(B) 1 : 3
(C) 3 : 1
(D) 2 : 1

38. [JEE Main 2019]

Let α and β be the roots of the quadratic equation $x^2 \sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$ $(0 < \theta < 45^\circ)$, and $\alpha < \beta$. Then $\sum_{n=0}^{\infty}\left(\alpha^n + \dfrac{(-1)^n}{\beta^n}\right)$ is equal to :

(A) $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1+\sin\theta}$
(B) $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\sin\theta}$
(C) $\dfrac{1}{1-\cos\theta} - \dfrac{1}{1+\sin\theta}$
(D) $\dfrac{1}{1+\cos\theta} - \dfrac{1}{1-\sin\theta}$

39. [JEE Main 2019]

Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is:

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{m+n}{6mn}$

(D) 1

40. [JEE Main 2019]

The product of three consecutive terms of a G.P. is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an A.P. Then the sum of the original three terms of the given G.P. is :

(A) 36

(B) 32

(C) 24

(D) 28

41. [JEE Main 2019]

Let $S_k = \frac{1+2+3+....+k}{k}$. If $S_1^2 + S_2^2 + ... + S_{10}^2 = \frac{5}{12}A$, then A equal to:

(A) 283

(B) 301

(C) 303

(D) 156

42. [JEE Main 2019]

If the sum of the first 15 terms of the series $\left(\frac{3}{4}\right)^3 + \left(1\frac{1}{2}\right)^3 + \left(2\frac{1}{4}\right)^3 + 3^3 + \left(3\frac{3}{4}\right)^3 + \dots$ is equal to $225k$, then k is equal to :

(A) 108

(B) 27

(C) 54

(D) 9

43. [JEE Main 2019]

The sum of all natural numbers 'n' such that $100 < n < 200$ and H.C.F $(91, n) > 1$ is:

(A) 3221

(B) 3302

(C) 3203

(D) 3121

44. [JEE Main 2019]

Then sum $\sum_{k=1}^{20} \frac{k}{2^k}$ is equal to:

(A) $2 - \frac{11}{2^{19}}$

(B) $1 - \frac{11}{2^{20}}$

(C) $2 - \frac{21}{2^{20}}$

(D) $2 - \frac{3}{2^{17}}$

45. [JEE Main 2019]

If three distinct number a, b, c are in G.P. and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct?

(A) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

(B) d, e, f are A.P.

(C) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.

(D) d, e, f are G.P.

46. [JEE Main 2019]

Let the sum of the first n terms of a non-constant A.P., a_1, a_2, a_3, … be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is common difference of this A.P., then the ordered pair (d, a_{50}) is equal to :

(A) (A, 50 + 46A)

(B) (A, 50 + 45A)

(C) (50, 50 + 45A)

(D) (50, 50 + 46A)

47. [JEE Main 2019]

Some identical balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row consists of two balls and so on. If 99 more identical balls are added to the total number of balls used in forming the equilateral triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than the number of balls each side of the triangle contains. Then the number of balls used to form the equilateral triangle is

(A) 190

(B) 262

(C) 225

(D) 157

48. [JEE Main 2019]

If the sum and product of the first three term in an A.P. are 33 and 1155, respectively, then a value of its 11th tern is

(A) –25

(B) 25

(C) –36

(D) –35

49. [JEE Main 2019]

The sum of the series $1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + \dots$ Upto 11th term is

(A) 915

(B) 946

(C) 945

(D) 916

50. [JEE Main 2019]

The sum $\frac{3 \times 1}{1^2} + \frac{5 \times (1^3 + 2^3)}{1^2 + 2^2} + \frac{7 \times (1^3 + 2^3 + 3^3)}{1^2 + 2^2 + 3^2} + \dots$ upto 10th term, is

(A) 620

(B) 660

(C) 680

(D) 600

51. [JEE Main 2019]

If $a_1, a_2, a_3 \dots a_n$ are in A.P. and $a_1 + a_4 + a_7 + \dots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to

(A) 76

(B) 64

(C) 98

(D) 38

52. [JEE Main 2019]

Let $a_1, a_2, a_3, \dots$ be in A.P. with $a_6 = 2$. Then the common difference of this A.P., which maximizes the product $a_1a_4a_5$ is

(A) $\frac{3}{2}$

(B) $\frac{8}{5}$

(C) $\frac{2}{3}$

(D) $\frac{6}{5}$

53. [JEE Main 2019]

The $1 + \frac{1^3+2^3}{1+2} + \frac{1^3+2^3+3^3}{1+2+3} + \dots + \frac{1^3+2^3+3^3+\dots+15^3}{1+2+3+\dots+15} - \frac{1}{2}(1+2+3+\dots..+15)$ is equal to

(A) 620 (B) 1860
(C) 1240 (D) 660

54. [JEE Main 2019]

Let a, b and c in G.P. with common ration r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. If $3a$, $7b$ and $15c$ are the first three terms of an A.P., then the 4th term of this A.P. is:

(A) a (B) $\frac{7}{3}a$
(C) $\frac{2}{3}a$ (D) $5a$

55. [JEE Main 2019]

If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\alpha^r + \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\beta^r$ is equal to :

(A) $\frac{7}{116}$ (B) $\frac{29}{358}$
(C) $\frac{1}{12}$ (D) $\frac{21}{346}$

56. [JEE Main 2019]

Let S_n denote the sum of the first n terms of an A.P.. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to :

(A) -410 (B) -260
(C) -320 (D) -380

57. [JEE Main 2019]

If $a_1, a_2, a_3, \ldots$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is :

(A) 200 (B) 280
(C) 150 (D) 120

58. [JEE Main 2019]

If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to :

(A) $\alpha\gamma$ (B) 0
(C) $\alpha\beta$ (D) $\beta\gamma$

59. [IIT JEE Main 2018]

Let $a_1, a_2, a_3, \ldots, a_{49}$ be in A.P. such that $\sum\limits_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots + a_{17}^2 = 140m$, then m is equal to:

(A) 68 (B) 34
(C) 33 (D) 66

60. [IIT JEE Main 2018]

Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series:
$1^2 + 2 \,.\, 2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ If $B - 2A = 100\lambda$, then λ is equal to:

(A) 248 (B) 464
(C) 496 (D) 232

61. [IIT JEE Main 2017]

For any three positive real numbers a, b and c, and $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$: Then

(A) a, b, and c are in G.P. (B) b, c, and a are in G.P.
(C) b, c, and a are in A.P. (D) a, b and c are in A.P,.

62. [IIT JEE Main 2016]

If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:

(A) 1 (B) $\frac{7}{4}$
(C) $\frac{8}{5}$ (D) $\frac{4}{3}$

63. [IIT JEE Main 2016]

If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots$, is $\frac{16}{5}m$, then m is equal to :

(A) 100 (B) 99
(C) 102 (D) 101

64. [IIT JEE Main 2015]

The sum of first 9 terms of the series. $\frac{1^3}{1} + \frac{1^3 + 2^3}{1 + 3} + \frac{1^3 + 2^3 + 3^3}{1 + 3 + 5} + \ldots$

(A) 142 (B) 192
(C) 71 (D) 96

65. [IIT JEE Main 2015]

If m is the A.M. of two distinct real numbers l and $n (l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals:

(A) $4\,lmn^2$ (B) $4\,l^2 m^2 n^2$
(C) $4\,l^2 mn$ (D) $4\,l\,m^2 n$

66. [IIT JEE Main 2014]

If $(10)^9 + 2(11)^1(10^8) + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to:

(A) 100 (B) 110
(C) $\frac{121}{10}$ (D) $\frac{441}{100}$

67. [IIT JEE Main 2014]

Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. then the common ratio of the G.P. is

(A) $2 - \sqrt{3}$ (B) $2 + \sqrt{3}$
(C) $\sqrt{2} + \sqrt{3}$ (D) $3 + \sqrt{2}$

68. [IIT JEE Main 2013]

The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, …, is

(A) $\frac{7}{81}(179 - 10^{-20})$ (B) $\frac{7}{9}(99 - 10^{-20})$
(C) $\frac{7}{81}(179 + 10^{-20})$ (D) $\frac{7}{9}(99 + 10^{-20})$

69. [IIT JEE Main 2012]

Statement-1 : The sum of the series $1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400)$ is 8000.

Statement-2 : $\sum_{k=1}^{n}\left(k^3-(k-1)^3\right)=n^3$, for any natural number n.

(A) Statement-1 is false, Statement-2 is true.
(B) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.
(D) Statement-1 is true, statement-2 is false.

70. [IIT JEE Main 2010]
A person is to count 4500 currency notes. Let a_n denote the number of notes the counts in the n^{th} minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in an AP with common difference -2, then the time taken by him to count all notes is

(A) 34 minutes	(B) 125 minutes
(C) 135 minutes	(D) 24 minutes

71. [IIT JEE Main 2009]
A man save ₹200 in each of the first three months of his service. In each of the subsequent months his saving increases by ₹40 more than the saving of immediately previous month. His total saving from the start of service will be ₹11040 after

(A) 19 months	(B) 20 months
(C) 21 months	(D) 18 months

72. [IIT JEE Main 2009]
The sum to infinite term of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$ is

(A) 3	(B) 4
(C) 6	(D) 2

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Single Correct Choice Type Questions

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE Advanced 2016]
Let $b_i > 1, 2, \ldots, 101$. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{53}$, then

(A) $s > t$ and $a_{101} > b_{101}$	(B) $s > t$ and $a_{101} < b_{101}$
(C) $s < t$ and $a_{101} > b_{101}$	(D) $s < t$ and $a_{101} < b_{101}$

2. [IIT JEE 2009]
If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms, is:

(A) $\frac{n(4n^2-1)c^2}{6}$	(B) $\frac{n(4n^2+1)c^2}{3}$
(C) $\frac{n(4n^2-1)c^2}{3}$	(D) $\frac{n(4n^2+1)c^2}{6}$

3. [IIT JEE 2009]
Let $a_1, a_2, a_3, \ldots$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is

(A) 22	(B) 23
(C) 24	(D) 25

Multiple Correct Choice Type Questions

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. [IIT JEE 2019]
Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1$, $b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$. $n \geq 2$. Then which of the following options is/are correct?

(A) $a_1 + a_2 + a_3 + \ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(B) $\sum_{n=1}^{\infty}\frac{a_n}{10^n} = \frac{10}{89}$

(C) $b_n = a^n + b^n$ for al $n \geq 1$

(D) $\sum_{n=1}^{\infty}\frac{b_n}{10^n} = \frac{8}{89}$

2. [JEE Advanced 2013]

Let $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s)

(A) 1056	(B) 1088
(C) 1120	(D) 1332

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let V_r denote the sum of the first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r - 1)$. Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for $r = 1, 2, \ldots$

1. [IIT JEE 2007]
The sum $V_1 + V_2 + \ldots + V_n$ is-

(A) $\frac{1}{12}n(n+1)(3n^2-n+1)$

(B) $\frac{1}{12}n(n+1)(3n^2+n+2)$

(C) $\frac{1}{2}n(2n^2-n+1)$

(D) $\frac{1}{3}(2n^3-2n+3)$

2. **[IIT JEE 2007]**
T_r is always -
(A) an odd number (B) in even number
(C) a prime number (D) a composite number

3. **[IIT JEE 2007]**
Which one of the following is a correct statement ?
(A) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 5
(B) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6
(C) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 11
(D) $Q_1 = Q_2 = Q_3 = \ldots$

Comprehension 2

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

4. **[IIT JEE 2007]**
Which one of the following statement is correct
(A) $G_1 > G_2 > G_3 > \ldots$
(B) $G_1 < G_2 < G_3 < \ldots$
(C) $G_1 = G_2 = G_3 = \ldots$
(D) $G_1 < G_3 < G_5 < \ldots$ and $G_2 > G_4 > G_6 > \ldots$

5. **[IIT JEE 2007]**
Which one of the following statements is correct?
(A) $A_1 > A_2 > A_3 > \ldots$
(B) $A_1 < A_2 < A_3 < \ldots$
(C) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 < \ldots$
(D) $A_1 < A_3 < A_5 < \ldots$ and $A_2 > A_4 > A_6 > \ldots$

6. **[IIT JEE 2007]**
Which one of the following statement is correct?
(A) $H_1 > H_2 > H_3 > \ldots$
(B) $H_1 < H_2 < H_3 < \ldots$
(C) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 < \ldots$
(D) $H_1 < H_3 < H_5 < \ldots$ and $H_2 > H_4 > H_6 > \ldots$

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct. Each question contains STATEMENT 1 and STATEMENT 2. You have to mark your answer as

Bubble (A) If both statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
Bubble (B) If both statements are TRUE but STATEMENT 2 is not the correct explanation of STATEMENT 1.
Bubble (C) If STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
Bubble (D) If STATEMENT 1 is FALSE but STATEMENT 2 is TRUE.

1. **[IIT JEE 2008]**
Suppose for distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.
Statement-1: The number b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.
Statement-2: The numbers b_1, b_2, b_3, b_4 are in H.P.
(A) Statement–1 is True, Statement–2 is True; Statement–2 is a correct explanation for Statement–1.
(B) Statement–1 is True, Statement–2 is True; Statement–2 is NOT a correct explanation for Statement–1.
(C) Statement–1 is True, Statement–2 is False.
(D) Statement–1 is False, Statement–2 is True.

Integer/Numerical Answer Type Question

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[IIT JEE 2020]**
Let m be the minimum possible value of $\log_3\left(3^{y_1}+3^{y_2}+3^{y_3}\right)$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ________

2. **[IIT JEE 2020]**
Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$. Be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c. for which the equality $2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$ holds for some positive integer n, is ________

3. **[IIT JEE 2019]**
Let A.P. $(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If A.P. $(1; 3) \cap$ A.P.$(2; 5) \cap$ A.P.$(3; 7)$ = A.P.$(a; d)$ then $a + d$ equals ________

4. **[IIT JEE 2018]**
Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, …, and Y be the sec consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, …, Then, the number of elements in the set $X \cup Y$ is ____.

5. **[IIT JEE 2017]**
The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ?

6. **[IIT JEE 2015]**
Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

7. **[IIT JEE 2014]**
Let a, b, c be positive integers such that $\frac{b}{a}$ is an integer. If a, b, c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is

8. **[IIT JEE 2013]**
A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then $k - 20 =$

9. **[IIT JEE 2011]**
Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$, $1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is.

10. **[IIT JEE 2011]**
The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is.

11. **[IIT JEE 2010]**
Let S_k, $k = 1, 2, \ldots, 100$, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|$ is –

12. **[IIT JEE 2010]**
Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \ldots, 11$.

If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$ is equal to

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** D	**3.** B	**4.** D	**5.** B	**6.** C	**7.** A	**8.** B	**9.** C	**10.** C
11. A	**12.** D	**13.** B	**14.** C	**15.** D	**16.** B	**17.** C	**18.** D	**19.** B	**20.** D
21. A	**22.** C	**23.** B	**24.** A	**25.** C	**26.** A	**27.** B	**28.** D	**29.** B	**30.** C
31. A	**32.** B	**33.** A	**34.** B	**35.** D	**36.** C	**37.** B	**38.** D	**39.** C	**40.** A
41. C	**42.** B	**43.** D	**44.** A	**45.** C	**46.** B	**47.** B	**48.** B	**49.** C	**50.** C
51. C	**52.** D	**53.** D	**54.** C	**55.** B	**56.** C	**57.** C	**58.** A	**59.** B	**60.** A
61. C	**62.** B	**63.** B	**64.** A	**65.** A	**66.** C	**67.** A	**68.** D	**69.** D	**70.** D
71. A	**72.** B	**73.** C	**74.** A	**75.** B	**76.** C	**77.** C	**78.** C	**79.** C	**80.** D
81. B	**82.** D	**83.** C	**84.** A	**85.** C	**86.** A	**87.** C	**88.** C	**89.** A	**90.** D
91. A	**92.** B	**93.** A	**94.** C	**95.** A	**96.** D	**97.** B	**98.** C	**99.** D	**100.** D
101. B	**102.** B	**103.** A	**104.** B	**105.** C					

Multiple Correct Choice Type Questions

1. B, C, D	**2.** A, B, C	**3.** A, B, C, D	**4.** A, B, C, D	**5.** A, B, C, D
6. A, B, C, D	**7.** A, B, D	**8.** A, C	**9.** A, B, C	**10.** B, C, D
11. A, B, D	**12.** B, D	**13.** A, D	**14.** A, C	**15.** C, D
16. A, B, D	**17.** A, C	**18.** A, B, C, D	**19.** A, C, D	**20.** A, B, C, D
21. B, D	**22.** A, C, D	**23.** A, C	**24.** A, C	**25.** C, D
26. B, C	**27.** A, B, C, D	**28.** A, B, C, D	**29.** A, B	**30.** A, D
31. A, C, D	**32.** A, D	**33.** A, B	**34.** A, B, C	**35.** A, B, C
36. A, C	**37.** A, B, C, D	**38.** A, C	**39.** A, B, C	**40.** A, B
41. A, C	**42.** A, B, C	**43.** A, B	**44.** A, B, C, D	**45.** B, C
46. A, B, C, D	**47.** A, C	**48.** B, C, D	**49.** A, D	

Linked Comprehension Type Questions

1. B	**2.** C	**3.** B	**4.** C	**5.** D	**6.** A	**7.** B	**8.** D	**9.** D	**10.** C
11. B	**12.** B	**13.** A	**14.** B	**15.** A	**16.** D	**17.** C	**18.** A	**19.** B	**20.** B
21. A	**22.** A	**23.** C	**24.** B	**25.** B	**26.** A	**27.** C	**28.** C	**29.** C	**30.** B
31. D	**32.** D								

Matrix Match/Column Match Type Questions

1. A → (r)	B → (q)	C → (q)	D → (p)
2. A → (p)	B → (s)	C → (r)	D → (q)
3. A → (r)	B → (s)	C → (p)	D → (q)
4. A → (r)	B → (p, t)	C → (s)	D → (q)
5. A → (r)	B → (q)	C → (q)	D → (p)
6. A → (r)	B → (r)	C → (r)	D → (q)
7. A → (q)	B → (r)	C → (s)	D → (t)

Integer/Numerical Answer Type Questions

1. 9	**2.** 12	**3.** 4	**4.** 3	**5.** 7	**6.** 9	**7.** 6	**8.** 1	**9.** 8	**10.** 8
11. 2	**12.** 6	**13.** 7	**14.** 4	**15.** 9	**16.** 2	**17.** 8	**18.** 6	**19.** 4	**20.** 3
21. 4	**22.** 3	**23.** 3	**24.** 0	**25.** 0	**26.** 4	**27.** 4	**28.** 3	**29.** 4	**30.** 9
31. 6	**32.** 1	**33.** 3	**34.** 8	**35.** 6	**36.** 6	**37.** 7	**38.** 1	**39.** 24	**40.** 16
41. 6	**42.** 2	**43.** 2	**44.** 405	**45.** 4	**46.** 100				

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1. D	**2.** C	**3.** A	**4.** D	**5.** B	**6.** B	**7.** B	**8.** C	**9.** D	**10.** B
11. C	**12.** A	**13.** D	**14.** D	**15.** C	**16.** A	**17.** C	**18.** C	**19.** B	**20.** D
21. B	**22.** A	**23.** C	**24.** B	**25.** C	**26.** 1540.00	**27.** 504	**28.** 14	**29.** 39	**30.** B
31. C	**32.** C	**33.** B	**34.** D	**35.** C	**36.** A	**37.** C	**38.** B	**39.** B	**40.** D
41. C	**42.** B	**43.** D	**44.** A	**45.** A	**46.** A	**47.** A	**48.** A	**49.** B	**50.** B
51. A	**52.** B	**53.** A	**54.** A	**55.** C	**56.** C	**57.** A	**58.** D	**59.** B	**60.** A
61. C	**62.** D	**63.** D	**64.** D	**65.** D	**66.** A	**67.** B	**68.** C	**69.** B	70. A
71. C	**72.** A								

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Single Correct Choice Type Questions

1. B	**2.** C	**3.** D

Multiple Correct Choice Type Questions

1. A, B, C	**2.** A, D

Linked Comprehension Type Questions

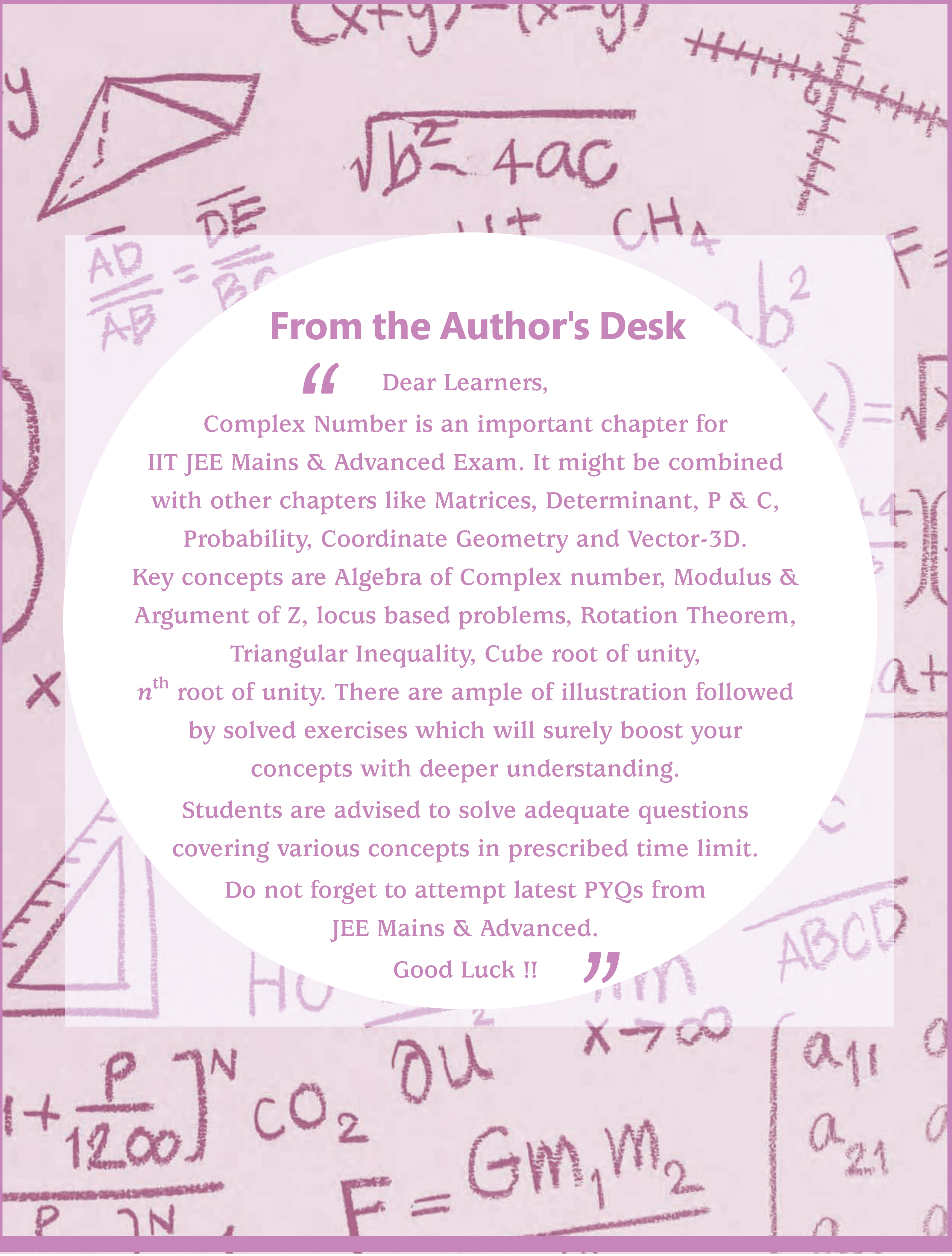
1. B	**2.** D	**3.** B	**4.** C	**5.** A	**6.** B

Reasoning Based Question

1. C

Integer/Numerical Answer Type Questions

1. 8	**2.** 1	**3.** 157.00	**4.** 3748	**5.** 6	**6.** 9	**7.** 4	**8.** 5	**9.** 9	**10.** 8
11. 3	**12.** 0								



From the Author's Desk

“ Dear Learners,

Complex Number is an important chapter for IIT JEE Mains & Advanced Exam. It might be combined with other chapters like Matrices, Determinant, P & C, Probability, Coordinate Geometry and Vector-3D.

Key concepts are Algebra of Complex number, Modulus & Argument of Z, locus based problems, Rotation Theorem, Triangular Inequality, Cube root of unity, n^{th} root of unity. There are ample of illustration followed by solved exercises which will surely boost your concepts with deeper understanding.

Students are advised to solve adequate questions covering various concepts in prescribed time limit.

Do not forget to attempt latest PYQs from JEE Mains & Advanced.

Good Luck !! ”

CHAPTER

4

Complex Numbers

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Definition and representation of complex number
(b) Algebra of complex numbers
(c) Computation of modulus and argument and properties
(d) Geometry concepts using complex number
(e) Triangular inequality and Rotation theorem
(f) Cube root of unity
(g) n^{th} root of unity

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

THE COMPLEX NUMBER SYSTEM

Definition: A number of the form $x+iy$, where $x, y \in R$ and $i=\sqrt{-1}$ is called a complex number and 'i' is called iota. A complex number is usually denoted by z and the set of complex numbers is denoted by C.

i.e., $C=\{x+iy : x \in R, y \in R, i=\sqrt{-1}\}$

For example, $5+3i, -1+i, 0+4i, 4+0i$, etc., are complex numbers.

Imaginary numbers do not follow the order property, i.e., $(a+ib)<(\text{or})>(c+id)$ is not defined. For example, the statement $(9+6i)>(3+2i)$ $2i>-i$ makes no sense.

For any positive real number a, we have

$$\sqrt{-a}=\sqrt{-1\times a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$$

$\sqrt{a}\sqrt{b}=\sqrt{ab}$ atleast one of a or b is non-negative

$$\sqrt{a}\sqrt{b}=-\sqrt{ab}\quad(a<0, b<0)$$

Integral powers of iota (i): Since $i=\sqrt{-1}$ hence we have $i^2=-1$, $i^3=-i$ and $i^4=1$. To find the value of $i^n\ (n>4)$, first divide n by 4.

$i^{4n}=1$
$i^{4n+1}=i$
$i^{4n+2}=-1$
$i^{4n+3}=-i$

where n is any integer.

Let q be the quotient and r be the remainder.

i.e., $n=4q+r$, where $0\le r\le 3$

$$i^n=i^{4q+r}=(i^4)^q\cdot(i)^r=(1)^q\cdot(i)^r=i^r$$

ILLUSTRATION 1

Simplify

(a) $\sqrt{-25}+3\sqrt{-4}+2\sqrt{-9}$ **(b)** $\sqrt{-2}\sqrt{-9/2}$

SOLUTION

(a) $\sqrt{-25}+3\sqrt{-4}+2\sqrt{-9}=5\sqrt{-1}+6\sqrt{-1}+6\sqrt{-1}$

$$=17\sqrt{-1}=17i$$

(b) $\sqrt{-2}\sqrt{-\dfrac{9}{2}}=\sqrt{2}i\times\sqrt{\dfrac{9}{2}}i=i^2\sqrt{2\cdot\dfrac{9}{2}}=-3$

ILLUSTRATION 2

Find the least positive value of n for which $\left(\dfrac{1+i}{1-i}\right)^n=1$

SOLUTION

$$\left(\frac{1+i}{1-i}\right)^n=\left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^n=\frac{(1+i)^{2n}}{(1-i^2)^n}$$

$$=\frac{(1+i^2+2i)^n}{(1+1)^n}=\left(\frac{2i}{2}\right)^n=i^n=1$$

$$i^n=1\Rightarrow n=4k$$

least positive $n=4$

ILLUSTRATION 3

Find the value of $\dfrac{i^n+i^{n+1}+i^{n+2}+i^{n+3}}{1-i+i^3-i^5}$ where $n\in N$

SOLUTION

$$i^n+i^{n+1}+i^{n+2}+i^{n+3}=i^n(1+i+i^2+i^3)$$

$$=i^n(1+i-1-i)=i^n(0)=0$$

The given number is zero as the denominator is $1-3i\neq 0$.

ILLUSTRATION 4

Find the value of sum $\sum\limits_{n=1}^{13}(i^n+i^{n+1})$, where $i=\sqrt{-1}$ equals

SOLUTION

$$\sum_{n=1}^{13}(i^n+i^{n+1})=\sum_{n=1}^{13}i^n(1+i)=(1+i)\sum_{n=1}^{13}i^n$$

$$=(1+i)(i+i^2+i^3+\ldots\ldots+i^{13})$$

$$=(1+i)\left[\frac{i(1-i^{13})}{1-i}\right]=(1+i)i$$

$$=-1+i$$

ILLUSTRATION 5

Simplify $i+2i^2+3i^3+\ldots+$ upto 100 terms (where $i=\sqrt{-1}$)

SOLUTION

$$S=i+2i^2+3i^3+\ldots+100i^{100} \qquad \ldots(1)$$

$$iS=0+i^2+2i^3+\ldots+99i^{100}+100i^{101} \qquad \ldots(2)$$

Apply (1) – (2)

$$S(1-i)=i+i^2+i^3+\ldots+i^{100}-100i^{101}$$

$$S(1-i)=\frac{i(1-i^{100})}{1-i}-100i^{101}$$

$$\Rightarrow\quad S(1-i)=\frac{i\left(1-(i^2)^{50}\right)}{1-i}-100(i^2)^{50}\cdot i$$

$$\Rightarrow\quad S(1-i)=\frac{i(1-1)}{1-i}-100i\ \Rightarrow S=\frac{-100i}{1-i}$$

ILLUSTRATION 6

Find $\dfrac{2^k}{(1+i)^{2k}}+\dfrac{(1+i)^{2k}}{2^k}$ depending upon k, whether it is odd or even.

SOLUTION

Here, $\dfrac{2^k}{(1+i)^{2k}}+\dfrac{(1+i)^{2k}}{2^k}=\dfrac{2^k}{(1+i^2+2i)^k}+\dfrac{(1+i^2+2i)^k}{2^k}$

$$=\frac{2^k}{(2i)^k}+\frac{(2i)^k}{2^k}=\frac{1}{i^k}+i^k=\frac{i^k}{i^{2k}}+i^k=\frac{i^k}{(-1)^k}+i^k$$

$$=i^k\left\{\frac{1}{(-1)^k}+1\right\}=i^k\{(-1)^k+1\}$$

$$\text{Required term}=\begin{cases}0, & \text{when } k \text{ is odd}\\ 2, & \text{when } k \text{ is even}, k=4n\\ -2, & \text{when } k \text{ is even}, k=4n+2\end{cases}$$

ILLUSTRATION 7

Find the value of $\sum\limits_{n=0}^{50}i^{(2n+1)!}$ (where $i=\sqrt{-1}$)

SOLUTION

$$\sum_{n=0}^{50}i^{(2n+1)!}=i+i^{3!}+i^{5!}+i^{7!}+\ldots\ldots+i^{101!}$$

$5!, 7!, 9!, \ldots\ldots, 101!$ are divisible by 4.

$$=i-1+1+1+\ldots\ldots\ldots+1$$

$$=-1+i+49=48+i$$

Test Your Concepts-I

(Solutions on page H.98)

1. Which of the following is not true about the set of purely imaginary numbers?

(A) Imaginary numbers follow property of order.

(B) $i>0$ is a meaningless statement.

(C) Addition of imaginary numbers are commutative in nature.

(D) Zero is purely imaginary but not imaginary.

2. Select the true statement from the following.

(A) $\sqrt{-16}+3\sqrt{-25}+\sqrt{-36}-\sqrt{-625}=0$

(B) $1+i^{14}+i^{18}+i^{22}$ is a real number

(C) $i^{3k}+i^{3k+1}+i^{3k+2}+i^{3k+3}=0$

(D) $6i^{54}+5i^{37}-i^{11}+6i^{68}=7i$

3. Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$

4. Find the value of $\left\{1-(-i)^{4n+3}\right\}(1-i)$, where $n \in N$

5. Find the sum of the terms in the series,

$i^2+i^4+i^6+\ldots\ldots+(2n \text{ terms})$

6. Find the value of $\sum\limits_{k=1}^{k=4m+1}\left(i^k+i^{2k}+i^{3k}+i^{4k}\right)$

7. Find the value of $\left(\sum\limits_{m=1}^{2n+1} i^{2m}\right)^{\sum\limits_{k=1}^{107} i^k}$

8. Find the sum of the following series

(i) $\sum\limits_{k=0}^{4n} i^{2k}$ (ii) $\sum\limits_{k=0}^{4n+1} i^{3k}$

9. Find the value of $\left(\dfrac{1+i}{1-i}\right)^{10001}+\left(\dfrac{1-i}{1+i}\right)^{10001}$

Answer Keys

1. (A) **2.** (A, B, C) **3.** –2 **4.** $-2i$ **5.** 0

6. $4m$ where $m \in I$ **7.** –1 **8.** (i) 1 (ii) $\dfrac{2}{1+i}$ **9.** Zero

REAL AND IMAGINARY PARTS OF A COMPLEX NUMBER

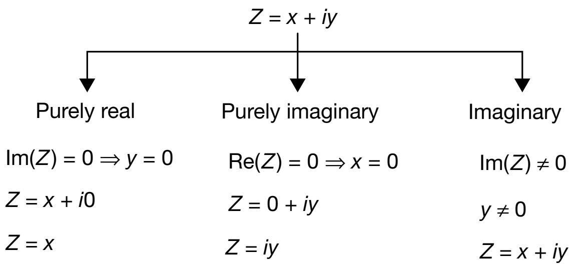
Complex number $Z = x + iy$, where $x, y \in R$

Real part of $Z = \text{Re}(Z) = x$ (Term without i)

Imaginary part of $Z = \text{Im}(z)$

$= y$ (coefficient of term present with i)

$\neq yi$



$Z = 0 = 0 + i0$

Zero is purely real, purely imaginary, real but not imaginary. Zero is also rational, integer, whole number, even number.

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

Let two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ where $a, b, c, d \in R$

Addition ($z_1 + z_2$):

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

Subtraction ($z_1 - z_2$):

$$(a+ib)-(c+id)=(a-c)+i(b-d)$$

Multiplication ($z_1 . z_2$):

$$(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$

Division $\left(\dfrac{z_1}{z_2}\right)$:

$$\frac{a+ib}{c+id}=\frac{(a+ib)}{(c+id)}\cdot\frac{(c-id)}{(c-id)} \quad \text{(Rationalisation)}$$

$$\frac{a+ib}{c+id}=\frac{(ac+bd)}{c^2+d^2}+\frac{i\,(bc-ad)}{c^2+d^2}$$

ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let z_1, z_2 and z_3 are any three complex numbers, then their algebraic operations should satisfy the following properties:

(i) Addition of complex numbers satisfies the commutative and associative properties,
i.e., $z_1+z_2=z_2+z_1$ and $(z_1+z_2)+z_3=z_1+(z_2+z_3)$.

(ii) Multiplication of complex numbers satisfies the commutative and associative properties,
i.e., $z_1z_2=z_2z_1$ and $(z_1z_2)z_3=z_1(z_2z_3)$.

(iii) Multiplication of complex numbers is distributive over addition, i.e., $z_1(z_2+z_3)=z_1z_2+z_1z_3$ and $(z_2+z_3)z_1=z_2z_1+z_3z_1$.

EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal iff their real and imaginary parts are separately equal.

i.e., $z_1 = z_2 \Leftrightarrow x_1+iy_1=x_2+iy_2 \Leftrightarrow x_1=x_2$ and $y_1=y_2$.

Additive Inverse

If z_1 and z_2 are additive inverse of each other, then $z_1 + z_2 = 0 \Rightarrow z_2 = -z_1$.

Multiplicative Inverse

If z_1 and z_2 are multiplicative inverse, then $z_1 z_2 = 1 \Rightarrow z_2 = \dfrac{1}{z_1}$.

SQUARE ROOT OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number, then

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b > 0$$

$$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b < 0.$$

ILLUSTRATION 8

Find the sum $(z_1 + z_2)$, product (z_1, z_2), difference $(z_1 - z_2)$ and division $\left(\dfrac{z_1}{z_2}\right)$ of the complex numbers $z_1 = 5 + 4i$ and $z_2 = -2 + 3i$.

SOLUTION

Sum $(z_1 + z_2) = (5 + 4i) + (-2 + 3i) = 3 + 7i$

Product $(z_1 \cdot z_2) = (5 + 4i)(-2 + 3i)$

$= (-10 - 12) + i(-8 + 15) = -22 + 7i$

Difference $(z_1 - z_2) = (5 + 4i) - (-2 + 3i) = 7 + i$

Division $\left(\dfrac{z_1}{z_2}\right) = \dfrac{5+4i}{-2+3i} \times \dfrac{-2-3i}{-2-3i}$

$$= \frac{(-10+12)+i(-8-15)}{4-9i^2} = \frac{2-23i}{13}$$

ILLUSTRATION 9

Express in terms of $x + iy$.

(a) $\dfrac{1+2i}{(2+i)^2}$ **(b)** $\dfrac{1}{i}$ **(c)** $\dfrac{(1+i)(1+i^{-7})}{2}$

SOLUTION

(a) $\dfrac{1+2i}{(2+i)^2} = \dfrac{1+2i}{4+i^2+4i} = \dfrac{1+2i}{3+4i}$

$$= \frac{1+2i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{3+8+i(6-4)}{3^2+4^2} = \frac{11+2i}{25}$$

(b) $\dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{-1} = 0 - i$

(c) $\dfrac{(1+i)(1+i^{-7})}{2} = \dfrac{(1+i)(i^7+1)}{2i^7} = \dfrac{(1+i)(-i+1)}{-2i}$

$$= \frac{1^2 - i^2}{-2i} = \frac{2}{-2i} \Rightarrow \frac{-1}{i} \times \frac{i}{i} = 0 + i$$

ILLUSTRATION 10

If $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$, where $x, y \in R$, then find (x, y).

SOLUTION

$$\frac{x-3}{3+i} + \frac{y-3}{3-i} = i = \frac{3x-9-xi+3i+3y-9+yi-3i}{3^2-i^2} = i$$

$\Rightarrow \quad 3x + 3y - 18 + (y - x)i = 10i$

Comparing real and imaginary part

$x + y = 6$ and $y - x = 10$

We get $y = 8$, $x = -2$

ILLUSTRATION 11

If $x, y \in R$, then find (x, y) for

$$(2 + 3i)x^2 - (3 - 2i)y = 2x - 3y + 5i.$$

SOLUTION

Rewriting given equation

$$(2x^2 - 3y) + i(3x^2 + 2y) = 2x - 3y + 5i$$

Comparing real and imaginary parts

$2x^2 - 3y = 2x - 3y$ and $3x^2 + 2y = 5$

$x(x - 1) = 0$ and $3x^2 + 2y = 5$

On solving we get,

$x = 0, y = 5/2$

$x = 1, y = 1$

$(x, y) = \left(0, \dfrac{5}{2}\right)$ and $(1, 1)$

ILLUSTRATION 12

Find the imaginary part of $\dfrac{(1+2i)^2 - (1-i)^3}{(3+2i)^3 - (2+i)^2}$

SOLUTION

$$z = \frac{(1+4i^2+4i)-(1-3i+3i^2-i^3)}{(27+3.9.2i+3.3.4i^2+8i^3)-(4+i^2+4i)}$$

$$z = \frac{(1-4-1+3)+i(4+3-1)}{(27-36-4+1)+i(54-8-4)}$$

$$z = \frac{-1+6i}{-12+42i} \Rightarrow z = \frac{1-6i}{12-42i} \times \frac{12+42i}{12+42i}$$

$$z = \frac{(12+6\times 42)+i(42-72)}{12^2+42^2} \Rightarrow \text{Im}(z) = \frac{-15}{954}$$

ILLUSTRATION 13

If $z = 3 + 2i$, then find additive inverse and multiplicative inverse.

SOLUTION

Additive inverse = $-z = -3 - 2i$

Multiplicative inverse

$$= \frac{1}{z} = \frac{1}{3+2i}\times\frac{3-2i}{3-2i} = \frac{3-2i}{9+4} = \frac{3}{13} - \frac{2i}{13}$$

ILLUSTRATION 14

Find square roots of $-7 - 24i$ (Without using formulae, just by manipulation only).

SOLUTION

Given expression = $-7 - 24i$

$$= -7 - 2(12i) = -7 - 2\sqrt{-144}$$

$$= -7 - 2\sqrt{-16\times 9}$$

$$= -16 + 9 - 2\sqrt{-16\times 9}$$

$$= (4i)^2 + (3)^2 - 2.3.4i = (3 - 4i)^2$$

Therefore, Square root = $\pm(3 - 4i)$

ILLUSTRATION 15

Find the square root of $\dfrac{25}{3-4i}$

SOLUTION

The given expression = $\dfrac{25}{3-4i}\times\dfrac{3+4i}{3+4i}$

$$= \frac{25(3+4i)}{25} = 3 + 4i$$

However, $\sqrt{3+4i} = x + yi$

$$x = \pm\left(\frac{\sqrt{3^2+4^2}+3}{2}\right)^{\frac{1}{2}} = \pm 2$$

$$y = \pm\left(\frac{\sqrt{3^2+4^2}-3}{2}\right)^{\frac{1}{2}} = \pm 1$$

Therefore, the required square roots are = $\pm(2 + i)$

Alternative method:

Let $\sqrt{3+4i} = x + iy$ where $x, y \in R$

$$3 + 4i = x^2 - y^2 + 2xyi$$

$$x^2 - y^2 = 3 \text{ and } 2xy = 4 \Rightarrow y = \frac{2}{x}$$

$$x^2 - \frac{4}{x^2} = 3 \;\Rightarrow x^4 - 3x^2 - 4 = 0$$

$$\Rightarrow\quad (x^2 - 4)(x^2 + 1) = 0$$

$$x = \pm 2;\ x = \pm i \text{(not acceptable) as } x \in R$$

$$x = \pm 2;\ y = \pm 1$$

Therefore, $\sqrt{3+4i} = \pm(2 + i)$

ILLUSTRATION 16

If $z^2 + 2(1 + 2i)z - (11 + 2i) = 0$, find z.

SOLUTION

As the given equation is quadratic equation,

$$z = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2(1+2i)\pm\sqrt{4(1+2i)^2+4(11+2i)}}{2}$$

$$= \frac{-2(1+2i)\pm 2\sqrt{1-4+4i+11+2i}}{2}$$

$$= -(1 + 2i) \pm \sqrt{8+6i}$$

Now $\sqrt{8+6i} = x + iy$, where $x, y \in R$

$$\Rightarrow\quad x^2 - y^2 + 2xyi = 8 + 6i$$

$$x^2 - y^2 = 8 \text{ and } 2xy = 6 \Rightarrow y = \frac{3}{x}$$

$$x^2 - \frac{9}{x^2} = 8 \Rightarrow x^4 - 8x^2 - 9 = 0$$

$$\Rightarrow\quad (x^2 - 9)(x^2 + 1) = 0$$

$$\Rightarrow\quad x^2 = 9 \Rightarrow x = \pm 3 \text{ or } x^2 = -1 \text{ (not possible)}$$

$$y = \frac{3}{x} = \pm 1$$

$$\therefore\quad \sqrt{8+6i} = \pm(3 + i)$$

$$z = -(1 + 2i) \pm (3 + i)$$

Therefore, $z = 2 - i, -4 - 3i$

Test Your Concepts-II

(Solutions on page H.98)

1. Simplify $\dfrac{1-2i}{2+i} + \dfrac{4-i}{3+2i}$
2. Find $\left(\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right)$
3. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + i\,y$, then find x and y
4. Find the reciprocal of $3 + \sqrt{7}i$

5. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find a and b

Answer Keys

1. $\frac{10}{13}-\frac{24}{13}i$

2. $\frac{1}{4}+\frac{9}{4}i$

3. $x=y=0$

4. $\frac{3}{16}-\frac{\sqrt{7}}{16}i$

5. $a=1, b=0$

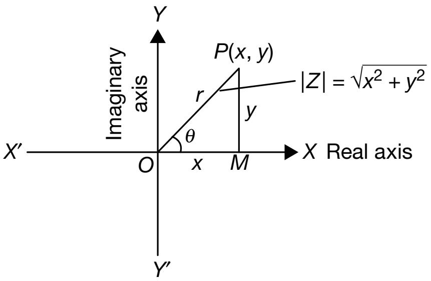
GEOMETRICAL REPRESENTATION OF COMPLEX NUMBER (Z)

The complex number $z = x + iy = (x, y)$ is represented by a point P whose coordinates are referred to rectangular axes XOX' and YOY' which are called real and imaginary axis, respectively. This plane is called Argand plane or complex plane.

Let $z = x + iy$, where $x, y \in R$

$$OP = \text{Modulus of } z = |z| = r = \sqrt{x^2+y^2}$$

Argument of z = Arg(z) = θ = angle made by ray OP with positive real axis = amp(z) = amplitude of z.



Where $x = OM = r\cos\theta$ and $y = PM = r\sin\theta$

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

Working Rule

$z = x + iy$	**(Cartesian Form)**
$z = r(\text{Cos}\theta + i\text{Sin}\theta)$ $z = r\text{CiS}(\theta)$	**(Polar Form)**
$z = re^{i\theta}$	**(Euler Form)**

Proof:

$$z = re^{i\theta} = r\left(1+\frac{i\theta}{1!}+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+.......\right)$$

$$= r\left(\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}.......\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}........\right)\right)$$

$$= r(\cos\theta + i\sin\theta)$$

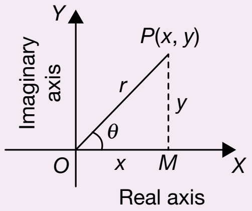
Computation of Modulus ($r = |z|$)

Let $z = x + iy$, when $x, y \in R$

Modulus of $z = |z| = r = \sqrt{x^2+y^2}$

$= \sqrt{(\text{Re}(z))^2+(\text{Im}(z))^2}$

= Distance of complex number (z) from origin



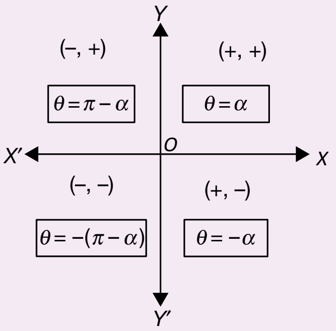
$|z| = 1$ is **Unimodular complex number**

Computation of Argument of z (θ = Arg(z))

Let $z = x + iy$, where $x, y \in R$.

STEP-1: First, get $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$ (acute angle).

STEP-2: Now locate the point (x, y) on Argand plane and calculate Arg(z).



- Amp(z) = Amplitude of z
  = Principal value of Arg(z) $\in (-\pi, \pi]$.
- Arg(z) is multivalued function.
- General Arg(z) = Principal value of Arg(z) $\pm 2m\pi$, $m \in I$.
- We can always add or subtract multiple of 2π in argument of any complex number.
- $z = 0 = 0 + i0$ is real, purely real, purely imaginary but not imaginary. Here, $|z| = 0$ but Arg(0) not defined.
- Every complex number can be represented on Argand plane uniquely using its modulus and argument.

$1 = e^{i0}$
$-1 = e^{i\pi}$
$i = e^{i\frac{\pi}{2}}$
$-i = e^{-i\frac{\pi}{2}}$

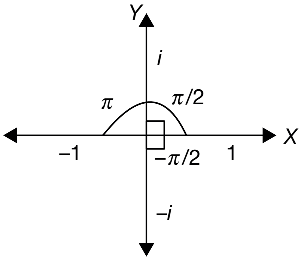


ILLUSTRATION 17

Compute $|z|$ and Arg(z) and convert into Euler form.

(i) $z = 1+\sqrt{3}i$ **(ii)** $z = -1+\sqrt{3}i$

(iii) $z = -1-\sqrt{3}i$ **(iv)** $z = 1-\sqrt{3}i$

SOLUTION

(i) $|z| = r = \sqrt{1+3} = 2\ \alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$(+1, +\sqrt{3})$ lies in Ist quadrant

$\theta = \alpha = \frac{\pi}{3} = \text{Arg}(z);\ z = 1+\sqrt{3}i = 2e^{i\pi/3}$

(ii) $z = -1+\sqrt{3}i$

$|z| = r = \sqrt{1+3} = 2;\ \alpha = \tan^{-1}\left|-\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$

$(-1, +\sqrt{3})$ lies in IInd quadrant

$\text{Arg}(z) = \theta = \pi - \alpha = \frac{2\pi}{3}$

$z = -1+\sqrt{3}i = 2e^{i2\pi/3}$

(iii) $z = -1-\sqrt{3}i$

$|z| = r = \sqrt{1+3} = 2;\ \alpha = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right| = \frac{\pi}{3}$

$(-1, -\sqrt{3})$ lies in IIIrd quadrant

$\text{Arg}(z) = \theta = -(\pi - \alpha) = -\frac{2\pi}{3}$

$z = -1-\sqrt{3}i = 2e^{-i2\pi/3}$

(iv) $z = 1-\sqrt{3}i$

$|z| = r = \sqrt{1+3} = 2;\ \alpha = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = \frac{\pi}{3}$

$(+1, -\sqrt{3})$ lies in IVth quadrant

$\text{Arg}(z) = \theta = -\alpha = -\frac{\pi}{3}$

$z = 1-\sqrt{3}i = 2e^{-i\pi/3}$

ILLUSTRATION 18

Compute $|z|$ and Arg(z) and convert into Euler form.

(i) $z = 1-i$ **(ii)** $z = 2(-1+i)$

SOLUTION

(i) $|z| = r = \sqrt{1+1} = \sqrt{2};\ \alpha = \tan^{-1}\left|\frac{-1}{1}\right| = \frac{\pi}{4}$

$(+1, -1)$ lies in IVth quadrant.

$\text{Arg}(z) = \theta = -\alpha = -\frac{\pi}{4}$

$z = 1-i = \sqrt{2}e^{-i\pi/4}$

(ii) $|z| = r = \sqrt{4+4} = 2\sqrt{2};\ \alpha = \tan^{-1}\left|\frac{1}{-1}\right| = \frac{\pi}{4}$

$(-1, +1)$ lies in IInd quadrant.

$\text{Arg}(z) = \theta = \pi - \alpha = \frac{3\pi}{4}$

$z = 2(-1+i) = 2\sqrt{2}e^{i3\pi/4}$

ILLUSTRATION 19

Convert into Euler form write principal argument.

(i) $z = -2(\cos 30° + i \sin 30°)$

(ii) $z = 6(\cos 310° - i \sin 310°)$

(iii) $z = \frac{(1+i)(1+\sqrt{3}i)^2}{1-i}$

SOLUTION

(i) $z = -2(\cos 30° + i \sin 30°)$

$$z = 2e^{i\pi}e^{i\frac{\pi}{6}} = 2e^{i\left(\pi+\frac{\pi}{6}-2\pi\right)}$$

(We can always add/subtract multiple 2π in argument)

$$= 2e^{-i\left(\frac{5\pi}{6}\right)}$$

(ii) $z = 6(\cos 310° - i \sin 310°) = 6e^{-i310°}$

$$= 6e^{-i310°+360°i} = 6e^{i50°}$$

(iii) $z = \frac{(1+i)(1+\sqrt{3}i)^2}{(1-i)}$ (change into Euler form)

$$= \frac{\sqrt{2}e^{i\pi/4}(2e^{i\pi/3})^2}{\sqrt{2}e^{-i\pi/4}} = e^{i\pi/2}4e^{i2\pi/3}$$

$$= 4e^{i(\pi/2+2\pi/3)} = 4e^{i7\pi/6}$$

$$= 4e^{i\left(\frac{7\pi}{6}-2\pi\right)} = 4e^{-i\left(\frac{5\pi}{6}\right)}$$

ILLUSTRATION 20

If $z = \frac{(1+i)(1+2i)(1+3i)}{(1-i)(2-i)(3-i)}$, then find the principal argument of z.

SOLUTION

$$z = \frac{(1+i)(1+2i)(1+3i)}{(1-i)(2-i)(3-i)}$$

$$= \frac{i^3\left(\frac{1}{i}+1\right)\left(\frac{1}{i}+2\right)\left(\frac{1}{i}+3\right)}{(1-i)(2-i)(3-i)} \quad \left(\because \frac{1}{i}=-i\right)$$

$$= \frac{-i(-i+1)(-i+2)(-i+3)}{(1-i)(2-i)(3-i)}$$

$$z=-i$$

Principal value of $\text{Arg}(z) = -\frac{\pi}{2}$

ILLUSTRATION 21

If $z = \frac{(2+i)^2}{2-i} - \frac{(2-i)^2}{2+i}$ find z, $|z|$ and $\text{Arg}(z)$

SOLUTION

$$z = \frac{(2+i)^3-(2-i)^3}{(2-i)(2+i)}$$

$$= \frac{8+i^3+6i(2+i)-(8-i^3-6i(2-i))}{4+1}$$

$$= \frac{2i^3+24i}{5} = \frac{22}{5}i$$

$$|z| = \frac{22}{5}, \ \text{Arg}(z) = \frac{\pi}{2}$$

ILLUSTRATION 22

If $x = 9^{\frac{1}{3}}.9^{\frac{1}{9}}.9^{\frac{1}{27}}\ldots\ldots\infty, y = 4^{\frac{1}{3}}.4^{-\frac{1}{9}}.4^{\frac{1}{27}}\ldots\ldots\infty$

$z = \sum_{r=1}^{\infty}(1+i)^{-r}$, then find the principal argument of $\omega = x+yz$.

SOLUTION

$$x = 9^{\frac{1}{3}}.9^{\frac{1}{9}}.9^{\frac{1}{27}}\ldots\ldots\infty = 9^{\frac{1}{3}}.9^{\frac{1}{3^2}}.9^{\frac{1}{3^3}}\ldots\ldots\infty = 9^{\frac{1/3}{1-\frac{1}{3}}} = 3$$

$$y = 4^{\frac{1}{3}}.4^{-\frac{1}{9}}.4^{\frac{1}{27}}\ldots\ldots\infty = 4^{\frac{1}{3}-\frac{1}{3^2}+\frac{1}{3^3}\ldots..\infty} = 4^{\frac{1/3}{1+\frac{1}{3}}} = \sqrt{2}$$

$$z = \sum \frac{1}{(1+i)^r} = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \ldots..$$

$$= \frac{1/1+i}{1-1/1+i} = -i$$

Therefore, $\omega = x+yz = 3-i\sqrt{2}$

As complex number is in 4th quadrant.

So, argument $\theta = -\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

ILLUSTRATION 23

Write in Euler form $z = \frac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$.

SOLUTION

$$z = \frac{i-1}{i\left(1-(1-2\sin^2\frac{\pi}{5})\right)+2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$$

$$= \frac{\sqrt{2}e^{i3\pi/4}}{2\sin\frac{\pi}{5}\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)} = \frac{\text{cosec}\frac{\pi}{5}e^{i3\pi/4}}{\sqrt{2}\ e^{i\pi/5}}$$

$$z = \left(\frac{1}{\sqrt{2}}\cos ec\frac{\pi}{5}\right)e^{i11\pi/20}$$

ILLUSTRATION 24

Write the complex number $\frac{\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\left(\sqrt{3}+i\right)}{i-1}$ in polar form.

SOLUTION

Convert all the complex number into Euler form.

$$\frac{\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\left(\sqrt{3}+i\right)}{i-1} = \frac{e^{-i\frac{\pi}{3}}2e^{i\frac{\pi}{6}}}{\sqrt{2}e^{i3\pi/4}} = \sqrt{2}e^{i\left(\frac{\pi}{6}-\frac{\pi}{3}-\frac{3\pi}{4}\right)}$$

$$= \sqrt{2}e^{-i11\pi/12}$$

$$= \sqrt{2}\,cis\left(\frac{-11\pi}{12}\right)$$

ILLUSTRATION 25

If $z = 1+\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$, then find $|z|$ and $\text{Arg}(z)$.

SOLUTION

$$z = 1+2\cos^2\frac{3\pi}{5}-1+2i\sin\frac{3\pi}{5}\cos\frac{3\pi}{5}$$

$$= 2\cos\frac{3\pi}{5}\left(\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$$

Here, $2\cos\frac{3\pi}{5}$ is negative. So, it can **NOT** be modulus.

$$z = 2\cos\left(\pi-\frac{2\pi}{5}\right)\left(\cos\left(\pi-\frac{2\pi}{5}\right)+i\sin\left(\pi-\frac{2\pi}{5}\right)\right)$$

$$= -2\cos\frac{2\pi}{5}\left(-\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)$$

$$= 2\cos\frac{2\pi}{5}\left(\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}\right)$$

$$= 2\cos\frac{2\pi}{5}e^{-i\frac{2\pi}{5}}$$

$$|z| = 2\cos\frac{2\pi}{5}; \ \text{Arg}(z) = \frac{-2\pi}{5}$$

ILLUSTRATION 26

If $\dfrac{z-1}{z+1}$ is purely imaginary, then find the locus of z.

SOLUTION

Let $z = x + iy$ Now $\dfrac{z-1}{z+1} = \dfrac{x-1+iy}{x+1+iy}$

$$\text{Re}\left(\frac{x-1+iy}{x+1+iy}\right) = 0 \Rightarrow \text{Re}\left(\frac{x-1+iy}{x+1+iy} \times \frac{x+1-iy}{x+1-iy}\right) = 0$$

$$\text{Re}\left(\frac{x^2-1+y^2+i(y(x+1)-y(x-1))}{(x+1)^2+y^2}\right) = 0$$

Therefore, $x^2 - 1 + y^2 = 0$

$\Rightarrow \quad x^2 + y^2 = 1$

So, locus of z is circle whose centre is origin and unit radius.

Test Your Concepts-III

(Solutions on page H.98)

1. Find the amplitude of $\sin\dfrac{\pi}{5} + i\left(1 - \cos\dfrac{\pi}{5}\right)$ is
2. If $z = \dfrac{(1+i)(1+2i)(1+3i)}{(1-i)(1-2i)(1-3i)}$ find $|z|$, Arg(z).
3. If $z = (\tan 1 - i)^2$ find $|z|$, then find Arg(z).
4. If $z = 1 + \cos\dfrac{10\pi}{9} + i\sin\dfrac{10\pi}{9}$, then find $|z|$ and Arg(z).
5. If Re(z) = 0, find Arg(z).
6. If $|z - i| = 1$ and Arg(z) = $\dfrac{\pi}{2}$, then find z.
7. Write $\dfrac{1+7i}{(2-i)^2}$ in the polar form
8. If Arg (Z) = 100°, find principal value of Arg(Z^{50}).
9. If $z = \dfrac{1+i\sqrt{3}}{2i\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)}$, find $|z|$ and Arg(z).
10. For any integer n, find Arg(z) = $\dfrac{(\sqrt{3}+i)^{4n+1}}{(1-i\sqrt{3})^{4n}}$
11. Find
    (i) Amp($z - \bar{z}$) (ii) Amp($-z$) – Amp(z)
12. Simplify $\left(\dfrac{1+i\sin\dfrac{\pi}{8}+\cos\dfrac{\pi}{8}}{1-i\sin\dfrac{\pi}{8}+\cos\dfrac{\pi}{8}}\right)^8$

Answer Keys

1. $\dfrac{\pi}{10}$

2. 1, 0

3. $\sec^2 1$, $2 - \pi$

4. $2\cos\dfrac{4\pi}{9}, \dfrac{-4\pi}{9}$

5. $\pm\pi/2$

6. $2i$

7. $\sqrt{2}\left[\cos\left(\dfrac{3\pi}{4}\right) + i\sin\dfrac{3\pi}{4}\right]$

8. –40°

9. $-\pi/2$

10. $\dfrac{\pi}{6}$

11. (i) $\pm\pi/2$ (ii) π

12. –1

VALUE OF ARGUMENTS OF COMPLEX NUMBERS

Complex Number	Value of Arguments
Positive purely real number	0
Negative purely real number	π
Positive purely imaginary number	$\dfrac{\pi}{2}$
Negative purely imaginary number	$-\dfrac{\pi}{2}$
$-z$	$\arg(z) + \pi \quad (\lvert -z \rvert = \lvert z \rvert)$
iz	$\text{Arg}(z) + \dfrac{\pi}{2} \quad (\lvert iz \rvert = \lvert z \rvert)$
$-iz$	$\text{Arg}(z) - \dfrac{\pi}{2} \quad (\lvert -iz \rvert = \lvert z \rvert)$

Remark(s)

Multiplication of *i*:

Since $z = r(\cos\theta + i\sin\theta)$ and $i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

then, $iz = \left[\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right]$

Hence, multiplication of z with i then vector for z rotates a right angle in the anticlockwise sense and modulus remains unchanged.

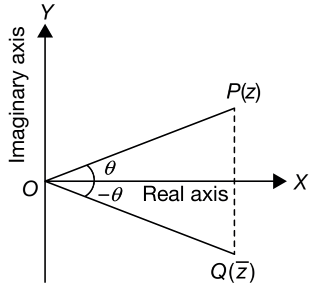
CONJUGATE OF A COMPLEX NUMBER

If there exists a complex number $z = x + iy$, $(x, y) \in R$, then its conjugate is defined as

$$\bar{z} = x - iy \qquad \text{(replace } i \text{ by } -i\text{)}$$

Hence, we have $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$.

Geometrically, the conjugate of z is the reflection or point image of z in the real axis.



Properties of Conjugate

If z, z_1 and z_2 are existing complex numbers, then we have the following results.

(i) $\overline{(\bar{z})} = z$

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(iv) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$,

In general $\overline{z_1.z_2.z_3.....z_n} = \bar{z}_1.\bar{z}_2.\bar{z}_3.....\bar{z}_n$

(v) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(vi) $(\bar{z})^n = (\overline{z^n})$

(vii) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(viii) Let $z = x + iy$, $i\bar{z} = x - iy$

Real part of $z = \text{Re}(z) = x = \frac{z+\bar{z}}{2}$

Imaginary part of $z = \text{Im}(z) = y = \frac{z-\bar{z}}{2i}$

(ix) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\text{Re}(z_1\bar{z}_2) = 2\text{Re}(\bar{z}_1 z_2)$

Reciprocal of a Complex Number

For an existing non-zero complex number $z = x + iy$, the reciprocal is given by $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z.\bar{z}} = \frac{\bar{z}}{|z|^2}$.

ILLUSTRATION 27

Write the conjugate of

(a) $z = \frac{3+i}{i(9+6i)}$ **(b)** $\omega = \frac{z-2+3i}{2i-\bar{z}}$

SOLUTION

(a) $\bar{z} = \left(\frac{\overline{3+i}}{i(9+6i)}\right) = \frac{\overline{3+i}}{\bar{i}\,\overline{(9+6i)}} = \frac{3-i}{-i(9-6i)} = \frac{3-i}{i(-9+6i)}$

(b) $\bar{\omega} = \left(\frac{\overline{z-2+3i}}{2i-\bar{z}}\right) = \frac{\bar{z}-2-3i}{-2i-z}$

ILLUSTRATION 28

If $z_1 = 9y^2 - 4 - 10xi$ and $z_2 = 8y^2 - 20i$, where z_1 and z_2 are conjugate of each other, find real 'x' and 'y'.

SOLUTION

$$z_1 = \bar{z}_2 \Rightarrow 9y^2 - 4 - 10xi = 8y^2 + 20i$$

$$9y^2 - 4 = 8y^2 \text{ and } -10x = 20$$

$$\Rightarrow \quad y^2 = 4;\ x = -2$$

$$\Rightarrow \quad y = \pm 2;\ x = -2$$

PROPERTIES OF MODULUS/ARGUMENTS

Let $z_1 = r_1 e^{i\theta_1};\ z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 e^{i\theta_1},\ r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$z_1 z_2 = |z_1|\,|z_2|\,e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{|z_1|\,e^{i\theta_1}}{|z_2|\,e^{i\theta_2}} = \frac{|z_1|}{|z_2|}e^{i(\theta_1-\theta_2)}$$

Properties of Modulus

(i) $|z| \geq 0 \Rightarrow |z| = 0$ if $z = 0$ and $|z| > 0$ if $z \neq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iii) $|\bar{z}| = |-z| = |-\bar{z}| = |zi| = |-zi| = |z|$

(iv) $|z_1 z_2| = |z_1|\,|z_2|$

In general,

$$|z_1z_2z_3......z_n| = |z_1||z_2||z_3|....|z_n|$$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $(z_2 \neq 0)$

(vi) $|z^n| = |z|^n, n \in N$

(vii) $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 \pm (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

or $|z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(z_1\bar{z}_2)$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary or $\mathrm{Re}\left(\dfrac{z_1}{z_2}\right) = 0$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$

(Law of parallelogram)

Properties of Arguments

Let k is an integer

(i) $\mathrm{Arg}(z_1z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2k\pi$, or

In general $\mathrm{Arg}(z_1z_2z_3.........z_n) = \mathrm{Arg}(z_1)$ $+\mathrm{Arg}(z_2) + \mathrm{Arg}(z_3) + ... + \mathrm{Arg}(z_n) + 2k\pi$,

(ii) $\mathrm{Arg}(z_1\bar{z}_2) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

(iii) $\mathrm{Arg}\left(\dfrac{z_1}{z_2}\right) = (\mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2) + 2k\pi$,

(iv) $\mathrm{Arg}\left(\dfrac{z}{\bar{z}}\right) = 2\mathrm{Arg}\, z + 2k\pi$,

(v) $\mathrm{Arg}(z^n) = n\,\mathrm{Arg}\, z + 2k\pi$,

(vi) If $\mathrm{Arg}\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\mathrm{Arg}\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$

(vii) $\arg(\bar{z}) = -\arg(z) = \arg\left(\dfrac{1}{z}\right)$

(viii) $\arg(z - \bar{z}) = \pm\dfrac{\pi}{2}$

(ix) $\arg(-z) = \arg(z) \pm \pi$

(x) $\arg(z) - \arg(\bar{z}) = \pm\pi$ (If z is purely imaginary)

(xi) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$,

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

(xii) The general value of $\arg(\bar{z}) = 2k\pi - \arg(z)$.

ILLUSTRATION 29

If $\dfrac{(a+i)^2}{2a-i} = p + qi$,

prove that $p^2 + q^2 = \dfrac{(a^2+1)^2}{(4a^2+1)}$ where $a, p, q \in R$.

SOLUTION

Take modules both side $\left|\dfrac{(a+i)^2}{(2a-i)}\right| = |p+qi|$

$$\Rightarrow \quad \frac{\left(\sqrt{a^2+1}\right)^2}{\sqrt{4a^2+1}} = \sqrt{p^2+q^2} \Rightarrow p^2+q^2 = \frac{(a^2+1)^2}{(4a^2+1)}$$

ILLUSTRATION 30

If $\dfrac{1-ix}{1+ix} = a - ib$, find '$x$' where $a, b, x \in R$.

SOLUTION

$$a - ib = \frac{1-ix}{1+ix}\cdot\frac{1-ix}{1-ix} = \frac{1-x^2-2xi}{1+x^2}$$

$$a = \frac{1-x^2}{1+x^2} \;\&\; -b = \frac{-2x}{1+x^2} \Rightarrow \frac{1}{1+x^2} = \frac{b}{2x}$$

$$1 + a = \frac{2}{1+x^2} = 2\left(\frac{b}{2x}\right) = \frac{b}{x} \Rightarrow x = \frac{b}{a+1}$$

ILLUSTRATION 31

Show that for every positive integers n_1, n_2 the value of expression $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$, where $i = \sqrt{-1}$ is a real number.

SOLUTION

$$(1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$$

$$= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{n_1} + \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{n_1} + \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{n_2} + \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{n_2}$$

$$= \left(\sqrt{2}\right)^{n_1}\left[e^{in_1\frac{\pi}{4}} + e^{-in_1\frac{\pi}{4}}\right] + \left(\sqrt{2}\right)^{n_2}\left[e^{in_2\frac{\pi}{4}} + e^{-in_2\frac{\pi}{4}}\right]$$

$$= \left(\sqrt{2}\right)^{n_1}\cdot 2\cos\frac{n_1\pi}{4} + \left(\sqrt{2}\right)^{n_2}\left(2\cos\frac{n_2\pi}{4}\right)$$

$= $ real number for $\forall n_1, n_2 \in N$

ILLUSTRATION 32

If $\left(\dfrac{z-1}{z+1}\right)$ is purely imaginary, then prove that $|z| = 1$.

SOLUTION

$$\mathrm{Re}\left(\frac{z-1}{z+1}\right) = 0 \Rightarrow \frac{z-1}{z+1} + \left(\overline{\frac{z+1}{z+1}}\right) = 0$$

$$\Rightarrow \quad \frac{z-1}{z+1}+\frac{\bar{z}-1}{\bar{z}+1}=0$$

$$\Rightarrow \quad (z-1)(\bar{z}+1)+(\bar{z}-1)(z+1)=0$$

$$\Rightarrow \quad z\bar{z}+z-\bar{z}-1+\bar{z}z+\bar{z}-z-1=0$$

$$\Rightarrow \quad 2(z\bar{z}-1)=0 \Rightarrow |z|=1$$

ILLUSTRATION 33

If $\frac{7z_2}{5z_1}$ is purely imaginary, find $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|$.

SOLUTION

let $\frac{7z_2}{5z_1}=x+iy$ (purely imaginary so $x=0$)

$$\Rightarrow \quad \frac{z_2}{z_1}=\frac{5}{7}yi$$

$$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|=\left|\frac{2+3\frac{z_2}{z_1}}{2-3\frac{z_2}{z_1}}\right|=\left|\frac{2+3(5/7yi)}{2-3(5/7yi)}\right|=1$$

ILLUSTRATION 34

Given that $|z-1|=1$, where z is a point on the Argand plane. Show that $\frac{z-2}{z}=i\tan \operatorname{Arg}(z)$.

SOLUTION

We have $|z-1|=1$

Let $z-1=\cos\theta+i\sin\theta$

$$z-2=-2\sin^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$z-2=2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \qquad \ldots(1)$$

and $\quad z=1+\cos\theta+i\sin\theta$

$$=2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$z=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) \qquad \ldots(2)$$

Therefore, $\arg z=\frac{\theta}{2}=i\tan(\arg z)$

From (1) and (2),

$$\frac{z-2}{z}=i\tan\frac{\theta}{2}=i\tan(\arg z)$$

ILLUSTRATION 35

If $|z_1|=1$, $|z_2|=2$, $|z_3|=3$ and $|9z_1z_2+4z_1z_3+z_2z_3|=12$, find the value of $|z_1+z_2+z_3|$.

SOLUTION

As $|z_1|=1 \Rightarrow z_1\bar{z}_1=1$

$$|z_2|=2 \Rightarrow z_2\bar{z}_2=4$$

$$|z_3|=3 \Rightarrow z_2\bar{z}_3=9$$

Therefore, $|9z_1z_2+4z_1z_3+1z_2z_3|=12$

$$\Rightarrow \quad |z_3\bar{z}_3z_1z_2+z_2\bar{z}_2z_1z_3+z_1\bar{z}_1z_2z_3|=12$$

$$\Rightarrow \quad |z_1z_2z_3|\,|\overline{z_1+z_2+z_3}|=12$$

$$\Rightarrow \quad |z_1|\,|z_2|\,|z_3|\,|z_1+z_2+z_3|=12 \quad (\because |\bar{z}|=|z|)$$

$$\Rightarrow \quad 1\cdot 2\cdot 3\cdot |z_1+z_2+z_3|=12$$

$$\Rightarrow \quad |z_1+z_2+z_3|=2$$

ILLUSTRATION 36

If z_1 and z_2 are two complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular, find $|z_1|$, if $|z_2|\neq 1$.

SOLUTION

As $\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1 \Rightarrow |z_1-2z_2|=|2-z_1\bar{z}_2|$

$$|z_1-2z_2|^2=|2-z_1\bar{z}_2|^2$$

$$\Rightarrow \quad (z_1-2z_2)(\bar{z}_1-2\bar{z}_2)=(2-z_1\bar{z}_2)(2-\bar{z}_1z_2)$$

$$\Rightarrow \quad |z_1|^2-2z_1\bar{z}_2-2z_2\bar{z}_1+4|z_2|^2$$

$$=4-2\bar{z}_1z_2-2z_1\bar{z}_2+|z_1|^2|z_2|^2$$

$$\Rightarrow \quad |z_1|^2|z_2|^2-|z_1|^2-4|z_2|^2+4=0$$

$$\Rightarrow \quad |z_1|^2\left(|z_2|^2-1\right)-4\left(|z_2|^2-1\right)=0$$

$$\Rightarrow \quad \left(|z_1|^2-4\right)\left(|z_2|^2-1\right)=0$$

As $|z_2|\neq 1$, So $|z_1|=2$

ILLUSTRATION 37

If $\omega=\frac{1-iz}{z-i}$ and $|\omega|=1$. Find the locus of z.

SOLUTION

As $|\omega|=1$, $\left|\frac{1-iz}{z-i}\right|=1 \Rightarrow |1-iz|=|z-i|$

Let $z=x+iy$

$$|1-i(x+iy)|=|x+iy-i|$$

$$\Rightarrow \quad (1+y)^2+x^2=x^2+(y-1)^2$$

$$\Rightarrow \quad y=0:\operatorname{Im}(z)=0$$

Therefore, locus of z is real axis.

ILLUSTRATION 38

Solve $z^2+|z|=0$

SOLUTION

Substitute $z=x+iy$

$$z^2+|z|=0 \Rightarrow (x+iy)^2+\sqrt{x^2+y^2}=0$$

$$\Rightarrow \quad (x^2-y^2+\sqrt{x^2+y^2})+2ixy=0+i0$$

On equating real and imaginary part

$$x^2-y^2+\sqrt{x^2+y^2}=0 \qquad \ldots(1)$$

and $xy=0$ $\ldots(2)$

From (2), either $x=0$ or $y=0$, substitute $x=0$ in (1)

$$\Rightarrow \quad -y^2+\sqrt{y^2}=0 \Rightarrow |y|=y^2$$

Which gives $y=0, 1, -1$

Similarly on putting $y=0 \Rightarrow x^2+\sqrt{x^2}=0$

$$\Rightarrow \quad x^2+|x|=0;\text{ we get } x=0,\ y=0$$

Therefore, final solution set is (0, 0) (0, 1) (0, –1).

ILLUSTRATION 39

Find all real number x which satisfy $|1+4i-2^{-x}| \le 5$, where $i=\sqrt{-1},\ x\in R$

SOLUTION

given $|1-2^{-x}+4i| \le 5$

$$\Rightarrow \quad (1-2^{-x})^2+16\le 25 \Rightarrow (1-2^{-x})^2\le 9$$

$$\Rightarrow \quad -3\le 1-2^{-x}\le 3 \Rightarrow -4\le -2^{-x}\le 2$$

$$\Rightarrow \quad -2\le 2^{-x}\le 4$$

$$\Rightarrow \quad 0<2^{-x}\le 4 \Rightarrow 2^{-\infty}<2^{-x}\le 2^2$$

we get $x\in[-2,\infty)$

ILLUSTRATION 40

If $\left|\dfrac{6z-i}{2+3iz}\right|\le 1$, then find the range of $|z|$.

SOLUTION

$$\left|\frac{6z-i}{2+3iz}\right|\le 1 \Rightarrow |6z-i|\le|2+3iz|$$

$$\Rightarrow \quad |6z-i|^2\le|2+3zi|^2 \ (\text{using } z\bar{z}=|z|^2)$$

$$\Rightarrow \quad (6z-i)(6\bar{z}+i)\le(2+3zi)(2-3\bar{z}i)$$

$$\Rightarrow \quad 36z\bar{z}+6zi-6i\bar{z}-i^2\le 4-6\bar{z}i+6zi+9z\bar{z}$$

$$\Rightarrow \quad 27z\bar{z}\le 3 \Rightarrow |z|^2\le\frac{1}{9}$$

$$0\le|z|\le\frac{1}{3}$$

ILLUSTRATION 41

If z is a complex number such that, $\dfrac{1+z+z^2}{1-z+z^2}$ is purely real. Find $|z|$. where z is imaginary number.

SOLUTION

$$\frac{1+z+z^2}{1-z+z^2}=1+\frac{2z}{1-z+z^2}=1+\frac{2}{z-1+\frac{1}{z}}$$

As the given expression is real, $z+\dfrac{1}{z}$ is also real.

Therefore, $z+\dfrac{1}{z}=\bar{z}+\dfrac{1}{\bar{z}} \Rightarrow z-\bar{z}=\dfrac{1}{\bar{z}}-\dfrac{1}{z}=\dfrac{z-\bar{z}}{z\bar{z}}$

$$(z-\bar{z})(|z|^2-1)=0$$

As z is an imaginary number $z\ne\bar{z}$ so we get $|z|=1$

ILLUSTRATION 42

The complex number whose real and imaginary parts integers satisfying the relation $z\bar{z}^3+z^3\bar{z}=350$ forms a rectangle on the Argand plane, find the length of diagonals.

SOLUTION

$$z\bar{z}^3+z^3\bar{z}=350 \Rightarrow z\bar{z}(\bar{z}^2+z^2)=350$$

$$\Rightarrow \quad |z|^2(\bar{z}^2+z^2)=350$$

Let $z=x+iy$

$$\Rightarrow \quad (x^2+y^2)(x^2-y^2-2xyi+x^2-y^2+2xyi)=350$$

$$\Rightarrow \quad (x^2+y^2)(x^2-y^2)=\frac{350}{2}=25.7$$

$\Rightarrow \quad x^2+y^2=25$ and $x^2-y^2=7$ as x and y both are integers

On solving, $x=\pm 4$ and $y=\pm 3$

Therefore, the vertices of rectangle are $(\pm 4, \pm 3)$

$A(4, 3)\ B(-4, 3)\ C(-4, -3)\ D(4, -3)$

So, length of diagonal $AC=\sqrt{64+36}=$ 10 units.

ILLUSTRATION 43

For complex numbers $z_1=x_1+i\,y_1$ and $z_2=x_2+iy_2$, we write $z_1\cap z_2$, if $x_1\le x_2$ and $y_1\le y_2$, then, show that for all complex numbers z with $1\cap z$, we have $\dfrac{1-z}{1+z}\cap 0$

SOLUTION

Let $z=x+iy$

$$\Rightarrow \quad 1\cap z \text{ gives } 1\cap x+iy$$

Or $1\le x$ and $0\le y$ $\ldots(1)$

Given $\dfrac{1-z}{1+z}\cap 0 \Rightarrow \dfrac{1-x-iy}{1+x+iy}\cap 0$

$$\Rightarrow \quad \frac{(1-x-iy)(1+x-iy)}{(1+x+iy)(1+x-iy)}\cap 0+0i$$

$$\Rightarrow \quad \frac{1-x^2-y^2}{(1+x)^2+y^2}-\frac{2iy}{(1+x)^2+y^2}\cap 0+0i$$

$\Rightarrow \quad x^2+y^2 \geq 1$

and $\quad -2y \leq 0 \Rightarrow y \geq 0$

which is true by equation (1).

ILLUSTRATION 44

If $\text{Arg}(z^{3/8}) = \frac{1}{2}\text{Arg}(z^2+\bar{z}z^{1/2})$, then show that $|z| = 1$ or $z = \bar{z}$.

SOLUTION

We have $\quad \arg(z^{3/8}) = \frac{1}{2}\text{Arg}(z^2+\bar{z}z^{1/2})$

$$\Rightarrow \quad 2\,\text{Arg}(z^{3/8}) = \text{Arg}(z^2+\bar{z}z^{1/2})$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z^2+\bar{z}z^{1/2}}{z^{3/4}}\right) = 0$$

$$\therefore \quad \frac{z^2+\bar{z}z^{1/2}}{z^{3/4}} \text{ is a real number}$$

$$\Rightarrow \quad z^{5/4} + \frac{\bar{z}}{z^{1/4}} = (\bar{z})^{5/4} + \frac{z}{(\bar{z})^{1/4}}$$

$$\Rightarrow \quad z^{5/4} + \frac{\bar{z}(\bar{z})^{1/4}}{|z|^{1/2}} = (\bar{z})^{5/4} + \frac{zz^{1/4}}{|z|^{1/2}}$$

$$\Rightarrow \quad z^{5/4} - (\bar{z})^{5/4} = \frac{z^{5/4} - (\bar{z})^{5/4}}{|z|^{1/2}}$$

$$\Rightarrow \quad \left\{z^{5/4} - (\bar{z})^{5/4}\right\}\left(1 - \frac{1}{|z|^{1/2}}\right) = 0$$

We get $z = \bar{z}$ or $|z| = 1$

ILLUSTRATION 45

If $|z_1| = |z_2| = \ldots\ldots = |z_n| = 1$ and $z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$, then prove that z is real number and also find the range of z.

SOLUTION

As $|z|^2 = z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}$ or $z = \frac{1}{\bar{z}}$

$$\text{Now } z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$$

$$= (z_1+z_2+\ldots+z_n)\left(\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right)$$

$$z = (z_1+z_2+\ldots+z_n)(\bar{z}_1+\bar{z}_2+\ldots\ldots+\bar{z}_n)$$

$$z = (z_1+z_2+\ldots.+z_n)\overline{(z_1+z_2+\ldots+z_n)}$$

$$z = |z_1+z_2+\ldots.+z_n|^2 \text{ is purely real}$$

$$|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n| = n$$

$$z = |z_1+z_2+\ldots.+z_n|^2 \Rightarrow z \leq n^2$$

Therefore, $z \in [0, n^2]$

ILLUSTRATION 46

If $\left(\frac{2iz_1-z_1-z_2}{2iz_1+z_1+z_2}\right) = \left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)$, then prove that $\frac{z_1}{z_2}$ is real.

SOLUTION

$$\left(\frac{2iz_1-z_1-z_2}{2iz_1+z_1+z_2}\right) = \left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)$$

Take modulus both sides

$$\Rightarrow \quad \left|\frac{2iz_1-z_1-z_2}{2iz_1+z_1+z_2}\right| = \left|\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right|$$

$$\Rightarrow \quad \left|\frac{2iz_1-z_1-z_2}{2iz_1+z_1+z_2}\right| = 1$$

Dividing numerator and denominator by z_1+z_2

$$\Rightarrow \quad \left|\frac{\frac{2z_1}{z_1+z_2}+i}{\frac{2z_1}{z_1+z_2}-i}\right| = 1 \Rightarrow \left|\frac{2z_1}{z_1+z_2}+i\right| = \left|\frac{2z_1}{z_1+z_2}-i\right|$$

Let $\frac{2z_1}{z_1+z_2} = x+iy$. On substitution we get,

$$|x+i(y+1)| = |x+i(y-1)|$$

$$\Rightarrow \quad x^2+(y+1)^2 = x^2+(y-1)^2$$

Which gives, $y = 0$

Therefore, $\frac{2z_1}{z_1+z_2}$ is purely real

So $\frac{2}{1+z_2/z_1}$ is purely real

Hence, $\frac{z_1}{z_2}$ is also purely real

ILLUSTRATION 47

Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then prove that $\frac{z_1+z_2}{z_1-z_2}$ may be purely imaginary or zero

SOLUTION

Given, $|z_1| = |z_2|$

Now, $\dfrac{z_1+z_2}{z_1-z_2} \times \dfrac{\bar{z}_1-\bar{z}_2}{\bar{z}_1-\bar{z}_2}$

$$= \frac{z_1\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_1 - z_2\bar{z}_2}{|z_1-z_2|^2}$$

$$= \frac{|z_1|^2 + (z_2\bar{z}_1 - z_1\bar{z}_2) - |z_2|^2}{|z_1-z_2|^2}$$

$$= \frac{z_2\bar{z}_1 - z_1\bar{z}_2}{|z_1-z_2|^2} \qquad (\text{Therefore, } |z_1|^2 = |z_2|^2)$$

As we know, $z - \bar{z} = 2i\,\mathrm{Im}(z)$

Therefore, $z_2\bar{z}_1 - z_1\bar{z}_2 = 2i\,\mathrm{Im}(z_2\bar{z}_1)$

Hence, $\dfrac{z_1+z_2}{z_1-z_2} = \dfrac{2i\,\mathrm{Im}(z_2\bar{z}_1)}{|z_1-z_2|^2}$

which is purely imaginary or zero.

Test Your Concepts-IV

(Solutions on page H.99)

1. Find $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right|$

2. For what real values of x and y the numbers $-3 + ix^2y$ and $x^2 + y + 4i$ complex conjugate?

3. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$

4. If $z = x + iy$ and $iz^2 - \bar{z} = 0$, then find $|z|$

5. If $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$, then find $(a_1^2+b_1^2)(a_2^2+b_2^2) \ldots\ldots (a_n^2+b_n^2)$

6. If $\left|\dfrac{z_1-z_2}{1-\bar{z}_1z_2}\right| < 1$, then prove that $|z_1| < 1$ and $|z_2| < 1$ or $|z_1| > 1$ and $|z_2| > 1$

7. (a) Find all non-zero complex numbers z satisfying $\bar{z} = iz^2$.

(b) If the complex numbers $z_1, z_2, \ldots\ldots z_n$ lie on the unit circle $|z| = 1$, then show that $|z_1 + z_2 + \ldots\ldots + z_n| = |z_1^{-1} + z_2^{-1} + \ldots + z_n^{-1}|$.

8. Let $\omega = \dfrac{1-iz}{z-i}$, such that $|\omega| = 1$, then z is purely real numbers. Prove whether it is true or false.

9. If $z = \dfrac{\sqrt{9+40i}+\sqrt{9-40i}}{\sqrt{9+40i}-\sqrt{9-40i}}$ simplify z, find $|z|$ and argument z.

10. Find the locus of z if $\mathrm{Re}\left(\dfrac{1}{z}\right) < \dfrac{1}{2}$

11. If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$ (where $z \neq -1$), then find $\mathrm{Re}(w)$

Answer Keys

1. 1
2. $x = \pm 1; y = -4$
3. 1
4. 0 or 1
5. $A^2 + B^2$
7. (a) $\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}, -\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}, i$
8. True
9. $\dfrac{5}{4}, -\pi/2$
10. $x^2 + y^2 - 2x > 0$
11. Zero

THEORY OF EQUATION

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 = 0$$

- n degree polynomial equation has exactly n roots.
  If n degree polynomial equation is satisfied by more than n distinct values, then it becomes an identity. Hence, it is true for all values which implies that coefficient of all variables are zero.
- If all the coefficients in equation are real, then imaginary roots are present with their conjugate.
- If not all the coefficients are real, then it is not necessary that imaginary roots are present with their conjugates.
  Let the roots are $x_1, x_2, x_3, \ldots, x_n$

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_1x + a_0 = 0$$

Sum of roots $= \sum x_i = -\dfrac{a_{n-1}}{a_n}$

$$\sum x_ix_2 = \frac{a_{n-2}}{a_n}; \quad \sum x_ix_2x_3 = -\frac{a_{n-3}}{a_n}$$

Product of roots $= x_1\,x_2\,x_3 \ldots\ldots x_n = (-1)^n \dfrac{a_0}{a_n}$

ILLUSTRATION 48

Show that the polynomial $x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ is divisible by $x^3 + x^2 + x + 1$, where $p, q, r, s \in N$.

SOLUTION

Let $f(x) = x^{4p} + x^{4q+1} + x^{4r+2} + x^{4s+3}$ and $x^3 + x^2 + x + 1$

$= (x^2 + 1)(x + 1) = (x + i)(x - i)(x + 1)$, where $i = \sqrt{-1}$

Now, $f(i) = i^{4p} + i^{4q+1} + i^{4r+2} + i^{4s+3} = 1 + i + i^2 + i^3 = 0$

(sum of four consecutive powers of i is zero)

$$f(-i) = (-i)^{4p} + (-i)^{4q+1} + (-i)^{4r+2} + (-i)^{4s+3}$$

$$= 1 + (-i)^1 + (-i)^2 + (-i)^3 = 1 - i - 1 + i = 0$$

and

$$f(-1) = (-1)^{4p} + (-1)^{4q+1} + (-1)^{4r+2} + (-1)^{4s+3}$$

$$= 1 - 1 + 1 - 1 = 0$$

Hence, by division theorem, $f(x)$ is divisible by

$$x^3 + x^2 + x + 1.$$

ILLUSTRATION 49

$P(z) = 2z^4 + az^3 + bz^2 + cz + 3$, where $a, b, c \in R$, if $P(z) = 0$, has two roots $2, i$, find 'a'.

SOLUTION

As the coefficients are real, imaginary roots are present with their conjugates. So, the roots of given equation are $2, i, -i$ and α.

Product of roots $= 2(i)(-i)\,\alpha = \dfrac{3}{2} \Rightarrow \alpha = \dfrac{3}{4}$

Sum of roots

$$= 2 + i - i + \alpha = \frac{-a}{2} \Rightarrow 2 + \frac{3}{4} = -\frac{a}{2} \Rightarrow a = -\frac{11}{2}$$

ILLUSTRATION 50

Let 'z' be complex number and $a, b, c, \in R$. If $az^2 + bz + c + i = 0$ has purely imaginary roots, then show that $a = b^2 c$.

SOLUTION

Let $z = x + iy$; $x = 0$ (purely imaginary). So, $z = yi$

$$a(iy)^2 + b(iy) + c + i = 0$$

$$c - ay^2 + i(by + 1) = 0$$

$$c = ay^2 \text{ and } y = -\frac{1}{b} \Rightarrow c = a\left(\frac{-1}{b}\right)^2 \Rightarrow b^2 c = a$$

ILLUSTRATION 51

In the quadratic equation $x^2 + (p + iq)\,x + 3i = 0$, $p, q, \in R$. If sum of squares of roots is 8, then find (p, q).

SOLUTION

Let α, β be roots of given quadratic equation

$$\alpha + \beta = -p - iq \text{ and } \alpha\beta = 3i$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 8$$

$\Rightarrow$ $p^2 - q^2 + 2ipq - 6i = 8 + i0$; we get $p^2 - q^2 = 8$; $pq = 3$

Which satisfy for $p = 3, q = 1$ and $p = -3, q = -1$

Therefore, $(p, q) = (\pm 3, \pm 1)$

ILLUSTRATION 52

Let z is a complex number satisfying the equation $z^2 - (3 + i)\,z + m + 2i = 0$, where $m \in R$. Suppose the equation has one real root, then find additive inverse of non-real root.

SOLUTION

Let $z = x + iy$; $y = 0$ as purely real root.

$$x^2 - (3 + i)x + m + 2i = 0$$

$$(x^2 - 3x + m) + i(2 - x) = 0$$

$$x^2 - 3x + m = 0 \text{ and } 2 - x = 0 \Rightarrow x = 2$$

$\Rightarrow$ $4 - 6 + m = 0 \Rightarrow m = 2$

$$z^2 - (3 + i)\,z + 2 + 2i = 0$$

Sum of roots $2 + \alpha = 3 + i$; $\alpha = 1 + i$.

Additive inverse of non-real root $= -\alpha = -1 - i$

ILLUSTRATION 53

If $z = (3 + 7i)\,(p + iq)$, where $p, q \in I - \{0\}$ is purely imaginary. Find the minimum value of $|z|^2$.

SOLUTION

Let $z = (3p - 7q) + i(3q + 7p)$ is purely imaginary.

$$3p - 7q = 0;\ q = \frac{3p}{7};\ |p_{\min}| = 7$$

$$|z| = |3q + 7p| = \left|\frac{9p}{7} + 7p\right| = \frac{58}{7}|p|$$

$$|z|_{\min} = \frac{58}{7}.7 = 58 \quad \Rightarrow \quad |z|^2_{\min} = 58^2$$

ILLUSTRATION 54

If $x^4 + ax^3 + bx^2 + cx + d = 0$ $(a, b, c, d \in R)$ has 4 non-real roots, such that, the sum of two roots is $3 + 4i$ and the product of other two roots is $13 + i$, find b.

SOLUTION

Coefficient $a, b, c, d \in R$, which it means non-real roots are present with their conjugates.

Let the roots are $\alpha, \bar{\alpha}, \beta, \bar{\beta}$

(Sum of the roots is imaginary. So, it cannot be pair of imaginary conjugate roots)

$$\alpha + \beta = 3 + 4i,\ \bar{\alpha}\,\bar{\beta} = 13 + i$$

b = sum of product roots taken two at a time

$$= \alpha\bar{\alpha} + \alpha\beta + \alpha\bar{\beta} + \bar{\alpha}\beta + \bar{\alpha}\bar{\beta} + \beta\bar{\beta}$$

$$= \alpha(\bar{\alpha}+\bar{\beta})+\beta(\bar{\alpha}+\bar{\beta})+\alpha\beta+\bar{\alpha}\bar{\beta}$$

$$= (\alpha+\beta)(\bar{\alpha}+\bar{\beta})+\alpha\beta+\bar{\alpha}\bar{\beta}$$

$$= (3+4i)\,(3-4i)+13-i+13+i$$

$$= 25+26 = 51$$

ILLUSTRATION 55

If $f(x) = x^4 - 8x^3 + 4x^2 + 4x + 39$ and $f(3 + 2i) = a + ib$, then find (a, b).

SOLUTION

Given $x = 3 + 2i \Rightarrow (x-3)^2 = -4 \Rightarrow x^2 - 6x + 13 = 0$

On dividing $f(x)$ by $x^2 - 6x + 13$, quotient is

$x^2 - 2x - 21$ and remainder is $-96x + 312$

Therefore, $f(x) = (x^2 - 6x + 13)\,(x^2 - 2x - 21) - 96x + 312$

$$f(3 + 2i) = 0\,(x^2 - 2x - 21) - 96\,(3 + 2i) + 312$$

$$a + ib = 24 - 192i \Rightarrow (a, b) = (24, -192)$$

ILLUSTRATION 56

If the equation $z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_1, a_2, a_3, a_4 are real coefficients different from zero has a pure imaginary root, then find the value of $\frac{a_3}{a_1a_2}+\frac{a_1a_4}{a_2a_3}$

SOLUTION

Let xi be the root such that $x \neq 0$ and $x \in R$ (as if $x = 0$ satisfies then $a_4 = 0$ which contradicts)

$$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$$

$$x^4 - a_2x^2 + a_4 = 0 \qquad \ldots(1)$$

and $a_1x^3 - a_3x = 0 \qquad \ldots(2)$

From equation (2),

$$a_1x^2 - a_3 = 0 \Rightarrow x^2 = \frac{a_3}{a_1}.$$

(as $x \neq 0$) substituting the value of x^2 in equation (1) putting the value of x^2 in equation (1)

$$\frac{a_3^2}{a_1^2} - \frac{a_2\,a_3}{a_1} + a_4 = 0 \text{ or } a_3^2 + a_4a_1^2 = a_1a_2a_3$$

or $\frac{a_3}{a_1\,a_2} + \frac{a_1a_4}{a_2\,a_3} = 1$ (dividing by $a_1a_2a_3$)

ILLUSTRATION 57

Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $\text{Re}(z) = 1$, then find range of β.

SOLUTION

Let $z = x + iy$, given $\text{Re}(z) = 1$

Therefore, $x = 1$

$\Rightarrow \quad z = 1 + iy$

Since, the complex roots are conjugate of each other, $z = 1 + iy$ and $1 - iy$ are the two distinct roots of $z^2 + \alpha z + \beta = 0$ $\quad (y \neq 0)$

Product of roots $= \beta$

$\Rightarrow \quad (1 + iy)\,(1 - iy) = \beta \qquad (\because y \neq 0;\, y^2 > 0)$

Therefore, $\beta = 1 + y^2 > 1 \Rightarrow \beta \in (1, \infty)$

De MOIVRE'S THEOREM

1. If n is any rational number, then
   $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
2. If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
   $(\cos\theta_3 + i\sin\theta_3)\ \ldots\ldots\ (\cos\theta_n + i\sin\theta_n)$
   then, $z = \cos(\theta_1 + \theta_2 + \theta_3 + \ldots.. + \theta_n)$
   $+ i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots.. + \theta_n)$
   where, $\theta_1, \theta_2, \theta_3 \ldots..\theta_n \in R$.
3. If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then
   $$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right],$$
   where $k = 0, 1, 2, 3, \ldots(n-1)$.

(i) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(ii) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

(iii) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

(iv) $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$

If $p, q \in z$ and $q \neq 0$, then

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\left(e^{i\theta}\right)^p\right)^{1/q}$$

$$= \left(e^{i(p\theta+2k\pi)}\right)^{1/q} = e^{\frac{i(2k\pi+p\theta)}{q}}$$

$$= \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$$

where, $k = 0, 1, 2, 3 \ldots (q - 1)$.

ILLUSTRATION 58

Prove that for $\forall \; n \in N$,

$$(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right).$$

SOLUTION

$$\text{LHS} = \left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)^n$$

$$= \left(e^{i(\pi/2-\theta)}\right)^n = e^{in(\pi/2-\theta)}$$

$$= \cos n\left(\frac{\pi}{2}-\theta\right)+i\sin\left(n\left(\frac{\pi}{2}-\theta\right)\right) = \text{RHS}$$

ILLUSTRATION 59

Prove that for $\forall\ n \in N$,

$$\frac{(1+\cos\theta+i\sin\theta)^n}{(1+\cos\theta-i\sin\theta)^n} = (\cos n\theta + i\sin n\theta)$$

SOLUTION

$$\text{LHS} = \frac{\left(1+2\cos^2\frac{\theta}{2}-1+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n}{\left(1+2\cos 2\frac{\theta}{2}-1-2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n}$$

$$= \frac{\left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right)^n}{\left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)\right)^n} = \left(\frac{e^{i\theta/2}}{e^{-i\theta/2}}\right)^n$$

$$= e^{in\theta} = \cos n\theta + i\sin\theta = \text{RHS}$$

ILLUSTRATION 60

If $z = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, find Im($z$).

SOLUTION

$$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$$

$\Rightarrow \quad z = \left(e^{i\pi/6}\right)^5 + \left(e^{-i\pi/6}\right)^5$ (Change in Euler form)

$$= e^{i5\pi/6} + e^{-i5\pi/6} = 2\cos 5\pi/6 = -\sqrt{3}$$

$$\text{Im}(z) = 0$$

ILLUSTRATION 61

If $n \in N$, prove that, $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right)$

SOLUTION

$$(1+i)^n + (1-i)^n$$

$$= \left(\sqrt{2}e^{i\pi/4}\right)^n + \left(\sqrt{2}e^{-i\pi/4}\right)^n$$

$$= \left(\sqrt{2}\right)^n e^{in\pi/4} + \left(\sqrt{2}\right)^n e^{-in\pi/4}$$

$$= 2^{n/2}\left[e^{in\pi/4}+e^{-in\pi/4}\right]$$

$$= 2^{n/2}\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\left(\frac{-n\pi}{4}\right)+i\sin\left(\frac{-n\pi}{4}\right)\right]$$

$$= 2^{n/2}\cdot 2\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

ILLUSTRATION 62

If α, β are the roots of $x^2 - 2x + 2 = 0$ and $\cot\theta = y + 1$, then evaluate $\frac{(y+\alpha)^n - (y+\beta)^n}{\alpha-\beta}$ in terms of θ.

SOLUTION

$$x^2 - 2x + 2 = 0 \Rightarrow (x-1)^2 = -1 = i^2 \Rightarrow x = 1 \pm i$$

Therefore, $\alpha = 1 + i$, $\beta = 1 - i$

Hence, $\frac{(y+\alpha)^n - (y+\beta)^n}{\alpha-\beta}$ becomes

$$\Rightarrow \quad \frac{[(\cot\theta-1)+(1+i)]^n - [(\cot\theta-1)+(1-i)]^n}{(1+i)-(1-i)}$$

$$\Rightarrow \quad \frac{(\cos\theta+i\sin\theta)^n - (\cos\theta-i\sin\theta)^n}{\sin^n\theta\cdot 2i}$$

By de Moivre's theorem

$$\Rightarrow \quad \frac{2i\sin(n\theta)}{2i\sin^n\theta} = \frac{\sin(n\theta)}{\sin^n\theta}$$

ILLUSTRATION 63

If α, β are the roots of $x^2 - 2x + 4 = 0$, then find the value of $\alpha^{18} + \beta^{18}$.

SOLUTION

$$x^2 - 2x + 4 = 0 \Rightarrow (x-1)^2 = -3 = 3i^2 \Rightarrow x = 1 \pm \sqrt{3}\,i$$

$$\alpha = 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\beta = 1 - i\sqrt{3} = 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\alpha^{18}+\beta^{18} = 2^{18}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{18} + 2^{18}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)^{18}$$

Using de Moivre's theorem

$$= 2^{18}\left[\cos 6\pi + i\sin 6\pi + \cos(-6\pi) + i\sin(-6\pi)\right]$$

$$= 2^{19}$$

ILLUSTRATION 64

Simplify $\frac{(\cos 3\theta + i\sin 3\theta)^{14}(\cos 5\theta - i\sin 5\theta)^8}{(\cos 4\theta + i\sin 4\theta)^{20}(\cos 13\theta - i\sin 13\theta)^6}$

SOLUTION

Convert into Euler form

$$\frac{(\cos 3\theta + i\sin 3\theta)^{14}(\cos 5\theta - i\sin 5\theta)^{8}}{(\cos 4\theta + i\sin 4\theta)^{20}(\cos 13\theta - i\sin 13\theta)^{6}}$$

$$= \frac{\left(e^{i3\theta}\right)^{14}\left(e^{-i5\theta}\right)^{8}}{\left(e^{i4\theta}\right)^{20}\left(e^{-i13\theta}\right)^{6}} = \frac{e^{i(42\theta-40\theta)}}{e^{i(80\theta-78\theta)}} = \frac{e^{i2\theta}}{e^{i2\theta}} = 1$$

ILLUSTRATION 65

Simplify $z = \dfrac{(\cos 2\theta - i\sin 2\theta)^{7}(-\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^{12}(-\cos 5\theta - i\sin 5\theta)^{-6}}$

SOLUTION

$$z = \frac{(\cos 2\theta - i\sin 2\theta)^{7}(-\cos 3\theta + i\sin 3\theta)^{-5}}{(\cos 4\theta + i\sin 4\theta)^{12}(-\cos 5\theta - i\sin 5\theta)^{-6}}$$

$$z = \frac{\left(e^{-i2\theta}\right)^{7}(-1)^{-5}\left(e^{-i3\theta}\right)^{-5}}{\left(e^{i4\theta}\right)^{12}(-1)^{-6}\left(e^{i5\theta}\right)^{-6}}$$

$$= \frac{e^{-i14\theta}(-1)\,e^{i15\theta}}{e^{i48\theta}\;e^{-i30\theta}} = \frac{e^{i\pi}}{e^{i17\theta}} = e^{i(\pi-17\theta)}$$

$$= \cos(\pi - 17\theta) + i\sin(\pi - 17\theta) = -\cos 17\theta + i\sin 17\theta$$

LOGARITHM OF A COMPLEX NUMBER

$$\log(x+iy) = \log_e(re^{i\theta}) = \log_e r + \log_e e^{i\theta}$$

$$= \log_e r + i\theta$$

$$= \log_e \sqrt{(x^2+y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\log_e(z) = \log_e|z| + i\,\text{Amp}(z)$$

EXAMPLE:

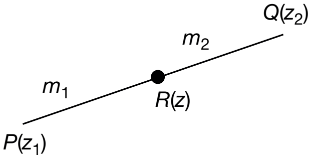
$\log(3-4i) = \log 5 + i\tan^{-1}(-4/3)$

COMPLEX NO. IN CO-ORDINATE GEOMETRY

1. **Section formula:** If $R(z)$ divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2 (m_1, m_2 > 0)$, then

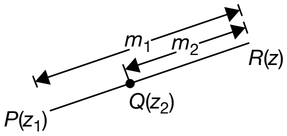
   (i) For internal division

   $$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$



   (ii) For external division

   $$z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$



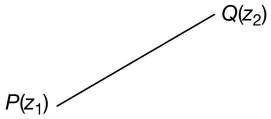
2. **Distance formula:** The distance between the two points $P(z_1)$ and $Q(z_2)$ is given by

   $PQ = |z_2 - z_1| = |\text{affix of } Q - \text{affix of } P|$

   $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

   $z_1 - z_2 = x_1 + iy_1 - x_2 - iy_2$

   $= (x_1 - x_2) + i(y_1 - y_2)$



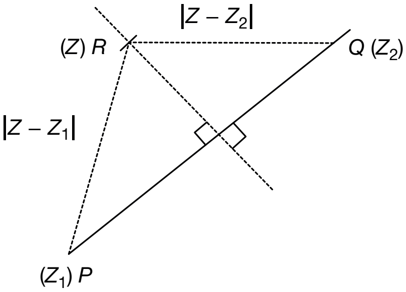
$$|z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$$

$= \text{Distance between } z_1 \text{ and } z_2$

$|z| = |z - 0| = \text{distance of } z \text{ from origin}$

EQUATION OF THE PERPENDICULAR BISECTOR

If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is moving point such that, it is always at equal distance from $P(z_1)$ and $Q(z_2)$, then the locus of $R(z)$ is the perpendicular bisector of PQ is perpendicular bisector of line segment joining Z_1 and Z_2.



$PR = QR$ or $|z - z_1| = |z - z_2|$

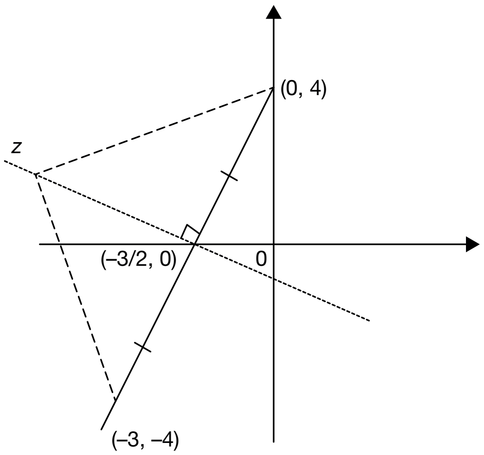
ILLUSTRATION 66

Plot

(i) $|z - 4i| = |z + 3 + 4i|$

SOLUTION

z lies on perpendicular line bisector of line segment joining point (–3, –4) and (0, 4)



(ii) $|z-2i| > |z+2|$

SOLUTION

z lies in shaded region shown in figure. Perpendicular line bisector joining (–2, 0) and (0, 2) is not included.

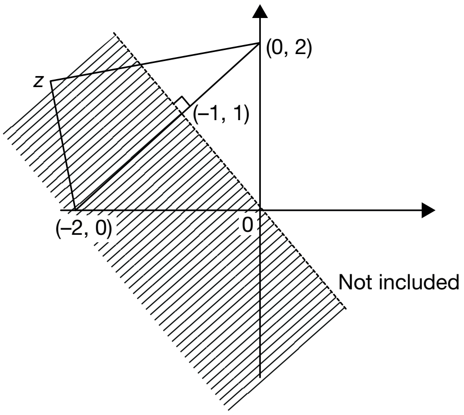
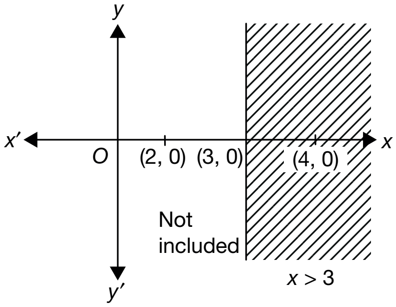


ILLUSTRATION 67

The inequality $|z-4| < |z-2|$ represents a region find it

SOLUTION

Given, $|z-4| < |z-2|$.
Since, $|z-z_1| > |z-z_2|$ represents the region on right side of the perpendicular bisector of z_1 and z_2.



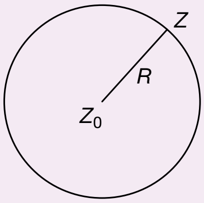
Therefore, $|z-2| > |z-4|$

$\Rightarrow$ $\text{Re}(z) > 3$ and $\text{Im}(z) \in R$

$x > 3$ and $y \in R$

Circle

Circle whose centre is Z_0 and radius R then its equation is $|Z - Z_0| = R$



(i) $|z-z_0| < r$ represents interior of a circle $|z-z_0| = r$

(ii) $|z-z_0| > r$ represents exterior of the circle $|z-z_o| = r$.

ILLUSTRATION 68

Plot on Argand plane $|z-3-4i| \le 5$.

SOLUTION

let $z = x + iy$

$$|x-3+i(y-4)| \le 5$$

$$\Rightarrow \quad (x-3)^2 + (y-4)^2 \le 25$$

Z lies interior region of circle, centre (3, 4), radius 5, including circumference

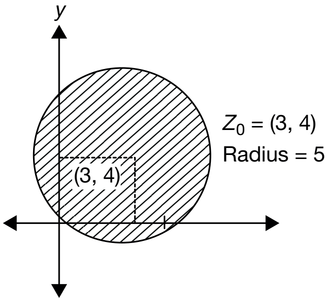


ILLUSTRATION 69

Plot on Argand plane $|z-1+i| > 2$.

SOLUTION

let $z = x + iy$

$$|x-1+i(y+1)| > 2$$

$$\Rightarrow \quad (x-1)^2 + (y+1)^2 > 4$$

Z lies outside of circle centre (1, –1), radius 2 excluding circumference

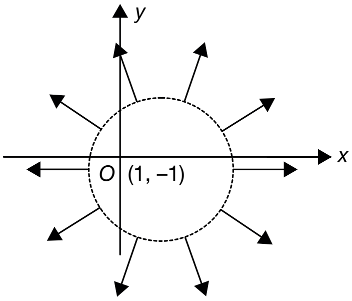


ILLUSTRATION 70

Plot on Argand plane $1 \le |z-1| < 3$.

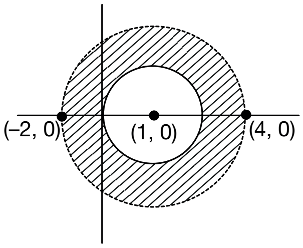
SOLUTION

let $z = x + iy$

$$| \le |x-1+iy| \le 3$$

$$| \le (x-1)^2 + y^2| \le 9$$

Z lies between two concentric circles of radius 1 and 3.

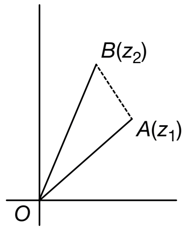


Inner circumference is included but outer circumference is excluded.

***In later section, we will discuss the circle in detail.**

TRIANGULAR INEQUALITY

Let us consider a triangle OAB whose vertices A and B are represented by complex numbers z_1 and z_2, respectively.



Now, length of the sides of ΔOAB are given as,

side $OA = |z_1 - 0| = |z_1|$

side $OB = |z_2 - 0| = |z_2|$

side $AB = |z_1 - z_2|$

In any triangle, the sum of any two sides is greater than the third side and the difference of any two side is less than the third side. By applying this basic concept, we get the following results.

$$|OA - OB| < AB < OA + OB$$

$$||z_1| - |z_2|| < |z_1 - z_2| < |z_1| + |z_2|$$

Replace z_2 by $-z_2$

$$|-z_2| = |z_2|$$

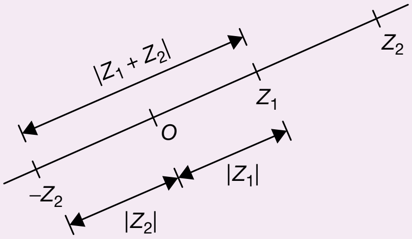
$$||z_1| - |-z_2|| < |z_1 + z_2| < |z_1| + |-z_2|$$

$$\boxed{\begin{array}{l} ||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2| \\ ||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2| \end{array}}$$

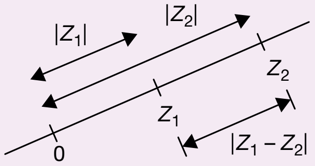
Let us consider when equality holds

Case (I)

$$|z_1 + z_2| = |z_1| + |z_2|$$



$$||z_1| - |z_2|| = |z_1 - z_2|$$

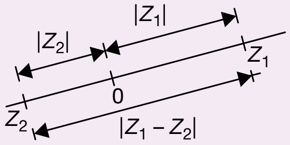


(i) Origin, z_1 and z_2 are collinear.

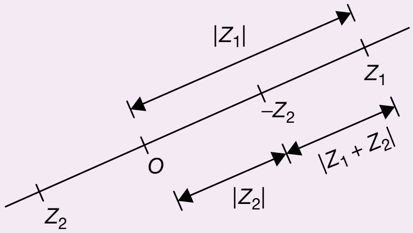
(ii) z_1 and z_2 are on the same side of origin.

Case (II)

$$|z_1 - z_2| = |z_1| + |z_2|$$



$$||z_1| - |z_2|| = |z_1 + z_2|$$



(i) Origin, z_1 and z_2 are collinear.

(ii) z_1 and z_2 lie on opposite side of origin.

Working Rule

TRICK

Same sign, then same side of origin
(opposite sign, then opposite side of origin).

The one and only one case in which $|z| + |z_2| + \dots + |z_n| = |z_1 + z_2 + \dots + z_n|$ is that, the numbers $z_1, z_2, \dots\dots z_n$ have the same amplitude.

ILLUSTRATION 71

Find the minimum value of $|1 - z| + |1 + z|$.

SOLUTION

Using Triangular Inequality

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$$

For this question we need $|z_1 + z_2| \le |z_1| + |z_2|$

Let $z_1 = 1 - z \quad z_2 = 1 + z$

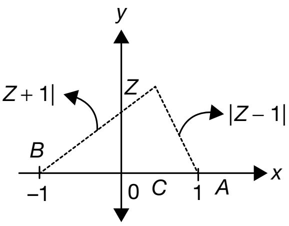
$$|1 - z + 1 + z| \le |1 - z| + |1 + z|$$

$$2 \le |1 - z| + 1\ |1 + z|$$

Minimum of $|1 - z| + |1 + z| = 2$

Alternative Method (Using Geometry)

$|1 - z| + |z + 1| = |z - 1| + |z - (-1)|$ (Since $|-z| = |z|$)



Minimum of sum of two sides AC and BC = third side AB (when triangle is not possible)

$$= 2$$

ILLUSTRATION 72

If $\left|z-\dfrac{2}{z}\right|=1$, find $|z|_{\max}$ and $|z|_{\min}$

SOLUTION

Using

$$||z_1|-|z_2|\le|z_1-z_2|\le|z_1|+|z_2|$$

Let $z_1=z$ and $z_2=\dfrac{2}{z}$

$$\left||z|-\frac{2}{|z|}\right|\le\left|z-\frac{2}{z}\right|\le|z|+\frac{2}{|z|}$$

$$\left||z|-\frac{2}{|z|}\right|\le1\le|z|+\frac{2}{|z|}\qquad(\text{Since, }|z|>0)$$

$$-1\le|z|-\frac{2}{|z|}\le1\qquad\ldots(1)$$

and $\quad 1\le|z|+\dfrac{2}{|z|}\Rightarrow|z|^2+2\ge|z|\qquad\ldots(2)$

Solving (2) we get, $|z|^2-|z|+2\ge0$

$D<0$ is always true for every $|z|>0$

Solving (1)

for $|z|\in R^+$

$$-|z|\le|z|^2-2\le|z|$$

$$|z|^2+|z|-2\ge0\text{ and }|z|^2-|z|-2\le0$$

$$(|z|+2)(|z|-1)\ge0\text{ and }(|z|-2)(|z|+1)\le0$$

$$|z|\in(-\infty,-2]\cup[1,\infty)\text{ and }-1\le|z|\le2$$

Common value $1\le|z|\le2$

Final answer common values of both conditions is

$$1\le|z|\le2$$

$$|z|_{\max}=2\text{ and }|z|_{\min}=1$$

ILLUSTRATION 73

If $|z-4-3i|=2$, find $|z|_{\max}$ and $|z|_{\min}$. Also find farthest and nearest points from origin on given curve.

SOLUTION

Using $||z_1|-|z_2||\le|z_1-z_2|\le|z_1|+|z_2|$

Let $z_1=z$ and $z_2=4+3i$

$$||z|-|4+3i||\le|z-4-3i|\le|z|+|4+3i|$$

$$||z|-5|\le2\le|z|+5$$

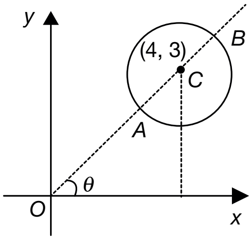
$$||z|-5|\le2\text{ and }|z|\ge-3\quad(\text{Always true})$$

$$-2\le|z|-5\le2\text{ and }3\le|z|\le7$$

$\therefore\quad |z|_{\min}=3,\ |z|_{\max}=7$

Alternate Method

$$|Z-(4+3i)|=2$$



$|Z|$ = distance of z from origin

$OC=|4+3i|=5$

Radius $(R)=2$

Minimum and maximum distance lies along normal.

$$|Z|_{\min}=OA=OC-R=5-2=3$$

$$|Z|_{\max}=OB=OC+R=5+2=7$$

$$\tan\theta=\frac{3}{4};\ \sin\theta=\frac{3}{5};\ \cos\theta=\frac{4}{5}$$

A: Nearest point from origin $=\left(3e^{\left(\tan^{-1}\frac{3}{4}\right)i}\right)$

$$=3(\cos\theta+i\sin\theta)=3\left(\frac{4}{5}+i\frac{3}{5}\right)$$

B: Farthest point from origin

$$=\left(7e^{i\tan^{-1}(3/4)}\right)=7(\cos\theta+i\sin\theta)=7\left(\frac{4}{5}+i\frac{3}{5}\right)$$

ILLUSTRATION 74

$|Z-15i|\le10$, find $|Z|_{\max/\min}$ and $(\operatorname{Arg}(Z))_{\max/\min}$.

Also find corresponding complex numbers.

SOLUTION

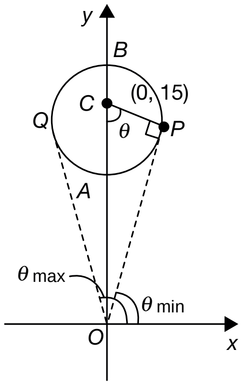
$|z|_{\max/\min}$ lie along normal passing through origin.

$$|Z|_{\max}=OC+R=25=OB$$

$$|Z|_{\min}=OC-R=5=OA$$

Point B (0, 25) $Z_B=25i$

Point A (0, 5) $Z_A=5i$



$$\cos\theta = \frac{10}{15} = \frac{2}{3}$$

$$(\text{Arg}(z))_{\min} = \cos^{-1}\frac{2}{3}$$

$$(\text{Arg}(z))_{\max} = \pi - \cos^{-1}\frac{2}{3}$$

$$CP^2 + OP^2 = OC^2 \quad \Rightarrow \quad 10^2 + OP^2 = 15^2$$

$$\Rightarrow \quad OP = 5\sqrt{5} = OQ$$

Point $P\left(5\sqrt{2}e^{i\theta_{\min}}\right) = 5\sqrt{2}e^{i\cos^{-1}\frac{2}{3}}$

Point $Q\left(5\sqrt{2}e^{i\theta_{\max}}\right) = 5\sqrt{2}e^{i\left(\pi-\cos^{-1}\frac{2}{3}\right)}$

ILLUSTRATION 75

If $|z_1| = 6$, $z_2 = 24 + 7i$, then find the maximum and minimum value of $|z_1 + z_2|$. Also find the corresponding of the complex number z_1.

SOLUTION

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$$

$$(|z_1| = 6,\ |z_2| = 25)$$

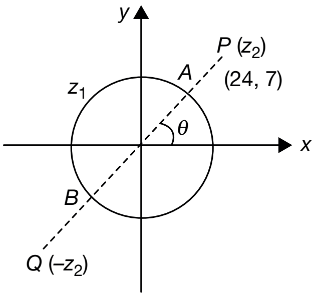
$$|6 - 25| \le |z_1 + z_2| \le 6 + 25$$

$$19 \le |z_1 + z_2| \le 31$$

Alternative Method: (Geometrical Method)

$$|z_1 + z_2| = |z_1 - (-z_2)|$$

It means, distance between z_1 and $-z_2$ is minimum and the maximum distance lie along normal. Point A is for maximum distance and point B is for minimum distance.



$$OA = OB = 6$$

$$\tan\theta = \frac{7}{24},\ \cos\theta = \frac{24}{25},\ \sin\theta = \frac{7}{25}$$

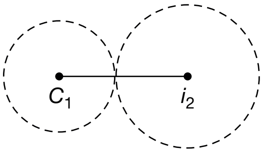
$$z_{1(A)} = OAe^{i\theta} = 6\left(\frac{24+7i}{25}\right)$$

and $\quad z_{1(B)} = -OBe^{i\theta} = -6\left(\frac{24+7i}{25}\right)$

ILLUSTRATION 76

If the circles $|z - 1| = 3$ and $|z - z_0| = 4$ touch externally, then find range of z_0.

SOLUTION



First circle $C_1(1, 0)$, $r_1 = 3$ second circle $C_2(2, 0)$ $r_2 = 4$

The circles touch if $C_1C_2 = r_1 + r_2$

$$|z_0 - 1| = 3 + 4 = 7$$

$$|z_0| = |z_0 - 1 + 1| \le |z_0 - 1| + 1 = 8$$

$$|z_0| = |z_0 - 1 + 1| \ge ||z_0 - 1| - 1| = 6$$

$$\therefore \quad 6 \le |z_0| \le 8$$

ILLUSTRATION 77

If $w = \dfrac{z}{z - \frac{i}{3}}$ and $|w| = 1$, then find locus of z

SOLUTION

Given that, $w = \dfrac{z}{z - \frac{i}{3}}$ and $|w| = 1$

$$\Rightarrow \quad \left|\frac{z}{z - \frac{i}{3}}\right| = 1 \Rightarrow |z| = \left|z - \frac{i}{3}\right|$$

So, z lies on the perpendicular bisector of $(0, 0)$ and $\left(0, \frac{1}{3}\right)$. Hence, z lies on a straight line.

ILLUSTRATION 78

If $|z| = 1$ and $z \ne \pm 1$, then show that all the values of $\dfrac{z}{1 - z^2}$ lie on imaginary axis

SOLUTION

Let $z = \cos\theta + i\sin\theta$ as $|z| = 1$

$$\Rightarrow \quad \frac{z}{1 - z^2} = \frac{\cos\theta + i\sin\theta}{1 - (\cos 2\theta + i\sin 2\theta)}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - 2i\sin\theta\cos\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{-2i\sin\theta(\cos\theta + i\sin\theta)} = \frac{i}{2\sin\theta}$$

Hence, $\dfrac{z}{1 - z^2}$ lies on the imaginary axis, i.e., y-axis.

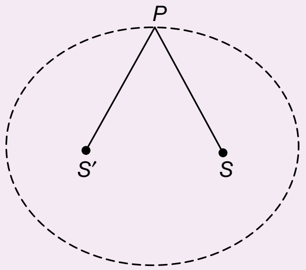
Alternative method

Let $\dfrac{z}{1 - z^2} = \dfrac{z}{z\bar{z} - z^2} = \dfrac{1}{\bar{z} - z}$ which is imaginary.

$\therefore \quad \dfrac{z}{1 - z^2}$ lies on imaginary axis

Ellipse

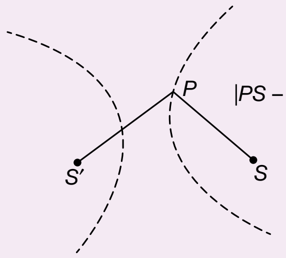
Locus of a point moving in a plane such that its sum of distances from two fixed points is constant.



$$PS + PS' = 2a = \ell_{\text{major axis}}$$
$$= \text{constant}$$

Hyperbola

Locus of a point moving in a plane such that its difference of distances from two fixed points is constant.



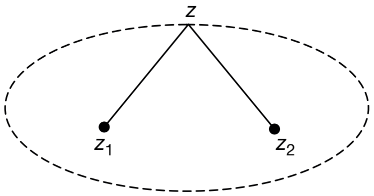
$$|PS - PS'| = 2a = \ell_{\text{transverse axis}}$$
$$= \text{constant}$$

Profile (I)

$|z - z_1| + |z - z_2| = K$

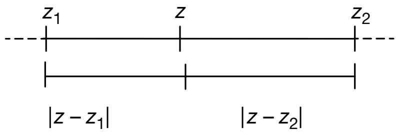
$K > |z_1 - z_2|$

Locus of $z \rightarrow$ Ellipse
(z_1 and z_2 are focii)



$K = |z_1 - z_2|$

z lies on line segment
joining z_1 and z_2



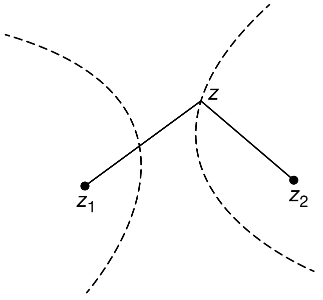
$K < |z_1 - z_2|$
$z \in \phi$

Profile (II)

$||z - z_1| - |z - z_2|| = K$

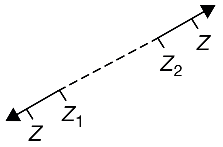
$K < |z_1 - z_2|$

Locus of $z \rightarrow$ Hyperbola with
focii z_1 and z_2



$K = |z_1 - z_2|$

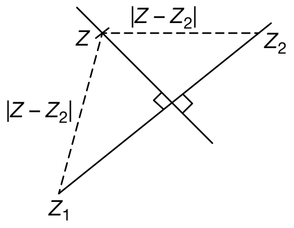
$z \rightarrow A$ pair of rays



$K = 0$

$|z - z_1| = |z - z_2|$

$z \rightarrow$ perpendicular bisector of
line joining z_1 & z_2



$K > |z_1 - z_2|$
$z \in \phi$

ILLUSTRATION 79

Identify the locus of z, if $|z-4i|+|z+4i|=10$.

SOLUTION

Let $z_1=4i;\ z_2=-4i$

Here, $|z_1-z_2|=|4i-(-4i)=|8\,i|=8$ (third side)

Sum of two sides > Third side

Therefore, locus of z is an ellipse, whose foci are $(0, 4)$ and $(0, -4)$.

ILLUSTRATION 80

Identify the locus of z, if $||z-3+4i|-|z+2-i||=1$.

SOLUTION

$$|z-(3-4i)|-|z-(-2+i)|=1$$

$$z_1=3-4i,\ z_2=-2+i$$

Here, $|z_1-z_2|=|3-4i-(-2+i)|$

$$=|5-5i|=5\sqrt{2}\ \text{(third side)}$$

Difference of two sides < Third side
Therefore, locus of z is hyperbola, whose focus $(3, -4)$ and $(-2, 1)$.

ILLUSTRATION 81

Identify locus of z, if $|z-8+2i|+|z-5+6i|=5$

SOLUTION

$$|z-(8-2i)|+|z-(5-6i)|=5$$

$$z_1=8-2i,\ z_2=5-6i$$

Here, $|z_1-z_2|=|8-2i-5+6i|=|3+4i|=5$

Therefore, $|z-z_1|+|z-z_2|=|z_1-z_2|$ (third side).

It means, z lies on line segment joining $(8, -2)$ and $(5, -6)$.

ILLUSTRATION 82

If $|(1+i)\,z+(1-i)\,\overline{z}+2|=|z-2|$, then interpret the locus of z.

SOLUTION

Let $z=x+iy$

$$|(1+i)\,(x+iy)+(1-i)\,(x-iy)+2|=|x+iy-2|$$

$$|x+iy+ix-y+x-iy-ix-y+2|=|x-2+iy|$$

$$|2x-2y+2|=|x-2+iy|$$

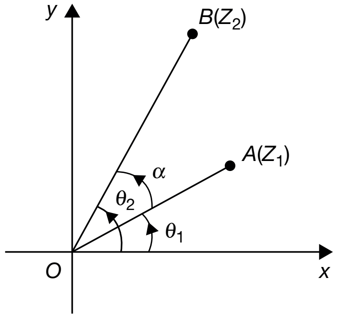
$$\Rightarrow\quad 2\sqrt{2}\left|\frac{x-y+1}{\sqrt{2}}\right|=\sqrt{(x-2)^2+y^2}$$

$$PS=ePM$$

Directrix is $x-y+1=0$ and focus $(2, 0)$.

Clearly, focus is not on directrix. So it can be conic section eccentricity $(e)=2\sqrt{2}$. Hence, locus of z is hyperbola.

ROTATION THEOREM (CONI METHOD)



Let consider $z_1=r_1\,e^{i\theta_1},\ z_2=r_2e^{i\theta_2}$

Ray OZ_1 is rotated by an angle α to become ray OZ_2 we have $\theta_2=\theta_1+\alpha$

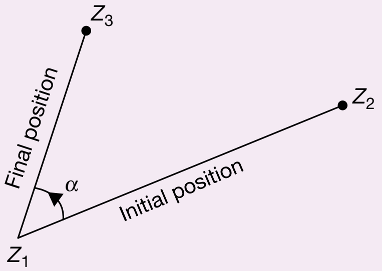
$$z_2=r_2e^{i(\theta_1+\alpha)}\Rightarrow\frac{z_2}{r_2}=e^{i\theta_1}.e^{i\alpha}=\frac{z_1}{r_1}e^{i\alpha}$$

$$\Rightarrow\quad\frac{z_2}{r_2}=\frac{z_1}{r_1}e^{i\alpha}$$

(i) Rotation About Origin

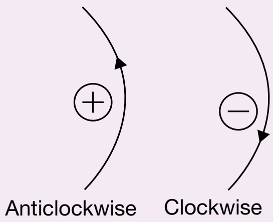
$$\frac{z_2-0}{|z_2-0|}=\frac{z_1-0}{|z_1-0|}e^{i\alpha}$$

(ii) Rotation About Z_1

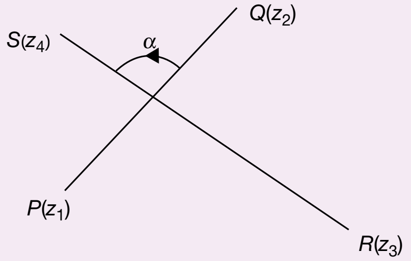


$$\boxed{\underbrace{\left(\frac{z_3-z_1}{|z_3-z_1|}\right)}_{\text{Final position}}=\underbrace{\left(\frac{z_2-z_1}{|z_2-z_1|}\right)}_{\text{Initial position}}e^{i\alpha}}$$

Where α is measured with sign according to following sign convention.
Anticlockwise rotation is considered as positive angle, whereas clockwise rotation is taken as negative angle.



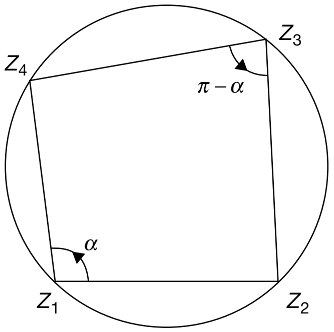
(iii) Vector *PQ*, after rotating by α angle becomes *RS* then we can write $\dfrac{z_4 - z_3}{|z_4 - z_3|} = \dfrac{z_2 - z_1}{|z_2 - z_1|} e^{i\alpha}$



(iv) If four points z_1, z_2, z_3 and z_4 are concyclic, then

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} = \text{real}$$

Proof:



Rotation theorem about z_1

$$\frac{z_4 - z_1}{|z_4 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} e^{i\alpha} \qquad \ldots(1)$$

Rotation theorem about z_3

$$\frac{z_2 - z_3}{|z_2 - z_3|} = \frac{z_4 - z_3}{|z_4 - z_3|} e^{i(\pi - \alpha)} \qquad \ldots(2)$$

Now

$$\frac{z_4 - z_1}{|z_4 - z_1|} \cdot \frac{z_2 - z_3}{|z_1 - z_3|} = \frac{z_2 - z_1}{|z_2 - z_1|} \frac{z_4 - z_3}{|z_4 - z_3|} e^{i(\alpha + \pi - \alpha)}$$

$$\Rightarrow \quad \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} = -\frac{|z_4 - z_1||z_2 - z_3|}{|z_2 - z_1||z_4 - z_3|}$$

$$= \text{purely real}$$

ILLUSTRATION 83

The vector $z = 2 + 3i$ is turned through an angle 270° in clockwise direction and stretched 3 times. Find the vector in new position.

SOLUTION

Rotation theorem about origin.

$$\frac{z_{new}}{|z_{new}|} = \frac{z_{old}}{|z_{old}|} e^{-i\frac{3\pi}{2}}$$

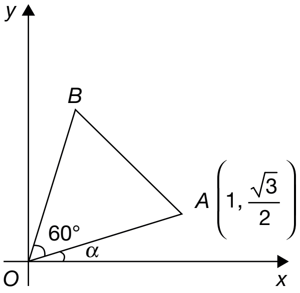
$$|Z_{\text{new}}| = 3|z_{\text{old}}|$$

$$z_{\text{new}} = 3(2 + 3i)(0 + i) = 3(-3 + 2i)$$

$$= -9 + 6i$$

ILLUSTRATION 84

OAB is equilateral triangle as shown in figure. Where *O* is origin and *A* is $1 + \dfrac{i\sqrt{3}}{2}$. Find the complex number represented for *B*.



SOLUTION

Rotation about origin

$$\frac{z_B - 0}{OB} = \frac{z_A - 0}{OA} e^{i\pi/3} \quad (OA = OB)$$

$$Z_B = \left(1 + \frac{\sqrt{3}}{2} i\right) e^{i\pi/3}$$

$$= \left(1 + \frac{\sqrt{3}}{2} i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \frac{-1}{4} + \frac{3\sqrt{3}}{4} i$$

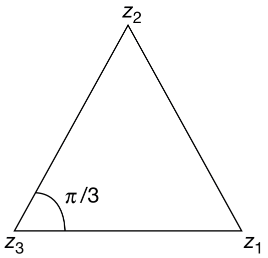
ILLUSTRATION 85

Show that the complex numbers z_1, z_2 and z_3 satisfying $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$ are the vertices of a equilateral triangle

SOLUTION

$$\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2} = \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$$

$$= \frac{1 - i^2 3}{2(1 + i\sqrt{3})} = \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$$



$$\Rightarrow \quad \frac{z_2-z_3}{z_1-z_3}=\frac{1+i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$\Rightarrow \quad \left|\frac{z_2-z_3}{z_1-z_3}\right|=1 \text{ and } \arg\left(\frac{z_2-z_3}{z_1-z_3}\right)=\frac{\pi}{3}$$

Hence, the triangle is an equilateral.

Remark

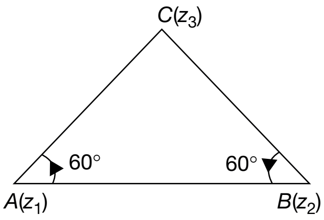
If z_1, z_2 and z_3 be the vertices of a triangle, then the triangle is equilateral iff

$$(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$$

or $\quad z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

or $\quad \dfrac{1}{z_1-z_2}+\dfrac{1}{z_2-z_3}+\dfrac{1}{z_3-z_1}=0$

Proof:



$$AB = BC = CA$$

Rotation about A $\dfrac{z_3-z_1}{CA}=\dfrac{z_2-z_1}{AB}e^{i\pi/3}$ ...(1)

Rotation about B $\dfrac{z_1-z_2}{AB}=\dfrac{z_3-z_2}{BC}e^{i\pi/3}$ ...(2)

Divide (1) and (2) $\dfrac{z_3-z_1}{z_1-z_2}=\dfrac{z_2-z_1}{z_3-z_2}$

On solving, $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

ILLUSTRATION 86

If $A(2+3i)$ and $B(3+4i)$ are two vertices of a square $ABCD$ (take in anticlockwise order), then find C and D.

SOLUTION

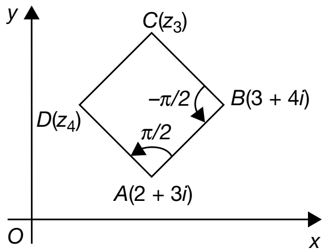
Let the affix of C and D are z_3 and z_4, respectively.

Considering $\angle DAB = 90°$ and $AD = AB$

On applying coni method at A,

we get $\dfrac{z_4-(2+3i)}{(3+4i)-(2+3i)}=\dfrac{AD}{AB}e^{\frac{i\pi}{2}}$

$\Rightarrow \quad z_4-(2+3i)=(1+i)i$



$\Rightarrow \quad z_4=2+3i+i-1=1+4i$

Apply coni at B

and $\quad \dfrac{z_3-(3+4i)}{(2+3i)-(3+4i)}=\dfrac{CB}{AB}e^{-i\frac{\pi}{2}}$

$\Rightarrow \quad z_3=3+4i-(1+i)(-i)$

$\Rightarrow \quad z_3=3+4i+i-1=2+5i$

ILLUSTRATION 87

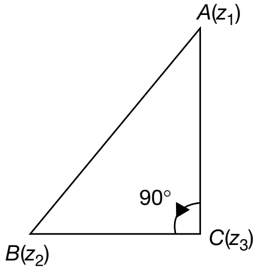
If z_1, z_2 and z_3 represents the vertices of an isosceles right angle triangle ABC respectively, right angled at C. Find the relation between z_1, z_2 and z_3. [IIT-1986]

SOLUTION

$BC = AC$ and $\angle C=\dfrac{\pi}{2}$

By rotation about C in anticlockwise sense,

$$\frac{z_2-z_3}{|z_2-z_3|}=\frac{z_1-z_3}{|z_1-z_3|}e^{i\frac{\pi}{2}} \Rightarrow \frac{z_2-z_3}{z_1-z_3}=\left|\frac{z_2-z_3}{z_1-z_3}\right|e^{i\frac{\pi}{2}}$$



$\Rightarrow \quad (z_2-z_3)=(z_1-z_3)\,e^{i\frac{\pi}{2}}=i(z_1-z_3)$

$\Rightarrow \quad (z_2-z_3)^2=-(z_1-z_3)^2 \Rightarrow (z_1-z_3)^2+(z_2-z_3)^2=0$

ILLUSTRATION 88

Prove that the roots of the equation $\dfrac{1}{z-z_1}+\dfrac{1}{z-z_2}+\dfrac{1}{z-z_3}=0$

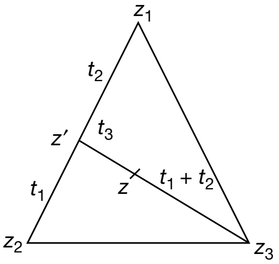
(where z_1, z_2 and z_3 are pairwise distinct complex numbers) corresponding to the points on a complex plane, which lie inside a triangle with vertices z_1, z_2 and z_3 excluding its boundaries.

SOLUTION

$$\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{1}{z-z_3}=0$$

$$\Rightarrow \quad \frac{\bar{z}-\bar{z}_1}{|z-z_1|^2}+\frac{\bar{z}-\bar{z}_2}{|z-z_2|^2}+\frac{\bar{z}-\bar{z}_3}{|z-z_3|^2}=0$$

$$\Rightarrow \quad t_1(\bar{z}-\bar{z}_1)+t_2(\bar{z}-\bar{z}_2)+t_3(\bar{z}-\bar{z}_3)=0 \qquad \ldots(1)$$



Let $|z-z_1|^2=\frac{1}{t_1}, |z-z_2|^2=\frac{1}{t_3}, |z-z_3|^2=\frac{1}{t_3}$

Where $t_1, t_2, t_3 \in R^+$

Taking conjugate of (1)

$$t_1(z-z_1)+t_2(z-z_2)+t_3(z-z_3)=0$$

$$(t_1+t_2+t_3)\,z=t_1z_1+t_2z_2+t_3z_3$$

$$\Rightarrow \quad z=\frac{t_1z_1+t_2z_2+t_3z_3}{t_1+t_2+t_3}$$

Let z' divides z_1 and z_2 in $t_2 : t_1$ ratio internally.

$$z'=\frac{t_1z_1+t_2z_2}{t_1+t_2}$$

$$z=\frac{t_1z_1+t_2z_2}{t_1+t_2}\cdot\frac{t_1+t_2}{t_1+t_2+t_3}+\frac{t_3z_3}{t_1+t_2+t_3}$$

$$=\frac{t_1+t_2}{t_1+t_2+t_3}z'+\frac{t_3z_3}{t_1+t_2+t_3}\Rightarrow z=\frac{(t_1+t_2)z'+t_3z_3}{t_1+t_2+t_3}$$

z divides line joining z_1 and z_3 in $t_3 : t_1+t_2$ ratio internally.

$\Rightarrow$ z lies inside the $\Delta z_1z_2z_3$

If $t_1=t_2=t_3 \Rightarrow z$ is the centroid of the triangle and also if $a=b=c$

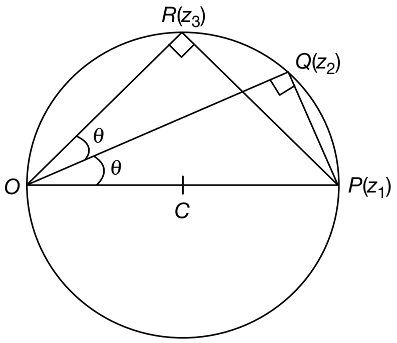
ILLUSTRATION 89

P is a points on the Argand diagram. On the circle with OP as diameter two points Q and R are chosen such that $\angle POQ=\theta$ and $\angle QOR=\theta$. If O is the origin and P, Q and R are represented by the complex numbers z_1, z_2 and z_3 respectively, prove that $z_1z_3\cos^2\theta = z_2^2\cos 2\theta$.

SOLUTION

We have $\angle POQ=\theta$ and $\angle POR=2\theta$

Also, $\angle OQP=\angle ORP=\frac{\pi}{2}$ (angle in the semicircle)



$$OP\cos\theta=OQ \quad\Rightarrow\quad |z_1|\cos\theta=|z_2|$$

$$OP\cos\theta=OR \quad\Rightarrow\quad |z_1|\cos 2\theta=|z_3|$$

rotation about origin

$$\frac{z_2-0}{|z_2-0|}=\frac{z_1-0}{|z_1-0|}e^{i\theta}\Rightarrow e^{i\theta}=\frac{z_2|z_1|}{z_1|z_2|}$$

$$\frac{z_3-0}{|z_3-0|}=\frac{z_1-0}{|z_1-0|}e^{i2\theta}=\frac{z_1}{|z_1|}\left(\frac{z_2|z_1|}{z_1|z_2|}\right)^2$$

$$\Rightarrow \quad z_1z_3|z_2|^2=z_2^2|z_3||z_1|$$

$$\Rightarrow \quad z_1z_3\left(|z_1|^2\cos^2\theta\right)=z_2^2|z_1|\cos 2\theta|z_1|$$

$$\Rightarrow \quad z_1z_3\cos^2\theta=z_2^2\cos 2\theta \quad \text{Hence proved}$$

ILLUSTRATION 90

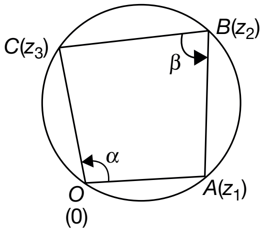
If z_1, z_2 and z_3 be three non-zero complex numbers which are in HP. then if z_1, z_2 and z_3 lies on circle, then show that circle also passes through origin.

SOLUTION

If $\frac{1}{z_1},\frac{1}{z_2}$ and $\frac{1}{z_3}$ are in AP.

Therefore, $\frac{1}{z_2}-\frac{1}{z_1}=\frac{1}{z_3}-\frac{1}{z_2}$

$$\Rightarrow \quad \frac{z_1-z_2}{z_1}=\frac{z_2-z_3}{z_3} \quad\Rightarrow\quad \frac{z_3}{z_1}\left(\frac{z_1-z_2}{z_3-z_2}\right)=-1$$



$$\Rightarrow \quad \text{Arg}\left(\frac{z_3}{z_1}\right) + \text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pi$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z_3 - 0}{z_1 - 0}\right) + \text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pi$$

$$\Rightarrow \quad \alpha + \beta = \pi$$

Therefore, z_1, z_2 and z_3 are concyclic.

Test Your Concepts-V

(Solutions on page H.100)

1. Find the greatest and least values of $|z|$, if z satisfies $\left|z - \frac{4}{z}\right| = 2$.

2. If (i) $|z_1 + z_2| = |z_1 - z_2|$, then find $\frac{z_1}{z_2}$

(ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$, then find $\frac{z_1}{z_2}$

(iii) $|z_1 + z_2| = |z_1| + |z_2|$, then find $\frac{z_1}{z_2}$

(iv) $|z_1 + z_2| = \big||z_1| - |z_2|\big|$, then find $\frac{z_1}{z_2}$

3. If $|z + 4| \le 3$, then find the maximum value of $|z + 1|$

4. ΔAOB is isosceles triangle such that $\angle AOB$ = 90°. If $O(0, 0)$ and $A(1, 2)$ are the vertices in Argand plane, find B in complex plane.

5. Let z be a complex number satisfying $|z - 5i| \le 1$ such that Amp(z) is minimum. Then find z.

6. If $|3z - 1| = 3|z - 2|$. Find locus of z.

7. If z be a complex number such that $|2z - 5| + |2z + 5| = 10$, then find the locus of z

Answer Keys

1. $\sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$

2. (i) purely imaginary (ii) purely imaginary (iii) purely real (iv) purely real

3. 6

4. (−2, 1), (2, −1)

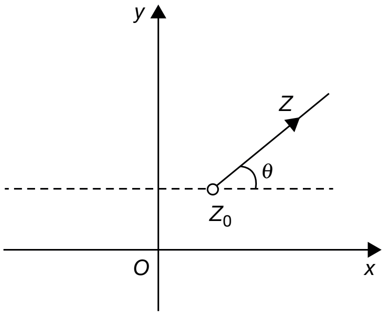
5. $\frac{2\sqrt{6}}{5} + \frac{24i}{5}$

6. a line parallel to y-axis

7. line segment joining point $\left(\frac{5}{2}, 0\right)$ and $\left(-\frac{5}{2}, 0\right)$

LOCUS PROBLEMS ON ARGUMENT

$\text{Arg}(z - z_0) = \theta$ = A ray originated from z_0 at an angle θ with positive real axis, where z_0 is excluded.



$\text{Arg}(z) = \text{Arg}(z - 0) = \theta$

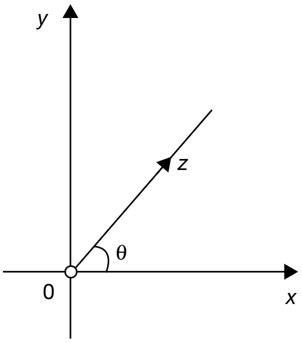


ILLUSTRATION 91

Plot locus of z on Argand plane, if Arg $(z-2+3i)=-\dfrac{\pi}{4}$.

SOLUTION

$$\arg(z-(2-3i))=-\frac{\pi}{4}$$

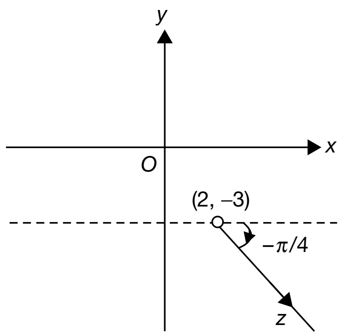
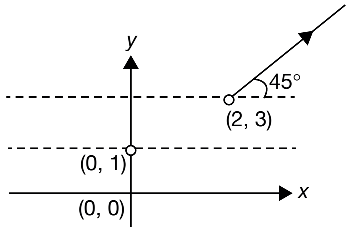


ILLUSTRATION 92

Find the point of intersection of Arg$(z-i)=\dfrac{\pi}{2}$ and Arg$(z-2-3i)=\dfrac{\pi}{4}$.

SOLUTION



Since these rays do not intersect, hence there is no point of intersection.

ILLUSTRATION 93

Plot locus of z on Argand plane if Arg $(z+3+4i)=\dfrac{2\pi}{3}$.

SOLUTION

$$\arg(z-(-3-4i))=\frac{2\pi}{3}$$

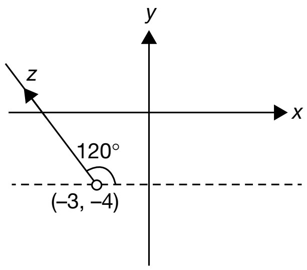


ILLUSTRATION 94

Draw the region satisfied by z on Argand plane, if Arg $(z+i)\geq\dfrac{\pi}{3}$.

SOLUTION

As we know that principal argument lies in $-\pi<\theta\leq\pi$

$$\frac{\pi}{3}\leq\arg(z-(-i))\leq\pi$$

Both boundaries are included

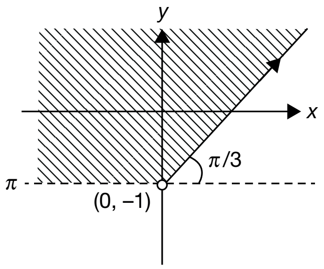


ILLUSTRATION 95

Draw the region satisfied by z on Argand plane if arg $(z-2+2i)<\dfrac{\pi}{4}$.

SOLUTION

As we know that principal argument lies in $-\pi<\theta\leq\pi$

$$\therefore\quad -\pi<\arg(z-(2-2i))<\frac{\pi}{4}$$

Both boundaries are not included

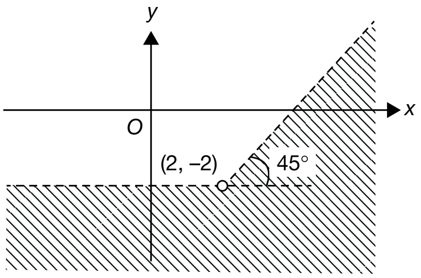


ILLUSTRATION 96

Draw the region satisfied by z on Argand plane if $\dfrac{\pi}{3}\leq$ Arg $(z-1-i)<\dfrac{5\pi}{6}$.

SOLUTION

$$\frac{\pi}{3}\leq\operatorname{Arg}(z-(1+i))<\frac{5\pi}{6}$$

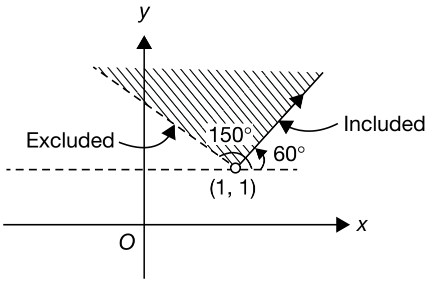


ILLUSTRATION 97

Draw the region satisfied by z on Argand plane if $|\text{Arg}\,(z-1+i)| \le \frac{\pi}{4}$.

SOLUTION

$$\frac{-\pi}{4} \le \text{Arg}(z-(1-i)) \le \frac{\pi}{4}$$

Both boundaries are included.

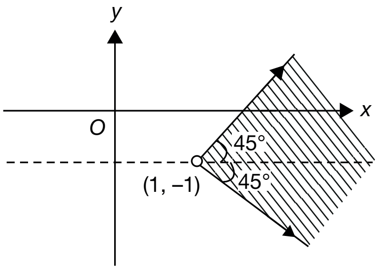


ILLUSTRATION 98

Draw the region $|z-3-4i| \le 5$ and

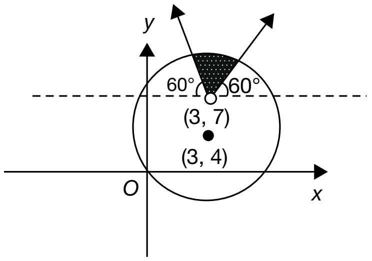
$$\frac{\pi}{3} \le \text{Arg}\,(z-3-7i) \le \frac{2\pi}{3}.$$

SOLUTION

$$|z-(3+4i)| \le 5$$

It is an interior region of circle, centre (3, 4), $r = 5$ including circumference.

$\frac{\pi}{3} \le \text{Arg}(z-(3+7i)) \le \frac{2\pi}{3}$; region is also shown.

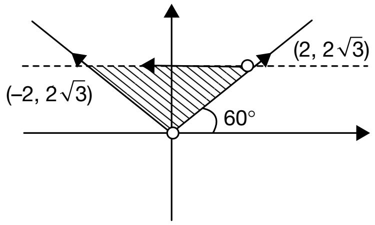


Common to both regions is shaded.

ILLUSTRATION 99

Find area enclosed by Arg $(z) = \frac{\pi}{3}$ and Arg$(z) = \frac{2\pi}{3}$, Arg $(z-2-2\sqrt{3}i) = \pi$.

SOLUTION



$$\text{Area} = \frac{1}{2}(2)(4\sqrt{3}) = 4\sqrt{3}$$

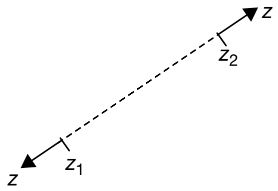
ARGUMENT BASED LOCUS PROBLEMS

Locus of z, if $\text{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \theta$

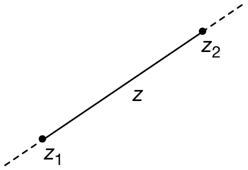
# $\text{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \theta$

$$\Rightarrow \arg\left(\frac{z_1-z}{z_2-z}\right) = \text{Arg}(\underset{\text{final}}{z_1} - z) - \text{Arg}(\underset{\text{initial}}{z_2} - z) = \theta$$

Case (I): $\theta = 0$; locus of z is pair of rays



Case (II): $\theta = \pi$; z locus of z is line segment joining z_1 and z_2.



Case (III): $0 < \text{Arg}\left(\frac{z-z_1}{z-z_2}\right) < \pi$; z locus of z is part of a circle.

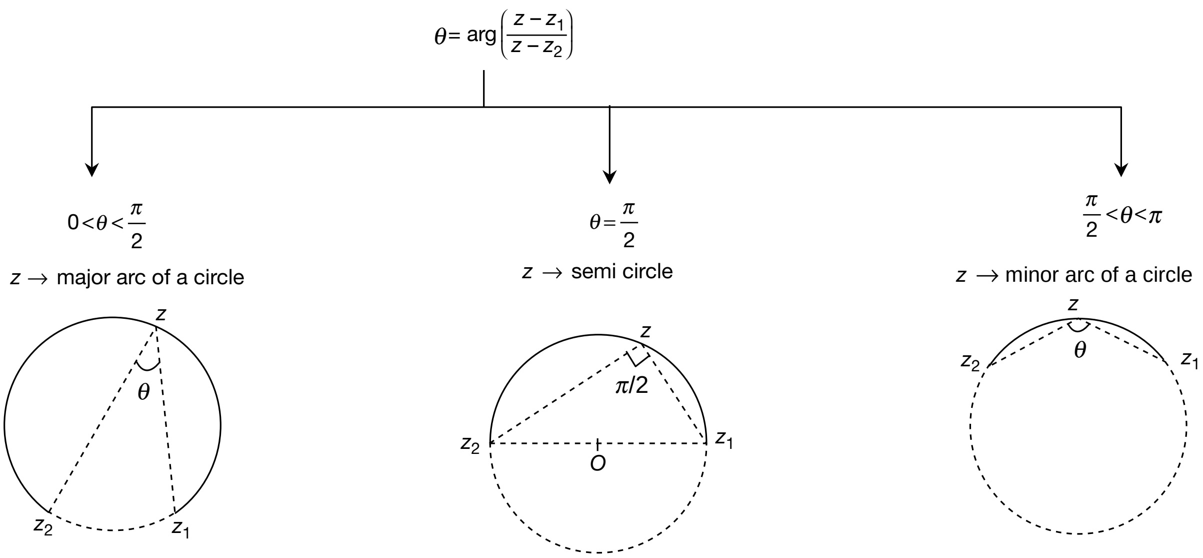

θ = arg((z − z1)/(z − z2))
0 < θ < π/2
z → major arc of a circle
θ = π/2
z → semi circle
π/2 < θ < π
z → minor arc of a circle
z
θ
z2
z1
π/2
O

ILLUSTRATION 100

Find locus of z, if $\operatorname{Arg}\left(\frac{z-3+4i}{z-2-5i}\right)=\frac{5\pi}{6}$.

SOLUTION

As $\theta=\frac{5\pi}{6}\in\left(\frac{\pi}{2},\pi\right)$

So locus of z is minor arc of a circle.

ILLUSTRATION 101

Find locus of z, if $\operatorname{Arg}\left(\frac{z-3+5i}{z-1-i}\right)=\pi$.

SOLUTION

$$\arg\left(\frac{z-(3-5i)}{z-(1+i)}\right)=\pi$$

z lies on line segment joining point (1, 1) and (3, –5)

ILLUSTRATION 102

Plot the locus of z, if $\operatorname{Arg}\left(\frac{z-i}{z+i}\right)=-\frac{\pi}{2}$.

SOLUTION

$$\arg\left(\frac{z-i}{z+i}\right)=\arg\left(\frac{i-z}{-i-z}\right)=\arg(i-z)-\arg(-i-z)=-\frac{\pi}{2}$$

z is semicircle of circle $x^2+y^2=1$ $(x\geq 0)$.

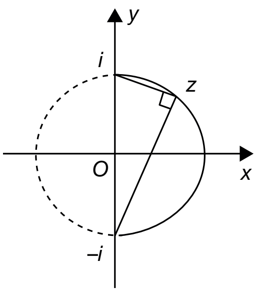


ILLUSTRATION 103

Find locus of the points z which satisfy the condition $\operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}$

SOLUTION

let $z=x+iy$

We have $\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{x-1+iy}{x+1+iy}\cdot\frac{x+1-iy}{x+1-iy}$

$$=\frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\Rightarrow\quad \operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

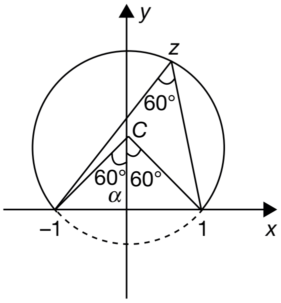
Therefore, $\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)=\frac{\pi}{3}$

$$\Rightarrow\quad \frac{2y}{x^2+y^2-1}=\tan\frac{\pi}{3}=\sqrt{3}\;\Rightarrow x^2+y^2-1=\frac{2}{\sqrt{3}}y$$

However, locus is major arc of the circle with respect to chord whose extremities are (1, 0) and (–1, 0).

Alternative Solution

$$\arg\left(\frac{z-1}{z+1}\right)=\arg\left(\frac{1-z}{-1-z}\right)=\frac{\pi}{3}$$



$$\tan 60^\circ = \frac{1}{\alpha} = \sqrt{3}$$

$$\Rightarrow \quad \alpha = \frac{1}{\sqrt{3}} \text{ centre of circle } C\left(0, \frac{1}{\sqrt{3}}\right)$$

$$r = \sqrt{\alpha^2 + 1} = \frac{2}{\sqrt{3}}$$

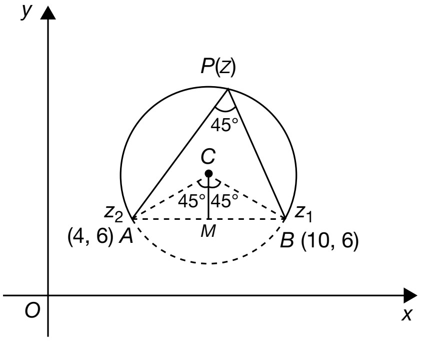
Required circle $\left|z - \frac{i}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$ major arc of this circle which is above x-axis.

ILLUSTRATION 104

Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{(z - z_1)}{(z - z_2)}$ *is* $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$. **[IIT-1991]**

SOLUTION

Since, $z_1 = 10 + 6i$, $z_2 = 4 + 6i$ and $\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$ represents locus of z is a circles shown as from the figure.



$$AM = BM = 3;\ CM = 3$$

$$r = AC = 3\sqrt{2};\ \text{centre } (7, 9)$$

$$\Rightarrow \quad \text{Equation of circle is } |z - (7 + 9i)| = 3\sqrt{2}.$$

ILLUSTRATION 105

Let $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$ where p, q, r are modulus of u, v, w complex numbers. Prove that $\text{Arg}\left(\frac{w}{v}\right) = \text{Arg}\left(\frac{w-u}{v-u}\right)^2$

SOLUTION

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0 \Rightarrow 3pqr - p^3 - q^3 - r^3 = 0$$

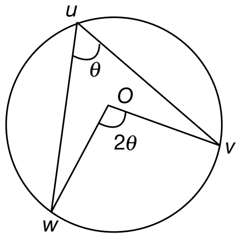
$$\Rightarrow \quad \frac{-(p+q+r)}{2}\left((p-q)^2 + (q-r)^2 + (r-p)^2\right) = 0$$

p, q and r are modulus so $p + q + r$ cannot be zero. It means, sum of squares must be zero which results that each square term is zero. $p = q = r$

$$p = q = r \Rightarrow |u| = |v| = |w|$$

$$\Rightarrow \quad |u - 0| = |v - 0| = |w - 0|$$

Origin is circumcentre of Δ formed by u,v,w



Angle subtended by a chord at centre is doubles angle subtended at circumference

$$\text{Arg}\left(\frac{w-0}{v-0}\right) = 2\,\text{Arg}\left(\frac{w-u}{v-u}\right) = \text{Arg}\left(\frac{w-u}{v-u}\right)^2$$

Hence Proved

ILLUSTRATION 106

If $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$ and $\text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{\pi}{6}$, then find value of $z_2(z_2 - 1) - z_3(z_2 + 1) + (z_3 + 1)(z_3 - 1)$.

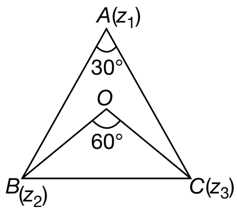
SOLUTION

As $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$, z_1, z_2 and z_3 are vertices of triangle ABC whose circumcentre is $O(1, 0)$.

As $\text{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{\pi}{6} \Rightarrow \angle BAC = 30^\circ$

Therefore, $\angle BOC = 60^\circ$ and $OB = OC$

So, ΔBOC is equilateral.



($\because$ if $P(u)$, $Q(v)$, $R(w)$ are forming equilateral triangle then $u^2 + v^2 + w^2 = uv + vw + wu$)

$$z_2^2 + z_3^2 + (1 + i0)^2 = z_2 z_3 + z_2(1 + i0) + z_3(1 + i0)$$

$$\Rightarrow \quad z_2^2 - z_2 - z_3(z_2 + 1) + z_3^2 + 1 = 0$$

$$\Rightarrow \quad z_2(z_2-1)-z_3(z_2+1)+z_3^2-1+2=0$$

$$\Rightarrow \quad z_2(z_2-1)-z_3(z_2+1)+(z_3-1)(z_3+1)=-2$$

STRAIGHT LINE

Equation of straight line in general form (In coordinate xy plane) is given by $ax+by+c=0$.

Slope of this line $m=-\dfrac{a}{b}$

Let $z=x+iy$; $\bar{z}=x-iy$

$$x=\frac{z+\bar{z}}{2};\ y=\frac{z-\bar{z}}{2i}$$

$$a\left(\frac{z+\bar{z}}{2}\right)+b\left(\frac{z-\bar{z}}{2i}\right)+c=0 \Rightarrow (a-ib)z+(a+ib)\bar{z}+2c=0$$

$$\bar{\alpha}z+\alpha\bar{z}+k=0 \text{ where } \alpha=a+ib$$

The general equation of a straight line is

$$\bar{\alpha}z+\alpha\bar{z}+k=0$$

where α is complex number and k is real number.

Complex slope(w) of line.

If z_1 and z_2 are two unequal complex number represented by points P and Q, then complex slope of line PQ is denoted by w and is defined as

$$w=\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} \qquad \ldots(1)$$

If z_1 and z_2 are two points on $\alpha\bar{z}+\bar{\alpha}z+b=0$, then

$$\alpha\bar{z}_1+\bar{\alpha}z_1+b=0 \qquad \ldots(2)$$

$$\alpha\bar{z}_2+\bar{\alpha}z_2+b=0 \qquad \ldots(3)$$

Eliminating b from (2) and (3)

$$\alpha(\bar{z}_1-\bar{z}_2)+\bar{\alpha}(z_1-z_2)=0$$

Therefore, $\dfrac{z_1-z_2}{\bar{z}_1-\bar{z}_2}=\dfrac{-\alpha}{\bar{\alpha}}$

Hence, Complex slope of $\alpha\bar{z}+\bar{\alpha}z+b=0$ is $w=\dfrac{-\alpha}{\bar{\alpha}}$.

So, from here we can say that **complex slope of line is always unimodular.** $\left(w=\dfrac{-\alpha}{\bar{\alpha}} \Rightarrow |w|=\dfrac{|-\alpha|}{|\bar{\alpha}|}=1\right)$

Real slope of the line

$$m=-\frac{a}{b}-\frac{\text{Re}(\alpha)}{\text{Im}(\alpha)}=-i\frac{(\alpha+\bar{\alpha})}{(\alpha-\bar{\alpha})}=\tan\theta \Rightarrow \frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}=i\tan\theta \qquad \ldots(4)$$

$$\Rightarrow \quad =\frac{\alpha}{\bar{\alpha}}=\frac{1+i\tan\theta}{1-i\tan\theta}=\frac{e^{i\theta}}{e^{-i\theta}}$$

$$\Rightarrow \quad w=e^{i2\theta}$$

Also if complex slope of PQ is w, then equation of line PQ can be written as $z-z_1=w(\bar{z}-\bar{z}_1)$.

Conclusion from results (1) and (4) is

The general equation of a straight line is of $\bar{\alpha}z+\alpha\bar{z}+k=0$. where α is complex number and k is real number.

Real slope of the line $m=-\dfrac{\text{Re}(\alpha)}{\text{Im}(\alpha)}=-i\dfrac{(\alpha+\bar{\alpha})}{(\alpha-\bar{\alpha})}=\tan\theta$.

The complex slope of the line $\bar{\alpha}z+\alpha\bar{z}+b=0$ is

$$w=-\frac{\alpha}{\bar{\alpha}}=-\frac{\text{coefficient of } \bar{z}}{\text{coefficient of } z}$$

$\boldsymbol{w=e^{i2\theta}}$ where θ is inclination angle of straight line made with real axis.

complex slope of linePQ $P(z_1)$ and $Q(z_2)$ is denoted by w and it is defined as $w=\dfrac{z_1-z_2}{\bar{z}_1-\bar{z}_2}$.

The two lines having complex slopes w_1 and w_2 are parallel iff $w_1=w_2$.

Proof:

Let the two lines be $z\bar{\alpha}+\bar{z}\alpha+b_1=0$ and $z\bar{\beta}++\bar{z}\beta+b_2=0$.

$$\text{Slope of first line} \Rightarrow m_1=\frac{-\text{Re}(\alpha)}{\text{Im}(\alpha)}$$

$$\text{Slope of second line} \Rightarrow m_2=\frac{-\text{Re}(\beta)}{\text{Im}(\beta)}$$

As the lines are parallel, $m_1=m_2 \Rightarrow \dfrac{-\text{Re}(\alpha)}{\text{Im}(\alpha)}=\dfrac{-\text{Re}(\beta)}{\text{Im}(\beta)}$

$$\Rightarrow \quad \frac{\alpha+\bar{\alpha}}{i(\alpha-\bar{\alpha})}=\frac{\beta+\bar{\beta}}{i(\beta-\bar{\beta})}$$

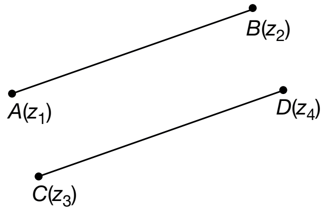
$$\Rightarrow \quad \alpha\beta-\alpha\bar{\beta}+\bar{\alpha}\beta-\overline{\alpha\beta}=\alpha\beta+\alpha\bar{\beta}-\bar{\alpha}\beta-\overline{\alpha\beta}$$

$$\Rightarrow \quad \alpha\bar{\beta}-\bar{\alpha}\beta=0 \Rightarrow \frac{\alpha}{\bar{\alpha}}=\frac{\beta}{\bar{\beta}} \Rightarrow w_1=w_2$$

Alternate:

If AB and CD two parallel lines, where $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$.

As $AB\,||\,CD$, so angle between AB and CD is 0 or π.



So, $\text{Arg}\left(\dfrac{z_1-z_2}{z_3-z_4}\right)=0 \Rightarrow \dfrac{z_1-z_2}{z_3-z_4}=\text{purely real}$

$$\Rightarrow \quad \overline{\frac{z_1-z_2}{z_3-z_4}} = \frac{z_1-z_2}{z_3-z_4} \Rightarrow \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} = \frac{z_3-z_4}{\bar{z}_3-\bar{z}_4}$$

$\Rightarrow \quad w_1 = w_2$ (by definition of complex slope)

Sum of complex slopes is zero for perpendicular lines $w_1 + w_2 = 0$.

Proof:

Let the two lines be $z\bar{\alpha} + \bar{z}\alpha + b_1 = 0$ and $z\bar{\beta} + +\bar{z}\beta + b_2 = 0$.

$$\text{Slope of 1st line} \Rightarrow m_1 = \frac{-\operatorname{Re}(\alpha)}{\operatorname{Im}(\alpha)}$$

$$\text{Slope of 2nd line} \Rightarrow m_2 = \frac{-\operatorname{Re}(\beta)}{\operatorname{Im}(\beta)}$$

As lines are perpendicular,

$$m_1 m_2 = -1 \Rightarrow \frac{\operatorname{Re}(\alpha)}{\operatorname{Im}(\alpha)} \cdot \frac{\operatorname{Re}(\beta)}{\operatorname{Im}(\beta)} = -1$$

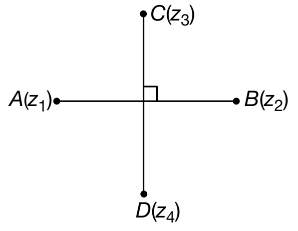
$$\Rightarrow \quad \frac{\alpha+\bar{\alpha}}{i(\alpha-\bar{\alpha})} \cdot \frac{\beta+\bar{\beta}}{i(\beta-\bar{\beta})} = -1$$

$$\Rightarrow \quad \alpha\beta + \alpha\bar{\beta} + \bar{\alpha}\beta + \bar{\alpha}\bar{\beta} = \alpha\beta - \alpha\beta - \bar{\alpha}\beta + \bar{\alpha}\bar{\beta}$$

$$\Rightarrow \quad \alpha\bar{\beta} + \beta\bar{\alpha} = 0 \Rightarrow \frac{\alpha}{\bar{\alpha}} + \frac{\beta}{\bar{\beta}} = 0 \Rightarrow w_1 + w_2 = 0$$

Alternate:

If AB and CD are two perpendicular lines,



$$\operatorname{Arg}\left(\frac{z_3-z_4}{z_1-z_2}\right) = \frac{\pi}{2} \Rightarrow \frac{z_3-z_4}{z_1-z_2} = \text{purely imaginary}$$

$$\Rightarrow \quad \overline{\frac{z_3-z_4}{z_1-z_2}} = -\left(\frac{z_3-z_4}{z_1-z_2}\right)$$

$$\Rightarrow \quad \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} + \frac{z_3-z_4}{\bar{z}_3-\bar{z}_4} = 0 \Rightarrow w_1 + w_2 = 0$$

EQUATION OF STRAIGHT LINE BY CONI METHOD

If $C(z)$ is any point on the line joining AB. Then

$\angle BCA = \pi, 0, -\pi$

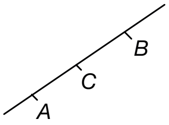
$$\text{So, } \operatorname{Arg}\left(\frac{z-z_1}{z_2-z_1}\right) = \pi, 0, -\pi$$

$$\text{So, } \left(\frac{z-z_1}{z_2-z_1}\right) = \text{purely real. Hence, } \frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$$

$$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1) - \bar{z}(z_2-z_1) - z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$$

PARAMETRIC FORM OF STRAIGHT LINE

Let $C(z)$ is any point on the line joining $A(z_1)$ and $B(z_2)$, which divides A and B in the ratio $\lambda : 1$.



$$z = \frac{\lambda z_2 + z_1}{\lambda + 1}$$

$$\Rightarrow z = \frac{\lambda}{\lambda+1} z_2 + \frac{1}{\lambda+1} z_1$$

$\Rightarrow \quad z = tz_1 + (1 - t)\, z_2$ is required equation of line.

Remark(s)

(a) If z_1, z_2 and z_3 are three complex number and $a, b, c \in R$ not all zero such that $az_1 + bz_2 + cz_3 = 0$ and $a + b + c = 0$, then the three points are collinear or points lie on line.

(b) If three complex numbers are in *AP*, then they lie on a straight line in the complex plane.

Determinant Form

Equation of a straight line joining the points having affixes

$$z_1 \text{ and } z_2 \text{ is } \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad z(\bar{z}_1-\bar{z}_2) - \bar{z}(z_1-z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0.$$

Remark(s)

CONDITION FOR COLLINEAR

(a) Three points z_1, z_2 and z_3 are collinear if,

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

(b) If three points $A(z_1), B(z_2), C(z_3)$ are collinear, then the slope of AB =slope of BC = slope of AC

$$\Rightarrow \frac{z_1-z_2}{\bar{z}_1-\bar{z}_2} = \frac{z_2-z_3}{\bar{z}_2-\bar{z}_3} = \frac{z_3-z_1}{\bar{z}_3-\bar{z}_1}.$$

Length of Perpendicular

The length of perpendicular from a point z_1 to the line $\bar{\alpha}z + \alpha\bar{z} + b = 0$ is given by $\dfrac{|\bar{\alpha}z_1 + \alpha\bar{z}_1 + b|}{2|\alpha|}$.

ILLUSTRATION 107

Consider the line $(1 + i)z - (1 - i)\,\bar{z} + 2i = 0$, find the distance of a point $2i$ from given line.

SOLUTION

Substitute $z = x + iy$

$$(1+i)z - (1-i)\,\bar{z} + 2i = 0$$

$$\Rightarrow \quad (1+i)(x+iy) - (1-i)(x-iy) + 2i = 0$$

$$\Rightarrow \quad x + iy + ix - y - x + iy + ix + y + 2i = 0$$

$$\Rightarrow \quad 2i\,(x+y+1) = 0 \Rightarrow x + y + 1 = 0 \qquad \text{(straight line)}$$

Distance of point (0,2) from line $= \left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

$$= \left|\frac{0+2+1}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}.$$

ILLUSTRATION 108

Find the equation of straight line passing through at $1+3i$ and $-1+i$ on Argand plane. Also find the point on this line which are at a distance 2 units from point $(1+3i)$.

SOLUTION

Point $A(1+3i)$ $A(1, 3)$

$B(-1+i)$ $B(-1, 1)$

$$m_{AB} = \frac{3-1}{1+1} = 1 = \tan\theta;\ \theta = 45^\circ$$

Parametric form of straight line

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r \Rightarrow \frac{x-1}{1/\sqrt{2}} = \frac{y-3}{1/\sqrt{2}} = \pm 2$$

Point $\left(1\pm\sqrt{2}, 3\pm\sqrt{2}\right)$.

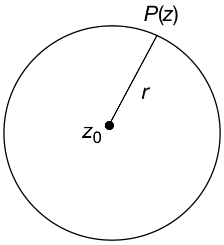
Straight line $y = x + 2$

let $z = x + iy;\ \bar{z} = x - iy$

$$\frac{z-\bar{z}}{2i} = \frac{z+\bar{z}}{2} + 2 \Rightarrow z(1+i) + (1-i)\bar{z} + 4 = 0$$

CIRCLE

(i) The equation of a circle whose centre is at point having affix z_o and radius r is $|z - z_0| = r$ centre (z_0) and radius $= r$.



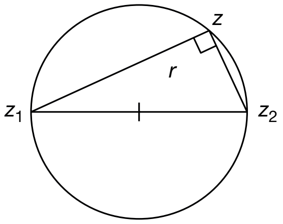
If the centre of the circle is at origin and radius r, then its equation is $|z| = r$.

(ii) $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_o| = r$.

(iii) **General equation of a circle:** The general equation of the circle is $z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + k = 0$ with centre $(-\alpha)$ and radius $= \sqrt{\alpha\bar{\alpha} - k}$

Working method (substitute $z = x + iy$)

(iv) **Equation of circle in diametric form:** If the endpoints of the diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ be any point on the circle then, $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$ on simplification $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ which is required equation of circle in diametric form.



(v) **Argument form of circle:** $\operatorname{Arg}\left(\dfrac{z-z_1}{z-z_2}\right) = \theta;\ \theta \neq 0, \pi$

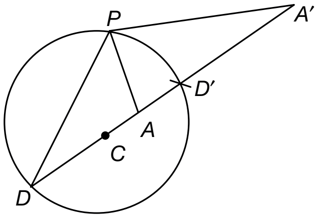
then z is a part of a circle $\operatorname{Arg}\left(\dfrac{z-z_1}{z-z_2}\right) = \pm\pi/2$ then

$z \to z$ is a semicircle with z_1 and z_2 as the extremities of diameter.

(vi) **Apollonius circle form:** $\left|\dfrac{z-z_1}{z-z_2}\right| = k$ where

$k \neq 1;\ k > 0$, then locus of z is circle

Working method (substitute $z = x + iy$)



Let PD and PD' are bisectors of $\angle APA'$, then $\angle DPD' = 90^\circ$

$$\left|\frac{z-z_1}{z-z_2}\right| = k \Rightarrow \frac{PA}{PA'} = k : 1$$

So, $P(z)$ describes a circle for which DD' is diameter.

As PD' is internal bisector, $AD' : D'A' = PA : PA'$, so D' divides AA' in $k : 1$ ratio internally.

Similarly, D divides AA' in $k : 1$ ratio externally.

So, coordinates of extremities of diameter are

$$D'\left(\frac{kz_2 + z_1}{k+1}\right), D\left(\frac{kz_2 - z_1}{k-1}\right)$$

Coordinates of centre $= C\left(\dfrac{k^2 z_2 - z_1}{k^2 - 1}\right)$

$$\text{Radius} = \frac{k}{k^2-1}|z_2 - z_1|$$

So, from here we can also prove that $CA \cdot CA' = CD'^2$

ILLUSTRATION 109

$z\bar{z} + (1-i)z + (1+i)\bar{z} - 7 = 0$, find the centre and radius.

SOLUTION

Given circle $z\bar{z} + (1-i)z + (1+i)\bar{z} - 7 = 0$ General circle is $z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + k = 0$ with centre $(-\alpha)$ and radius $= \sqrt{\alpha\bar{\alpha} - k}$

Now, compare the centre $(-\alpha) = (-1 - i)$

$$r = \sqrt{\alpha\bar{\alpha} - k} = \sqrt{2-(-7)} = 3$$

Alternate

Given circle $z\bar{z} + (1-i)z + (1+i)\bar{z} - 7 = 0$

Let $z = x + iy$

$\Rightarrow \quad (x+iy)(x-iy) + (1-i)(x+iy) + (1+i)(x-iy) - 7 = 0$

$\Rightarrow \quad x^2 + y^2 + 2x + 2y - 7 = 0,$

centre $(-1, -1)$, radius $= \sqrt{1+1+7} = 3,$

Circle $|z + 1 + i| = 3$

ILLUSTRATION 110

If $z\bar{z} + (-3+4i)\bar{z} - (3+4i)z + k = 0$. Find k for which it represents circle.

SOLUTION

For circle radius $= \sqrt{\alpha\bar{\alpha} - k}$

$$\alpha\bar{\alpha} - k > 0 \Rightarrow (-3+4i)(-3-4i) - k > 0$$

$\Rightarrow \quad 25 - k > 0 \Rightarrow k < 25$

ILLUSTRATION 111

If $\left|\frac{Z+4i}{Z-i}\right| = 2$, find centre and radius of the circle.

SOLUTION

Let $z = x + iy$ $|z + 4i| = 2|z - i|$

$\Rightarrow \quad |x + iy + 4i| = 2|x + iy - i|$

$\Rightarrow \quad x^2 + (y+4)^2 = 4(x^2 + (y-1)^2)$

$\Rightarrow \quad 16 + 8y = 3x^2 + 3y^2 + 4 - 8y$

$\Rightarrow \quad 3x^2 + 3y^2 - 16y - 12 = 0$

$\Rightarrow \quad x^2 + y^2 - \frac{16}{3}y - 4 = 0$

Centre $(0, 8/3)$; radius $r = \sqrt{0 + \frac{64}{9} + 4} = \frac{10}{3}$

Circle $\left|z - \frac{8}{3}i\right| = \frac{10}{3}$

ILLUSTRATION 112

Find the equation of circle which touches the line $iz + \bar{z} + i + 1 = 0$ and the lines $(2-i)z = (2+i)\bar{z}$ and $(2+i)z + (i-2)\bar{z} - 4i = 0$ are normal to circle.

SOLUTION

At first, we should convert equation of normal in cartesian form (substitute $z = x + iy$).

$(2-i)z = (2+i)\bar{z}$

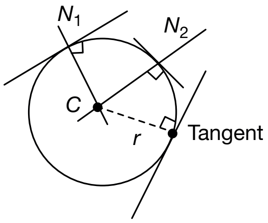
$\Rightarrow \quad (2-i)(x+iy) = (2+i)(x-iy) \Rightarrow 2y = x$

$(2+i)z + (i-2)\bar{z} - 4i = 0$

$\Rightarrow \quad (2+i)(x+iy) + (i-2)(x-iy) - 4i = 0$

$\Rightarrow \quad 4yi + 2ix - 4i = 0 \Rightarrow 2y + x - 2 = 0$

Point of intersection of normals = Centre $\left(1, \frac{1}{2}\right)$



Tangent $iz + \bar{z} + i + 1 = 0$ (substitute $z = x + iy$)

$\Rightarrow \quad i(x+iy) + x - iy + i + 1 = 0$

$\Rightarrow \quad (1+i)x - y(1+i) + (1+i) = 0 \quad \Rightarrow \quad x - y + 1 = 0$

Length of perpendicular from centre to tangent $(P) = R$ (radius)

$$\Rightarrow \quad \left|\frac{1 - \frac{1}{2} + 1}{\sqrt{2}}\right| \Rightarrow r = \frac{3}{2\sqrt{2}}$$

Circle $\left|z - 1 - \frac{i}{2}\right| = \frac{3}{2\sqrt{2}}$ (complex form)

ILLUSTRATION 113

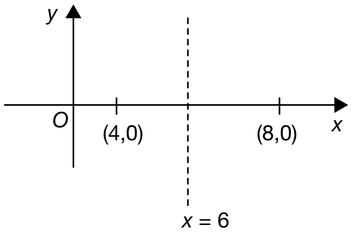
Find 'z' if $\left|\frac{z-4}{z-8}\right| = 1$ and $\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}$

SOLUTION

$$|z - 4| = |z - 8|$$

Let understand this equation geometrically. So, locus of z is the perpendicular bisector of line joining the points $(4, 0)$ and $(8, 0)$.

Let $z = x + iy$ we get $x = 6$, $z = 6 + iy$



Now, $\frac{|z-12|}{|z-8i|}=\frac{5}{3} \Rightarrow \left|\frac{iy-6}{6+i(y-8)}\right|=\frac{5}{3}$

$\Rightarrow \quad 3|-6+iy| = 5|6+i(y-8)|$

$\Rightarrow \quad 9(36+y^2) = 25\,(36+(y-8)^2)$

$\Rightarrow \quad y^2-25y+136=0$

$\Rightarrow \quad (y-17)(y-8)=0 \quad \Rightarrow \quad y=17, 8$

Possible values are $z=6+17i$ and $z=6+8i$.

ILLUSTRATION 114

If $2|z-1|=|z-2|$, then find the locus of z.

SOLUTION

Let $z=x+iy$

$$2|z-1|=|z-2|$$

So, $2|x-1+iy| = |x-2+iy|$

$\Rightarrow \quad 4[(x-1)^2+y^2]=(x-2)^2+y^2$

$\Rightarrow \quad 4x^2+4y^2-8x+4=x^2+y^2-4x+4$

$\Rightarrow \quad 3x^2+3y^2=4x$

$\Rightarrow \quad x^2+y^2-\frac{4}{3}x=0$ is required locus of z is a circle centre $\left(\frac{2}{3},0\right)$, $r=\frac{2}{3}$

ILLUSTRATION 115

If z is complex number and t be a real parameter. Find the curve represented by $z=2+it+(1+i)t^2$.

SOLUTION

Let $z=x+iy$.

So, $x+iy=2+t^2+i(t+t^2)$

On comparing real and imaginary terms of LHS and RHS,

$$x=2+t^2 \text{ and } y=t+t^2$$

Taking difference

$$x-y=2-t \quad \Rightarrow \quad t=2-x+y$$

$$x=2+(2-x+y)^2$$

$\Rightarrow \quad x=z+4+x^2+y^2-4x-2xy+4y$

$\Rightarrow \quad x^2+y^2-5x+4y-2xy+6=0$

ILLUSTRATION 116

Solve $2z=|z|+2i$.

SOLUTION

Let $z=x+iy$.

Given equation becomes $2(x+iy)=\sqrt{x^2+y^2}+2i$ on comparing real and imaginary part of LHS and RHS

$$2x=\sqrt{x^2+y^2} \quad (\because x \geq 0) \qquad \ldots(1)$$

and $y=1$

Substitute $y=1$ in (1) $\Rightarrow 2x=\sqrt{x^2+1} \Rightarrow x=\pm\frac{1}{\sqrt{3}}$

(As x can't be negative)

Therefore, $z=\frac{1}{\sqrt{3}}+i$

ILLUSTRATION 117

Find the general equation of circles which are orthogonal to the circle $|z|=1$ and $|z-1|=4$.

SOLUTION

Two given circles are

$$x^2+y^2=1 \text{ and } x^2+y^2-2x-15=0$$

Let the circle orthogonal to both be $x^2+y^2+2gx+2fy+c=0$.

By using the condition of orthogonality

$$2(g_1g_2+f_1f_2)=c_1+c_2 \text{ we get,}$$

$$2g\times 0+2f\times 0=c-1 \Rightarrow c=1$$

$$2g\times(-1)+2f\times 0=c-15 \Rightarrow -2g=c-15$$

Given $c=1$ and $g=7$.

Hence, the circle is $x^2+y^2+14x+2fy+1=0$.

$$|z+7+if| = \sqrt{48+f^2}$$

Centre $(-7,-f)$, $r=\sqrt{48+f^2}$

Test Your Concepts-VI

(Solutions on page H.102)

1. Find the points representing the complex number z for which $|z+3i|^2 - |z-3|^2 = 6$ lie on the line where $z = x + iy$
2. Show that in Argand diagram, the locus given by $\text{Arg}[(1+2i)z - 1 + 3i] = \frac{\pi}{4}$ is a straight line through the points $(2-2i)$ and $(5-3i)$
3. Find the point of intersection of the curves $\text{Arg}(z-3i) = \frac{3\pi}{4}$ and $\text{Arg}(2z+1-2i) = \frac{\pi}{4}$
4. If $\left|\frac{z-2}{z+2}\right| = \frac{\pi}{6}$, then find the locus of z
5. Let $z_1 = 6 + i$ and $z_2 = 4 - 3i$. Let z be a complex number such that $\text{Arg}\left(\frac{z-z_1}{z_2-z}\right) = \frac{\pi}{2}$, then show that $|z-(5-i)| = \sqrt{5}$
6. If $z = x + iy$, such that $|z+1| = |z-1|$ and Amp $\frac{z-1}{z+1} = \frac{\pi}{4}$, then find x and y
7. If z_1, z_2, z_3 and z_4 be the vertices of a quadrilateral taken in order such that $z_1 + z_3 = z_2 + z_4$ and $|z_1 - z_3| = |z_2 - z_4|$ then find $\text{Arg}\left(\frac{z_1-z_2}{z_3-z_2}\right)$

Answer Keys

1. $x + y = 1$ **3.** No intersection **4.** circle **6.** $x = 0, y = \sqrt{2} + 1$ **7.** $\pm\frac{\pi}{2}$

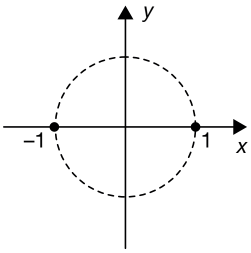
ROOTS OF A COMPLEX NUMBER

Square Roots of Unity

Square root of unity $z^2 - 1 = 0$

$\Rightarrow$ $(z-1)(z+1) = 0$

$\Rightarrow$ $z = -1, 1$



Cube Root of Unity

$$z^3 - 1 = 0 \Rightarrow (z-1)(z^2+z+1) = 0,$$

$$z = 1$$

$$z^2 + z + 1 = 0 \Rightarrow z = -\left(\frac{1 \pm \sqrt{3}i}{2}\right)$$

$$z = 1;\ z = \frac{-1+\sqrt{3}i}{2} = e^{i2\pi/3} = \omega$$

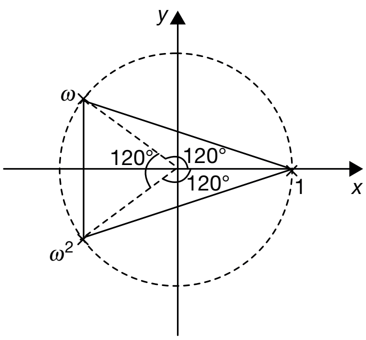
$$z = \frac{-1-\sqrt{3}i}{2} = e^{-i2\pi/3} = e^{i4\pi/3} = \bar{\omega} = \omega^2$$

Properties of Cube Roots of Unity

- $1 + \omega + \omega^2 = 0$
- $\omega^3 = 1$

$$\boxed{\begin{array}{l} \omega^{3n} = 1 \\ \omega^{3n+1} = \omega \\ \omega^{3n+2} = \omega^2 \end{array}}\ n \in N$$

- Cube roots of unity are in GP with common ratio ω.



- All three roots lie on a circle and equally divides the circle. Hence, they construct an equilateral triangle.
- $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{if } n \text{ is not a multiple of } 3 \\ 3, \text{if } n \text{ is a multiple of } 3 \end{cases}$
- A complex number $a + ib$, for which $|a:b| = 1:\sqrt{3}$ or $\sqrt{3}:1$, can always be expressed in terms of $1, \omega, \omega^2$.
- Cube root of -1 are $-1, -\omega, -\omega^2$.

Remark(s)

- Equation $t^2 + t + 1 = 0$ has roots ω, ω^2
- Equation $t^2 - t + 1 = 0$ has roots $-\omega, -\omega^2$
- $x^2 + x + 1 = (x-\omega)(x-\omega^2)$
- $x^2 - x + 1 = (x+\omega)(x+\omega^2)$
- $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- $x^2 + y^2 = (x+iy)(x-iy)$
- $x^3 + y^3 = (x+y)(x+y\omega)(x+y\omega^2)$
- $x^3 - y^3 = (x-y)(x-y\omega)(x-y\omega^2)$
- $x^2 + y^2 + z^2 - xy - yz - zx$
  $= (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
- $x^3 + y^3 + z^3 - 3xyz$
  $= (x+y+z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

ILLUSTRATION 118

If ω is imaginary cube root of unity, then find the value of

(a) $(1 + \omega^2 - \omega)(1 + \omega - \omega^2)$
(b) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5$

(c) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(d) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$ ………. upto $2n$ terms

(e) $(1+5\omega^2+\omega^4)(1+5\omega^4+\omega^2)(5\omega^3+\omega+\omega^2)$

(f) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ …. upto n terms

SOLUTION

(a) Using $1+\omega+\omega^2=0$; $\omega^{3n}=1$,

$(1+\omega^2-\omega)(1+\omega-\omega^2)=(-\omega-\omega)(-\omega^2-\omega^2)$

$=(-2\omega)(-2\omega^2)=4\omega^3=4$

(b) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5$

$=(-\omega-\omega)^5+(-\omega^2-\omega^2)^5=-32\omega^5-32\omega^{10}$

$=-32(\omega^2+\omega)=32$

(c) $(1-\omega)(1-\omega^2)(1-\omega^3\cdot\omega)(1-\omega^6\cdot\omega^2)=(1-\omega)^2(1-\omega^2)^2$

$=((1-\omega)(1-\omega^2))^2=(1-\omega-\omega^2+\omega^3)^2$

$=(1+1+1)^2=9$

(d) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$ ………. upto $2n$ terms $=(1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)$ ………. Upto $2n$ here

$=(1-\omega+\omega^2)^n(1-\omega^2+\omega)^n$

$=(-\omega-\omega)^n(-\omega^2-\omega^2)^n$

$=((-2\omega).(-2\omega^2))^n=(4\omega^3)^n=4^n$

(e) $(1+5\omega^2+\omega^4)(1+5\omega^4+\omega^2)(5\omega^3+\omega+\omega^2)=(1+\omega^2+\omega+4\omega^2)(1+\omega+\omega^2+4\omega)(1+\omega+\omega^2+4\omega^3)$

$=(4\omega^2)(4\omega)(4\omega^3)=64\omega^6=64$

(f) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ …. upto n terms $=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)$ ……… up to n terms

Case(I) (*n* is even)	**Case(II) (*n* is odd)**
$=((1+\omega)(1+\omega^2))^{n/2}$ $=(1+\omega+\omega^2+\omega^3)^{n/2}$ $=(0+1)^{n/2}=1$	$=(1+\omega)^{\frac{n+1}{2}}(1+\omega^2)^{\frac{n-1}{2}}$ $=(1+\omega)\left((1+\omega)(1+\omega^2)\right)^{\frac{n-1}{2}}$ $=-\omega^2\left(1+\omega+\omega^2+\omega^3\right)^{\frac{n-1}{2}}$ $=-\omega^2(1)^{\frac{n-1}{2}}=-\omega^2$

ILLUSTRATION 119

If ω is imaginary cube root of unity, then find $\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

SOLUTION

$$\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=\omega^2\frac{\left(\frac{a}{\omega^2}+\frac{b}{\omega}+c\right)}{c+a\omega+b\omega^2}\qquad\left(\because\frac{1}{\omega}=\omega^2;\frac{1}{\omega^2}=\omega\right)$$

$$=\omega^2\frac{(a\omega+b\omega^2+c)}{(c+a\omega+b\omega^2)}=\omega^2$$

ILLUSTRATION 120

If ω is imaginary cube root of unity, then evaluate $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

SOLUTION

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$

$$=\frac{1}{\omega^2}\left(\frac{a\omega^2+b\omega^3+c\omega^4}{b+c\omega+a\omega^2}\right)+\frac{1}{\omega}\left(\frac{a\omega+b\omega^2+c\omega^3}{c+a\omega+b\omega^2}\right)$$

$$=\frac{1}{\omega^2}(1)+\frac{1}{\omega}(1)=\omega(1)+\omega^2(1)=-1$$

ILLUSTRATION 121

If ω is imaginary cube root of unity, then evaluate $\dfrac{1}{1+2\omega}+\dfrac{1}{2+\omega}-\dfrac{1}{1+\omega}$

SOLUTION

$$\frac{2+\omega+1+2\omega}{(1+2\omega)(2+\omega)}-\frac{1}{1+\omega}=\frac{3(1+\omega)}{2+4\omega+\omega+2\omega^2}-\frac{1}{(-\omega^2)}$$

$$=\frac{-3\omega^2}{2+5\omega-2-2\omega}+\frac{1}{\omega^2}=\frac{-3\omega^2}{3\omega}+\frac{\omega}{\omega^3}=-\omega+\omega=0$$

ILLUSTRATION 122

Evaluate $\sum_{r=0}^{10}(1+\omega^r+\omega^{2r})$, where ω is imaginary cube root of unity.

SOLUTION

$$\sum_{r=0}^{10}(1+\omega^r+\omega^{2r})=\sum_{r=0}^{10}1+\sum_{r=0}^{10}\omega^r+\sum_{r=0}^{10}(\omega^2)^r$$

$$=(1+1+\ldots+1)+(1+\omega+\omega^2+\ldots+\omega^{10})+(1+\omega^2+\omega^4+\ldots+\omega^{20})$$

$$=11+1.\left(\frac{1-\omega^{11}}{1-\omega}\right)+1.\left(\frac{1-(\omega^2)^{11}}{1-\omega^2}\right)$$

$$=11+\left(\frac{1-\omega^2}{1-\omega}\right)+\left(\frac{1-\omega}{1-\omega^2}\right)=11+1+\omega+\frac{1}{1+\omega}$$

$$=11+1+\omega+\frac{1}{-\omega^2}$$

$$=12+\omega-\omega=12$$

ILLUSTRATION 123

If ω is imaginary cube root of unity, then evaluate $(1-\omega+\omega^2)(1+\omega-\omega^2)(-1+\omega+\omega^2)$

SOLUTION

As $1+\omega+\omega^2=0 \Rightarrow \omega+\omega^2=-1,\ 1+\omega=-\omega^2$ and $1+\omega^2=-\omega$

Therefore, $(1-\omega+\omega^2)(1+\omega-\omega^2)(-1+\omega+\omega^2)$

$$=(-\omega-\omega)\,(-\omega^2-\omega^2)\,(-1-1)=-8\omega^3=-8$$

ILLUSTRATION 124

If α, β, γ are roots of $x^3-3x^2+3x+7=0$, then find the value of $\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}$

SOLUTION

We have $x^3-3x^2+3x+7=0$

$$\Rightarrow \quad (x-1)^3+8=0$$

Therefore, $(x-1)^3=(-2)^3$

$$\Rightarrow \quad \left(\frac{x-1}{-2}\right)^3=1$$

$$\Rightarrow \quad \frac{x-1}{-2}=(1)^{1/3}=1,\ \omega,\ \omega^2 \text{ (cube roots of unity)}$$

Therefore, $\quad x=-1,\ 1-2\omega,\ 1-2\omega^2$

Here $\alpha=-1,\ \beta=1-2\omega,\ \gamma=1-2\omega^2$

Hence, $\alpha-1=-2,\ \beta-1=-2\omega,\ \gamma-1=-2\omega^2$

Then,

$$\frac{\alpha-1}{\beta-1}+\frac{\beta-1}{\gamma-1}+\frac{\gamma-1}{\alpha-1}=\left(\frac{-2}{-2\omega}\right)+\left(\frac{-2\omega}{-2\omega^2}\right)+\left(\frac{-2\omega^2}{-2}\right)$$

$$=\frac{1}{\omega}+\frac{1}{\omega}+\omega^2=3\omega^2$$

ILLUSTRATION 125

If $x^2-x+1=0$, then find the value of $\sum_{n=1}^{15}\left(x^n+\frac{1}{x^n}\right)^2$

SOLUTION

$$x^2-x+1=0 \Rightarrow x=\frac{1\pm i\sqrt{3}}{2}$$

Therefore, $x=-\omega,\ -\omega^2$

$$\sum_{n=1}^{15}\left(x^n+\frac{1}{x^n}\right)^2=\sum_{n=1}^{15}\left(x^{2n}+\frac{1}{x^{2n}}+2\right)$$

$$=\sum_{n=1}^{15}\left(\omega^{2n}+\frac{1}{\omega^{2n}}+2\right)$$

When n is multiple of 3 then $\omega^{2n}+\frac{1}{\omega^{2n}}+2=4$ else 1

Therefore, $\sum_{n=1}^{15}\left(\omega^{2n}+\frac{1}{\omega^{2n}}+2\right)=5\times4+10\times1=30$

ILLUSTRATION 126

Evaluate $(2-\omega)(2-\omega^2)+2\,(3-\omega)(3-\omega^2)+\ldots\ldots\ldots+(n-1)(n-\omega)(n-\omega^2)$, where ω is imaginary cube root of unity and $n \in N$. **[IIT 1996]**

SOLUTION

For the given expression,

$$T_n=(n-1)\,(n-\omega)\,(n-\omega^2)$$

$$T_n=(n-1)(n^2-n(\omega+\omega^2)+\omega^3)$$

$$T_n=(n-1)(n^2+n+1)=n^3-1$$

So $\quad S_n=\sum_{n=1}^{n}T_n \quad \Rightarrow \quad S_n=\sum_{n=1}^{n}n^3-\sum_{n=1}^{n}1$

$$\Rightarrow \quad S_n=\frac{n^2(n+1)^2}{4}-n \quad \Rightarrow \quad S_n=\frac{n\left(n(n+1)^2-4\right)}{4}$$

ILLUSTRATION 127

If $a, b, c \in I$ and not all equal. If ω is imaginary cube root of unity then find minimum value of $|a+b\omega+c\omega^2|$. **[IIT-2005]**

SOLUTION

As $|a+b\omega+c\omega^2| = \sqrt{(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)}$

(As $|z|^2=z.\bar{z}$)

$$|a+b\omega+c\omega^2| = \sqrt{a^2+b^2+c^2-ab-bc-ca}$$

$$=\frac{1}{\sqrt{2}}\sqrt{(a-b)^2+(b-c)^2+(c-a)^2}$$

As all a, b and c together not equal but two of them at a time can be equal.
For minimum value of expression $a=b=1$ and $c=1+1$
Therefore, $\min(|a+b\omega+c\omega^2|)=1$

4th ROOT OF UNITY

$$z^4-1=0$$

$$\Rightarrow \quad (z-1)(z+1)(z-i)(z+i)=0$$

We get $z=1, -1, i, -$i

Alternate approach

$$z^4=1 \Rightarrow z=(1)^{1/4}$$

$$\Rightarrow \quad z=\left(e^{i0}\right)^{1/4} \text{ (change in Euler form)}$$

$$=\left(e^{i2m\pi}\right)^{1/4} \text{ (add } 2m\pi \text{ in argument)}$$

$$z = e^{i\frac{m\pi}{2}}$$

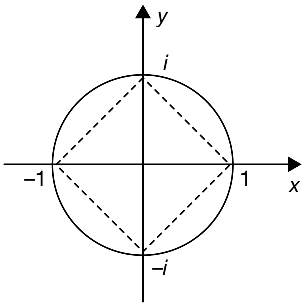
$m = 0, 1, 2, 3, \ldots$

$m = 0 \quad z = e^{i0} = 1$

$m = 1 \quad z = e^{i\pi/2} = i$

$m = 2 \quad z = e^{i\pi} = -1 = i^2$

$m = 3 \quad z = e^{i3\pi/2} = -i = i^3$



(i) all roots lie on unit radii circle with centre at origin.

(ii) all roots are equiplaced on circle and construct square.

(iii) all roots are forming GP with common ratio (i).

n^{th} ROOT OF UNITY

$z^n = 1 \Rightarrow z = (1)^{1/n} = (e^{i0})^{t/n}$ (change Euler form)

$z = (e^{i2m\pi})^{1/n}$ (add $2m\pi$ in argument)

$z = e^{\frac{i2m\pi}{n}}$ where, $m = 0, 1, 2, \ldots.. n-1$

$m = 0 \quad z = 1$

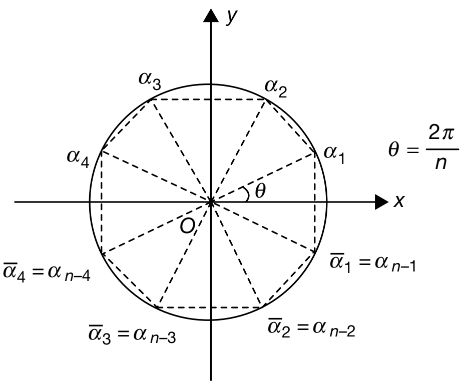
$m = 1 \quad z = e^{i2\pi/n} = \alpha_1 = \alpha$

$m = 2 \quad Z = e^{i4\pi/n} = \alpha_2 = \alpha^2$

$m = 3 \quad z = e^{i6\pi/n} = \alpha_3 = \alpha^3$

$m = n-1 \quad z = e^{i\frac{2\pi(n-1)}{n}} = \alpha_{n-1} = \alpha^{n-1}$

- n^{th} root of unity are in GP with common ratio (α)
- All n roots lie on unit radii circle with centre at origin.
- All n roots are equidistant on circle. That means they divide circle into n equal parts.
- All n roots construct regular n sided polygon
- All imaginary roots are present with their conjugates.


y
x
O
θ
$\theta = \frac{2\pi}{n}$
α_1
α_2
α_3
α_4
$\bar{\alpha}_1 = \alpha_{n-1}$
$\bar{\alpha}_2 = \alpha_{n-2}$
$\bar{\alpha}_3 = \alpha_{n-3}$
$\bar{\alpha}_4 = \alpha_{n-4}$

Equation $z^n - 1 = 0$ has roots $1, \alpha_1, \alpha_2, \alpha_3 \ldots. \alpha_{n-1}$

Sum of roots $= 1 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots.. + \ldots\alpha_{n-1} = 0$

Product of roots $= 1 \cdot \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \ldots\ldots.. \cdot \alpha_{n-1} = (-1)^{n-1}$

$$1^P + \alpha_1^P + \alpha_2^P + \alpha_3^P + \ldots.\alpha_{n-1}^P = \begin{cases} n, \text{ if } P \text{ is multiple of } n \\ 0, \text{ if } P \text{ is not multiple of } n \end{cases}$$

Proof:

Case I: $P = n\lambda \quad \lambda \in I$

$\alpha^n = 1$

$\text{LHS} = 1^P + \alpha^P + (\alpha^2)^P + (\alpha^3)^P + \ldots. + (\alpha^{n-1})^P$

$= 1 + \alpha^{\lambda n} + \alpha^{2n\lambda} + \alpha^{3n\lambda} + \ldots.. + \alpha^{(n-1)n\lambda}$

$= 1 + (\alpha^n)^\lambda + (\alpha^n)^{2\lambda} + (\alpha^n)^{3\lambda} + \ldots\ldots + (\alpha^n)^{(n-1)\lambda}$

$= 1 + (1)^\lambda + (1)^{2\lambda} + (1)^{3\lambda} + \ldots. + (1)^{(n-1)\lambda}$

$= 1 + 1 + 1 + \ldots. + 1 = n$

Case II: $\mathbf{P} \neq \lambda n$

$\text{LHS} = 1^P + \alpha^P + \alpha^{2P} + \alpha^{3P} + \ldots.. + \alpha^{(n-1)P}$

(It is the sum of GP)

$$= A\left(\frac{1-R^N}{1-R}\right) = 1\left(\frac{1-(\alpha^P)^n}{1-\alpha^P}\right) = \frac{1-(\alpha^n)^P}{1-\alpha^P} = \frac{1-(1)^P}{1-\alpha^P} = 0$$

ILLUSTRATION 128

If ω is imaginary fifth root of unity, find $\omega^4 + \omega^8 + \omega^{12} + \omega^{16}$

SOLUTION

ω is a non real root.

so, $\omega \neq 1 \quad \omega^5 = 1; 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

given expression $= \omega^4 + \omega^8 + \omega^{12} + \omega^{16}$

$= \omega^4 + \omega^5 \cdot \omega^3 + \omega^{10} \cdot \omega^2 + \omega^{15} \cdot \omega$

$= \omega^4 + \omega^3 + \omega^2 + \omega = -1.$

ILLUSTRATION 129

If ω is real fifth root of unity, find $\omega^4 + \omega^8 + \omega^{12} + \omega^{16}$

SOLUTION

ω is a real 5^{th} root of unity. So, $\omega = 1$

given expression $= \omega^4 + \omega^8 + \omega^{12} + \omega^{16} = 1 + 1 + 1 + 1 = 4$

ILLUSTRATION 130

Find value of $\displaystyle\sum_{\lambda=1}^{12}\left(\sin\frac{2\lambda\pi}{13} - i\cos\frac{2\lambda\pi}{13}\right)$

SOLUTION

$$\sum_{\lambda=1}^{12}\left(\sin\frac{2\lambda\pi}{13} - i\cos\frac{2\lambda\pi}{13}\right)$$

$$= (-i)\sum_{\lambda=1}^{12}\left(i\sin\frac{2\lambda\pi}{13}+\cos\frac{2\lambda\pi}{13}\right) = -i\sum_{\lambda=1}^{12} e^{\frac{i2\lambda\pi}{13}}$$

$$= -i\sum_{\lambda=1}^{12}\alpha^{\lambda}$$ Where α is imaginary 13th root of unity

$$= -i(\alpha+\alpha^2+\ldots+\alpha^{12}) = -i(-1) = i$$

ILLUSTRATION 131

If ω imaginary root of n^{th} root of unity then evaluate $\sum_{r=1}^{n}(ar+b)\omega^{r-1}$

SOLUTION

$$\sum_{r=1}^{n}(ar+b)\omega^{r-1}$$

$$= (a+b)+(2a+b)\omega+(3a+b)\omega^2+\ldots+(na+b)\omega^{n-1}$$

$$= b\underbrace{(1+\omega+\omega^2+\ldots\ldots\ldots..+\omega^{n-1})}_{zero}+$$

$$a(1+2\omega+3\omega^2+\ldots..n\omega^{n-1})$$

$$= a(1+2\omega+3\omega^2+\ldots+n\omega^{n-1})$$

$$= aS$$

Now, $S = 1+2\omega+\ldots\ldots\ldots\ldots+n\omega^{n-1}$

$$\begin{aligned} S\omega &= \quad \omega+2\omega^2+\ldots\ldots+(n-1)\,\omega^{n-1}+n\omega^n \\ - \quad & \quad - \quad - \quad - \quad\quad - \quad\quad - \end{aligned}$$

$$S(1-\omega) = \underbrace{(1+\omega+\omega^2+\ldots..\omega^{n-1})}_{zero} - n\omega^n = -n \quad (\text{as } \omega^n = 1)$$

$$S = \frac{n}{\omega-1} \Rightarrow \text{Required term} = \frac{na}{\omega-1}$$

ILLUSTRATION 132

Evaluate: $\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^p$

SOLUTION

$$\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}(-i)\left(i\sin\frac{2q\pi}{11}+\cos\frac{2q\pi}{11}\right)\right)^p$$

$$= \sum_{p=1}^{32}(3p+2)\left((-i)\sum_{q=1}^{10}(\alpha^q)\right)^p$$

$$= \sum_{p=1}^{32}(3p+2)(-i)(\alpha+\alpha^2+\ldots+\alpha^{10})^p$$

$$S = \sum_{p=1}^{32}(3p+2)((-i)(-1))^p$$

$$= \sum_{p=1}^{32}(3p+2)\,i^p = 3\sum_{p=1}^{32}pi^p + 2\sum_{p=1}^{32}i^p$$

$$S = 3S_1 + 2\left(i\frac{(1-i^{32})}{1-i}\right) = 3S_1 + 0 = 3S_1$$

$$S_1 = i+2i^2+3i^3+4i^4+\ldots\ldots+32i^{32}$$

$$-iS_1 = 0-i^2-2i^3-3i^4-\ldots\ldots\ldots\ldots-31i^{32}-32i^{33}$$

$$(1-i)S_1 = i+i^2+i^3+i^4+\ldots..+i^{32}-32i^{33}$$

$$(1-i)S_1 = \frac{i(1-i^{32})}{1-i} - 32i = 0-32i$$

$$\Rightarrow \quad S_1 = \frac{-32i}{1-i} = \frac{-32i(1+i)}{2}$$

$$S_1 = -16i(1+i) = -16i+16$$

Therefore, $S = 3S_1 = 48(1-i)$

ILLUSTRATION 133

Find the roots of the equation $z^6 + 64 = 0$, whose real part is positive.

SOLUTION

$$z^6 = -64 \Rightarrow z^6 = 2^6(-1)$$

$$z^6 = 2^6.e^{i(2n+1)\pi}$$

$$\Rightarrow \quad z = 2e^{i(2n+1)\frac{\pi}{6}} \text{ where } n = 0, 1, 2, 3, 4, 5$$

$$z = 2e^{i\frac{\pi}{6}}, 2e^{i\frac{\pi}{2}}, e^{i\frac{5\pi}{6}}\ 2e^{i\frac{7\pi}{6}}, 2e^{i\frac{3\pi}{2}}, 2e^{i\frac{11\pi}{6}}$$

Therefore, roots with positive real part are $= 2e^{i\frac{\pi}{6}}, 2^{i\frac{11\pi}{6}}$

ILLUSTRATION 134

Locate the roots of $(z+1)^4 - 81 = 0$.

SOLUTION

$$(z+1)^4 = 81 = 3^4$$

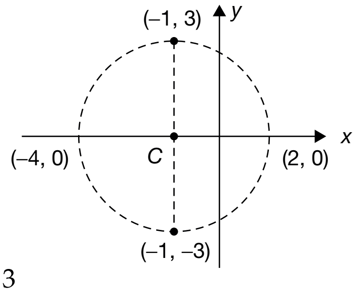
$$z+1 = 3(1)^{1/4}$$

$$= 3, 3i, -3i, -3$$

$$z = 2, -1+3i, -1-3i, -4$$

$$(z+1)^4 = 81 = 3^4$$

$$|z+1| = 3$$



all roots are equiplaced on circle with centre (–1, 0), $r = 3$

ILLUSTRATION 135

Find number of roots having negative real part of equation $z^{10} - z^5 - 992 = 0$.

SOLUTION

$$z^{10} - z^5 - 992 = (z^5 - 32)\,(z^5 + 31) = 0$$

$$z^5 = 32 \quad \Rightarrow \quad z = 2(1)^{1/5}$$

$$\Rightarrow \qquad = 2e^{\frac{i2k\pi}{5}} \text{ where } k = 0, 1, 2, 3, 4$$

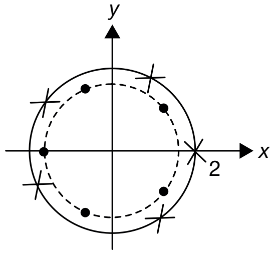
(root are on circle $c(0, 0)$ $r = 2$ shown in diagram with ×)

$$z^5 = -31 \quad \Rightarrow \quad z = (-31)^{1/5}$$

$$= (31)^{1/5}(-1)^{1/5}$$

$$= (31)^{1/5} e^{i(2m+1)\pi/5} \text{ where } m = 0, 1, 2, 3, 4$$

(root are on circle $c(0, 0)$ $r = (31)^{1/5}$ shown in diagram with •)



Number of negative real part of $z = 5$.

ILLUSTRATION 136

If $z_n = \cos\left(\dfrac{\pi}{(2n+1)(2n+3)}\right) + i\sin\left(\dfrac{\pi}{(2n+1)(2n+3)}\right)$

Then evaluate $\lim\limits_{n\to\infty} z_1, z_2, z_3, \ldots, z_n$.

SOLUTION

$$z_n = CiS\left(\frac{\pi}{(2n+1)(2n+3)}\right) = e^{i\frac{\pi}{2}\left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)}$$

$$z_1 z_2 z_3 \ldots z_n$$

$$= \prod_{n=1}^{n} z_n = e^{i\frac{\pi}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\ldots.+\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right]}$$

$$= e^{i\frac{\pi}{2}\left(\frac{1}{3}-\frac{1}{2n+3}\right)} = e^{i\left(\frac{\pi}{2}\frac{2n}{3(2n+3)}\right)}$$

$$\lim_{n\to\infty} \prod z_n = \lim_{n\to\infty} e^{i\frac{\pi n}{3(2n+3)}} = \lim_{n\to\infty} e^{\frac{i\pi}{3(2+3/n)}}$$

$$= e^{i\frac{\pi}{6}} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

ILLUSTRATION 137

If ω is the fifth root of 2 and $x = \omega + \omega^2$,

prove that $x^5 = 10x^2 + 10x + 6$.

SOLUTION

$$\omega^5 = 2 \quad \Rightarrow \quad \omega = 2^{1/5}(1)^{1/5}$$

$$x = \omega + \omega^2$$

$$x^5 = (\omega + \omega^2)^5$$

$$= {}^5C_0\omega^5 + {}^5C_1\omega^4\omega^2 + {}^5C_2\omega^3\omega^4 + {}^5C_3\omega^2\omega^6 + {}^5C_4\omega\omega^8 + {}^5C_5\omega^{10}$$

$$x^5 = 2 + 10\omega + 20\omega^2 + 20\omega^3 + 10\omega^4 + 4$$

$$x^5 = 6 + 10(\omega + \omega^4) + 20\,(\omega^2 + \omega^3)$$

$$= 6 + 10\,(\omega + 2\omega^2 + 2\omega^3 + \omega^4) \qquad \ldots(1)$$

$$x^2 + x = \omega^2 + \omega^4 + 2\omega^3 + \omega + \omega^2 = \omega^4 + 2\omega^2 + 2\omega^3 + \omega$$

From condition (1) we get

$$x^5 = 6 + 10(x^2 + x)$$

Hence proved

Miscellaneous Problems Based on n^{th} Root of Unity

Equation $z^n - 1 = 0$ has roots $1, \ \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$.

We can write it as,

$$z^n - 1 = (z-1)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3) \ldots.\ (z-\alpha_{n-1})$$

$$(z-\alpha_1)(z-\alpha_2)(z-\alpha_3) \ldots\ldots\ldots\ldots (z-\alpha_{n-1})$$

$$= \frac{z^n - 1}{z-1} = 1 + z + \ldots + z^{n-1}$$

$$(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\ldots\ldots(z-\alpha_{n-1})$$

$$= \frac{z^n - 1}{z-1} = 1 + z + \ldots. + z^{n-1}$$

$$(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\ldots\ldots(z-\alpha_{n-1}) = 1 + z + z^2 + \ldots + z^{n-1}$$

Substitute $z = 1$

$$(1-\alpha_1)\,(1-\alpha_2)\,(1-\alpha_3)\ldots\ldots..(1-\alpha_{n-1})$$

$$= 1 + 1 + 1 + \ldots + 1 = n$$

Substitute $z = -1$

$$(-1-\alpha_1)(-1-\alpha_2)(-1-\alpha_3)\ldots.(-1-\alpha_{n-1})$$

$$= 1 - 1 + 1 - 1 + \ldots. + (-1)^{n-1}$$

$$(-1)^{n-1}(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots\ldots(1+\alpha_{n-1})$$

$$= \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

$$(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots\ldots(1+\alpha_{n-1}) = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

ILLUSTRATION 138

If $1, \alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are n roots of n^{th} root of unity, then evaluate the following results.

(a) $(5-\alpha_1)(5-\alpha_2)\ \ldots\ldots\ (5-\alpha_{n-1})$

(b) $\dfrac{1}{5-\alpha_1}+\dfrac{1}{5-\alpha_2}+\ldots+\dfrac{1}{5-\alpha_{n-1}}$

SOLUTION

As $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n roots of n^{thv} root of unity So,

$$z^n-1=(z-1)\,(z-\alpha_1)\,(z-\alpha_2)\ \ldots..\ (z-\alpha_{n-1}) \qquad \ldots(1)$$

(a) To obtain this result, substitute $z=5$ in (1).

Therefore, $5^n-1=(5-1)\,(5-\alpha_1)\,(5-\alpha_2)\ \ldots\ (5-\alpha_{n-1})$

So, $(5-\alpha_1)\,(5-\alpha_2)\ \ldots.\ (5-\alpha_{n-1})=\dfrac{5^n-1}{4}$

(b) Take logarithm of equation (1) on both sides

$\log(z^n-1)=\log(z-1)+\log(z-\alpha_1)+\ldots+\log(z-\alpha_{n-1})$

Now, differentiate both sides of above equation

$$\frac{nz^{n-1}}{z^n-1}=\frac{1}{z-1}+\frac{1}{z-\alpha_1}+\ldots.+\frac{1}{z-\alpha_{n-1}}$$

Now, to obtain the result substitute $z=5$.

$$\frac{n5^{n-1}}{5^n-1}=\frac{1}{4}+\frac{1}{5-\alpha_1}+\ldots.+\frac{1}{5-\alpha_{n-1}}$$

$$\therefore\quad \frac{1}{5-\alpha_1}+\frac{1}{5-\alpha_2}+\ldots.+\frac{1}{5-\alpha_{n-1}}=\frac{n.5^{n-1}}{5^n-1}-\frac{1}{4}$$

ILLUSTRATION 139

If z_1 and $\bar{z}_1$ represents vertices of regular polygon of n sides whose centre at origin. Find the value of n if

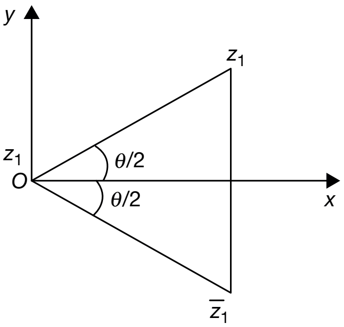
(a) $\dfrac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1)}=\sqrt{2}-1$ **(b)** $\dfrac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1)}=2-\sqrt{3}$

SOLUTION

Let $z_1=x+iy$, then

$$\frac{\mathrm{Im}(z_1)}{\mathrm{Re}(z_1)}=\frac{y}{x}=\tan\frac{\theta}{2}$$

where θ is angle subtended by any side at centre of regular polygon $\left(\because \theta=\dfrac{2\pi}{n}\right)$



Isosceles triangle

(a) $\dfrac{y}{x}=\tan\dfrac{\theta}{2}=\sqrt{2}-1=\tan 22\dfrac{1}{2}^{\circ}$

Therefore, $\theta=45° \Rightarrow n=\dfrac{2\pi}{\theta}=\dfrac{360°}{45°}=8$

(b) $\dfrac{y}{x}=\tan\dfrac{\theta}{2}=2-\sqrt{3}=\tan 15°$

Therefore, $\theta=30° \Rightarrow n=\dfrac{360°}{30°}=12$

ILLUSTRATION 140

If z_r; $r=1,2,3,,\ldots.\ 50$ are roots of equation $\displaystyle\sum_{r=0}^{50} z^r=0$ then find $\displaystyle\sum_{r=1}^{50}\frac{1}{z_r-1}$.

SOLUTION

Equation $\displaystyle\sum_{r=0}^{50} z^r=0$

$\Rightarrow\quad 1+z^1+z^2+\ldots.+z^{50}=0$

$\Rightarrow\quad \dfrac{z^{51}-1}{z-1}=0 \Rightarrow z^{51}-1=0$

Equation has roots $1, z_1, z_2, \ldots., z_{50}$

Replace z by $\dfrac{1}{z'-1} \Rightarrow z'-1=\dfrac{1}{z}$

$$z'=\frac{1}{z}+1$$

$$\left(1+\frac{1}{z'}\right)^{51}-1=0 \Rightarrow (z'+1)^{51}-z'^{51}=0$$

$\Rightarrow\quad 51(z')^{50}+\dfrac{51.50}{2}(z')^{49}+\ldots..+1=0$

has roots $\dfrac{1}{z_1-1},\dfrac{1}{z_2-1},\dfrac{1}{z_3-1},\ldots..,\dfrac{1}{z_{50}-1}$

Sum of roots $=\displaystyle\sum_{r=0}^{50}\frac{1}{z_{r-1}}=\frac{-51.50}{2(51)}=-25.$

ILLUSTRATION 141

Find number of ordered pair(s) of (z, ω), where z and ω are complex numbers which satisfy

$$z^3+\bar{\omega}^7=0 \text{ and } z^5\omega^{11}=1$$

SOLUTION

$$z^3+\bar{\omega}^7=0 \Rightarrow z^3=-\bar{\omega}^7 \Rightarrow |z|^3=|\omega|^7$$

$\Rightarrow\quad |z|^{15}=|\omega|^{35} \qquad \ldots(1)$

Again, $z^5\omega^{11}=1 \Rightarrow |z|^5\ |\omega|^{11}=1$

$$\Rightarrow \quad |z|^{15}|\omega|^{33}=1 \qquad \ldots(2)$$

From (1) and (2), $|z| = |\omega| = 1$

From the given equation, $z^{15}=-\bar{\omega}^{35}=\dfrac{1}{\omega^{33}}\left(z^{5}=\dfrac{1}{\omega^{11}}\right)$

(since, $|\omega|^2=\omega\bar{\omega}$)

$$\Rightarrow \quad \left(|\omega|^2\right)^{33}\bar{\omega}^2=-1\Rightarrow\bar{\omega}^2=-1\Rightarrow\bar{\omega}^2=i^2$$

Therefore, $\omega=\pm i$

Substitute this ω in any of given equation $z=\pm i$

Therefore, (i, i) and $(-i, -i)$ are ordered pair of (z, ω).

ILLUSTRATION 142

If 1, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n roots of n^{th} root of unity, then evaluate

(a) $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1})$

(b) $\sin\dfrac{\pi}{n}\sin\dfrac{2\pi}{n}\ldots\ldots\sin\dfrac{(n-1)\pi}{n}$ **[IIT-1984]**

SOLUTION

As 1, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are n roots of n^{th} root of unity.

$$z^n-1=(z-1)(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1}) \qquad \ldots(1)$$

$$\alpha=e^{\frac{i2r\pi}{n}}$$

(a) From (1),

$$\frac{z^n-1}{z-1}=(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1}) \qquad \ldots(2)$$

Now take limit $z\to1$ on both sides of equation (2)

$$\lim_{z\to1}\frac{z^n-1}{z-1}=\lim_{z\to1}(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_{n-1})$$

Therefore, $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1})=n$

(b) From the result of part (a) we can say that, $\prod\limits_{r=1}^{n-1}|1-\alpha_r|=n$

$$\Rightarrow \prod\left|1-\cos\frac{2\pi r}{n}-i\sin\frac{2\pi r}{n}\right|=n$$

$$\Rightarrow \prod\left|2\sin^2\frac{\pi r}{n}-i2\sin\frac{\pi r}{n}\cos\frac{\pi r}{n}\right|=n$$

$$\Rightarrow \prod 2\sin\frac{\pi r}{n}\prod\left|\sin\frac{\pi r}{n}-i\cos\frac{\pi r}{n}\right|=n$$

As second modulus is 1, so

$$2^{n-1}\sin\frac{\pi}{n}.\sin\frac{2\pi}{n}.\sin\frac{3\pi}{n}\ldots\ldots\sin\frac{\pi(n-1)}{n}=n$$

Therefore, $\sin\dfrac{\pi}{n}.\sin\dfrac{2\pi}{n}.\sin\dfrac{3\pi}{n}\ldots\ldots\sin\dfrac{\pi(n-1)}{n}=\dfrac{n}{2^{n-1}}$

Test Your Concepts-VII

(Solutions on page H.102)

1. If $\omega\,(\neq1)$ is a cube root of unity, then find the value of $\omega^{1994}+\omega^{1995}$
2. If a, b and c are the roots of $y^3-3y^2+3y+26=0$ and ω is cube roots of unity, then find the value of $\dfrac{a-1}{b-1}+\dfrac{b-1}{c-1}+\dfrac{c-1}{a-1}$
3. If $(\omega\neq1)$ is a cube root of unity, then find
$$\begin{vmatrix}1 & 1+i+\omega^2 & \omega^2\\ 1-i & -1 & \omega^2-1\\ -i & -1+\omega-i & -1\end{vmatrix}$$
4. If $z^2+z+1=0$, where z is a complex number then the find value of $\left(z+\dfrac{1}{z}\right)^2+\left(z^2+\dfrac{1}{z^2}\right)^2+\left(z^3+\dfrac{1}{z^3}\right)^2+\ldots+\left(z^6+\dfrac{1}{z^6}\right)^2$
5. Evaluate $\sum\limits_{k=1}^{14}\sin\left(\dfrac{2\pi k}{15}\right)-i\cos\left(\dfrac{2\pi k}{15}\right)$
6. If $z_r=\cos\dfrac{r\alpha}{n^2}+i\sin\dfrac{r\alpha}{n^2}$, where $r=1, 2, 3, \ldots., n$, then find $\lim\limits_{n\to\infty} z_1z_2z_3\ldots z_n$ is equal to
7. If $1, \omega, \omega^2 \ldots \omega^{n-1}$ roots of unity, then find $(7-\omega)(7-\omega^2)\ldots(7-\omega^{n-1})$
8. If $\alpha\neq1$ is any nth root of unity, then find $S=1+3\alpha+5\alpha^2\ldots.$ up to n terms

Answer Keys

1. $-\omega$
2. $3\omega^2$
3. 0
4. 12
5. i
6. $e^{i\alpha/2}$
7. $\dfrac{7^n-1}{6}$
8. $-\dfrac{2n}{1-\alpha}$

SOLVED PROBLEMS

PROBLEM 1

Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$ where p and q are distinct prime. Show that, either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$ but not both together. [IIT-2002]

SOLUTION

Given $z^{p+q} - z^p - z^q + 1 = 0$ …(1)

$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$

Since, α is root of equation (1),

either $\alpha^p - 1 = 0$ or $\alpha^q - 1 = 0$.

$$\alpha^p - 1 = 0 \Rightarrow \alpha^p = 1$$

Sum of roots $= 1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$

$$\alpha^q - 1 = 0 \Rightarrow \alpha^q = 1$$

Sum of roots $= 1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$

However, $\alpha^p - 1 = 0$ and $\alpha^q - 1 = 0$ cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1 = \alpha^p = \alpha^q$.

PROBLEM 2

If ω is the n^{th} imaginary root of unity and z_1 and z_2 any two complex numbers prove that

$$\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = n\{|z_1|^2 + |z_2|^2\} \quad (n \in \text{N})$$

SOLUTION

$1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ are the n, n^{th} roots of unity, then $\sum_{k=0}^{n-1} \omega^k = 0$

$$\text{LHS} \sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = \sum_{k=0}^{n-1} (z_1 + \omega^k z_2)(\bar{z}_1 + (\bar{\omega})^k \bar{z}_2)$$

$$= \sum_{k=0}^{n-1} \left(z_1\bar{z}_1 + z_1\bar{z}_2(\bar{\omega})^k + \bar{z}_1 z_2 \omega^k + z_2\bar{z}_2(\omega^k)(\bar{\omega})^k \right)$$

$$= \sum_{k=0}^{n-1} |z_1|^2 + \sum_{k=0}^{n-1} z_1\bar{z}_2(\bar{\omega})^k + \sum_{k=0}^{n-1} \bar{z}_1 z_2 \omega^k + \sum_{k=0}^{n-1} |z_2|^2$$

$$= n|z_1|^2 + 0 + 0 + n|z_2|^2$$

$$= n\{|z_1|^2 + |z_2|^2\} = \text{R.H.S}$$

PROBLEM 3

Find number of real or complex values of z which simultaneously satisfy the two given equations

$$1 + z + z^2 + \ldots + z^{17} = 0 \text{ and } 1 + z + z^2 + \ldots + z^{13} = 0$$

SOLUTION

$$1 + z + z^2 + \ldots + z^{17} = 0 \Rightarrow \frac{z^{18} - 1}{z - 1} = 0$$

$\Rightarrow \quad z^{18} = 1 \Rightarrow z = (1)^{1/18}$

$$z = \left(e^{i2m\pi}\right)^{1/18} = e^{i\frac{m\pi}{9}} \text{ where } m = 1, 2, \ldots, 17$$

$$1 + z + z^2 + \ldots + z^{13} = 0$$

$\Rightarrow \quad \dfrac{z^{14} - 1}{z - 1} = 0 \Rightarrow z^{14} = 1 \Rightarrow z = (1)^{1/14}\ z \neq 1$

$$z = \left(e^{i2r\pi}\right)^{1/14} = e^{i\frac{r\pi}{7}}$$

where $r = 1, 2, \ldots, 13$ for common values $e^{i\frac{m\pi}{9}} = e^{i\frac{r\pi}{7}}$

$\Rightarrow \quad \dfrac{m}{9} = \dfrac{r}{7}$ where $m = 1, 2, \ldots, 17$ and $r = 1, 2, \ldots, 13$

$m = \dfrac{9r}{7}$ $r = 7; m = 9$ only satisfied data so only one common value of z.

PROBLEM 4

If A, B and C be the angles of a triangle, then prove that $\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$ is purely real.

SOLUTION

$A + B + C = \pi$, $e^{\pi i} = \cos\pi + i\sin\pi = -1 = e^{-\pi i}$,

$$e^{i(\text{B} + \text{C})} = e^{i(\pi - \text{A})} = e^{\pi i}\, e^{-i\text{A}} = -e^{-i\text{A}}$$

Therefore, $\quad e^{-i(\text{B}+\text{C})} = -e^{i\text{A}}$

$$\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$$

Take $e^{i\text{A}}, e^{i\text{B}}$ and $e^{i\text{C}}$ common from R_1, R_2 and R_3, respectively.

$$e^{i(A+B+C)}\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(B+A)} \\ e^{-i(B+C)} & e^{-i(C+A)} & e^{iC} \end{vmatrix}$$

$$= -1\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

Take e^{iA}, e^{iB} and e^{iC} common from C_1, C_2 and C_3

Therefore, $\Delta = (-1)\ (e^{i(A+B+C)})\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$

Using $e^{i(A+B+C)} = e^{i\pi} = -1$

$$\Delta = (-1)\ (-1)\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4 = \text{Purely real.}$$

PROBLEM 5

If $\sin\alpha + 2\sin\beta + 3\sin\gamma = \cos\alpha + 2\cos\beta + 3\cos\gamma = 0$, then evaluate the following.

(a) $\cos 3\alpha + 8\cos 2\beta + 27\cos 3\gamma$

(b) $\sin(\beta+\gamma) + 2\sin(\gamma+\alpha) + 3\sin(\alpha+\beta)$

SOLUTION

As $\quad \cos\alpha + 2\cos\beta + 3\cos\gamma = 0 \qquad \ldots(1)$

$\quad i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0 \qquad \ldots(2)$

(a) On adding (1) and (2), $\text{cis}\alpha + 2\text{cis}\beta + 3\text{cis}\gamma = 0$

$A + 2B + 3C = 0$ (where $\text{cis}\alpha = A$, $\text{cis}\beta = B$, $\text{cis}\gamma = C$)

So, $A^3 + 8B^3 + 27C^3 = 18ABC$

$\text{cis}3\alpha + 8\text{cis}3\beta + 27\text{cis}3\gamma = 18\text{cis}(\alpha+\beta+\gamma)$

On comparing real part,

$\text{cis}3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha+\beta+\gamma)$

(b) As $A + 2B + 3C = 0$

So, $\dfrac{1}{A} + \dfrac{2}{B} + \dfrac{3}{C} = 0 \Rightarrow BC + 2CA + 3AB = 0$

Therefore, $\text{cis}\beta\,\text{cis}\gamma + 2\text{cis}\gamma\,.\,\text{cis}\alpha + 3\text{cis}(\alpha+\beta) = 0$

$\text{cis}(\beta+\gamma) + 2\text{cis}(\gamma+\alpha) + 3\text{cis}(\alpha+\beta) = 0$

On comparing imaginary part,

$\sin(\beta+\gamma) + 2\sin(\gamma+\alpha) + 3\sin(\alpha+\beta) = 0$

PROBLEM 6

If $(1+x+x^2)^{3n} = \sum\limits_{r=0}^{6n} a_r x^r$, then compute the value of $a_0 + a_6 + a_{12} + \ldots\ldots + a_{6n}$

SOLUTION

$$(1 + x + x^2)^{3n} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \ldots\ldots\ldots + a_{6n}x^{6n}$$

In the above equation we put x, as 6th root of unity

i.e., $\quad x = 1, e^{i\pi/3}, e^{i2\pi/3}, -1, e^{i4\pi/3}, e^{i8\pi/3}, e^{i5\pi/3}$

On substituting these values and then adding the resulting equations, we get

$$6\left[a_0 + a_6 + a_{12} + \ldots + a_{6n}\right] = 3^{3n} + 1 + \left(2e^{i\pi/3}\right)^{3n} + \left(2e^{-i\pi/3}\right)^{3n}$$

$$a_0 + a_6 + a_{12} + \ldots + a_{6n} = \frac{3^{3n} + 1 + 2^{3n}\left[e^{in\pi} + e^{-in\pi}\right]}{6}$$

$$= \frac{3^{3n} + 1 + 2^{3n}\cdot 2\cos n\pi}{6}$$

$$= \frac{3^{3n} + 1 + 2^{3n+1}(-1)^n}{6}$$

PROBLEM 7

If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots. + a_{2n}x^{2n}$ and $E_1 = a_0 + a_3 + a_6 + \ldots.$, $E_2 = a_1 + a_4 + a_7 + \ldots\ldots.$, $E_3 = a_2 + a_5 + a_8 + \ldots$, then prove that $E_1 = E_2 = E_3 = 3^{n-1}$

SOLUTION

We have $(1+x+x^2)^n = (a_0 + a_3x^3 + a_6x^6 + \ldots..) + x(a_1 + a_4x^3 + a_7x^6 + \ldots.) + x^2(a_2 + a_5x^3 + a_8x^6 + \ldots.)$

Substituting $x = 1, \omega, \omega^2$ respectively, we get

$3^n = E_1 + E_2 + E_3 \qquad \ldots(1)$

$0 = E_1 + \omega E_2 + \omega^2 E_3 \qquad \ldots(2)$

$0 = E_1 + \omega^2 E_2 + \omega E_3 \qquad \ldots(3)$

Adding (1), (2) and (3) we get

$$3^n = 3E_1 + (1 + \omega + \omega^2)E_2 + (1 + \omega^2 + \omega)E_3 \Rightarrow E_1 = 3^{n-1}$$

Multiplying (1) by 1, (2) by ω^2 and (3) by ω and adding, we get

$$3^n = (1 + \omega^2 + \omega)E_1 + (1 + \omega^3 + \omega^3)E_2 + (1 + \omega^4 + \omega^2)E_3 = 3E_2$$

$\Rightarrow \quad E_2 = 3^{n-1}$

Similarly, multiplying (1) by 1, (2) by ω and (3) by ω^2 and adding we obtain $E_3 = 3^{n-1}$

Hence, $E_1 = E_2 = E_2 = 3^{n-1}$

PROBLEM 8

If $\cot^2\dfrac{\pi r}{2n+1}$, where $r = 1, 2, \ldots, n$ are roots of

$$x^n - \frac{{}^{2n+1}C_3}{{}^{2n+1}C_1}x^{n-1} + \frac{{}^{2n+1}C_5}{{}^{2n+1}C_1}x^{n-2} - \ldots = 0,$$

then evaluate $\lim\limits_{n\to\infty} 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2}$.

SOLUTION

As we know that, $\sin\theta < \theta < \tan\theta$ for all $\theta \in \left(0, \frac{\pi}{2}\right)$.

So, $\operatorname{cosec}\theta > \frac{1}{\theta} > \cot\theta \Rightarrow \cot^2\theta < \frac{1}{\theta^2} < \operatorname{cosec}^2\theta.$

$$\text{So, } \sum_{r=1}^{n} \cot^2\frac{\pi r}{2n+1} < \sum_{r=1}^{n}\left(\frac{2n+1}{\pi r}\right)^2 < \sum_{r=1}^{n} \operatorname{cos}ec^2\frac{\pi r}{2n+1} \qquad \ldots(1)$$

As $\cot^2\frac{\pi r}{2n+1}$ are roots of the given equation, (1) becomes

$$\frac{{}^{2n+1}C_3}{{}^{2n+1}C_1} < \left(\frac{2n+1}{\pi}\right)^2\left(1+\frac{1}{2^2}+\ldots..\frac{1}{n^2}\right) < n+\frac{{}^{2n+1}C_3}{{}^{2n+1}C_1}$$

$$\frac{\frac{(2n+1)(2n)(2n-1)}{6(2n+1)}}{\left(\frac{(2n+1)}{\pi}\right)^2} < 1+\frac{1}{2^2}+\ldots+\frac{1}{n^2} < \frac{n+\frac{(2n+1)(2n)(2n-1)}{6(2n+1)}}{\left(\frac{2n+1}{\pi}\right)^2}$$

$$\Rightarrow \frac{\pi^2}{6}\frac{\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2} < 1+\frac{1}{2^2}+\ldots+\frac{1}{n^2} < \frac{n^2}{n\left(2+\frac{1}{n}\right)^2}+\frac{\pi^2\left(1-\frac{1}{2n}\right)}{6\left(1+\frac{1}{2n}\right)^2}$$

as $n \to \infty$

$$\frac{\pi^2}{6} \le \lim_{n\to\infty}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right) \le 0+\frac{\pi^2}{6}$$

(by sandwich theorem)

$$\lim_{n\to\infty}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)=\frac{\pi^2}{6}$$

PROBLEM 9

If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, then evaluate

(a) $\sum\cos 3\alpha$ **(b)** $\sum\cos^2\alpha$

SOLUTION

As $\cos\alpha+\cos\beta+\cos\gamma=0 \qquad \ldots(1)$

$i\sin\alpha+i\sin\beta+i\sin\gamma=0 \qquad \ldots(2)$

(a) Add (1) and (2)

$(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)+(\cos\gamma+i\sin\gamma)=0$

i.e., $A+B+C=0$ (where $\operatorname{cis}\alpha=A$, $\operatorname{cis}\beta=B$, $\operatorname{cis}\gamma=C$)

Therefore, $A^3+B^3+C^3=3$ ABC

$\Rightarrow (\operatorname{cis}\alpha)^3+(\operatorname{cis}\beta)^3+(\operatorname{cis}\gamma)^3=3\operatorname{cis}\alpha.\operatorname{cis}\beta.\operatorname{cis}\gamma$

$\Rightarrow (\cos3\alpha+i\sin3\alpha)+(\cos3\beta+i\sin3\beta)$

$+(\cos3\gamma+i\sin3\gamma)$

$=3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$

$$\sum\cos3\alpha+i\sum\sin3\alpha=$$
$$3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$$

On comparing real parts of LHS and RHS

$$\sum\cos3\alpha=3\cos(\alpha+\beta+\gamma)$$

(b) On applying (1) – (2),

$(\cos\alpha-i\sin\alpha)+(\cos\beta-i\sin\beta)+(\cos\gamma-i\sin\gamma)=0$

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}=A^{-1}+B^{-1}+C^{-1}=0$$

$$\Rightarrow \frac{\sum AB}{ABC}=0 \Rightarrow AB+BC+CA=0$$

$$(A+B+C)^2=\sum A^2+2\sum AB$$

$$\Rightarrow \sum A^2=(A+B+C)^2=0$$

$$\Rightarrow \sum\left(e^{i\alpha}\right)^2=0 \Rightarrow \sum e^{i2\alpha}=0$$

Therefore, $\sum\cos2\alpha=0$

$$\sum(2\cos^2\alpha-1)=0 \quad \Rightarrow \quad \sum\cos^2\alpha=\frac{3}{2}$$

PROBLEM 10

$A(z_a)$, $B(z_b)$ and $C(z_c)$ be the vertices of ΔABC taken in anti-clockwise direction whose circumcircle is $|z|=r$. If the internal angular bisector of angle A meets the circumcircle again at $D(z_d)$, then find z_d

SOLUTION

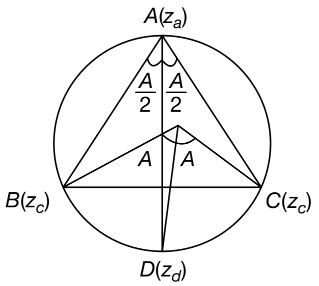
$$\frac{z_c-0}{R}=\frac{z_b-0}{R}e^{i2A} \Rightarrow z_c=z_be^{i2A} \qquad \ldots(1)$$

$$\frac{z_d-0}{R}=\frac{z_b-0}{R}e^{iA}$$

$$\Rightarrow \quad z_d^2=z_b^2e^{i2A}=z_b^2.\frac{z_c}{z_b}=z_bz_c$$

Alternate:

Let D represents the complex number $z=(z_d)$



$$\angle BAD=\angle CAD=\frac{A}{2}$$

Rotation about A

$$\frac{z-z_a}{|z-z_a|}=\frac{z_b-z_a}{|z_b-z_a|}e^{\frac{iA}{2}}$$

$$\frac{(z-z_a)^2}{|z-z_a|^2}=\frac{(z_b-z_a)^2}{|z_b-z_a|^2}e^{iA} \qquad \ldots(1)$$

Similarly $\dfrac{(z_c-z_a)^2}{|z_c-z_a|^2}=\dfrac{(z-z_a)^2}{|z_c-z_a|^2}e^{iA}$ $\qquad \ldots(2)$

From (1) and (2)

$$\frac{z-z_a}{\bar{z}-\bar{z}_a}=\frac{z_b-z_a}{\bar{z}_b-\bar{z}_a}e^{iA},\ \frac{z_c-z_a}{\bar{z}_c-\bar{z}_a}=\frac{z-za}{\bar{z}-\bar{z}_a}e^{iA}$$

$$\Rightarrow \quad \frac{z-z_a}{\bar{z}-\bar{z}_a}\times\frac{\bar{z}_c-\bar{z}_a}{z_c-z_a}=\frac{z_b-z_a}{\bar{z}_b-\bar{z}_a}\,\frac{\bar{z}-\bar{z}_a}{z-z_a}$$

$$\Rightarrow \quad \left(\frac{z-z_a}{\bar{z}-\bar{z}_a}\right)^2=\frac{z_b-z_a}{\bar{z}_b-\bar{z}_a}\cdot\frac{z_c-z_a}{\bar{z}_c-\bar{z}_a}$$

$$\Rightarrow \quad \left(\frac{z-z_a}{\frac{r^2}{z}-\frac{r^2}{z_a}}\right)^2=\left(\frac{z_b-z_a}{\frac{r^2}{z_b}-\frac{r^2}{z_a}}\right)\left(\frac{z_c-z_a}{\frac{r^2}{z_c}-\frac{r^2}{z_a}}\right)$$

$$\left(\begin{array}{l}\because z_a, z_b, z_c \text{ and } z \text{ lie on } |z|=r\\ \therefore |z_a|=|z_b|=|z_c|=r\end{array}\right)$$

$$\Rightarrow \quad (zz_a)^2=(z_az_b)(z_az_c)$$

$$\Rightarrow \quad z^2=z_bz_c \Rightarrow z_d^2=z_bz_c$$

PROBLEM 11

A, *B* and *C* are the vertices of a triangle inscribed in the circle $|z|=1$. Altitude from A meets the circumcircle again at *D*. If *A*, *B* and *C* represents the complex number z_1, z_2, z_3, respectively, then find the complex number representing the reflection of *D* in the line *BC*

SOLUTION

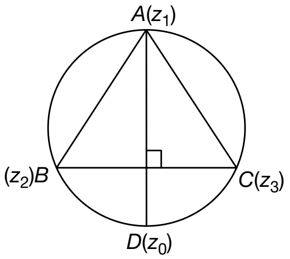


Image of *D* w.r.t sides of triangle is orthocentre $(z_0+z_2+z_3)$

$$AD \perp BC \Rightarrow \frac{z_0-z_1}{\bar{z}_0-\bar{z}_1}+\frac{z_2-z_3}{\bar{z}_2-\bar{z}_3}=0 \qquad (\text{using } z\bar{z}=1)$$

$$\Rightarrow \quad \frac{z_0-z_1}{\frac{1}{z_0}-\frac{1}{z_1}}+\frac{z_2-z_3}{\frac{1}{z_2}-\frac{1}{z_3}}=0$$

$$\Rightarrow \quad -z_0z_1-z_2z_3=0$$

$$z_0=-\frac{z_2z_3}{z_1}$$

$$\therefore \quad \text{Orthocentre}\left(z_2+z_3-\frac{z_2z_3}{z_1}\right)$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If z_1, z_2, z_3 and z_4 be the consecutive vertices of a square, then $z_1^2 + z_2^2 + z_3^2 + z_4^2$ equals
(A) $z_1z_2 + z_2z_3 + z_3z_4 + z_4z_1$
(B) $z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4$
(C) 0
(D) None of the above

2. If z_1, z_2 and z_3 are the vertices of an isosceles right angled triangle, right angled at the vertex z_2, then $(z_1 - z_2)^2 + (z_3 - z_2)^2$ equals
(A) 0
(B) $(z_1 - z_3)^2$
(C) $\left(\frac{z_1 + z_3}{2}\right)^2$
(D) None

3. Let $C = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$ and $S = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$, then
(A) $C = \frac{1}{2}$
(B) $S = \frac{1}{2}$
(C) $C = \frac{\sqrt{7}}{2}$
(D) $S = \frac{\sqrt{7}}{2}$

4. If z_1, z_2, z_3 are the non-zero complex numbers representing the points A, B, C such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Then
(A) A, B, C are collinear.
(B) circle passes through the points A, B, C and has centre at origin O.
(C) circle passes through A, B, C and passes through origin.
(D) none of these.

5. If $|2z - 4 - 2i| = |z| \sin\left(\frac{\pi}{4} - \arg(z)\right)$, then the locus of z is/an
(A) ellipse
(B) circle
(C) parabola
(D) pair of straight line

6. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then $\frac{\sin 3\alpha + \sin 3\beta + \sin 3\gamma}{\sin(\alpha + \beta + \gamma)}$ is equal to
(A) 0
(B) 1
(C) 2
(D) 3

7. If z is a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then the set of possible values of $|z|$ is
(A) {1, 2}
(B) {1}
(C) {1, 2, 3}
(D) {1, 2, 3, 4}

8. Let A, B and C represents the complex numbers z_1, z_2 and z_3 in the Argand plane. If the circumcentre of the ΔABC is at the origin, then the complex number corresponding to the orthocentre is
(A) $\frac{1}{4}(z_1 + z_2 + z_3)$
(B) $\frac{1}{3}(z_1 + z_2 + z_3)$
(C) $\frac{1}{2}(z_1 + z_2 + z_3)$
(D) $(z_1 + z_2 + z_3)$

9. If $z = x + iy$, then the equation $\left|\frac{2z - i}{z + 1}\right| = m$ does not represent a circle when
(A) $m = \frac{1}{2}$
(B) $m = 1$
(C) $m = 2$
(D) $m = 3$

10. The complex number 3 + 4i is rotated about the origin by an angle $\frac{\pi}{4}$ in anticlockwise direction and then stretched twice. The complex number corresponding to the new position is
(A) $\sqrt{2}(-3 + 4i)$
(B) $\sqrt{2}(-1 + 7i)$
(C) $\sqrt{2}(3 - 4i)$
(D) $\sqrt{2}(-1 - 7i)$

11. If $(a + ib)^5 = \alpha + i\beta$, then $(b + ia)^5$ is equal to
(A) $\beta - i\alpha$
(B) $\beta + i\alpha$
(C) $\alpha - \beta$
(D) $-\alpha - i\beta$

12. The complex numbers $a + i$, $a - i$, $1 + ai$ and $1 - ai$, where $a \in$ R taken in that order on the Argand plane represent the vertices of a parallelogram if
(A) $a = 1$
(B) $a = -1$
(C) $a = 0$
(D) none of these

13. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = \alpha + i\beta$, then $2 \cdot 5 \cdot 10 \ldots (1 + n^2)$ (where $\alpha, \beta, n \in R$) is equal to
(A) $\alpha - i\beta$
(B) $\alpha^2 - \beta^2$
(C) $\alpha^2 + \beta^2$
(D) none of these

14. If $z = \left(1 + \sqrt{3}i\right)^{10} + \left(1 - \sqrt{3}i\right)^{10}$, then Arg($z$) may be
(A) $\frac{\pi}{2}$
(B) π
(C) $\frac{\pi}{4}$
(D) none of these

15. If the complex numbers z_1, z_2 and z_3 satisfy $3z_1 = 5z_2 - 2z_3$, then z_1, z_2 and z_3 lie in a
(A) circle
(B) parabola
(C) line
(D) hyperbola

16. If z_1 and z_2 are the two complex numbers satisfying the equation $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$, then $\frac{z_1}{z_2}$ is a number which is
(A) positive real
(B) negative real
(C) zero or purely imaginary
(D) none of these

17. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|z_1 + z_2 + z_3| = 5$, then $|4z_2z_3 + 9z_3z_1 + 16z_1z_2| =$
(A) 20
(B) 24
(C) 48
(D) 120

18. Find the value of $\sin\left[\log_e\left\{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^z\right\}\right]$, where z satisfies the equation $|z - 2i| = 1$ and has least modulus
(A) 1
(B) 0
(C) –1
(D) $\frac{1}{2}$

19. If $\omega \neq 1$ be a cube root of unity and $(1 + \omega)^7 = l + m\omega$, then the value of $l + m$ is
(A) 0
(B) 1
(C) 2
(D) –1

20. One vertex of an equilateral triangle is at the origin and the other two vertices are the roots of $2z^2 + 2z + k = 0$, then the value of k is
(A) 1
(B) $\frac{1}{3}$
(C) $\frac{2}{3}$
(D) $\frac{1}{2}$

21. If $\log_{\tan 30^\circ}\left(\frac{2|z|^2 + 2|z| - 3}{|z| + 1}\right) < -2$ then
(A) $|z| < \frac{3}{2}$
(B) $|z| > \frac{3}{2}$
(C) $|z| > 2$
(D) $|z| < 2$

22. The number of common roots of the equations $x^3 + 2x^2 + 2x + 1 = 0$ and $x^{2012} + x^{2014} + 1 = 0$ is
(A) 1
(B) 2
(C) 3
(D) 4

23. If $|z| = \min\{|z - 1|, |z + 1|\}$, then the value of $|z + \bar{z}|$ is
(A) 1
(B) 2
(C) 3
(D) 4

24. If the roots of $z^3 + iz^2 + 2i = 0$ represent the vertices of a ΔABC in the Argand plane, then the area of the triangle is (in square units)
(A) 3
(B) 1
(C) 4
(D) 2

25. Let $z = \cos\theta + i\sin\theta$, then the value of $\sum_{m=1}^{15} lm(z^{2m-1})$ at $\theta = 2^\circ$ is
(A) $\frac{1}{2^\circ}$
(B) $\frac{1}{3\sin 2^\circ}$
(C) $\frac{1}{2\sin 2^\circ}$
(D) $\frac{1}{4\sin 2^\circ}$

26. If z_1 is a root of the equation $a_0z^n + a_1z^{n-1} + \ldots + a_{n-1}z + a_n = 3$, where $|a_i| < 2$ for $i = 0, 1, \ldots n$. Then
(A) $|z_1| > \frac{1}{3}$
(B) $|z_1| < \frac{1}{4}$
(C) $|z_1| > \frac{1}{4}$
(D) $|z| < \frac{1}{3}$

27. If $x = a + ib$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$, where $i = \sqrt{-1}$, then $a + b$ is
(A) 1
(B) 2
(C) 3
(D) 4

28. If the complex number z satisfies, $z + |z| = 2 + 8i$, then the value of $|z|$ will be
(A) 8
(B) 17
(C) 15
(D) 24

29. If $|z + 2 - i| = 5$, then the maximum value of $|3z + 9 - 7i|$ is
(A) 20
(B) 15
(C) 5
(D) 16

30. The greatest positive argument of z satisfying $|z - 4| = \text{Re}(z)$ is
(A) $\frac{\pi}{3}$
(B) $\frac{2\pi}{3}$
(C) $\frac{\pi}{2}$
(D) $\frac{\pi}{4}$

31. If z and w are the two complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\text{Arg}(zw) = \pi$, then Arg(z) is
(A) $\frac{\pi}{4}$
(B) $\frac{\pi}{2}$
(C) $\frac{3\pi}{4}$
(D) $\frac{5\pi}{4}$

32. If $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of a triangle such that $z_3 = \left(\frac{z_2 - iz_1}{1 - i}\right)$ and $|z_1| = 3$, $|Z_2| = 4$ and $|z_2 + iz_1| = |z_1| + |z_2|$, then the area of the ΔABC is
(A) $\frac{5}{2}$
(B) 0
(C) $\frac{25}{2}$
(D) $\frac{25}{4}$

33. The radius of the circle given by $\text{Arg}\left(\frac{Z - 5 + 4i}{Z + 3 - 2i}\right) = \frac{\pi}{4}$ is
(A) $5\sqrt{2}$
(B) 5
(C) $\frac{5}{\sqrt{2}}$
(D) $\sqrt{2}$

34. If $f(x) = 2x^3 + 2x^2 - 7x + 72$, then $f\left(\frac{3 - 5i}{2}\right)$ is
(A) 1
(B) 2
(C) 3
(D) 4

35. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
(A) $\frac{1}{2}$
(B) $\frac{3}{2}$
(C) 4
(D) 1

36. If $|z - |z + 1|| = |z + |z - 1||$, where z is a complex number on the complex plane, then which of the following lies on the locus of z
(A) line $y = 0$
(B) line $x = 2$
(C) circle $x^2 + y^2 = 1$
(D) line $x = 0$ or on a line segment joining (–1, 0) and (1, 0)

37. If z_1, z_2 are the two complex numbers such that $|z_1| = 1$ and $|z_2| = 1$, then the maximum value of $|z_1 + z_2| + |z_1 - z_2|$ is
(A) 2
(B) $2\sqrt{2}$
(C) 4
(D) none of these

38. If $|z-2| = \min\{|z-1|, |z-5|\}$, where z is a complex number then
(A) $\text{Re}(z) = \frac{3}{2}$ only
(B) $\text{Re}(z) = \frac{7}{2}$ only
(C) $\text{Re}(z) \in \left\{\frac{3}{2}, \frac{7}{2}\right\}$
(D) $\text{Re}(z) \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$

39. If z is a non-real complex number, then the minimum value of $\frac{\text{Im}\, z^5}{\text{Im}^5 z}$ is
(A) –1
(B) –2
(C) –4
(D) –5

40. Let $z_r (1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$. Then the value of k equals if, $|30z_1 + 20z_2 + 15z_3 + 12z_4| = k|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$
(A) $|z_1z_2z_3|$
(B) $|z_2z_3z_4|$
(C) $|z_4z_1z_2|$
(D) None of these

41. If $P(z)$ and $A(z_1)$ are the variable points such that $zz_1 = |z|^2$ and $|z - \bar{z}| + |z_1 + \bar{z}_1| = 10$, then the area enclosed by the curve formed by them will be
(A) 25π
(B) 20π
(C) 50
(D) 100

42. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. The complex number corresponding to point R in the Argand plane is
(A) $(3 + 5i)$
(B) $(3 + 7i)$
(C) $(3 + 8i)$
(D) $(3 + 9i)$

43. If $|z_1| = 2$, $|z_2| = 3$, $|z_3| = 4$ and $|2z_1 + 3z_2 + 4z_3| = 4$, then the absolute value of $8z_2z_3 + 27z_3z_1 + 64z_1z_2$ equals
(A) 24
(B) 48
(C) 72
(D) 96

44. If the ratio $\frac{1-z}{1+z}$ is purely imaginary, then
(A) $0 < |z| < 1$
(B) $|z| = 1$
(C) $|z| > 1$
(D) bounds for $|z|$ cannot be decided

45. If P and Q are represented by the numbers z_1 and z_2 such that $\left|\frac{1}{z_2} + \frac{1}{z_1}\right| = \left|\frac{1}{z_2} - \frac{1}{z_1}\right|$, then the circumcentre of ΔOPQ, (where O is the origin) is
(A) $\frac{z_1 - z_2}{2}$
(B) $\frac{z_1 + z_2}{2}$
(C) $\frac{z_1 + z_2}{3}$
(D) $z_1 + z_2$

46. If α is the non-real root of $x^7 = 1$, then $1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \ldots + 13\alpha^6$ is equal to
(A) 0
(B) $\frac{14}{1-\alpha}$
(C) $\frac{14}{\alpha - 1}$
(D) none of these

47. If a complex number 'z' lies in the interior or on the boundary of a circle of radius 3 and the centre at (–4, 0), then the greatest and the least values of $|z + 1|$ are respectively
(A) 5, 0
(B) 6, 1
(C) 6, 0
(D) 5, 1

48. The least positive integer 'n' for which $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right)$, where $x > 0$ is
(A) 2
(B) 4
(C) 8
(D) 12

49. If 'a' is a complex number such that $|a| = 1$, then find the values of 'a', so that the equation $az^2 + z + 1 = 0$ has one purely imaginary root.
(A) $a = \cos\theta + i\sin\theta,\ \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$
(B) $a = \sin\theta + i\cos\theta,\ \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{2}\right)$
(C) $a = \cos\theta + i\sin\theta,\ \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$
(D) $a = \sin\theta + i\cos\theta,\ \theta = \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)$

50. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} - i\cos\frac{2k\pi}{11}\right)$ is
(A) –1
(B) 0
(C) $-i$
(D) i

51. If 'z' lies on the circle $|z - 2i| = 2\sqrt{2}$, then the value of $\text{Arg}\left(\frac{z-2}{z+2}\right)$ is equal to
(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{6}$
(D) $\frac{\pi}{2}$

52. If 'z' is a complex number such that the equation $|z - a^2| + |z - 2a| = 3$ always represents an ellipse, then the range of $a\ (\in R^+)$ is
(A) $(1, \sqrt{2})$
(B) $(1, \sqrt{3})$
(C) $(-3, 1)$
(D) $(0, 3)$

53. ω is a non-real complex cube root of unity and $\alpha, \beta \in R$. If ω, ω^2 are roots of $\frac{1}{a+x} + \frac{1}{b+x} = \frac{3}{x}$, then a and b are the roots of
(A) $3x^2 - 6x + 2 = 0$
(B) $6x^2 - 3x + 2 = 0$
(C) $2x^2 - 3x + 6 = 0$
(D) $6x^2 - 2x + 3 = 0$

54. If z is a complex number such that $|z - 1| = 1$, then $\text{Arg}\left(\frac{1}{z} - \frac{1}{2}\right)$ may be

(A) $\frac{\pi}{6}$ (B) $-\frac{\pi}{2}$

(C) $\frac{\pi}{4}$ (D) $-\frac{\pi}{4}$

55. $\theta \in (0, 2\pi)$ and z_1, z_2 and z_3 are three complex numbers such that they are collinear and $(1 + |\sin\theta|)\, z_1 + (|\cos\theta| - 1)\, z_2 - \sqrt{2}\, z_3 = 0$. If at least one of these complex numbers (z_1, z_2 and z_3) is non-zero, then the number of possible values of θ is

(A) Infinite (B) 4

(C) 2 (D) 8

56. Let $A(z_1)$, $B(z_2)$ and $C(z_3)$ be the vertices of a triangle oriented in anticlockwise direction. If $BC : CA : AB = 2 : \sqrt{2} : \sqrt{3} + 1$, then the imaginary part of $\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^4$ is

(A) 0 (B) $-7 + 2\sqrt{6}$

(C) $7 - 2\sqrt{6}$ (D) cannot be determined

57. A point P representing the complex number z moves in the Argand plane so that it always lies in the region defined by $|z - 1| \le |z - i|$ and $|z - 2 - 2i| \le 1$. If P describes the boundary of this region, then the value of $|z|$ when the Arg(z) has the least value is

(A) $\sqrt{5}$ (B) 7

(C) $\sqrt{7}$ (D) 5

58. Let $P(z)$ be a variable point in the complex plane such that $|z| = \min\{|z - 1|, |z + 1|\}$, then the value of $(z + \bar{z})$ is

(A) 1 if $\text{Re}(z) > 0$ (B) 1 if $\text{Re}(z) < 0$

(C) 0 if $\text{Re}(z) > 0$ (D) 0 if $\text{Re}(z) < 0$

59. If z is a complex number satisfying $|z|^2 + 2(z + \bar{z}) + 3i(z - \bar{z}) + 4 = 0$, $i = \sqrt{-1}$, then the complex number $z + 3 + 2i$ will lie on a circle with

(A) centre $1 - 5i$, radius 4 (B) centre $1 + 5i$, radius 4

(C) centre $1 + 5i$, radius 3 (D) centre $1 - 5i$, radius 3

60. The value of $i\log_e(x - i) + i^2\pi + i^3\log_e(x + i) + i^4(2\tan^{-1}x)$, $x > 0$, $i = \sqrt{-1}$, is

(A) 0 (B) 1

(C) 2 (D) 3

61. If $\left|\frac{z_1}{z_2}\right| = 1$ and $\text{Arg}(z_1 z_2) = 0$, then

(A) $z_1 = z_2$ (B) $|z_2|^2 = z_1 z_2$

(C) $z_1 z_2 = 1$ (D) $z_1 z_2 = 2$

62. Let z be a complex number and a_k, b_k ($k = 1, 2, 3$) are real numbers, then the value of

$$\begin{vmatrix} a_1 z + b_1\bar{z} & a_2 z + b_2\bar{z} & a_3 z + b_3\bar{z} \\ b_1 z + a_1\bar{z} & b_2 z + a_2\bar{z} & b_3 z + a_3\bar{z} \\ b_1 z + a_1 & b_2 z + a_2 & b_3 z + a_3 \end{vmatrix} \text{ will be}$$

(A) $(a_1a_2a_3 + b_1b_2b_3)|z|^2$ (B) $(a_1a_2a_3 - b_1b_2b_3)|z|^2$

(C) $a_1^2 - a_1^2$ (D) $|z|^2$

63. If $|z - 1| \le 2$ and $|\omega z - 1 - \omega^2| = a$ (where 'ω' is a cube root of unity), then the complete set of values of a is

(A) $0 \le a \le 2$ (B) $\frac{1}{2} \le a \le \frac{\sqrt{3}}{2}$

(C) $\frac{\sqrt{3}}{2} - \frac{1}{2} \le a \le \frac{1}{2} + \frac{\sqrt{3}}{2}$ (D) $0 \le a \le 4$

64. If z is a complex number having the least absolute value and $|z - 2 + 2i| = 1$, then the value of z is

(A) $\left(2 - \frac{1}{\sqrt{2}}\right)(1 - i)$ (B) $\left(2 - \frac{1}{\sqrt{2}}\right)(1 + i)$

(C) $\left(2 + \frac{1}{\sqrt{2}}\right)(1 - i)$ (D) $\left(2 + \frac{1}{\sqrt{2}}\right)(1 + i)$

65. Let z be a complex number having the argument θ, $0 < \theta < \frac{\pi}{2}$ and satisfying the equation, $|z - 3i| = 3$. Then the value of $\cot\theta - \frac{6}{z}$ will be

(A) i (B) $-i$

(C) $2i$ (D) $-2i$

66. Let z be a complex number satisfying $|z + 16| = 4|z + 1|$ then,

(A) $|z| = 4$ (B) $|z| = 5$

(C) $|z| = 6$ (D) $4 < |z| < 6$

67. Let z_1 and z_2 be any two complex numbers then, $\left|z_1 + \sqrt{z_1^2 - z_2^2}\right| + \left|z_1 - \sqrt{z_1^2 - z_2^2}\right|$ is equal to

(A) $\left|z_1^2 - z_2^2\right| + \left|z_1^2 + z_2^2\right|$ (B) $|z_1 - z_2| + \left|z_1^2 + z_2^2\right|$

(C) $|z_1 + z_2| + \left|z_1^2 + z_2^2\right|$ (D) $|z_1 + z_2| + |z_1 - z_2|$

68. Both the roots of the equation $z^2 + az + b = 0$ are of unit modulus if

(A) $|a| \le 2, |b| = 1, \text{Arg}(b) = 2\text{Arg}(a)$

(B) $|a| \le 2, |b| = 1, \text{Arg}(b) = \text{Arg}(a)$

(C) $|a| \ge 2, |b| = 2, \text{Arg}(b) = 2\text{Arg}(a)$

(D) $|a| \ge 2, |b| = 2, \text{Arg}(b) = \text{Arg}(a)$

69. Let $z = \frac{z_1 - z_2}{z_1\bar{z}_2 - 1}$, $z_1 \ne \frac{1}{\bar{z}_2}$, $0 < |z_2| < 1$. If $|z| \le 1$, then

(A) $|z_1| > 1$ (B) $|z_1| \le 1$

(C) $2 < |z_1| < 3$ (D) $2 < |z_1| < 8$

70. Let 'z' be a complex number satisfying $|z - 2 - i| \le 5$. Then $|z - 14 - 6i|$ lies in

(A) $(8, 18)$ (B) $(2, 8)$

(C) $(0, 2)$ (D) $(3, 7)$

71. If z_1 and z_2 are the complex numbers such that $z_1^3 - 3z_1z_2^2 = 2$ and $3z_1^2z_2 - z_2^3 = 11$, then the value of $\left|z_1^2 + z_2^2\right|$ is

(A) 3 (B) 4

(C) 5 (D) 6

72. The circle $|z| = 2$ intersects the curve whose equation is $z^2 = (\bar{z})^2 + 4i$ in the points A, B, C and D. If z_1, z_2, z_3, z_4 represent the affixes of these points, then

(A) $z_1 z_2 z_3 z_4 = 1$

(B) $\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \bar{z}_4 = 0$

(C) $z_1 + z_2 + z_3 + z_4 = 2$

(D) $\text{Arg}(z_1) + \text{Arg}(z_2) + \text{Arg}(z_3) + \text{Arg}(z_4) = 2k\pi$, $k = 0, 1$ or -1

73. If the tangents at z_1 and z_2 on the circle $|z - z_0| = r$ intersect at z_3, then $\dfrac{(z_3 - z_1)(z_0 - z_2)}{(z_0 - z_1)(z_3 - z_2)}$ will be

(A) 1
(B) –1
(C) i
(D) $-i$

74. If z is a complex number, then the number of complex numbers satisfying the equation $z^{2009} = \bar{z}$ is

(A) 3
(B) 2009
(C) 2010
(D) 2011

75. If $a_1, a_2, \ldots, a_n$ are real numbers with $a_n \neq 0$ and $\cos\alpha + \sin\alpha$ is a root of $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z + a_n = 0$. Then the sum $a_1 \cos a + a_2 \cos 2a + a_3 \cos 3a + \ldots + a_n \cos na$ is

(A) 0
(B) 1
(C) –1
(D) $\dfrac{1}{2}$

76. If ω is a cube root of unity, then $\omega + \omega^{\left(\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\frac{27}{128}+\ldots\ldots\infty\right)}$ is equal to

(A) i
(B) i^2
(C) 0
(D) i^3

77. If z_1, z_2 and z_3 are any three complex numbers on Argand plane, then $Z_1(\text{Im}(Z_2Z_3)) + Z_2(\text{Im}(Z_3Z_1)) + Z_3(\text{Im}(Z_1Z_2))$ is equal to

(A) 0
(B) $z_1 + z_2 + z_3$
(C) $z_1 z_2 z_3$
(D) $\left(\dfrac{z_1 + z_2 + z_3}{z_1 z_2 z_3}\right)$

78. If the equation $z^2 + z + \alpha = 0$ has a purely imaginary root and α lies on the circle $|z| = 1$ then the imaginary part of that root is/are

(A) $\pm\sqrt{2}$
(B) 0
(C) $\pm\sqrt{2 - \sqrt{2}}$
(D) $\pm\sqrt{\dfrac{\sqrt{5} - 1}{2}}$

79. Let $z(\alpha, \beta) = \cos\alpha + e^{i\beta}\sin\alpha$ ($\alpha, \beta \in R$, $i = \sqrt{-1}$) then the exhaustive set of values of the modulus of $z(\theta, 2\theta)$, as θ varies is

(A) $(0, 1)$
(B) $(0, \sqrt{2})$
(C) $(1, 2)$
(D) $(\sqrt{2}, 2)$

80. If $|z| = 1$ and $z \neq \pm 1$, then one of the possible values of $\text{Arg}(z) - \text{Arg}(z + 1) - \text{Arg}(z - 1)$ is

(A) $-\dfrac{\pi}{6}$
(B) $\dfrac{\pi}{3}$
(C) $-\dfrac{\pi}{2}$
(D) $\dfrac{\pi}{4}$

81. If z_1, z_2 and z_3 are three distinct complex numbers and a, b, c are three positive real numbers such that $\dfrac{a}{|z_2 - z_3|} = \dfrac{b}{|z_3 - z_1|} = \dfrac{c}{|z_1 - z_2|}$ then, $\dfrac{a^2}{z_2 - z_3} + \dfrac{b^2}{z_3 - z_1} + \dfrac{c^2}{z_1 - z_2}$ is

(A) $3abc$
(B) $(abc)^3$
(C) $a + b + c$
(D) 0

82. Let P be a point on the circumcircle of the triangle whose vertices A, B and C (P, A, B, C are in order) are represented by the complex numbers ω^2, $2i\omega$ and -4 (ω is a non-real cube root of unity), respectively such that $PA \cdot BC = PC \cdot AB$, then the complex number associated with the midpoint of PB is

(A) $\omega - 1$
(B) 0
(C) $-i$
(D) $\omega - \omega^2$

83. The complex numbers satisfying $(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2$ is

(A) $\dfrac{\sqrt{33} - 5}{4}$
(B) $\dfrac{\sqrt{33} + 5}{24}$
(C) $\dfrac{-i\sqrt{23} - 5}{24}$
(D) $\dfrac{-i\sqrt{23} + 5}{24}$

84. If $|z| = 1$ and $z' = \dfrac{1 + z^2}{z}$, then

(A) z' lie on a line passing through origin
(B) $|z'| = \sqrt{2}$
(C) $\text{Re}(z') = 0$
(D) $\text{Im}(z') = 0$

85. Let $f(x)$ be the remainder obtained on dividing $x^{2007} - 1$ by $(x^2 + 1)(x^2 + x + 1)$, then $f(x)$ is a polynomial of degree

(A) 0
(B) 1
(C) 2
(D) 3

86. If $\alpha \neq 1$ is any of 7th roots of unity, then real part of $\alpha^{2009} + 3\alpha^{2010} + 5\alpha^{2011} + \ldots +$ up to 7 terms is

(A) 7
(B) 14
(C) –7
(D) –14

87. All the complex numbers z that satisfy the equation $z^{10} = (1 - z)^{10}$ lie on

(A) $x = \dfrac{1}{2}$
(B) $x = -\dfrac{1}{2}$
(C) $y = \dfrac{1}{2}$
(D) $y = -\dfrac{1}{2}$

88. If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the n, nth roots of unity, then the value of $\displaystyle\sum_{i=0}^{n-1} \frac{\alpha_i}{(3 - \alpha_i)}$ is equal to

(A) $\dfrac{n}{3^{n-1}}$
(B) $\dfrac{n - 1}{3^n - 1}$
(C) $\dfrac{n + 1}{3^n - 1}$
(D) $\dfrac{n + 2}{3^n - 1}$

89. Let $|z_1 - 1| = 1, |z_2 + 4| = 2$, then the maximum value of $|z_1 - z_2|$ is

(A) 8
(B) 5
(C) 4
(D) 2

90. Let $|z - (1 + \text{i})| < 2$ then, $|iZ + 1 + 2i|$ will be

(A) < 7
(B) < 9
(C) < 5
(D) < 10

91. If $x^6 = 2x^3 - 1$ and x is not real, then $\sum_{r=1}^{50}(x^r + x^{2r})^3$ is equal to

(A) 0 (B) 256
(C) 76 (D) 94

92. If $A(z_1), B(z_2), C(z_3)$ be the vertices of the ΔABC in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$ then z_2 is equal to

(A) $z_3 + i(z_1 + z_3)$ (B) $z_3 - i(z_1 - z_3)$
(C) $z_3 + i(z_1 - z_3)$ (D) $z_3 - i(z_2 - z_1)$

93. z_1 and z_2 be two complex numbers with α and β as their principal arguments, such that $\alpha + \beta > \pi$, then principal $\operatorname{Arg}(z_1 z_2)$ is

(A) $\alpha + \beta + \pi$ (B) $\alpha + \beta - \pi$
(C) $\alpha + \beta - 2\pi$ (D) $\alpha + \beta$

94. If the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16}$ is $\pm\left(\frac{x}{y} + \frac{y}{x} - \frac{i}{m}\right)$ then m is

(A) 2 (B) 3
(C) 4 (D) 5

95. Let $A_1 z_1)$, $A_2(\bar{z}_1)$ are the adjacent vertices of a regular polygon. If $\frac{\operatorname{Im}(\bar{z}_1)}{\operatorname{Re}(z_1)} = 1 - \sqrt{2}$, then number of sides of the polygon will be

(A) 6 (B) 8
(C) 16 (D) 12

96. If exactly one root of $z^2 + az + b = 0$, where $a, b \in C$ is purely imaginary, then

(A) $\left(\bar{b} - b\right)^2 = -\left(a\bar{b} + \bar{a}b\right)\left(a + \bar{a}\right)$
(B) $\left(\bar{b} - b\right)^2 = -\left(a\bar{b} + \bar{a}b\right)\left(a - \bar{a}\right)$
(C) $\left(\bar{b} - b\right)^2 = -\left(a\bar{b} - \bar{a}b\right)\left(a + \bar{a}\right)$
(D) $\left(\bar{b} - b\right)^2 = -\left(a\bar{b} - \bar{a}b\right)\left(a - \bar{a}\right)$

97. If the imaginary part of $\frac{2z+1}{iz+1}$ is –4, then the locus of the point representing z in the complex plane is

(A) a straight line (B) a parabola
(C) a circle (D) an ellipse

98. If $Z_k = \cos\frac{k\pi}{10} + i\sin\frac{k\pi}{10}$, then $z_1 z_2 z_3 z_4$ is equal to

(A) –1 (B) 1
(C) –2 (D) 2

99. If a, b, c, p, q, r are three non-zero complex numbers such that $\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 + i$ and $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$, then the value of $\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$ is

(A) 0 (B) –1
(C) $2i$ (D) $-2i$

100. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is

(A) equal to 1 (B) less than 1
(C) greater than 3 (D) equal to 3

101. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(A) of area $\sqrt{3}$
(B) right angled and isosceles
(C) equilateral
(D) obtuse angled and isosceles

102. The vertices B and D of a parallelogram are $1 - 2i$ and $4 + 2i$, respectively. If the diagonals are at right angles and $|AC| = 2|BD|$, then the complex number representing A is

(A) $\frac{3}{2}i + \frac{1}{2}$ (B) $3i - 4$
(C) $3i - \frac{3}{2}$ (D) $\frac{5}{2}$

103. For all z satisfying $|z + 1 - i| = 1$, we have

(A) $\frac{\pi}{2} \le \operatorname{Arg}(z) \le \pi$ (B) $-\pi \le \operatorname{Arg}(z) \le \frac{-\pi}{2}$
(C) $-\pi < \operatorname{Arg}(z) \le \frac{\pi}{2}$ (D) None of these

104. If $z^5 = (z - 1)^5$, then the roots are represented in the Argand plane by the points that are

(A) Collinear
(B) Concylic
(C) Vertices of a parallelogram
(D) None of these

105. Let $|z - 5 + 12\,i| = 1$ and the least and greatest values of $|\,z\,|$ are m and n and if l be the least positive value of $\frac{x^2 + 24x + 1}{x}$ $(x > 0)$, then l is

(A) $\frac{m+n}{2}$ (B) $m + n$
(C) m (D) n

106. If $x = 2 + 5i$, then the value of $x^3 - 5x^2 + 33x - 19$ is equal to

(A) –5 (B) –7
(C) 7 (D) 10

107. The complex numbers z_1, z_2 and z_3 are the vertices of equilateral triangle. If z_0 is the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2$ is

(A) z_0^2 (B) $3\,z_0^2$
(C) z_0^3 (D) $3\,z_0^3$

108. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(A) $x = n\pi$ (B) $x = 0$
(C) $x = \left(n + \frac{1}{2}\right)\pi$ (D) no value of x

109. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then z_2 may be

(A) $1 + i\sqrt{3}, 2$
(B) $3 - i\sqrt{2}, 4$
(C) $1 - i\sqrt{3}, -2$
(D) $2 - i\sqrt{3}, 1$

110. If $a = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$, then the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$ is

(A) $x^2 + x + 2 = 0$
(B) $x^2 - 5x + 7 = 0$
(C) $x^2 - x + 2 = 0$
(D) $x^2 + x - 2 = 0$

111. Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \pi$, then z will be

(A) w
(B) $\bar{w}$
(C) $-\bar{w}$
(D) $2w$

112. If $|z_1 - 1| \le |z_2 - 2| \le |z_3 - 3| \le 3$, then the greatest value of $|z_1 + z_2 + z_3|$ is

(A) 6
(B) 7
(C) 9
(D) 12

113. If $\dfrac{\pi}{2} < \alpha < \dfrac{3\pi}{2}$, then the modulus and argument of $(1 + \cos 2\alpha) + i\sin 2\alpha$ is

(A) $-2\sin\alpha, \dfrac{\pi}{6}$
(B) $-2\cos\alpha, \alpha - \pi$
(C) $-2\sin\alpha, \alpha - \pi$
(D) None of these

114. If $|z - 1| + |z + 3| \le 8$, then the minimum and maximum values of $|z - 4|$, respectively is

(A) 1, 9
(B) 10, 15
(C) 16, 22
(D) 13, 18

115. If $|z - 25i| \le 15$, then

$|\text{maximum } \operatorname{Arg}(z) - \text{minimum } \operatorname{Arg}(z)|$ equals

(A) $2\cos^{-1}\dfrac{3}{5}$
(B) $2\cos^{-1}\dfrac{4}{5}$
(C) $\dfrac{\pi}{2} + \cos^{-1}\dfrac{3}{5}$
(D) $\sin^{-1}\dfrac{3}{5} - \cos^{-1}\dfrac{3}{5}$

116. Let z_1 and z_2 be two non-zero complex numbers such that $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_1} = 1$, then the origin and the points represented by z_1 and z_2 will

(A) lie on a straight line
(B) form a right triangle
(C) form an equilateral triangle
(D) None of these

117. If $z = (\lambda + 3) - i\sqrt{5 - \lambda^2}$ then the locus of z is

(A) an ellipse
(B) a semicircle
(C) a parabola
(D) a straight line

118. If $x^2 + x + 1 = 0$, then the value of

$$\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + \left(x^{27} + \frac{1}{x^{27}}\right)^2 \text{ is}$$

(A) 27
(B) 72
(C) 45
(D) 54

119. If the centre of a regular hexagon is at the origin and one of the vertices on Argand diagram is $1 + 2i$, then its perimeter is

(A) $2\sqrt{5}$
(B) $6\sqrt{2}$
(C) $4\sqrt{5}$
(D) $6\sqrt{5}$

120. Let $A(Z_1)$, $B(Z_2)$ and $C(Z_3)$ be the vertices of an equilateral ΔABC, then the value of $\operatorname{Arg}\left(\dfrac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right)$ is equal to

(A) $\dfrac{\pi}{3}$
(B) $\dfrac{\pi}{4}$
(C) $\dfrac{\pi}{2}$
(D) $\dfrac{\pi}{6}$

121. If $|z - 1 - i| = 1$, then the locus of a point represented by the complex number $5(z - i) - 6$ is

(A) a circle with centre (1, 0) and radius 3
(B) a circle with centre (–1, 0) and radius 5
(C) a line passing through origin
(D) a line passing through (–1, 0)

122. If $|z - 1| = 1$ and $\arg z = \theta (z \ne 0)$ and $0 < \theta < \dfrac{\pi}{2}$, then $1 - \dfrac{2}{z}$ is equal to

(A) $\tan\theta$
(B) $i\tan\theta$
(C) $\tan\dfrac{\theta}{2}$
(D) $i\tan\dfrac{\theta}{2}$

123. If $x_1, x_2, x_3 \ldots, x_n$ are the roots of $x^n + ax + b = 0$, then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)\ldots(x_1 - x_n)$ is

(A) $nx_1^n + a$
(B) $nx_1^{n-1} + a$
(C) $nx_1 + a^{n-1}$
(D) $nx_1 + a^n$

124. In triangle ABC, $A(z_1)$, $B(z_2)$ and $C(z_3)$ is inscribed in the circle $|z| = 5$. If $H(z_H)$ is the orthocentre of ΔABC, then Z_H is equal to

(A) $\dfrac{2}{3}(z_1 + z_2 + z_3)$
(B) $\dfrac{4}{3}(z_1 + z_2 + z_3)$
(C) $(z_1 + z_2 + z_3)$
(D) $3(z_1 + z_2 + z_3)$

125. If the tangents drawn to circle $|Z| = 2$ at $A(Z_1)$ and $B(Z_2)$ meet at $P(Z_P)$, then

(A) $Z_P = \left(\dfrac{z_1 + z_2}{2}\right)$
(B) $Z_P = \dfrac{2(z_1 + z_2)}{\sqrt{z_1 z_2}}$
(C) $Z_P = \dfrac{2z_1 z_2}{z_1 + z_2}$
(D) $Z_P^2 = z_2 z_1$

126. Let n is of the form $3P$, where P is an odd integer then, ${}^nC_0 + {}^nC_3 + {}^nC_6 + {}^nC_9 + \ldots + {}^nC_n$ equals

(A) $\dfrac{1}{3}(2^n - 2)$
(B) $\dfrac{2}{3}(2^n - 2)$
(C) $\dfrac{1}{3}(2^{n-1} - 2)$
(D) $\dfrac{2}{3}(2^n + 2)$

127. Points $A(z_1)$, $B(z_2)$ and $C(z_3)$ form a triangle with centroid z_0. If the triangles XCB, CYA and BAZ are similar to ΔABC are outwardly drawn on the sides of ΔABC, then the centroid of ΔXYZ is

(A) $3z_0$
(B) $-z_0$
(C) z_0
(D) $-2z_0$

128. The roots of $1+z+z^3+z^4=0$ are represented by the vertices of
(A) a square
(B) an equilateral triangle
(C) a rhombus
(D) a rectangle

129. The reflection of the complex number $\dfrac{6+10i}{(1+i)^2}$ in the straight line $i\bar{z} = z$ is
(A) $-3+5i$
(B) $-3-5i$
(C) $3-5i$
(D) $3+5i$

130. If $z = \cos\alpha + i\sin\alpha$, $0 < \alpha < \dfrac{\pi}{6}$ then the argument of $\dfrac{z^4-1}{z^3+1}$ is
(A) $\dfrac{\pi}{2} + \dfrac{\alpha}{2}$
(B) $\dfrac{\pi}{2} - \dfrac{\alpha}{2}$
(C) $\dfrac{3\alpha}{2}$
(D) $2\alpha - \dfrac{\pi}{2}$

131. The number of complex numbers z satisfying $|z+\bar{z}| + |z-\bar{z}| = 4$ and $|z+2i| + |z-2i| = 4$ is/are
(A) 0
(B) 1
(C) 2
(D) 4

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If a and b are complex numbers, then $|a+\sqrt{a^2-b^2}| + |a-\sqrt{a^2-b^2}|$ is
(A) equal to $|a+b| + |a-b|$
(B) equal to $|a+b| - |a-b|$
(C) $\le \dfrac{1}{2}(|a|^2 + |b|^2)$
(D) $\le 2(|a| + |b|)$

2. Let z be a complex number satisfying $|z+1| < |z-2|$. If $\omega = 3z+2+i$, then
(A) $|\omega+1| < |\omega-8|$
(B) $|\omega+1+i| < |\omega-8+i|$
(C) $|\omega+5| < |\omega-4|$
(D) $|\omega-12+i| < |\omega-3+i|$

3. If $\alpha \ne 1$ is a 5th root of 1 and $\beta \ne -1$ is a 5th root of -1. Which of the following assertions is/are true?
(A) $\alpha\beta$ is a 5th root of -1
(B) $\alpha^2\beta$ is a 5th root of 1
(C) $|\alpha\beta| = 1$
(D) $|\alpha+\beta| < 2$

4. Equation of tangent drawn to circle $|z| = r$ at the point $A(z_0)$ is
(A) $\text{Re}\left(\dfrac{z}{z_0}\right) = 1$
(B) $z\bar{z}_0 + z_0\bar{z} = 2r^2$
(C) $\text{Im}\left(\dfrac{z}{z_0}\right) = 1$
(D) $\text{Im}\left(\dfrac{z_0}{z}\right) = 1$

5. Let z_1, z_2 and z_3 in GP be the roots of the equation $z^3 - bz^2 + 3z - 1 = 0$ then,
(A) $z_2 = 1$
(B) $z_2 = 2$
(C) $b = 3$
(D) b can be -3

6. If $|z| = 1$, then the locus of z_1 which moves such that $z_1 = \dfrac{5-4z}{4z-2}$ in the complex plane is
(A) circle centred at $\left(\dfrac{1}{2}, 0\right)$
(B) circle of radius $4\dfrac{1}{2}$
(C) circle centred at $\left(-\dfrac{1}{2}, 0\right)$ and of radius 1
(D) circle passing through all quadrants

7. If $A(z_1)$, $B(z_2)$ and $C(z_3)$ are three points in Argand plane where $|z_1+z_2| = |z_1| - |z_2|$ and $|(1-i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$ then
(A) A, B, C lie on a fixed circle with centre $\dfrac{z_2+z_3}{2}$
(B) A, B and C form a right triangle
(C) A, B and C form an equilateral triangle
(D) A, B and C form an obtuse triangle

8. If from a point $P(Z_1)$ on the curve $|Z| = 2$, a pair of tangents are drawn to the curve $|Z| = 1$ meeting at $Q(Z_2)$, $R(Z_3)$ then
(A) complex number $\dfrac{z_1+z_2+z_3}{3}$ will lie on the curve $|Z| = 1$
(B) $\left(\dfrac{4}{\bar{Z}_1} + \dfrac{1}{\bar{Z}_2} + \dfrac{1}{\bar{Z}_3}\right)\left(\dfrac{4}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3}\right) = 9$
(C) orthocentre and circumcentre of ΔPQR will coincide
(D) $\text{Arg}\left(\dfrac{Z_2}{Z_3}\right) = \dfrac{2\pi}{3}$

9. Let z be a complex number such that $\left|z + \dfrac{1}{z}\right| = 2$. If $|z| = r_1$ and r_2
(A) $|r_1 - r_2| = \sqrt{2}$, for $\text{Arg}(z) = \dfrac{\pi}{4}$
(B) Range of $|r_1 - r_2| = [0, 2]$, as $\text{Arg}(z)$ varies
(C) $|r_1 - r_2| = 3$, if $\text{Arg}(z) = \dfrac{\pi}{3}$
(D) $|r_1 - r_2| = 2$, if $\text{Arg}(z) = \dfrac{\pi}{2}$

10. If z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$, then
(A) $3 \le |z_1 - 2z_2| \le 5$
(B) $1 \le |z_1 + z_2| \le 3$
(C) $|z_1 + 3z_2| \ge 5$
(D) $|z_1 - z_2| \ge 1$

11. Let A and B are two different points denoting the complex numbers α and β, respectively. A complex number z lies between A and B, where $z \ne \alpha$, $z \ne \beta$. Which of the following relation(s) hold good?

(A) $|\alpha - z| + |z - \beta| = |\alpha - \beta|$

(B) $\exists$ a positive real number 't' such that $z = (1 - t)\alpha + t\beta$

(C) $\begin{vmatrix} z - \alpha & \bar{z} - \bar{\alpha} \\ \beta - \alpha & \bar{\beta} - \bar{\alpha} \end{vmatrix} = 0$

(D) $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$

12. If z_1, z_2 and z_3 be three uni-modular complex numbers such that $\frac{z_1^2}{z_2 z_3} + \frac{z_2^2}{z_1 z_3} + \frac{z_3^2}{z_1 z_2} + 1 = 0$, then $|z_1 + z_2 + z_3|$ can take which of the following values?

(A) 4
(B) 3
(C) 2
(D) 1

13. Let z_1, $\bar{z}_1$, z_2, $\bar{z}_2$, z_3, $\bar{z}_3$, z_4, $\bar{z}_4$, z_5, $\bar{z}_5$ be the roots of the equation $x^{10} + (13x - 1)^{10} = 0$. If $N = \sum_{i=1}^{5} \frac{1}{|z_i|^2}$, then

(A) the digit in the units place of N is 0
(B) the sum of the digits in N is 13
(C) the digit in the 10's place of N is 5
(D) the product of digits of N is 0

14. For any complex number $z = x + iy$, define $(z) = |x| + |y|$. If z_1 and z_2 are any complex numbers, then

(A) $(z_1 + z_2) \le (z_1) + (z_2)$
(B) $(z_1 + z_2) = (z_1) + (z_2)$
(C) $(z_1 + z_2) \ge (z_1) + (z_2)$
(D) $|(z_1 + z_2)| \le |(z_1)| + |(z_2)|$

15. Let $z_1 = 1 + i$, $z_2 = -1 - i$ and z_3 be the complex numbers such that z_1, z_2 and z_3 form an equilateral triangle. Then z_3 is equal to

(A) $\sqrt{3}(1 + i)$
(B) $\sqrt{3}(1 - i)$
(C) $\sqrt{3}(i - 1)$
(D) $\sqrt{3}(-1 - i)$

16. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at M and satisfy $BD = 2AC$. If the points D and M are represented by the complex numbers $1 + i$ and $2 - i$, respectively, then A is represented by

(A) $3 - \frac{i}{2}$
(B) $3 + \frac{i}{2}$
(C) $1 + \frac{3i}{2}$
(D) $1 - \frac{3i}{2}$

17. Z_1, Z_2, Z_3, Z_4, Z_5 and Z_6 are the vertices of a regular hexagon in anticlockwise direction. If Z_1, Z_2 are given, then

(A) $Z_3 = -\frac{1}{2}\left(1 + \sqrt{3}\,i\right)Z_1 + \frac{1}{2}\left(3 + \sqrt{3}i\right)Z_2$

(B) $Z_4 = -\sqrt{3}i\,Z_1 + \left(1 + \sqrt{3}i\right)Z_2$

(C) $Z_5 = \left(1 - \sqrt{3}i\right)Z_1 + \sqrt{3}i\,Z_2$

(D) $Z_6 = \frac{1}{2}\left(3 - \sqrt{3}i\right)Z_1 - \frac{1}{2}\left(1 - \sqrt{3}i\right)Z_2$

18. If $z_1 = 5 + 12i$ and $|z_2| = 4$, then

(A) maximum $\left(|z_1 + iz_2|\right) = 17$

(B) minimum $\left(|Z_1 + (1 + i)z_2|\right) = 13 - 4\sqrt{2}$

(C) minimum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{4}$

(D) maximum $\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3}$

19. If $iz^3 + z^2 - z + i = 0$, where $i = \sqrt{-1}$ and 'z' is a complex number, then

(A) 'z' lies on an ellipse
(B) 'z' has finite number of solutions
(C) 'z' lies on a straight line only
(D) 'z' is at unit distance from origin

20. Let $P_k (k = 1, 2, 3, 4, \ldots, n)$ be the nth roots of unity. If $z = a + ib$ and $A_k = \text{Re}(z)\text{Re}(P_k) + i\{\text{Im}(z)\text{Im}(P_k)\}$, then

(A) locus of A_k is an ellipse
(B) locus of A_k is a hyperbola
(C) if S be the focus of locus of A_k, then $\sum_{k=1}^{n} A_k S = na$
(D) if S be the focus of locus of A_k, then $\sum_{k=1}^{n} (A_k S)^2 = \frac{n(3a^2 - b^2)}{2}$

21. If $P(z_1), Q(z_2), R(z_3)$ and $S(z_4)$ are the vertices of a rhombus taken in order, then

(A) $\frac{z_1 - z_4}{z_2 - z_3}$ is purely real

(B) $\text{Amp}\,\frac{z_1 - z_4}{z_2 - z_4} = \text{Amp}\,\frac{z_2 - z_4}{z_3 - z_4}$

(C) $\frac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

(D) $|z_1 - z_3| \ne |z_2 - z_4|$

22. If $\text{Im}(z^2) = 64$, where $z = x + iy$, then locus of z is a conic with eccentricity e, then

(A) length of latus rectum is 16.
(B) $e^2 = 2$
(C) foci $\left(\pm 8\sqrt{2}, 0\right)$
(D) foci (8, 8) and (–8, –8)

23. If z_1 and z_2 are the two complex numbers such that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then,

(A) $\frac{z_1}{z_2}$ is purely real
(B) $\frac{z_1}{z_2}$ is purely imaginary
(C) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$
(D) $\text{Arg}\left(\frac{z_1}{z_2}\right) = 0$

24. If $A(z_1), B(z_2)$ and $C(z_3)$ are three points in Argand plane, where $|z_1 + z_2| = |z_1| - |z_2|$ and $|(1 - i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$, then

(A) A, B and C lie on fixed circle with centre $\left(\frac{z_2 + z_3}{2}\right)$
(B) A, B and C form a right angle triangle
(C) A, B and C form an equilateral triangle
(D) A, B and C form an obtuse angle triangle

25. If $\left|\log_{\frac{1}{\sqrt{3}}}\left(\frac{|z|^2-|z|+1}{2+|z|}\right)\right|<2$, then

(A) $\frac{1}{3}<|z|<1$ (B) $|z|<\frac{1}{3}$

(C) $|z|>5$ (D) $1<|z|<5$

26. A complex number Z is rotated in anticlockwise direction by an angle α and we get Z' and if the same complex number Z is rotated by an angle α in clockwise direction, we get Z''. Then

(A) Z', Z and Z'' are in GP

(B) Z', Z and Z'' are in HP

(C) $Z'+Z''=2Z\cos\alpha$

(D) $Z'^2+Z''^2=2Z^2\cos 2\alpha$

27. If Z_1, Z_2, Z_3 and Z_4 are the vertices of a square in that order, then

(A) $|Z_1-Z_2| = |Z_2-Z_3| = |Z_3-Z_4| = |Z_4-Z_1|$

(B) $Z_1+Z_3=Z_2+Z_4$

(C) $|Z_1-Z_3| = |Z_2-Z_4|$

(D) $\frac{|Z_1-Z_3|}{|Z_2-Z_4|}$ is purely imaginary

28. If $\alpha=1+i$ and $\beta=-i$ and $P(z)$ is a variable point in the Argand plane then,

(A) least value of $|z-\alpha|+|z-\beta|$ is $\sqrt{5}$

(B) least value of $|z-\alpha|+2|z-\beta|$ is $\sqrt{5}$

(C) least value of $2|z-\alpha|+|z-\beta|$ is $\sqrt{5}$

(D) greatest value of $\big||z-\alpha|-|z-\beta|\big|$ is $\sqrt{5}$

29. If $1, \alpha_r, r=1,2,3,\dots,8$ be the 9th roots of unity, then

(A) $\alpha_1+\alpha_2+\dots+\alpha_8=-1$ (B) $\alpha_1+\alpha_2+\dots+\alpha_8=0$

(C) $\sum_{r=1}^{8}\frac{1}{\alpha_r^8}=-1$ (D) $\alpha_1^8+\alpha_2^8+\dots+\alpha_8^8=-1$

30. Let a, b and c be distinct complex numbers with $|a|=|b|=|c|=1$ and z_1, z_2 be the roots of the equation $az^2+bz+c=0$ with $|z_1|=1$. Also, let P and Q represents the complex numbers z_1 and z_2 in the complex plane with $\angle POQ=\theta$, with O being the origin then

(A) $b^2=ac, \theta=\frac{2\pi}{3}$ (B) $\theta=\frac{2\pi}{3}, PQ=\sqrt{3}$

(C) $PQ=2\sqrt{3}, b^2=ac$ (D) $\theta=\frac{\pi}{3}, b^2=ac$

31. Let z satisfies Arg$(z-3-4i)$ = Arg$(z-1+2i)$ + Arg(iz) + Arg$(\bar{z})$ and z_1 is a fixed complex number such that $|z-z_1|$ is constant then

(A) z lies on a semicircle (B) $|z_1^8|$ is equal to 625

(C) Arg(z_1) is $\frac{\pi}{4}$ (D) Arg(z_1-2) is $\frac{\pi}{2}$

32. For points z_1,z_2,z_3,z_4 in complex plane such that $|z_1|<1, |z_2|=1$ and $|z_3|\le 1$ and $z_3=\frac{z_2(z_1-z_4)}{\bar{z}_1 z_4-1}$, then $|z_4|$ can be

(A) 2 (B) $\frac{2}{5}$

(C) $\frac{1}{3}$ (D) $\frac{5}{2}$

33. For distinct complex numbers $z_1,z_2,\dots,z_n$, the value of $\frac{|z_2-z_1|^2+|z_3-z_2|^2+\dots\dots+|z_n-z_{n-1}|^2}{|z_n-z_1|^2}$ cannot be less than

(A) $\frac{1}{n-1}$ (B) $\frac{1}{n}$

(C) $\frac{1}{n+1}$ (D) $\frac{1}{n-2}$

34. If the complex number z satisfies the equation, $|z^2-1|=|z|^2+1$ and z and $(0,0)$ are the two vertices of an equilateral triangle of side length 2, then the third vertex can be

(A) $(\sqrt{3},1)$ (B) $(-\sqrt{3},1)$

(C) $(-\sqrt{3},-1)$ (D) $(\sqrt{3},-1)$

35. One vertex of the triangle of maximum area that can be inscribed in the curve $|z-2i|=2$, is $2+2i$, the remaining vertex is/are

(A) $-1+i(2+\sqrt{3})$ (B) $-1-i(2+\sqrt{3})$

(C) $-1+i(2-\sqrt{3})$ (D) $-1-i(2-\sqrt{3})$

36. If $z_1,z_2\in C$ are such that $\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|=1$, then

(A) $|z_1|=1$ (B) $|z_2|=1$

(C) $z_1=e^{i\theta}, \theta\in R$ (D) $z_2=e^{i\phi}, \phi\in R$

37. If $z\in C$ and $\log_{|z+i|}|3+4i|<\log_{|z+i|}|5+12i|$, then z lies

(A) Inside a circle passing through the origin

(B) Outside a circle passing through the origin

(C) Inside the disc $|z+i|<5$

(D) Outside the curve $z=-i+e^{i\theta}, \theta\in R$

38. If $ABCDEF$ is a regular hexagon in anticlockwise sense with the affix of A at z and if the centre of hexagon is origin O, then

(A) the affix of centroid of OAC is $\frac{z}{3}(1+\omega)$

(B) if an equilateral triangle is constructed on side AB away from the origin, then the affix of its centroid is $\frac{2}{3}(1-\omega^2)z$

(C) area of hexagon is $\frac{3\sqrt{3}}{2}|z|^2$

(D) the area of the rectangle $ABCD$ is $\sqrt{3}|z|^2$

39. If Z satisfies the inequality $|z-1-2i|\le 1$, then

(A) $\min(\arg(z))=\tan^{-1}\left(\frac{3}{4}\right)$

(B) $\max(\arg(z))=\frac{\pi}{2}$

(C) $\max(\text{mod } z)=\sqrt{5}+1$

(D) $\min(\text{mod } z)=\sqrt{5}-1$

40. If z_1, z_2 and z_3 denote the vertices of an equilateral triangle, then

(A) $z_1^2+z_2^2+z_3^2=z_1z_2+z_3z_1+z_1z_2$

(B) $(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$

(C) $\sum(z_1-z_2)(z_2-z_3)=0$

(D) $\dfrac{1}{z_1-z_2}+\dfrac{1}{z_2-z_3}+\dfrac{1}{z_3-z_1}=0$

41. If $z_1=a+i\,b;\ z_2=c+id$ are complex numbers such that $|z_1|=|z_2|=1$ and then $\text{Re}(z_1\bar{z}_2)=0$ the pair of complex numbers $\omega_1=a+ic\ \textit{and}\ \omega_2=b+id$ satisfies

(A) $|\omega_1|=1$

(B) $|\omega_2|=1$

(C) $\text{Re}(\omega_1\bar{\omega}_2)=0$

(D) $\text{Re}(\omega_1\bar{\omega}_2)=1$

42. If $|z|=2\sqrt{2}$ and $\arg\left(\dfrac{z-(5+i)}{z-(1+i)}\right)=\dfrac{\pi}{4}$. Then Z is

(A) $\left(\dfrac{3+\sqrt{7}}{2}\right)+i\left(\dfrac{3-\sqrt{7}}{2}\right)$

(B) $\left(\dfrac{3-\sqrt{7}}{2}\right)+i\left(\dfrac{3+\sqrt{7}}{2}\right)$

(C) $\left(\dfrac{3+\sqrt{7}}{2}\right)+i\left(\dfrac{3+\sqrt{7}}{2}\right)$

(D) $2-2i$

43. If $|z_1|=15$ and $|z_2-3-4i|=5$, then

(A) $|z_1-z_2|_{\min}=5$

(B) $|z_1-z_2|_{\min}=10$

(C) $|z_1-z_2|_{\max}=20$

(D) $|z_1-z_2|_{\max}=25$

44. If the points A and B are represented by non-zero complex numbers z_1 and z_2 on the Argand plane such that $|z_1+z_2|=|z_1-z_2|$ and 0 is the origin, then

(A) orthocentre of ΔOAB lies at O

(B) circumcentre of ΔOAB is $\dfrac{z_1+z_2}{2}$

(C) $Arg\left(\dfrac{z_1}{z_2}\right)=\pm\dfrac{\pi}{2}$

(D) ΔOAB is isosceles

45. The adjacent vertices of a regular polygon of n sides, whose centre is at origin, are given by $(1+\sqrt{2},1)$ and $(1+\sqrt{2},-1)$. Then the value of n is

(A) 8

(B) 4

(C) 12

(D) 6

46. If α is a variable complex number such that $|\alpha|>1$ and $z=\alpha+\dfrac{1}{\alpha}$ lies on a conic then

(A) eccentricity of the conic is $\dfrac{2|\alpha|}{1+|\alpha|^2}$

(B) distance between foci is 4

(C) length of latus rectum is $\dfrac{2\left(|\alpha|^2-1\right)}{|\alpha|^2+1}$

(D) distance between directrices is $\left(|\alpha|+\dfrac{1}{|\alpha|}\right)^2$

47. If a, b and c are non-zero complex numbers of equal moduli and satisfy $az^2+bz+c=0$, then

(A) $\min|z|=\dfrac{\sqrt{5}-1}{2}$

(B) $\min|z|=0$

(C) $\min|z|$ does not exist

(D) $\max|z|=\dfrac{\sqrt{5}+1}{2}$

48. Let x_1 and x_2 are the roots of the quadratic equation $x^2+ax+b=0$, where a and b are complex numbers and y_1 and y_2 are the roots of the quadratic equation, $y^2+|a|y+|b|=0$. If $|x_1|=|x_2|=1$, then

(A) $|y_1|=1$

(B) $|y_2|=1$

(C) $|y_1|\neq|y_2|$

(D) $|y_1|=|y_2|=2$

49. If $|z-3|=\min\{|z-1|,|z-5|\}$, then $\text{Re}(z)$ is

(A) 2

(B) $\dfrac{5}{2}$

(C) $\dfrac{7}{2}$

(D) 4

50. Suppose three real numbers a, b and c are in GP. Let $z=\dfrac{a+ib}{c-ib}$ then,

(A) $Z=\dfrac{ib}{c}$

(B) $Z=\dfrac{ia}{b}$

(C) $Z=\dfrac{ia}{c}$

(D) $z=0$

51. A complex number z satisfies the equation $|z^2-9|+|z^2|=41$, then the correct statement among the following is

(A) $|z+3|+|z-3|=10$

(B) $|z+3|+|z-3|=8$

(C) maximum value $|z|$ is 5

(D) maximum value $|z|$ is 6

52. Let z_2 be the reflection of z_1 in $a\bar{z}+\bar{a}z+b=0$. Then,

(A) $\text{Re}\{a(\bar{z}_1+\bar{z}_2)\}=-\dfrac{b}{2}$

(B) $\text{Re}\{a(\bar{z}_1+\bar{z}_2)\}=-2b$

(C) $\text{Arg}(z_2-z_0)=-\text{Arg}(z_1-z_0)$, where $a\bar{z}_0+\bar{a}z_0+b=0$

(D) $\left(\dfrac{z_1-z_2}{\bar{z}_1-\bar{z}_2}\right)-\dfrac{a}{\bar{a}}=0$

53. Let z_1 and z_2 are non-zero (given) complex numbers and k be any positive real number. Consider the system of equations

$$|3z-z_1-2z_2|=|z_1-z_2| \text{ and } \arg\left(\frac{z_1-z_2}{z-kz_1-(1-k)z_2}\right)=\pm\frac{\pi}{2}.$$

Then

(A) The system of equations has no solution, if $k\in\left(\dfrac{2}{3},\infty\right)$

(B) The system of equations have more than one solutions, if $k\in\left(0,\dfrac{2}{3}\right)$

(C) The system of equations have no solution, if $k\in\left(0,\dfrac{2}{3}\right)$

(D) The system of equations have more than one solution, if $k\in\left(\dfrac{2}{3},\infty\right)$

54. One vertex of the triangle of maximum area that can be inscribed in the curve $|z-2i|=2$ is $2+2i$, then the remaining vertices are

(A) $-1+i(2+\sqrt{3})$

(B) $-1-i(2+\sqrt{3})$

(C) $-1+i(2-\sqrt{3})$

(D) $-1-i(2-\sqrt{3})$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Consider the following sets, where Z is a complex number and K is a real number.

$A:\{Z:|\text{Im}(Z)|=K-|\text{Re}(Z)-K|\}$

$B:\{Z:|Z-K|>|Z-2K|\}$

C: Circle inscribed in the geometrical figure formed by A

1. Area bounded by A is

(A) $\frac{3K^2}{2}$
(B) K^2
(C) $2K^2$
(D) $\frac{K^2}{2}$

2. Radius of C is

(A) $\frac{K}{2}$
(B) $\frac{K}{\sqrt{2}}$
(C) K
(D) $\frac{3K}{2}$

3. Number of points of contact of C with A, that belong to B is

(A) 0
(B) 2
(C) 3
(D) 4

Comprehension 2

Z is a complex number satisfying $|Z-2-3i|=\alpha$. Z_m and Z_M are the corresponding complex numbers for which, $|Z+1+i|$ is minimum and maximum, respectively.

4. If $\text{Arg}\left(\frac{Z_m+1+i}{Z_M-2-3i}\right)=0$, then

(A) $0<\alpha<5$
(B) $5<\alpha<10$
(C) $10<\alpha<20$
(D) $\alpha=10$

5. If $\text{Arg}\left(\frac{Z_m+1+i}{Z_M-2-3i}\right)=\pi$, then

(A) $0<\alpha<1$
(B) $\alpha>5$
(C) $\alpha=1$
(D) $1<\alpha<5$

6. If Z is a complex number satisfying $|Z-2-3i|=\alpha$, what can be the range of α for which the maximum value of principal (Arg(Z)) can be obtained.

(A) $0<\alpha<1$
(B) $1<\alpha<5$
(C) $\sqrt{5}<\alpha<\sqrt{13}$
(D) $\alpha>\sqrt{13}$

Comprehension 3

Answer the following questions by considering the region R in the Argand plane described by the complex numbers and Z satisfying the inequalities, $|Z-2|\le|Z-4|$, $|Z-3|\le|Z+3|$, $|Z-i|\le|Z-3i|$, $|Z+i|\le|Z+3i|$.

7. The maximum value of $|Z|$ for any Z in R is

(A) 5
(B) 3
(C) 1
(D) $\sqrt{13}$

8. Area of the largest circle that can be inscribed in R is

(A) 5π
(B) $\frac{9\pi}{4}$
(C) 9π
(D) 25π

9. Maximum of $|Z_1-Z_2|$ given that Z_1 and Z_2 are any two complex numbers lying in the region R is

(A) 5
(B) 14
(C) $\sqrt{13}$
(D) 12

Comprehension 4

The points $D(z_1)$, $E(z_2)$ and $F(z_3)$ lie on a circle centred at the origin O. The tangents to the circle at D, E and F intersect at A, B and C. A, B and C are present opposite to D, E and F, respectively. Based on this, solve the following problems.

10. If A is represented by z, then $\text{Arg}\left(\frac{z_3-z}{z_3}\right)$ equals to

(A) $-\frac{\pi}{2}$
(B) $-\frac{\pi}{3}$
(C) $-\pi$
(D) $-\frac{2\pi}{3}$

11. The complex number z is given by

(A) $\frac{2z_1z_3}{z_1+z_3}$
(B) $\frac{2z_2z_3}{z_2+z_3}$
(C) $\frac{2z_1z_2}{z_1+z_2}$
(D) none of these

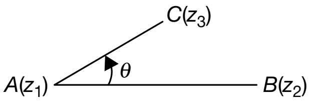
12. The complex representation of M (point of intersection of AO and DE) is given by

(A) $\left(\frac{z_1+z_3}{z_1+z_2}\right)z_1$
(B) $\left(\frac{z_1-z_3}{z_1-z_2}\right)z_1$
(C) $\left(\frac{z_1+z_2}{z_1+z_3}\right)z_3$
(D) $\frac{z_1z_3}{z_1+z_3}$

Comprehension 5

Let z_1, z_2 and z_3 be represented by the three points A, B and C on the Argand plane such that $\angle BAC=\theta$. Then $\text{Arg}\left(\frac{z_1-z_3}{z_1-z_2}\right)$ is the angle through which AB must be turned in the anticlockwise direction in order that it may lie along AC, i.e.,

$$\text{Arg}\left(\frac{z_1-z_3}{z_1-z_2}\right)=\theta$$



We can also write, $\frac{z_1-z_3}{z_1-z_2}=\left|\frac{z_1-z_3}{z_1-z_2}\right|e^{i\theta}$

13. In a ΔABC, where $A\equiv A(z_1), B\equiv B(z_2), C\equiv C(z_3)$ and $\text{Arg}\left(\frac{z_3}{z_2}\right)=2A$, $\text{Arg}\left(\frac{z_1}{z_3}\right)=2B$, $\text{Arg}\left(\frac{z_2}{z_1}\right)=2C$, then the origin coincides with

(A) orthocentre
(B) circumcentre
(C) incentre
(D) centroid

14. In a ΔABC, whose angles are in AP, the minimum angle C of the measure $45°$ has its circumcentre at origin and vertex C at $\left(\sqrt{3},0\right)$. The coordinates of the vertices A and B are respectively

(A) $\left(\frac{1}{2},-\frac{1}{2}\right),\left(\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\right)$
(B) $\left(-\frac{3}{2},-\frac{\sqrt{3}}{2}\right),\left(\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\right)$
(C) $(1,-1),(-1,1)$
(D) $\left(-\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{-3}{2},-\frac{\sqrt{3}}{2}\right)$

15. $OABC$ is an isosceles trapezium with $OA = OC = AB$ and $\angle AOC = \frac{2\pi}{3}$, with O being the origin $B \equiv B(z_1)$, then the complex number corresponding to C

(A) cannot be found
(B) is $i\sqrt{3}z_1$
(C) is $\frac{i}{\sqrt{3}}z_1$
(D) is $\frac{\sqrt{3}}{i}z_1$

Comprehension 6

Let $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ be the vertices of a trapezium in an Argand plane. Let $|z_1 - z_2| = 4, |z_3 - z_4| = 10$ and the diagonals AC and BD intersect at P. It is given that, $\text{Arg}\left(\frac{Z_4 - Z_2}{Z_3 - Z_1}\right) = \frac{\pi}{2}$ and $\text{Arg}\left(\frac{Z_3 - Z_2}{Z_4 - Z_1}\right) = \frac{\pi}{4}$.

16. Area of the trapezium $ABCD$ is equal to

(A) $\frac{130}{3}$
(B) $\frac{160}{3}$
(C) $\frac{190}{3}$
(D) $\frac{140}{3}$

17. Area of the ΔPCB is equal to

(A) $\frac{100}{21}$
(B) $\frac{200}{21}$
(C) $\frac{100}{7}$
(D) $\frac{400}{21}$

18. $|CP - DP|$ is equal to

(A) $\frac{10}{\sqrt{21}}$
(B) $\frac{16}{\sqrt{21}}$
(C) $\frac{17}{\sqrt{21}}$
(D) $\frac{19}{\sqrt{21}}$

Comprehension 7

R' is the radius of a circle passing through the centre of inscribed circle of ΔABC and through the endpoint of base BC where $B(1, 1)$, $C(3, 5)$ and $\angle BAC = 60°$, R is the radius of circle with satisfy z, $\text{Arg}\left(\frac{z-(3+5i)}{z-(1+i)}\right) = \frac{\pi}{3}$

19. Value of R is equal to

(A) $\frac{2\sqrt{5}}{\sqrt{3}}$
(B) $\frac{2\sqrt{3}}{\sqrt{5}}$
(C) $\frac{\sqrt{5}}{\sqrt{3}}$
(D) $\frac{\sqrt{3}}{\sqrt{5}}$

20. Value of R' is equal to

(A) $\sqrt{3}$
(B) $\sqrt{13}$
(C) $\sqrt{11}$
(D) $\frac{2\sqrt{5}}{\sqrt{3}}$

21. Ratio of R' and R is

(A) $\frac{\sqrt{13}\sqrt{5}}{2\sqrt{3}}$
(B) $\frac{\sqrt{13}\sqrt{5}}{\sqrt{3}}$
(C) $\frac{\sqrt{13}}{2\sqrt{5}}$
(D) 1

Comprehension 8

Consider the equation $az^2 + z + 1 = 0$ having purely imaginary roots, where $a = \cos\theta + I\sin\theta$, $I = \sqrt{-1}$ and $f(x) = x^3 - 3x^2 + 3(1 + \cos\theta)x + 5$

Then, answer the following.

22. Which of the following is true about $f(x)$?
(A) $f(x)$ decreases for $x \in [2n\pi, (2n+1)\pi]$, $n \in z$
(B) $f(x)$ decreases for $x \in \left[(2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right]$, $n \in z$
(C) $f(x)$ is a non-monotonic function
(D) $f(x)$ increases for $x \in R$

23. Which of the following is true?
(A) $f(x) = 0$ has three real distinct roots
(B) $f(x) = 0$ has all positive real root
(C) $f(x) = 0$ has one negative real root
(D) $f(x) = 0$ has three but not distinct roots

24. Number of roots of the equation $\cos 2\theta = \cos\theta$ in $(0, 4\pi)$ which also satisfy the above data is
(A) 2
(B) 3
(C) 4
(D) 6

Comprehension 9

The roots of $x = 1^{\frac{1}{3}}$ are called the cube roots of unity and they are usually denoted by $1, \omega, \omega^2$. It follows that, $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$, ω and ω^2 are called complex cube roots of unity.

25. If $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2w^2$ and $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega$, then $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$ is
(A) 0
(B) 1
(C) 2
(D) $\frac{1}{2}$

26. If $x + y + z \neq 0$ and $\begin{vmatrix} \frac{x}{1+\omega} & \frac{y}{\omega+\omega^2} & \frac{z}{\omega^2+1} \\ \frac{y}{\omega+\omega^2} & \frac{z}{\omega^2+1} & \frac{x}{1+\omega} \\ \frac{z}{\omega^2+1} & \frac{x}{1+\omega} & \frac{y}{\omega+\omega^2} \end{vmatrix} = 0$, then
(A) $x^2 + y^2 + z^2 = 0$
(B) $x + y\omega + z\omega^2 = 0$ or $x = y = z$
(C) $x \neq y \neq z = 0$
(D) None of these

27. $\dfrac{1}{a+\omega}+\dfrac{1}{b+\omega}+\dfrac{1}{c+\omega}=\dfrac{1}{\omega}$, where a, b and c are real, then $\sum\dfrac{1}{a^2-a+1}$ is

(A) 1 (B) 0

(C) –1 (D) 2

Comprehension 10

For complex numbers z_1, z_2 is z moves such that

(i) $|z-z_1|=K\,|z-z_2|$ is a circle (or) straight line accordingly as $K\neq 1$ (or) $K=1$

(ii) $|z-z_1|+|z-z_2|=k$ is an ellipse if $k>|z_1-z_2|$

28. If $\left|\dfrac{2iz_1-z_1-z_2}{2iz_1+z_1+z_2}\right|=\left|\dfrac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right|$, then $\dfrac{z_1}{z_2}$ is

(A) purely real

(B) purely imaginary

(C) has positive argument

(D) neither purely real nor purely imaginary

29. If z moves such that $|z-(10+12i)|+|z-(14+12i)|=24$, then the maximum amplitude of z is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{2}$

(C) $\dfrac{2\pi}{3}$ (D) $\dfrac{5\pi}{6}$

30. If z moves such that $0<\text{Arg}\left(\dfrac{2z^2-5z+3}{3z^2-z-2}\right)\le\dfrac{\pi}{2}$, then the set of all points of z gives a region whose area is

(A) $>\pi$ (B) $<\dfrac{\pi}{2}$

(C) $>\dfrac{\pi}{2}$ (D) $>2\pi$

Comprehension 11

Consider a ΔABC, whose vertices are $A(z_1)$, $B(z_2)$ and $C(z_3)$, respectively. Perpendiculars are drawn from each of the vertices to the opposite side so as to cut the circumcircle at $D(z_4)$, $E(z_5)$ and $F(z_6)$ respectively. If the circumcentre of ΔABC triangle is at origin (0, 0), then answer the following questions.

31. $z_1z_2z_3+z_4z_5z_6$ is

(A) –1

(B) 1

(C) dependent on the angle of ΔABC

(D) none

32. If the perpendicular from vertex A to the opposite side is also the angle bisector of $\angle A$, then

(A) $2z_1=z_2+z_3$ (B) $z_1^2=z_2z_3$

(C) $\dfrac{2}{z_1}=\dfrac{1}{z_1}+\dfrac{1}{z_3}$ (D) $z_2^2=z_1z_3$

33. If the foot of perpendicular from A on the opposite side BC is $L(z)$ and $\angle C=\dfrac{\pi}{4}$, then

(A) $z=\dfrac{z_2+z_4-i(z_2-z_4)}{2}$ (B) $z=\dfrac{z_2+z_4+i(z_2-z_4)}{2}$

(C) $z=\dfrac{i(z_2+z_4)-(z_2-z_3)}{2}$ (D) none of these

Comprehension 12

Let the vertices of a ΔA, B and C are represented by complex numbers z_1, z_2 and z_3, where $|z_i|=1$; for $i=1, 2, 3$. If the triangle has some speciality, then there will be some pure relation among z_1, z_2 and z_3 free from modulus and argument symbols. For example, if the triangle is in equilateral, then

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

34. If the ΔABC is isosceles with $AB=AC$, then

(A) $z_1^2=z_2z_3$ (B) $z_2^2=z_1z_3$

(C) $z_3^2=z_1z_2$ (D) none of these

35. The orthocentre of the ΔABC must be represented by the complex number

(A) $\sqrt{z_1^2+z_2^2+z_3^2}$

(B) $\sqrt{z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1}$

(C) $z_1+z_2+z_3$

(D) $\dfrac{z_1+z_2+z_3}{3}$

36. If perpendicular AE from A on BC is produced to a point D on the circle $|z|=1$, then D must have affix

(A) $-\dfrac{z_1z_2}{z_3}$ (B) $-\dfrac{z_1z_3}{z_2}$

(C) $-\dfrac{z_2z_3}{z_1}$ (D) $\dfrac{z_2z_3}{z_1}$

Comprehension 13

If z_1 and z_2 are two fixed points and z varies such that $|z-z_1|+|z-z_2|=k$, where $k>|z_1-z_2|$ then, locus of z is an ellipse with eccentricity $=\dfrac{|z_1z_2|}{k}$

37. If z_1 lies on $|z-3|+|z+3|=8$ such that $\text{Arg}(z_1)=\dfrac{\pi}{6}$, then $[|z_1|^2]$ equals where [ ] is GIF is

(A) 9 (B) 10

(C) 11 (D) 12

38. If e is the eccentricity of the ellipse $|z-z_1|+|z-z_2|=k$, where $k>|z_1-z_2|$ and if z_3 is an end of a latus rectum, then $|z_3|$ can be

(A) $k\sqrt{e^4-e^2+1}$ (B) $\dfrac{k}{2}\sqrt{e^4-e^2+1}$

(C) $k\sqrt{e^4+e^2+1}$ (D) $\dfrac{k}{2}\sqrt{e^4+e^2+1}$

39. If z moves such that $|z-z_1|+|z-z_2|=k$, where $k>|z_1-z_2|$ and if z_3 is such that $|z_3|=\dfrac{k}{2}$, then the value of $|z_1-z_3|^2+|z_2-z_3|^2$ equals

(e is eccentricity)

(A) $k^2(e^2+1)$ (B) $\frac{k^2}{2}(e^2+1)$

(C) $k^2(1-e^2)$ (D) $\frac{k^2}{2}(1-e^2)$

Comprehension 14

Consider the set of complex numbers A, B, C and S that are defined as,

$A = \{z : ||z+2| - |z-2|| = 2\}$; $B = \{z : \operatorname{Arg}\left(\frac{z-1}{z}\right) = \frac{\pi}{2}\}$

$C = \{z : \operatorname{Arg}(z-1) = \pi\}$; $S = \{z : \operatorname{Re}\left(\frac{z-1}{z+1}\right) = 0\}$

40. $A \cap B \cap C =$

(A) ϕ (B) $(0, \infty)$

(C) $(0, 1)$ (D) $(-1, 0)$

41. If z_1, z_2, $z_3 \in S$, then the minimum value of $|z_1+z_2|^2 + |z_2+z_3|^2 + |z_3+z_1|^2$ is

(A) 5 (B) 3

(C) 2 (D) 9

42. If z lies on S, then $\operatorname{Arg}\left(\frac{2z+i-1}{2z+i+1}\right)$ equals

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{-\pi}{4}$

Comprehension 15

If α is any of 7th roots of unity, then $\alpha = \cos\left(\frac{2K\pi}{7}\right)$ $(K = 0 \text{ to } 6)$ and $\sum_{i=1}^{6} \alpha^i = -1 (\alpha \neq 1)$ and $\alpha^7 = 1$.

By using this, answer the following.

43. The equation whose roots are $\alpha + \alpha^2 + \alpha^4$ and $\alpha^3 + \alpha^5 + \alpha^6$ is $(\alpha \neq 1)$ is

(A) $x^2 + x - 2 = 0$ (B) $x^2 + x + 2 = 0$

(C) $x^2 - x + 2 = 0$ (D) $x^2 - x - 2 = 0$

44. If $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6$ then for $a \neq 1$, $f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + \ldots + f(\alpha^6 x)$ equals

(A) 42 (B) 21

(C) 14 (D) 7

45. $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7}$ is equal to

(A) $\frac{\sqrt{7}}{16}$ (B) $\frac{\sqrt{7}}{8}$

(C) $\frac{\sqrt{7}}{32}$ (D) $\frac{\sqrt{7}}{64}$

Comprehension 16

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{(2k\pi+\theta)}{n} + i\sin\frac{(2k\pi+\theta)}{n}\right]$$

Where $k = 0, 1, 2, 3, \ldots (n-1)$, given n, n^{th} roots of the complex number z.

46. If $1, \omega, \omega^2 \ldots \omega^{n-1}$ are the n, n^{th} roots of unity and z_1 and z_2 are any two complex numbers, then $\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2$

(A) $n[|z_1|^2 + |z_2|^2]$ (B) $(n-1)[|z_1|^2 + |z_2|^2]$

(C) $(n+1)[|z_1|^2 + |z_2|^2]$ (D) $(n+2)[|z_1|^2 + |z_2|^2]$

47. If $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ are the n^{th} roots of unity, then $\sum_{i=1}^{n-1} \frac{1}{2-\alpha^i}$ is equal to

(A) $\frac{(n-2)2^{n-1}+1}{2^n-1}$ (B) $(n-2).2^n$

(C) $\frac{(n-2)\cdot 2^{n-1}}{2^n-1}$ (D) $\frac{(n+2)\cdot 2^{n-1}}{2^n-1}$

48. If $1, \omega, \omega^2 \ldots \omega^{n-1}$ are the n^{th} roots of unity, then $(1-\omega)(1-\omega^2) \ldots (1-\omega^{n-1})$ is equal to

(A) 0 (B) 1

(C) n (D) n^2

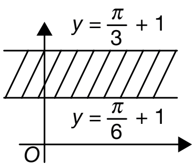
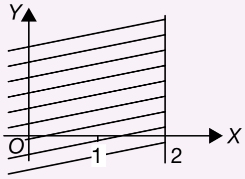
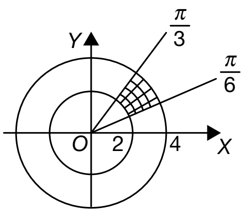
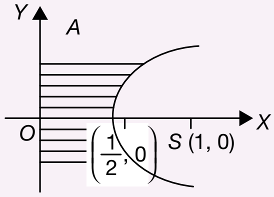
MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match **COLUMN-I** (in which Argand diagram is given) with **COLUMN-II** (in which corresponding locus is given)

COLUMN-I	**COLUMN-II**
(A) 	(p) $\frac{\pi}{6}<\arg(Z)<\frac{\pi}{3}$, $2<\lvert Z\rvert<4$
(B) 	(q) $\text{Re}(Z)<\lvert Z-1\rvert$
(C) 	(r) $\lvert Z-1\rvert<\lvert Z-3\rvert$
(D)  Y A X O $\left(\frac{1}{2},0\right)$ S (1, 0)	(s) $\left\lvert\arg(\overline{Z})+\frac{\pi}{4}\right\rvert<\frac{\pi}{12}$, $4<Z\overline{Z}<16$
	(t) $\frac{\pi}{6}<\text{Im}(Z-i)<\frac{\pi}{3}$

2. Z_1, Z_2 and Z_3 are the vertices of a triangle. Match the condition in **COLUMN-I** with type of triangle in **COLUMN-II**

COLUMN-I	**COLUMN-II**
(A) $Z_1^2+Z_2^2+Z_3^2=Z_2Z_3+Z_3Z_1+Z_1Z_2$	(p) Right angled
(B) $\text{Re}\left(\frac{Z_3-Z_1}{Z_3-Z_2}\right)=0$	(q) Obtuse angled
(C) $\text{Re}\left(\frac{Z_3-Z_1}{Z_3-Z_2}\right)<0$	(r) Isosceles and right angled
(D) $\frac{Z_3-Z_1}{Z_3-Z_2}=i$	(s) Equilateral

3.

COLUMN-I	**COLUMN-II**
(A) Let w be a non-real cube root of unity, then the number of distinct elements in the set $\left\{\left(1+w+w^2+\ldots+w^n\right)^m \mid m,n\in N\right\}$ is	(p) 4
(B) Let 1, w, w^2 be the cube root of unity. The least possible degree of a polynomial with real coefficient having roots $2w,(2+3w),\left(2+3w^2\right)$ and $\left(2-w-w^2\right)$ is	(q) 5
(C) $\alpha=6+4i$ and $\beta=(2+4i)$ are two complex numbers on the complex plane. A complex number z satisfying $\text{amp}\left(\frac{z-\alpha}{z-\beta}\right)=\frac{\pi}{6}$ moves on a major segment of a circle whose radius is	(r) 6
	(s) 8

4. The complex numbers z_1, z_2, z_3, …, z_n taken in order represent the consecutive vertices of a regular polygon of n sides and $z_3+z_n=Az_1+\overline{A}z_2$, where A is a complex number. Let [.] denotes the greatest integer function.

COLUMN-I (n)	**COLUMN-II** [|A|]
(A) 4	(p) 0
(B) 6	(q) 1
(C) 8	(r) 2
(D) 12	(s) 3

5. Match the following

COLUMN-I	**COLUMN-II**
(A) If z_1, z_2 and z_3 are the vertices of an equilateral triangle, then	(p) $(z_1-z_2)^2+3(z_3-z_2)^2=0$

(*Continued*)

COLUMN-I	COLUMN-II
(B) If z_1, z_2 and z_3 are the vertices of A, B and C, respectively of an isosceles right angled triangle with right angle at C, then	(q) $(z_2 - z_3)^2 = 3(z_1 - z_2)(z_3 - z_1)$
(C) If z_1, z_2 and $z_3$are the vertices of A, B and C, respectively of an isosceles triangle and angles B and C are each equal to $\frac{\pi}{6}$, then	(r) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$
(D) If z_1, z_2 and $z_3$are the vertices of A, B and C, respectively of a triangle ΔABC in which $\angle A = \frac{\pi}{6}$, $\angle B = \frac{\pi}{2}$ and $\angle C = \frac{\pi}{3}$, then	(s) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$

6. Match the following

COLUMN-I (Equation of circle)	COLUMN-II (Radius)
(A) $\lvert z-1\rvert^2 + \lvert z+1\rvert^2 = 4$	(p) 3
(B) $\text{Amp}\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{2}$	(q) 4
(C) $z\bar{z} - z(1-2i) - \bar{z}(1+2i) = 4$	(r) 2
(D) $\lvert z+3\rvert = 2\lvert z-3\rvert$	(s) 1

7. Match Column I with Column II

COLUMN-I	COLUMN-II
(A) Locus of the point ‘z’ satisfying the equation $\text{Re}(z^2) = \text{Re}(z + \bar{z})$ is	(p) a straight line
(B) Locus of the point ‘z’ satisfying the equation $\lvert z - z_1\rvert + \lvert z - z_2\rvert = l, l \in R^+$ and Z_1, Z_2 are two fixed complex numbers such that $l > \lvert z_1 - z_2\rvert$ is	(q) a circle
(C) Locus of the point ‘z’ satisfying the equation $\left\lvert\frac{2z-i}{z+1}\right\rvert = m,$ where $i = \sqrt{-1}, m \in R^+$ is	(r) an ellipse
(D) If $\lvert\bar{z}\rvert = 25$, then the locus of the points represented by $-1 + 75\bar{z}$ is	(s) a hyperbola

8.

COLUMN-I	COLUMN-II
(A) The minimum value of $e^x - x - \frac{x^3}{3}$ is	(p) 1
(B) Let z_j for $1 \le j \le 5$ denote the five distinct 5th roots of 1 in the complex plane. Let $w_j = \frac{z_j}{1+z_j^2} + \frac{z_j^2}{1+z_j^4} + \frac{z_j^3}{1+z_j} + \frac{z_j^4}{1+z_j^3}$, for $1 \le j \le 5$. Then $\sum_{j=1}^{5} w_j$ is equal to	(q) 6
(C) If z is a complex number such that $z \in C - R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then $\lvert z\rvert$ is	(r) 4
(D) Let ABC be an equilateral triangle with the circumradius as 1. For any point P on the circumcircle, $PA^2 + PB^2 + PC^2$ equals	(s) 10

9. Match the following

COLUMN-I (Equation)	COLUMN-II (Roots)
(ω is complex cube roots of unity, $i = \sqrt{-1}$)	
(A) $x^6 + 1 = 0$	(p) ± 1
(B) $x^8 + x^4 + 1 = 0$	(q) $\pm i$
(C) $x^{12} - 1 = 0$	(r) $\pm\omega, \pm\omega^2$
(D) $x^6 - 1 = 0$	(s) $\pm i\omega, \pm i\omega^2$

10. Match the following

COLUMN-I	COLUMN-II
(A) The maximum of value of $\lvert\lvert z - w\rvert - \lvert z - \bar{w}\rvert\rvert^2$ is (where $\lvert z\rvert = 5$ and ω is complex cube roots of unity)	(p) 0
(B) In the ΔABC, $A(z_1)$, $B(z_2)$ and $C(z_3)$ has its circumcentre at origin. If the perpendicular from A to BC intersect the circumcenter at z_4, then $z_1z_2 + z_2z_3$	(q) 6
(C) Tangent drawn to circle $(x-1)^2 + (y-1)^2 = 5$ at a point P meets the line $2x + y + 6 = 0$ at Q on the x-axis, then the value of $\frac{PQ^2}{2}$ is	(r) 12
(D) Total number of common terms of 3, 7, 11, 15, 19, … up to 60 terms and –3, –1, 1, 3, … up to 26 terms is	(s) 3

11.

COLUMN-I	COLUMN-II
(A) If $\lvert z-2i\rvert+\lvert z-7i\rvert=k$, then the locus of z is an ellipse, if k equal to	(p) 7
(B) If $\left\lvert\dfrac{2z-3}{3z-2}\right\rvert=k$, then locus of z is a circle if $\dfrac{2}{3}$ is a point inside the circle and $\dfrac{3}{2}$ is outside the circle. k is equal to	(q) 8
(C) If $\lvert z-3\rvert-\lvert z-4i\rvert=k$, then locus of z is a hyperbola if k is	(r) 2
(D) If $\lvert z-(3+4i)\rvert=\dfrac{k}{50}\lvert a\bar{z}+\bar{a}z+b\rvert$ where $a=3+4i$, then locus of z is a hyperbola with k equal to	(s) 4
	(t) 5

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If $|z_1|=2$, $|z_2|=3$, $|z_3|=4$ and $|2z_1+3z_2+4z_3|=4$, then find $\dfrac{|8z_2z_3+27z_3z_1+64z_1z_2|}{16}$ (where z_1, z_2, z_3 are complex numbers).

2. If $n=12m$ $(m\in N)$ ${}^nC_0-\dfrac{{}^nC_2}{(2+\sqrt{3})^2}+\dfrac{{}^nC_4}{(2+\sqrt{3})^4}-\dfrac{{}^nC_6}{(2+\sqrt{3})^6}+\ldots=(-1)^m\left(\dfrac{\sqrt{P}}{1+\sqrt{Q}}\right)^n$, then find $P–Q$.

3. If $z+\dfrac{1}{z}=1$ and $a=z^{2005}+\dfrac{1}{z^{2005}}$ and b is the last digit of the number $2^{2^n}-1$ when the integer $n>1$ and $a^2+b^2=13\lambda$, then find λ.

4. If the imaginary part of z^2 is 16, then the locus of 'z' is a conic, whose latus rectum will be?

5. ΔABC has its vertices as $A(z_1)$, $B(z_2)$ and $C(z_3)$. DE is parallel to BC. Here, $D(z_4)$ and $E(z_5)$. If AB is divided in the ratio 2:3 and $|z_2-z_3|=10$ unit, then find $|z_4-z_5|$.

6. If Z_1 lies on the circle $|Z|=3$ and $x+iy=Z_1+\dfrac{1}{Z_1}$, then $\dfrac{x^2}{100}+\dfrac{y^2}{64}=\dfrac{1}{k}$. Find k.

7. Two lines $zi-\bar{z}i+2=0$ and $z(1+i)+\bar{z}(1-i)+2=0$ intersect at point 'p'. There is a complex number $\alpha=x+iy$ at a distance of 2 units from the point p which lies on $z(1+i)+\bar{z}(1-i)+2=0$. Then find $[|x|]$ ([ ] is the greatest integer function).

8. If $iz^3+z^2-z+i=0$, then find $[|z|]$ (where [.] is GIF).

9. If $x=a+b\omega+c\omega^2$, $y=a\omega+b\omega^2+c$ and $z=a\omega^2+b+c\omega$, then find $\dfrac{x^2}{yz}+\dfrac{y^2}{zx}+\dfrac{z^2}{xy}$. where ω is imaginary cube root of unity.

10. If the equation $z^4+a_1z^3+a_2z^2+a_3z+a_4=0$ (where a_1, a_2, a_3 and a_4 are the real coefficients different from zero) has a purely imaginary root, then find the value of the expression $\dfrac{a_3}{a_1a_2}+\dfrac{a_1a_4}{a_2a_3}$.

11. If $z_1z_2\in C$, $z_1^2+z_2^2\in R$, $z_1(z_1^2-3z_2^2)=2$ and $z_2(3z_1^2-z_2^2)=11$, then find the value of $z_1^2+z_2^2$.

12. The complex numbers $1+i$ and $1+2i$ are the roots of the equation $x^5-6x^4+\alpha x^3+\beta x^2+\gamma x+\delta=0$, where $\alpha, \beta, \gamma, \delta\in R$. Then find the value of $(\delta+25)$.

13. If z is a complex number satisfying the equation $|z+i|+|z-i|=8$, then find the maximum value of $|z|$.

14. If the minimum possible value of $|z|^2+|z-3|^2+|z-6i|^2$ is 'N', then find the value of '$N-28$'.

15. If $|z_2+iz_1|=|z_1|+|z_2|$ and $|z_1|=3$, $|z_2|=4$ and the area of ΔABC whose vertices A, B and C are z_1, z_2 and $\dfrac{z_2-iz_1}{1-i}$, respectively is Δ^2, then find $[\Delta]$ (where [ ] is GIF).

16. If 'z' is a complex number, then find the maximum value of $|z-1|^2+|z-2|^2+|z-3|^2$.

17. If $w=\cos\left(\dfrac{2\pi}{7}\right)+i\sin\left(\dfrac{2\pi}{7}\right)$ and $\alpha=w+w^2+w^4$, $\beta=w^3+w^5+w^6$, then find the value of $\alpha\beta$.

18. The area of the region of the Argand plane described by Z satisfying, $\dfrac{\pi}{6}\le \operatorname{Arg}(Z)\le\dfrac{2\pi}{3}$ and $3\le|Z|\le 5$ is $k^2\pi$, then find k.

19. Let α and β are the complex numbers satisfying $|\alpha-1+i|=2$, $|\beta-4+5i|=3$ and $2|\beta|_{\max}-3|\alpha|_{\max}=\sqrt{k_1}-\sqrt{k_2}$, then find $\left[\dfrac{k_1}{k_2}\right]$ (where [.] is greater integer function).

20. If from a point P representing the complex number z_1 on the curve $|z|=2$, the tangents are drawn to the curve $|z|=1$ meeting at $Q(z_2)$ and $R(z_3)$, then find $\left(\dfrac{4}{\bar{z}_1}+\dfrac{1}{\bar{z}_2}+\dfrac{1}{\bar{z}_3}\right)\left(\dfrac{4}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right)$

21. The minimum distance between the circles $|z|=1$ and $\operatorname{Arg}\left(\dfrac{z-z_1}{z-z_2}\right)=\dfrac{\pi}{4}$, $z_1=9+5i$, $z_2=3+5i$ is d, then find $[d]$ (where [.] is GIF).

22. If $\omega \neq 1$ is a cube root of unity and $a + b = 21$, $a^3 + b^3 = 105$, then find the value of $(a\omega^2 + b\omega)(a\omega + b\omega^2)$.

23. If $a = \cos\left(\frac{\pi}{2n}\right) + i\sin\left(\frac{\pi}{2n}\right)$ and

$$z = \underset{n\to\infty}{Lt}\frac{\pi}{2n}(1 + a + a^2 + \ldots + a^{n-1})$$, then find $|z|^2$.

24. If z is a complex number, then find the number of common points whose affixes satisfying $|z + 2| + |z - 2| = 8$ and $|z + 1| + |z - 1| = 2$.

25. $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ are the n roots of unity.
Let $S = (3 - \alpha)(3 - \alpha^2) \ldots (3 - \alpha^{n-1})$. If $n = 6$, find $\frac{S}{91}$.

26. If $z_1 = e^{i\theta}$ and $1, z_1, z_1{}^2, z_1{}^3 \ldots z_1{}^{n-1}$ are the vertices of a regular polygon such that $\frac{\text{Im}(z_1^2)}{\text{Re}(z_1)} = \frac{\sqrt{5}-1}{2}$, then find the value of $[\sqrt{n}]$ (where [ ] is GIF)

27. Find the least value of r for which the two curves Arg(z) = $\frac{-\pi}{6}$ and $|z + 2\sqrt{3}i| = r$ have at least one common point.

28. If $Z \neq 1$ is a 5th root of unity, then find $\log_{\sqrt{3}}\left|1 + z + z^2 + z^3 - \frac{2}{z}\right|$.

29. Let $1, \alpha_1, \alpha_2, \ldots, a_{10}$ be the 11th roots of unity. Let $\lambda = \sum_{r=1}^{10} r(\alpha_r + \alpha_{11-r})$ the value of $\frac{\lambda + 11}{11}$ equals to

30. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is the locus of the complex number 'Z' in the Argand plane satisfying $|Z - (3 + 2i)| = \left|z\cos\left(\frac{\pi}{4} - arc\ z\right)\right|$, then find $a + b - 2h$.

31. If the equation $|z - z_1|^2 + |z - z_2|^2 = k$ represents the equation of a circle where $z_1 = 2 + 3i$, $z_2 = 4 + 3i$ are the extremities of a diameter, then find the value of k.

32. Find the value of $1.(2 - w)(2 - w^2) + 2(3 - w)(3 - w^2) + 3.(4 - w)(4 - w^2) + 4.(5 - w)(5 - w^2) = 44k$, k
(where w is the cube root of unity).

33. Let $z = x + iy$ be any point in the Argand plane, such that $z^2 + (\bar{z})^2 + 2|z|^2 = 8i(\bar{z} - z)$. If the locus of z is a conic, then find the length of the latus rectum of the conic.

34. Sum of common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is K. Then find $|k|$.

35. Consider a triangle having vertices at the points $A\left(\frac{2}{\sqrt{3}}e^{i\pi/2}\right)$, $B\left(\frac{2}{\sqrt{3}}e^{i\pi/6}\right)$, $C\left(\frac{2}{\sqrt{3}}e^{-i5\pi/6}\right)$. Let P be any point on its incircle, then find the value of $AP^2 + BP^2 + CP^2$.

36. If $\alpha = (1)^{\frac{1}{5}}$ and $|1 + \alpha + \alpha^2| = \frac{a + \sqrt{b}}{2}$; $a, b \in R^+$ then find $a + b$.

37. If $(3 + x^{2008} + x^{2009})^{2010} = a_0 + a_1x + a_2x^2 + \ldots$ then the value of $a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 \ldots$ is a^b, then sum of digits in the number $(a + b)$ is K, then find K (where a is a prime number).

38. Find the number of complex numbers z, satisfying $|z - 3 - i| = |z - 9 - i|$ and $|z - 3 + 3i| = 3$.

39. The sum of the real parts of the complex numbers satisfying the equations $\left|\frac{z - 4i}{z - 2i}\right| = 1$ and $\left|\frac{z - 8 + 3i}{z + 3i}\right| = \frac{3}{5}$ is k. Find $\frac{k}{5}$.

40. Let λ, z_0 be two complex numbers. $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of a triangle such that $z_1 = z_0 + \lambda$, $z_2 = z_0 + \lambda e^{i\pi/4}$, $z_3 = z_0 + \lambda e^{i17\pi/11}$ and $\angle ABC = \frac{3k\pi}{22}$, then find the value of k.

41. The complex number z satisfying $|z + 2 + i| + |z - 2 + i| = 4$, $0 \leq \text{Arg}(z + 2 + 2i) \leq \frac{\pi}{4}$ and $3\frac{\pi}{4} \leq \text{Arg}(z - 2 + 2i) \leq \pi$ will lie on a line segment of the length k. Find k.

42. If $Z = \frac{1}{2}(\sqrt{3} - i)$, then the least positive integral value of 'n' such that $(Z^{101} + i^{109})^{106} = Z^n$ is 'k' then find $\frac{2}{5}k$.

43. If $Z_n = \left(\cos\left(\frac{\pi}{n(n+1)(n+2)}\right) + i\sin\left(\frac{\pi}{n(n+1)(n+2)}\right)\right)$ for $n = 1, 2, 3, \ldots$ and the principle argument value of $\lim_{n\to\infty}(z_1 z_{2\ldots\ldots} z_n)$ is $\frac{k\pi}{24}$, then find the value of k.

44. Suppose that w is the imaginary $(2009)^{\text{th}}$ roots of unity. If $(2^{2009} - 1)\sum_{r=1}^{2008}\frac{1}{2 - w^r} = (a)(2^b) + c$, where $a, b, c, \in N$, and the least value of $(a + b + c)$ is $(2008)K$. Find the numerical value of K.

45. A triangle with vertices represented by Z_1, Z_2, Z_3 has opposite sides of lengths in the ratio 2: $\sqrt{19}$:3, respectively. Then the value of $4(z_1 - z_2)^2 + 6(z_1 - z_2)(z_3 - z_2) + 9(z_3 - z_2)^2$ is k. Find k.

46. Let $1, \omega$ and ω^2 be the cube root of unity. Find the least possible degree of a polynomial with real coefficients having roots $2w$, $(2 + 3w)$, $(2 + 3w^2)$ and $(2 - w - w^2)$.

ARCHIVE: JEE MAIN

1. **[JEE Main 2020]**

If $\left(\frac{1+i}{1-i}\right)^{\frac{m}{2}} = \left(\frac{1+i}{i-1}\right)^{\frac{n}{3}} = 1$, $(m, n \in N)$ then the greatest common divisor of the least values of m and n is _____.

2. **[JEE Main 2020]**

If Re$\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x, y) lies on a:

(A) straight line whose slope is $-\dfrac{2}{3}$

(B) circle whose diameter is $\dfrac{\sqrt{5}}{2}$

(C) straight line whose slope is $\dfrac{3}{2}$

(D) circle whose centre is at $\left(-\dfrac{1}{2},-\dfrac{3}{2}\right)$

3. [JEE Main 2020]

If $\dfrac{3+i\sin\theta}{4-i\cos\theta}$, $\theta\in[0,2\pi]$, is a real number, then an argument of $\sin\theta+i\cos\theta$ is:

(A) $\pi-\tan^{-1}\left(\dfrac{3}{4}\right)$ (B) $\pi-\tan^{-1}\left(\dfrac{4}{3}\right)$

(C) $\tan^{-1}\left(\dfrac{4}{3}\right)$ (D) $\tan^{-1}\left(\dfrac{3}{4}\right)$

4. [JEE Main 2020]

If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then:

(A) $b^2-b=42$ (B) $b^2+b=12$

(C) $b^2+b=72$ (D) $b^2-b=30$

5. [JEE Main 2020]

Let $\alpha=\dfrac{-1+i\sqrt{3}}{2}$. If $a=(1+\alpha)\displaystyle\sum_{k=0}^{100}\alpha^{2k}$ and $b=\displaystyle\sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation:

(A) $x^2+101x+100=0$ (B) $x^2+102x+101=0$

(C) $x^2-102x+101=0$ (D) $x^2-101x+100=0$

6. [JEE Main 2020]

Let z be a complex number such that $\left|\dfrac{z-i}{z+2i}\right| = 1$ and $|z| = \dfrac{5}{2}$. Then the value of $|z+3i|$ is:

(A) $\sqrt{10}$ (B) $\dfrac{7}{2}$

(C) $2\sqrt{3}$ (D) $\dfrac{15}{4}$

7. [JEE Main 2020]

If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be:

(A) $\sqrt{7}$ (B) $\sqrt{\dfrac{17}{2}}$

(C) $\sqrt{8}$ (D) $\sqrt{10}$

8. [JEE Main 2020]

The imaginary part of $\left(3+2\sqrt{-54}\right)^{1/2}-\left(3-2\sqrt{-54}\right)^{1/2}$ can be:

(A) $-2\sqrt{6}$ (B) 6

(C) $\sqrt{6}$ (D) $-\sqrt{6}$

9. [JEE Main 2020]

The value of $\left(\dfrac{1+\sin\dfrac{2\pi}{9}+i\cos\dfrac{2\pi}{9}}{1+\sin\dfrac{2\pi}{9}-i\cos\dfrac{2\pi}{9}}\right)^3$ is:

(A) $\dfrac{1}{2}\left(\sqrt{3}-i\right)$ (B) $-\dfrac{1}{2}\left(\sqrt{3}-i\right)$

(C) $-\dfrac{1}{2}\left(1-i\sqrt{3}\right)$ (D) $\dfrac{1}{2}\left(1-i\sqrt{3}\right)$

10. [JEE Main 2020]

If a and b are real numbers such that $(2+\alpha)^4 = a + b\alpha$, where $\alpha=\dfrac{-1+i\sqrt{3}}{2}$, then $a+b$ is equal to:

(A) 57 (B) 33

(C) 24 (D) 9

11. [JEE Main 2020]

If z_1, z_2 are complex numbers such that $\text{Re}(z_1) = |z_1-1|$. $\text{Re}(z_2) = |z_2-1|$ and $\arg(z_1-z_2) = \dfrac{\pi}{6}$, then $\text{Im}(z_1+z_2)$ is equal to:

(A) $\dfrac{\sqrt{3}}{2}$ (B) $\dfrac{2}{\sqrt{3}}$

(C) $\dfrac{1}{\sqrt{3}}$ (D) $2\sqrt{3}$

12. [JEE Main 2020]

Let $u=\dfrac{2z+i}{z-ki}$, $z = x+iy$ and $k>0$. If the curve represented by $\text{Re}(u)+\text{Im}(u)=1$ intersects the y-axis at the points P and Q where $PQ=5$, then the value of k is:

(A) 3/2 (B) 4

(C) 2 (D) 1/2

13. [JEE Main 2020]

The value of $\left(\dfrac{-1+i\sqrt{3}}{1-i}\right)^{30}$ is:

(A) $2^{15}i$ (B) -2^{15}

(C) $-2^{15}i$ (D) 6^5

14. [JEE Main 2020]

If the four complex numbers z, $\bar{z}$, $\bar{z}-2\text{Re}(\bar{z})$ and $z-2\,\text{Re}(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to:

(A) 4 (B) 2

(C) $4\sqrt{2}$ (D) $2\sqrt{2}$

15. [JEE Main 2020]

Let $z = x+iy$ be a non-zero complex number such that $z^2 = i|z|^2$, where $i=\sqrt{-1}$, then z lies on the:

(A) imaginary axis (B) real axis

(C) line, $y=x$ (D) line, $y=-x$

16. [JEE Main 2020]

The region represented by $\{z = x+iy \in C : |z| - \text{Re}(z) \le 1\}$ is also given by the inequality:

(A) $y^2 > x+1$ (B) $y^2 \ge 2(x+1)$

(C) $y^2 \le x+\dfrac{1}{2}$ (D) $y^2 \le 2\left(x+\dfrac{1}{2}\right)$

17. [JEE Main 2019]

Let z_1 and z_2 be two complex numbers satisfying $|z_1| = 9$ and $|z_2-3-4i| = 4$. Then the minimum value of $|z_1-z_2|$ is:

(A) 0
(B) 1
(C) 2
(D) $\sqrt{2}$

18. [JEE Main 2019]
If α and β be the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is:
(A) 3
(B) 5
(C) 4
(D) 2

19. [JEE Main 2019]
If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ $(i = \sqrt{-1})$, then $(1 + iz + z^5 + iz^8)^9$ is equal to:
(A) 0
(B) –1
(C) $(-1 + 2i)^9$
(D) 1

20. [JEE Main 2019]
All the points in the set $S = \left\{\frac{\alpha + i}{\alpha - i}; \alpha \in R\right\}$ $(i = \sqrt{-1})$ lie on a:
(A) circle whose radius is 1.
(B) straight line whose slope is 1.
(C) straight line whose slope is – 1.
(D) circle whose radius is $\sqrt{2}$.

21. [JEE Main 2019]
Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$ then:
(A) $4\,\text{lm}\,(\omega) > 5$
(B) $5\,\text{Re}\,(\omega) > 1$
(C) $5\,\text{Re}\,(\omega) < 1$
(D) $5\,\text{lm}\,(\omega) > 4$

22. [JEE Main 2019]
If $a > 0$ and $z = \frac{(1+i)^2}{a-i}$, has magnitude $\sqrt{\frac{2}{5}}$, then $\bar{z}$ is equal to:
(A) $-\frac{1}{5} + \frac{3}{5}i$
(B) $-\frac{1}{5} - \frac{3}{5}i$
(C) $\frac{1}{5} - \frac{3}{5}i$
(D) $-\frac{3}{5} - \frac{1}{5}i$

23. [JEE Main 2019]
If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then:
(A) $z\bar{w} = \frac{1-i}{\sqrt{2}}$
(B) $\bar{z}w = i$
(C) $\bar{z}w = -i$
(D) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

24. [JEE Main 2019]
The equation $|z - i| = |z - 1|$, $i = \sqrt{-1}$, represents:
(A) a circle of radius 1.
(B) the line through the origin with slope-1.
(C) a circle of radius $\frac{1}{2}$.
(D) the line through the origin with slope 1.

25. [JEE Main 2019]
Let $z \in C$ with $\text{lm}(z) = 10$ and it satisfies $\frac{2z - n}{2z + n} = 2i - 1$ for some natural number n. Then:
(A) $n = 20$ and $\text{Re}(z) = -10$
(B) $n = 40$ and $\text{Re}(z) = 10$
(C) $n = 40$ and $\text{Re}(z) = -10$
(D) $n = 20$ and $\text{Re}(z) = 10$

26. [JEE Main 2019]
Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to
(A) –256
(B) 512
(C) –512
(D) 256

27. [JEE Main 2019]
Let $A = \left\{\theta \in \left(-\frac{\pi}{2}, \pi\right) : \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{is purely imaginary}\right\}$. Then the sum of the elements in A is
(A) $\frac{5\pi}{6}$
(B) $\frac{3\pi}{4}$
(C) $\frac{2\pi}{3}$
(D) π

28. [JEE Main 2019]
Let z_0 be a root of the quadratic equation $x^2 + x + 1 = 0$. If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then $\text{Arg}(z)$ will be
(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{3}$
(C) 0
(D) $\frac{\pi}{4}$

29. [JEE Main 2019]
Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 4|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$, then
(A) $\text{Im}(z) = 0$
(B) $|z| = \sqrt{\frac{5}{2}}$
(C) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$
(D) $\text{Re}(z) = 0$

30. [JEE Main 2019]
Let $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$. If $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and imaginary parts of z, respectively, then
(A) $\text{Re}(z) = -3$
(B) $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$
(C) $\text{Im}(z) = 0$
(D) $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$

31. [JEE Main 2019]
Let $\left(-2 - \frac{1}{3}i\right)^3 = \frac{x + iy}{27}$ $\left(i = \sqrt{-1}\right)$, where x and y are real numbers, then y-x equals
(A) –85
(B) 85
(C) –91
(D) 91

32. [JEE Main 2019]
Let z be a complex number such that $|z| + z = 3 + i$ (where $i = \sqrt{-1}$). Then $|z|$ is equal to
(A) $\frac{\sqrt{34}}{3}$
(B) $\frac{5}{3}$
(C) $\frac{5}{4}$
(D) $\frac{\sqrt{41}}{4}$

33. [JEE Main 2019]
If $\frac{z - \alpha}{z + \alpha}$ $(\alpha \in R)$ is a purely imaginary number and $|z| = 2$, then the value of α is

(A) $\frac{1}{2}$ (B) $\sqrt{2}$

(C) 2 (D) 1

34. [JEE Main 2018]

If $\alpha, \beta \in C$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ will be

(A) 2 (B) –1

(C) 0 (D) 1

35. [JEE Main 2017]

Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

(A) $-z$ (B) z

(C) –1 (D) 1

36. [JEE Main 2016]

A value of θ for which $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary is

(A) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{6}$ (D) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

37. [JEE Main 2015]

A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on

(A) a circle of radius 2

(B) a circle of radius $\sqrt{2}$

(C) a straight line parallel to x-axis

(D) a straight line parallel to y-axis

38. [JEE Main 2014]

If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(A) is equal to $\frac{5}{2}$

(B) lies in the interval (1, 2)

(C) is strictly greater than $\frac{5}{2}$

(D) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

39. [JEE Main 2013]

If z is a complex number of unit modulus and argument θ, then $\text{Arg}\left(\frac{1+z}{1+\bar{z}}\right)$ is equal to

(A) $-\theta$ (B) $\frac{\pi}{2} - \theta$

(C) θ (D) $\pi - \theta$

40. [AIEEE 2012]

If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies

(A) on the imaginary axis

(B) either on the real axis or on a circle passing through the origin

(C) on a circle with centre at the origin

(D) either on the real axis or on a circle not passing through the origin

41. [AIEEE 2011]

Let α and β are the real and z is the complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $\text{Re}(z) = 1$, then it is necessary that

(A) $|\beta| = 1$ (B) $\beta \in (1, \infty)$

(C) $\beta \in (0, 1)$ (D) $\beta \in (-1, 0)$

42. [AIEEE 2011]

If $(\omega \neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals

(A) (1, 0) (B) (–1, 1)

(C) (0, 1) (D) (1, 1)

43. [AIEEE 2010]

The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals is

(A) 0 (B) 1

(C) 2 (D) ∞

44. [AIEEE 2010]

If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ will be

(A) –2 (B) –1

(C) 1 (D) 2

45. [AIEEE 2009]

If $\left|Z - \frac{4}{Z}\right| = 2$, then the maximum value of $|Z|$ will be

(A) 2 (B) $2 + \sqrt{2}$

(C) $\sqrt{3} + 1$ (D) $\sqrt{5} + 1$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE Advanced 2019]

Let S be the set of all complex number z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z_0 - 1|}; x \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

(A) $-\frac{\pi}{2}$ (B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

2. **[JEE Advanced 2013]**

Let the complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on the circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

3. **[JEE Advanced 2012]**

Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a cannot take the value

(A) -1 (B) $\frac{1}{3}$

(C) $\frac{1}{2}$ (D) $\frac{3}{4}$

4. **[JEE 2011]**

Let $\omega = e^{i\pi/3}$ and a, b, c, x, y and z are non-zero complex numbers such that

$a + b + c = x,$

$a + b\omega + c\omega^2 = y$ and

$a + b\omega^2 + c\omega = z,$

then the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is

5. **[JEE 2009]**

Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is

(A) $\frac{1}{\sin 2°}$ (B) $\frac{1}{3\sin 2°}$

(C) $\frac{1}{2\sin 2°}$ (D) $\frac{1}{4\sin 2°}$

6. **[JEE 2009]**

Let $z = x + iy$ be a complex number, where x and y are integers. Then, the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is

(A) 48 (B) 32

(C) 40 (D) 80

7. **[JEE 2008]**

A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1, the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by

(A) $6 + 7i$ (B) $-7 + 6i$

(C) $7 + 6i$ (D) $-6 + 7i$

8. **[JEE 2007]**

A man walks a distance of 3 units from the origin towards the northeast (N 45° E) direction. From there, he walks a distance of 4 units towards the northwest (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane is

(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)\, e^{i\pi/4}$

(C) $(4 + 3i)\, e^{i\pi/4}$ (D) $(3 + 4i)\, e^{i\pi/4}$

9. **[JEE 2007]**

If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\frac{z}{1-z^2}$ lie on

(A) a line not passing through the origin

(B) $|z| = \sqrt{2}$

(C) the x-axis

(D) the y-axis

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[JEE Advanced 2020]**

Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(A) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(B) $|z| \le 2$ for all $z \in S$

(C) $\left|z + \frac{1}{2}\right| \ge \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

2. **[JEE Advanced 2018]**

For a non-zero complex number z, let Arg(z) denote the principal argument with $-\pi < \text{Arg}(z) \le \pi$. Then, which of the following statement(s) is/are false?

(A) $\text{Arg}(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$.

(B) The function $f : R \to (-\pi, \pi)$ defined by $f(t) = \text{Arg}(-1 + it)$ for all $t \in R$, is continuous at all points of R, where $i = \sqrt{-1}$.

(C) For any two non-zero complex number z_1 and z_2, $\text{Arg}\left(\frac{z_1}{z_2}\right) - \text{Arg}(z_1) + \text{Arg}(z_2)$ is an integer multiple of 2π.

(D) For any three given distinct complex number z_1, z_2 and z_3, the locus of the point z satisfying the condition $\text{Arg}\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line.

3. **[JEE Advanced 2018]**

Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy(x, y \in R, I = \sqrt{-1})$ of equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is/are true?

(A) If L has exactly one element, then $|s| \neq |t|$.
(B) If $|s| = |t|$, then L has infinitely many elements.
(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2.
(D) If L has more than one element, then L has infinitely many elements.

4. **[JEE Advanced 2017]**
Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\dfrac{az+b}{z+1}\right) = y$, then which of the following is/are possible value(s) of x?

(A) $-1 - \sqrt{1+y^2}$
(B) $1 + \sqrt{1+y^2}$
(C) $1 - \sqrt{1+y^2}$
(D) $-1 + \sqrt{1-y^2}$

5. **[JEE Advanced 2016]**
Let $a, b \in R$ and $a^2 + b^2 \neq 0$. Suppose
$S = \left\{z \in C : z = \dfrac{1}{a+ibt}, t \in R, t \neq 0\right\}$, where $i = \sqrt{-1}$.
If $z = x + iy$ and $z \in S$, then (x, y) lies on

(A) the circle with radius $\dfrac{1}{2a}$ and centre $\left(\dfrac{1}{2a}, 0\right)$; for $a > 0$, $b \neq 0$
(B) the circle with radius $-\dfrac{1}{2a}$ and centre $\left(-\dfrac{1}{2a}, 0\right)$; for $a < 0$, $b \neq 0$
(C) the x-axis; for $a \neq 0$, $b = 0$
(D) the y-axis; for $a = 0$, $b \neq 0$

6. **[JEE Advanced 2013]**
Let w and $P = \{w^n : n = 1, 2, 3, \ldots\ldots.\}$.
If $H_1 = \left\{z \in C : \text{Re}\, z > \dfrac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < -\dfrac{1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2$ will be

(A) $\dfrac{\pi}{2}$
(B) $\dfrac{\pi}{6}$
(C) $\dfrac{2\pi}{3}$
(D) $\dfrac{5\pi}{6}$

7. **[JEE 2010]**
Let z_1 and z_2 be two distinct complex numbers and $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a non-zero complex number w, then

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$,
$S_2 = \left\{z \in C : \text{Im}\left[\dfrac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \text{Re}\, z > 0\}$.

1. **[JEE Advanced 2013]**
$\min\limits_{z \in S} |1 - 3i - z|$ equals

(A) $\dfrac{2-\sqrt{3}}{2}$
(B) $\dfrac{2+\sqrt{3}}{2}$
(C) $\dfrac{3-\sqrt{3}}{2}$
(D) $\dfrac{3+\sqrt{3}}{2}$

2. **[JEE Advanced 2013]**
Area of S will be

(A) $\dfrac{10\pi}{3}$
(B) $\dfrac{20\pi}{3}$
(C) $\dfrac{16\pi}{3}$
(D) $\dfrac{32\pi}{3}$

Comprehension 2

Let A, B and C are the three sets of complex numbers as defined below.
$A = \{z : \text{Im}(z) \geq 1\}$, $B = \{z : |z - 2 - i| = 3$ and
$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$

3. **[JEE 2008]**
The number of elements in the set $A \cap B \cap C$ is

(A) 0
(B) 1
(C) 2
(D) ∞

4. **[JEE 2008]**
Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29
(B) 30 and 34
(C) 35 and 39
(D) 40 and 44

5. **[JEE 2008]**
Let z be any point in $A \cap B \cap C$ and w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) -6 and 3
(B) -3 and 6
(C) -6 and 6
(D) -3 and 9

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. **[JEE Advanced 2014]**

Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots, 9$.

(P) For each z_k, there exists a z_j such that $z_k \cdot z_j = 1$	(1) True
(Q) There exists a $k \in \{1, 2, \ldots\ldots 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	(2) False
(R) $\frac{\lvert 1-z_1\rvert\,\lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	(3) 1
(S) $1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	(4) 2

Codes:

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

2. **[JEE 2010]**

Match the statements in **COLUMN-I** with those in **COLUMN-II**.

(**Note**: Here z takes the values in the complex plane and Im(z) and Re(z) denote the imaginary and real parts of z, respectively.)

COLUMN-I	COLUMN-II
(A) The set of points z satisfying $\lvert z - i\lvert z\rvert\rvert = \lvert z + i\lvert z\rvert\rvert$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\lvert z+4\rvert + \lvert z-4\rvert = 10$ is contained in or equal to	(q) the set of points z satisfying $\text{Im}(z) = 0$
(C) If $\lvert w\rvert = 2$, then the set of points satisfying $z = w - \frac{1}{w}$ is contained in or equal to	(r) the set of points z satisfying $\lvert\text{Im}(z)\rvert \leq 1$
(D) If $\lvert w\rvert = 1$, then the set of points satisfying $Z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\lvert\text{Re}(z)\rvert \leq 2$
	(t) the set of points z satisfying $\lvert z\rvert \leq 3$

(*Continued*)

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[JEE Advanced 2020]**
For a complex number Z, let Re(Z) denote the real part of Z. Let S be the set of all complex numbers Z satisfying $Z^4 - \lvert Z\rvert^4 = 4iZ^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $\lvert Z_1 - Z_2\rvert^2$, where $Z_1, Z_2 \in S$ with $\text{Re}(Z_1) > 0$ and $\text{Re}(Z_2) < 0$, is ….

2. **[JEE Advanced 2019]**
Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{\lvert a + b\omega + c\omega^2\rvert^2 : a, b, c$ distinct non-zero integers$\}$ equals

3. **[JEE Advanced 2015]**
For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression
$$\frac{\sum_{k=1}^{12} \lvert \alpha_{k+1} - \alpha_k\rvert}{\sum_{k=1}^{3} \lvert \alpha_{4k-1} - \alpha_{4k-2}\rvert}$$
will be

4. **[JEE 2011]**
If z is any complex number satisfying $\lvert z - 3 - 2i\rvert \leq 2$, then the minimum value of $\lvert 2z - 6 + 5i\rvert$ is

5. **[JEE 2010]**
Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then, the number of distinct complex numbers z satisfying
$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to}$$

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** A	**3.** D	**4.** C	**5.** A	**6.** D	**7.** B	**8.** D	**9.** C	**10.** B
11. B	**12.** B	**13.** C	**14.** B	**15.** C	**16.** C	**17.** D	**18.** C	**19.** C	**20.** C
21. C	**22.** B	**23.** A	**24.** D	**25.** D	**26.** A	**27.** C	**28.** B	**29.** A	**30.** D
31. C	**32.** D	**33.** A	**34.** D	**35.** B	**36.** D	**37.** B	**38.** C	**39.** C	**40.** C
41. C	**42.** B	**43.** D	**44.** B	**45.** B	**46.** C	**47.** C	**48.** B	**49.** A	**50.** D
51. B	**52.** D	**53.** B	**54.** B	**55.** B	**56.** A	**57.** C	**58.** A	**59.** C	**60.** A
61. B	**62.** C	**63.** D	**64.** A	**65.** A	**66.** A	**67.** D	**68.** A	**69.** B	**70.** A
71. C	**72.** B	**73.** B	**74.** D	**75.** C	**76.** B	**77.** A	**78.** D	**79.** B	**80.** C
81. D	**82.** B	**83.** C	**84.** D	**85.** D	**86.** C	**87.** A	**88.** A	**89.** A	**90.** C
91. D	**92.** C	**93.** C	**94.** C	**95.** B	**96.** A	**97.** C	**98.** A	**99.** C	**100.** A
101. C	**102.** C	**103.** A	**104.** A	**105.** B	**106.** D	**107.** B	**108.** D	**109.** C	**110.** A
111. C	**112.** D	**113.** B	**114.** A	**115.** B	**116.** C	**117.** B	**118.** D	**119.** D	**120.** C
121. B	**122.** B	**123.** B	**124.** C	**125.** C	**126.** A	**127.** C	**128.** B	**129.** A	**130.** A
131. C									

Multiple Correct Choice Type Questions

1. A, D	**2.** A, B	**3.** A, C, D	**4.** A, B	**5.** A, C
6. C, D	**7.** A, B	**8.** A, B, C, D	**9.** A, B, D	**10.** A, B, C, D
11. A, B, C, D	**12.** C, D	**13.** A, B, C, D	**14.** A, D	**15.** B, C
16. A, D	**17.** A, B, C, D	**18.** A, B, D	**19.** B, D	**20.** A, C, D
21. A, B, C, D	**22.** A, B, D	**23.** B, C	**24.** A, B	**25.** B, D
26. A, C, D	**27.** A, B, C	**28.** A, B, C, D	**29.** A, C, D	**30.** A, B
31. A, B, D	**32.** B, C	**33.** A, B, C	**34.** A, B, C, D	**35.** A, C
36. A, B, C, D	**37.** B, D	**38.** A, B, C, D	**39.** A, B, C, D	**40.** A, B, C, D
41. A, B, C	**42.** A, B	**43.** A, D	**44.** A, B, C	**45.** A
46. A, B, D	**47.** A, D	**48.** A, B	**49.** A, D	**50.** A, B
51. A, C	**52.** D	**53.** A, B	**54.** A, C	

Linked Comprehension Type Questions

1. C	**2.** B	**3.** A	**4.** A	**5.** B	**6.** D	**7.** D	**8.** B	**9.** A	**10.** A
11. B	**12.** C	**13.** B	**14.** D	**15.** C	**16.** D	**17.** B	**18.** A	**19.** A	**20.** D
21. D	**22.** D	**23.** C	**24.** C	**25.** C	**26.** B	**27.** A	**28.** A	**29.** B	**30.** C
31. A	**32.** B	**33.** A	**34.** A	**35.** C	**36.** C	**37.** D	**38.** B	**39.** B	**40.** A
41. B	**42.** B	**43.** B	**44.** D	**45.** B	**46.** A	**47.** A	**48.** C		

Matrix Match/Column Match Type Questions

1. A → (t)	B → (r)	C → (p, s)	D → (q)
2. A → (s)	B → (p)	C → (q)	D → (r)
3. A → (r)	B → (q)	C → (p)	
4. A → (r)	B → (r)	C → (q)	D → (q)
5. A → (s)	B → (r)	C → (q)	D → (p)
6. A → (t)	B → (r)	C → (p)	D → (q)

7. A → (s)	B → (r)	C → (p, q)	D → (q)
8. A → (p)	B → (s)	C → (p)	D → (q)
9. A → (q), (s)	B → (r, s)	C → (p, q, r, s)	D → (p, r)
10. A → (s)	B → (p)	C → (q)	D → (r)
11. A → (p, q)	B → (p, q, r, s, t)	C → (r, s)	D → (p, q)

Integer/Numerical Answer Type Questions

1. 6	**2.** 5	**3.** 2	**4.** 8	**5.** 4	**6.** 9	**7.** 1	**8.** 1	**9.** 3	**10.** 1
11. 5	**12.** 5	**13.** 4	**14.** 2	**15.** 2	**16.** 2	**17.** 2	**18.** 2	**19.** 9	**20.** 9
21. 4	**22.** 5	**23.** 2	**24.** 0	**25.** 4	**26.** 4	**27.** 3	**28.** 2	**29.** 0	**30.** 4
31. 4	**32.** 5	**33.** 4	**34.** 1	**35.** 5	**36.** 6	**37.** 5	**38.** 1	**39.** 5	**40.** 5
41. 2	**42.** 4	**43.** 6	**44.** 2	**45.** 0	**46.** 5				

ARCHIVE: JEE MAIN

1. 4	**2.** B	**3.** B	**4.** D	**5.** C	**6.** B	**7.** A	**8.** A	**9.** B	**10.** D
11. D	**12.** C	**13.** C	**14.** D	**15.** C	**16.** D	**17.** A	**18.** C	**19.** B	**20.** A
21. B	**22.** B	**23.** C	**24.** D	**25.** C	**26.** A	**27.** C	**28.** D	**29.** Bonus	**30.** C
31. D	**32.** B	**33.** C	**34.** D	**35.** A	**36.** A	**37.** A	**38.** B	**39.** C	**40.** B
41. B	**42.** D	**43.** B	**44.** C	**45.** D					

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Single Correct Choice Type Questions

1. A	**2.** C	**3.** D	**4.** Bonus	**5.** D	**6.** A	**7.** B	**8.** D	**9.** D

Multiple Correct Choice Type Questions

1. B, C	**2.** A, B, D	**3.** A, C, D	**4.** A, D	**5.** A, C, D
6. C, D	**7.** A, C, D			

Linked Comprehension Type Questions

1. C	**2.** B	**3.** C	**4.** B or C or D	**5.** D

Matrix Match/Column Match Type Question

1. C

2. A → (q, r)	B → (p)	C → (p, s, t)	D → (q, r, s, t)

Integer/Numerical Answer Type Questions

1. 8	**2.** 3	**3.** 4	**4.** 5	**5.** 1

From the Author's Desk

“ Dear Learners,

From Binomial Theorem chapter atleast one question is asked in JEE Mains and Advanced. Problems based on Terms and Coefficient, Divisibility/ Remainder, Multinomial Theorem, Series formed by binomial coefficients are key concepts to focus on. It is a chapter that involves lots of calculations, so more practice is required. Students are advised to practice adequate questions covering various concepts in prescribed time frame. Do not forget to attempt PYQs from JEE Mains & Advanced.

Good Luck!! ”

CHAPTER

5 Binomial Theorem

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Binomial theorem definiton, general term and coefficient related concepts
(b) Divisibility/Remainder problems
(c) $\left(\sqrt{a}+b\right)^n = I+f$ format problems
(d) Numerically greatest term
(e) Multinomial theorem
(f) Binomial theorem for negative integer and fractional index
(g) Series of binomial coefficients

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

The term and notation of ***factorial*** often be used in term in Algebra so let's understand it first.

FACTORIAL

The continued product of first n natural numbers is called the factorial of n and is denoted by $n!$ or $\underline{|n}$

The definition of factorial

$$n! = \underline{|n} = \begin{cases} 1 & \text{when } n = 0 \\ 1.2.3\ldots n & \text{when } n \neq 0 \end{cases}$$

$0! = 1$, $1! = 1$
$2! = 1.2 = 2$, $3! = 1.2.3 = 6$
$4! = 1.2.3.4 = 24$, $5! = 1.2.3.4.5 = 120$
$6! = 1.2.3.4.5.6 = 720$, $7! = 1.2.3.4.5.6.7 = 5040$

$20! = 1.2.3.4\ldots18.19.20$
$= 16!.17.18.19.20 = 17!.18.19.20 = 18!.19.20 = 19!.20$

that means we can short it down to required number

ILLUSTRATION 1

Prove that

(i) $\dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!} = \dfrac{37}{6!}$

(ii) $1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)(2n+1)\, 2^n = \dfrac{(2n+1)!}{n!}$

SOLUTION

(i) Consider LHS $= \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!}$

$$= \frac{1}{4!}\left(1+\frac{1}{5}+\frac{1}{6\times5}\right) = \frac{1}{4!}\left(\frac{30+6+1}{6\times5}\right)$$

$$= \frac{37}{6!} = \text{RHS}$$

(ii) Consider,

LHS $= 1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1) \cdot 2^n$

$$= \frac{1\cdot2\cdot3\cdot4\ldots(2n-1)(2n)(2n+1)}{2\cdot4\cdot6\ldots(2n)} \cdot 2^n$$

[On multiplying Nr and Dr by $2 \cdot 4 \cdot 6 \ldots (2n)$]

$$= \frac{(2n+1)!}{2^n(1\cdot2\cdot3\ldots n)} \cdot 2^n = \frac{(2n+1)!}{n!} = \text{RHS}$$

ILLUSTRATION 2

Solve for x

(i) $(x)! = 132\,(x-2)!$

(ii) $\dfrac{(x+3)!(2x-1)!}{(2x+1)!(x+2)!} = \dfrac{7}{72}$

SOLUTION:

(i) We have,

$(x)! = 132\,(x-2)!$
$\Rightarrow$ $x(x-1)\,(x-2)! = 132\,(x-2)!$
$\Rightarrow$ $x(x-1) = 132$
$\Rightarrow$ $x(x-1) = 12 \times (12-1)$
$\therefore$ $x = 12$

(ii) We have,

$$\frac{(x+3)!(2x-1)!}{(2x+1)!(x+2)!} = \frac{7}{72}$$

$$\Rightarrow \frac{(x+3)(x+2)!(2x-1)!}{(2x+1)(2x)(2x-1)!(x+2)!} = \frac{7}{72}$$

$$\Rightarrow \frac{x+3}{(2x+1)(2x)} = \frac{7}{72}$$

$\Rightarrow$ $72x + 216 = 7(4x^2 + 2x)$

$\Rightarrow$ $72x + 216 = 28x^2 + 14x$

$\Rightarrow$ $28x^2 - 58x - 216 = 0 \quad \Rightarrow \quad 14x^2 - 29x - 108 = 0$

$\Rightarrow$ $14x^2 - 56x + 27x - 108 = 0$

$\Rightarrow$ $14x(x - 4) + 27(x - 4) = 0$

$\therefore$ $(x - 4)(14x + 27) = 0$

$\therefore$ $x = 4$ [since, x cannot be negative]

BINOMIAL EXPRESSION

An expression consisting of two terms with positive or negative sign between them is called a binomial expression.

$$(x+44y), \left(x^{21} - \frac{11}{x^{1/3}}\right), \left(\frac{4p^2}{x^3} + \frac{2q^{3/2}}{x^7}\right), (x^4 + 32y^{2/3})$$

Consider the following standard expansions

$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5.xy^4 + y^5$

To generalize this expansion lets learn Binomial Theorem.

Binomial Theorem

For every positive integer n

$$\boxed{\begin{aligned}(X+Y)^n &= {}^nC_0X^nY^0 + {}^nC_1X^{n-1}Y + \ldots + {}^nC_rX^{n-r}Y^r \\ &\quad + \ldots + {}^nC_nX^0Y^n\end{aligned}}$$

$$(X+Y)^n = \sum_{r=0}^{n} {}^nC_rX^{n-r}Y^r$$

$$(X+Y)^n = \sum_{n=0}^{n} {}^nC_r(\text{first term})^{n-r}(\text{second term})^r$$

CONCLUSIONS

(i) Total terms in binomial expansion are $(n + 1)$.

(ii) The first term contains X^n while last term contains Y^n. As we move ahead in expansion, power of first term(X) decreased by one and power of second term (Y) is increased by one.

(iii) In each term the sum of the exponents of variables X and Y remains constant throughout the expansion as n.

(iv) ${}^nC_r$ is called as binomial coefficient where

$${}^nC_r = \frac{n!}{r!.(n-r)!} = \binom{n}{r}.$$

${}^nC_r$ is also written as $\binom{n}{r}$

(v) Binomial coefficients of equidistant terms from beginning and last are equal ${}^nC_r = {}^nC_{n-r}$

(vi) We can write general term of expansion as

$$\boxed{\begin{aligned}T_{r+1} &= {}^nC_rX^{n-r}Y^r \\ &= {}^nC_r(\text{first term})^{n-r}(\text{second term})^r\end{aligned}}$$

(vii) **p^{th} term from end**

p^{th} term from the end in the expansion of $(x + y)^n$ = $(n - p + 2)^{th}$ term from beginning.

(viii) **Pascal's Triangle**:

The numbers represented in triangular diagram give the coefficients for the expansion of $(x + y)^n$. The first row is for $n = 0$, the second row is for $n = 1$ and goes on. Each row has 1 as its first and last number, other numbers are generated by adding the two numbers present at left and right position in the above row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Properties of Binomial Coefficient

(i) $\boxed{{}^nC_r = {}^nC_{n-r}}$

Proof:

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = {}^nC_{n-r}$$

(ii) $\boxed{{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}}$

Proof:

$${}^nC_r + {}^nC_{r+1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}$$

$$= \frac{n!(r+1+n-r)}{(r+1)!(n-r)!} = \frac{(n+1)n!}{(r+1)!(n-r)!}$$

$$= \frac{(n+1)!}{(r+1)!(n-r)!} = {}^{n+1}C_{r+1}$$

(iii) $$\boxed{\begin{aligned}{}^nC_r &= \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\cdot{}^{n-2}C_{r-2}\\ &= \frac{n}{r}\cdot\frac{n-1}{r-1}\cdot\frac{n-2}{r-2}\,{}^{n-3}C_{r-3}\end{aligned}}$$

Proof:

$${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n}{r}\left(\frac{(n-1)!}{(r-1)!(n-r)!}\right) = \frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$= \frac{n(n-1)}{r(r-1)}\left(\frac{(n-2)!}{(r-2)!(n-r)!}\right) = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2}$$

(iv) $$\boxed{\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}}$$

Proof:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \left(\frac{n!}{r!(n-r)!}\right)\Big/\left(\frac{n!}{(r-1)!(n-r+1)!}\right)$$

$$= \frac{(n-r+1)!(r-1)!}{(n-r)!r!}$$

$$= \frac{(n-r+1)(n-r)!(r-1)!}{(n-r)!r\cdot(n-1)!} = \frac{n-r+1}{r}$$

ILLUSTRATION 3

Find the value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$.

SOLUTION

$${}^{50}C_4 + \left({}^{50}C_3 + {}^{51}C_3 + {}^{52}C_3 + \dots\dots + {}^{55}C_3\right)$$

[As ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

$$= {}^{51}C_4 + {}^{51}C_3 + {}^{52}C_3 + \dots\dots + {}^{55}C_3$$

$$= {}^{52}C_4 + {}^{52}C_3 + \dots\dots + {}^{55}C_3$$

$$= {}^{53}C_4 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 = {}^{54}C_4 + {}^{54}C_3 + {}^{55}C_3$$

$$= {}^{55}C_4 + {}^{55}C_3 = {}^{56}C_4$$

ILLUSTRATION 4

Find the value of expression

$${}^{n+1}C_2 + 3({}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2).$$

SOLUTION

$${}^{n+1}C_2 + 3({}^3C_3 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2)$$

(Using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

$$= {}^{n+1}C_2 + 3({}^{n+1}C_3) = \frac{(n+1)n}{2} + \frac{3\cdot(n+1)n(n-1)}{6}$$

$$= \frac{n^2(n+1)}{2}$$

ILLUSTRATION 5

Let $\binom{n}{k} = {}^nC_r$, then find the value of sum $\binom{99}{97} + \binom{98}{96} + \binom{97}{95} + \dots + \binom{3}{1} + \binom{2}{0}$

SOLUTION

(writing in reverse order)

We have S = ${}^2C_0 + {}^3C_1 + {}^4C_2 + {}^5C_3 + \dots + {}^{99}C_{97}$

$$S = \underbrace{{}^3C_0 + {}^3C_1}\, + {}^4C_2 + {}^5C_3 + \dots + {}^{97}C_{95} + {}^{98}C_{96} + {}^{99}C_{97}$$

$$S = \underbrace{{}^4C_1 + {}^4C_2}\, + {}^5C_3 + \dots + {}^{97}C_{95} + {}^{98}C_{96} + {}^{99}C_{97}$$

(Using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)

$$S = {}^5C_2 + {}^5C_3 + \dots + {}^{97}C_{95} + {}^{98}C_{96} + {}^{99}C_{97}$$

Similarly continuing in the same manner, we get.

$$S = {}^{99}C_{96} + {}^{99}C_{97} = {}^{100}C_{97}$$

ILLUSTRATION 6

Find the sum of the series

$$\frac{{}^{101}C_1}{{}^{101}C_0} + \frac{2\cdot{}^{101}C_2}{{}^{101}C_1} + \frac{3\cdot{}^{101}C_3}{{}^{101}C_1} + \dots. + \frac{101\cdot{}^{101}C_{101}}{{}^{101}C_{100}}$$

SOLUTION

Consider the general term

$T_r = \dfrac{r\cdot{}^{101}C_r}{{}^{101}C_{r-1}}$ which simplifies to $(102 - r)$

$$\text{Hence sum} = \sum_{r=1}^{101}(102-r)$$

$$= 101 + 100 + 99 + \dots + 2 + 1$$

$$= \frac{101\cdot(101+1)}{2} = 5151$$

ILLUSTRATION 7

Find the value of $\sum_{r=1}^{10} r^2\left(\dfrac{{}^{26}C_r}{{}^{26}C_{r-1}}\right)$

SOLUTION

$$\sum_{r=1}^{10} r^2\cdot\left(\frac{{}^{26}C_r}{{}^{26}C_{r-1}}\right) = \sum_{r=1}^{10} r^2\left(\frac{26-r+1}{r}\right)$$

$$\left(\because \text{using } \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right)$$

$$= \sum_{r=1}^{10} r(27-r)$$

$$= 27\sum_{r=1}^{10} r - \sum_{r=1}^{10} r^2 = 27\left(\frac{10\cdot 11}{2}\right) - \frac{10\cdot 11\cdot 21}{6} = 1100$$

ILLUSTRATION 8

If $\sum_{k=1999}^{2015} {}^{k}C_x = {}^{a}C_{x+1} - {}^{b}C_{x+1}$, then find $|a-b|$.

SOLUTION

$$\sum_{k=1999}^{2015} {}^{k}C_x = {}^{1999}C_x + {}^{2000}C_x + \ldots\ {}^{2005}C_x$$

$$= {}^{1999}C_{x+1} + {}^{1999}C_x + {}^{2000}C_x + \ldots + {}^{2015}C_x - {}^{1999}C_x$$

$+\ 1$

$$= {}^{2016}C_{x+1} - {}^{1999}C_{x+1}$$

$\Rightarrow \quad a = 2016,\ b = 1999$

$|a-b| = 17$

ILLUSTRATION 9

If $C_0, C_1, C_2 \ldots C_n$ are binomial coefficients such that C_{n-3}, C_{n-2}, C_{n-1} are in A.P. (where $C_r = {}^nC_r$), then find n.

SOLUTION

$\because \quad {}^nC_1 = {}^nC_{n-1},\ {}^nC_2, = {}^nC_{n-2}$ and ${}^nC_3 = {}^nC_{n-3}$

$\Rightarrow \quad C_1, C_2, C_3$ are in A.P.

$$\therefore \quad \frac{2.n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$$

$$n[n^2 - 3n + 2 + 6 - 6n + 6] = 0 \qquad (\because n \neq 0)$$

$$n^2 - 9n + 14 = 0 \Rightarrow (n-7)(n-2) = 0$$

$$\Rightarrow \quad n = 2, 7 \qquad (\because n > 2)$$

$$\therefore \quad n = 7$$

ILLUSTRATION 10

Expand and simplify the expansion $(2x^3 - \sqrt{5}x^{-1})^4$.

SOLUTION

Using formula $(X+Y)^n$ with $n = 4$, $X = 2x^3$; $Y = -\sqrt{5}x^{-1}$

Now $(2x^3 - \sqrt{5}x^{-1})^4 = \left[2x^3 + \left(-\sqrt{5}x^{-1}\right)\right]^4$

$$= (2x^3)^4 + 4.(2x^3)^3(-\sqrt{5}x^{-1})$$

$$+6(2x^3)^2(-\sqrt{5}x^{-1})^2 + 4(2x^3)(-\sqrt{5}x^{-1})^3 + (-\sqrt{5}x^{-1})^4$$

$$= 16x^{12} - 32\sqrt{5}x^8 + 120x^4 - 40\sqrt{5} + 25x^{-4}$$

ILLUSTRATION 11

Simplify:

$$\frac{(18^3 + 7^3 + 3.18.7.25)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}$$

SOLUTION

The numerator is of the form

$a^3 + b^3 + 3ab(a+b) = (a+b)^3$

Where $a = 18$, and $b = 7$

$\therefore\ N^r = (18+7)^3 = (25)^3$

Denominator can be written as

$$3^6 + {}^6C_1.3^5.2^1 + {}^6C_2.3^4.2^2 + {}^6C_3.3^3.2^3 + {}^6C_4.3^2.2^4$$
$$+ {}^6C_5.3.2^5 + {}^6C_6 2^6$$

$$= (3+2)^6 = 5^6 = (25)^3$$

$$\therefore \text{ Given expression} = \frac{(25)^3}{(25)^3} = 1$$

ILLUSTRATION 12

If 2nd, 3rd and 4th terms in the expansion of $(x+y)^n$ be 240, 720 and 1080, then find n.

SOLUTION

2nd term: ${}^nC_1\, x^{n-1}y = 240$

3rd term: ${}^nC_2 x^{n-2}y^2 = 720$

4th term: ${}^nC_3 x^{n-3}y^3 = 1080$

Taking ratio of first two relations

$$\frac{n}{\frac{n(n-1)}{2}}.\frac{x}{y} = \frac{240}{720} \Rightarrow \frac{2}{n-1}.\frac{x}{y} = \frac{1}{3} \qquad \ldots(1)$$

Taking ratio of last two relations

$$\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)(n-2)}{6}}.\frac{x}{y} = \frac{720}{1080} \Rightarrow \frac{3}{n-2}.\frac{x}{y} = \frac{2}{3} \qquad \ldots(2)$$

Solving (1) and (2)

$$\frac{2}{n-1} \times \frac{n-2}{3} = \frac{1}{2} \Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = 5$$

ILLUSTRATION 13

Find the 7th term in the expansion of $\left(\frac{p^2}{x^3} + \frac{q^{3/2}}{x^7}\right)^{14}$.

SOLUTION

$$T_7 = {}^{14}C_6.\left(\frac{p^2}{x^3}\right)^{14-6}.\left(\frac{q^{3/2}}{x^7}\right)^6 = {}^{14}C_6.p^{16}.q^9.x^{-66}$$

ILLUSTRATION 14

Find the 3rd term from the end in the expansion of $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$.

SOLUTION

3rd term from the end = (8 – 3 + 2)th term from beginning = 7th term from beginning.

$$= T_7 = {}^8C_6\left(x^{-2/3}\right)^{8-6}.\left(\frac{-3}{x^2}\right)^2 = 252.x^{-16/3}$$

ILLUSTRATION 15

In the expansion of $\left(x^2-\frac{1}{3x}\right)^9$ find

(i) The coefficient of x^9 **(ii)** The term independent of x

SOLUTION

$T_{r+1} = {}^nC_r$ (first term)$^{n-r}$ (second term)^r.

In the expansion of $\left(x^2-\frac{1}{3x}\right)^9$

$$\therefore\ T_{r+1} = {}^9C_r\,(x^2)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$= (-1)^r\,{}^9C_r\,x^{18-2r}\times\frac{1}{3^r\times x^r}$$

$$\therefore\ T_{r+1} = (-1)^r\,{}^9C_r\cdot\left(\frac{1}{3^r}\right)\cdot x^{18-3r} \qquad \ldots(1)$$

(i) To find coefficient of x^9. Let T_{r+1} be the term containing x^9.

From (1), $18-3r = 9 \quad \therefore\ r = 3\in N$

Put $r = 3$ in (1), $T_4 = (-1)^3\,{}^9C_3\left(\frac{1}{3^3}\right)\cdot x^9$

$$\therefore\ \text{Coefficient of } x^9 = (-1)^3\,{}^9C_3\left(\frac{1}{3^3}\right) = -\frac{28}{9}$$

(ii) Let T_{r+1} be the term independent of x.

From (1),

$\therefore\ 18-3r = 0 \quad \Rightarrow r = 6\in N$

$$\therefore\ T_7 = (-1)^6\,{}^9C_6\times\frac{1}{3^6}\,x^0 = \frac{9!}{3!\,6!}\times\frac{1}{3^6} = \frac{28}{243}$$

$\therefore\ T_7 = \frac{28}{243}$ is the term independent of x

ILLUSTRATION 16

Given the coefficient of x^{21} in the expansion of $\left(x^2+\frac{3a}{x}\right)^{15}$ is 3640, then find the value of $3a$.

SOLUTION

$$T_{r+1} = {}^{15}C_r(x^2)^{15-r}\left(\frac{3a}{x}\right)^r = {}^{15}C_r3^ra^rx^{30-3r}$$

$\Rightarrow 30-3r = 21$

$\Rightarrow 3r = 9 \qquad \Rightarrow r = 3$

$\Rightarrow$ Coefficient is ${}^{15}C_3 3^3a^3 = 5.7.13.3^3a^3 = 3640$

$a = \frac{2}{3} \Rightarrow 3a = 2$

ILLUSTRATION 17

Find the value of 'r' for which x^{-17} is contained by the rth term in the expansion of $(x^4-x^{-3})^{15}$.

SOLUTION

$$T_r = {}^{15}C_{r-1}(x^4)^{15-r+1}\left(-\frac{1}{x^3}\right)^{r-1} = {}^{15}C_{r-1}(-1)^{r-1}\,x^{67-7r}$$

$67-7r = -17 \Rightarrow 7r = 84;\ r = 12$

ILLUSTRATION 18

Find the coefficients of

(i) x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ **(ii)** x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$

(iii) Find the relation between a and b, so that these coefficients are equal

SOLUTION

(i) In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$

$$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\frac{a^{11-r}}{b^r}\times\frac{x^{22-2r}}{x^r}$$

$$T_{r+1} = {}^{11}C_r = \frac{a^{11-r}}{b^r}\,x^{22-3r}$$

To find the coefficient of x^7

$\therefore\ 22-3r = 7$

$\therefore\ r = 5\in N$

$$\therefore\ T_6 = {}^{11}C_5\frac{a^6}{b^5}x^7$$

$$\therefore\ \text{Coefficient of } x^7 = {}^{11}C_5\frac{a^6}{b^5}$$

(ii) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

$$T_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r = (-1)r\ {}^{11}C_r\frac{a^{11-r}}{b^r}x^{11-3r}$$

To find the coefficient of x^{-7},

$\therefore\ 11-3r = -7$

$\therefore\ r = 6 \in \text{N}$

$\therefore\ T_7 = (-1)^6\ {}^{11}C_6\,\frac{a^5}{b^6}\,x^{-7}$

$\therefore$ Coefficient of $x^{-7} = (-1)^6\ {}^{11}C_6\,\frac{a^5}{b^6} = {}^{11}C_6\,\frac{a^5}{b^6}$

(iii) These two coefficients are equal if and only if

${}^{11}C_5\,\frac{a^6}{b^5} = {}^{11}C_6\,\frac{a^5}{b^6}$ [From (i) and (ii)]

$\therefore\ a = \frac{1}{b} \therefore\ ab = 1$ $({}^{11}C_5 = {}^{11}C_6)$

ILLUSTRATION 19

Find the ratio of coefficient of x^2 to coefficient x^{10} in the expansion of $\left(x^5 + 4.3^{-\log_{\sqrt{3}}\sqrt{x^3}}\right)^{10}$

SOLUTION

$$\left(x^5 + 4.3^{-\log_{\sqrt{3}}\sqrt{x^3}}\right)^{10} = (x^5 + 4.x^{-3})^{10}$$

$T_{r+1} = {}^{10}C_r(x^5)^{10-r}\,(4x^{-3})^r$

$= {}^{10}C_r\,4^r x^{50-5r-3r} = {}^{10}C_r 4^r x^{50-8r}$

Coefficient of x^2, $50 - 8r = 2 \Rightarrow r = 6$

Coefficient of x^{10}, $50 - 8r = 10 \Rightarrow r = 5$

Required ratio $= \frac{{}^{10}C_6\,4^6}{{}^{10}C_5\,4^5} = \frac{10}{3}$

ILLUSTRATION 20

Find the largest real value of x such that

$$\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right) = \frac{8}{3}$$

SOLUTION

$$\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right) = \sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)\frac{4!}{4!}$$

$$= \sum_{k=0}^{4}\frac{{}^4C_k \cdot 5^{4-k} \cdot x^k}{4!} = \frac{(5+x)^4}{4!}$$

Now $\frac{(5+x)^4}{4!} = \frac{8}{3}$ $\Rightarrow (5 + x)^4 = 64$

$x + 5 = 2^{3/2} = 2\sqrt{2}$ $\Rightarrow x = 2\sqrt{2} - 5$

ILLUSTRATION 21

Find the term independent of x in $\left(x^3 - \frac{4}{x^2}\right)^{15}$

SOLUTION

$T_{r+1} = {}^{15}C_r(-4)^r\,x^{45-5r}$

$\therefore\ 45 - 5r = 0;\ r = 9.$

$\therefore$ Constant term $= -4^9.{}^{15}C_9.$

ILLUSTRATION 22

Find the sum of all possible value(s) of ‘x’ for which the sixth term of $\left[\left(2^{\log(10-3^x)}\right)^{\frac{1}{2}} + \left(2^{(x-2)\log 3}\right)^{\frac{1}{5}}\right]^m$ is equal to 21 and binomial coefficients of second, third and fourth terms are the first, third and fifth terms of an arithmetic progression.
[Take everywhere base of log as 10]

SOLUTION

Let $a = 2^{\log(10-3^x)}$ and $b = 2^{(x-2)\log 3}$

Hence $T_{r+1} = {}^mC_r\,a^{\frac{m-r}{2}}\,b^{\frac{r}{5}}$ …(1)

Now ${}^mC_1$, ${}^mC_2$, ${}^mC_3$ are in A.P.

$\therefore\ T_6 = {}^mC_5\,a^{\frac{m-r}{2}}\,b^{\frac{r}{5}}$

$\Rightarrow m = 7$ or -2 (rejected)

Substituting $m = 7$ in equation (1)

${}^7C_5 \cdot a \cdot b = 21$

$\therefore\ ab = 1$

$\Rightarrow 2^{\log(10-3^x)}.\ 2^{(x-2)\log 3} = 1$

$\Rightarrow \log(10 - 3^x) + (x - 2)\log 3 = 0$

$\Rightarrow \log(10 - 3^x)\,(3^{x-2}) = 0$

$\Rightarrow \frac{(10-3^x)3^x}{9} = 1 \ \Rightarrow\ 3^x = 1$ or $3^x = 9 \ \Rightarrow\ x = 0$ or 2

Hence sum = 2

ILLUSTRATION 23

If the 4th term in expansion of $\left(\sqrt{x^{\frac{1}{\log_{10}x+1}}} + \sqrt{x^{\frac{1}{12}}}\right)^6$, where $x > 1$ is 200, then find the value of x.

SOLUTION

$T_4 = 200$

$$\Rightarrow {}^6C_3\left(\sqrt{x^{\frac{1}{\log_{10}x+1}}}\right)^3\left(\sqrt{x^{\frac{1}{12}}}\right)^3 = 200$$

$$\Rightarrow \frac{3}{2}\frac{1}{\log_{10}x+1}+\frac{1}{4}=\log_x 10$$

Let $\log_{10}x = t$

$$\frac{3}{2}\frac{1}{t+1}+\frac{1}{4}=\frac{1}{t} \quad \Rightarrow \quad \frac{6+t+1}{4(t+1)}=\frac{1}{t}$$

$$\Rightarrow t(t+7)=4t+4 \quad \Rightarrow t^2+3t-4=0$$

$$\Rightarrow (t+4)(t-1)=0 \quad \therefore \quad t=-4 \text{ or } t=1$$

$$\log_{10}x=-4 \text{ or } \log_{10}x=1$$

$$\Rightarrow \quad x=10^{-4} \text{ or } x=10 \text{ but } x>1$$

$$\therefore x=10$$

ILLUSTRATION 24

If the coefficient of first three terms in the expansion $\left(\sqrt{x}+\frac{1}{2\cdot\sqrt[4]{x}}\right)^n$ where $n \in$ N form an arithmetic progression, then find the number of terms in the expansion having integral powers of x.

SOLUTION

$$T_{r+1}={}^nC_r\left(\sqrt{x}\right)^{n-r}\cdot\left(\frac{1}{2\cdot\sqrt[4]{x}}\right)^r$$

Also, ${}^nC_0$, $\frac{{}^nC_1}{2}$, $\frac{{}^nC_2}{4}$ (in that order) are in A.P.

$$\Rightarrow 2\left(\frac{{}^nC_1}{2}\right)={}^nC_0+\frac{{}^nC_2}{4}$$

$$\Rightarrow \ n=1+\frac{n(n-1)}{8}\Rightarrow 8(n-1)=n(n-1)\Rightarrow \quad n=8$$

$$\therefore T_{r+1}=\frac{{}^8C_r}{2^r}\cdot(x)^{\frac{16-3r}{4}};\ r=0,1,2,3,4,5,6,7,8$$

But $\left(\frac{16-3r}{4}\right)$ is an integer for $r = 0, 4, 8$.

ILLUSTRATION 25

Sum of the series $\frac{1.{}^{10}C_1}{2^9}-\frac{2.{}^{10}C_2.3^1}{2^8}+\frac{3.{}^{10}C_3.3^2}{2^7}-\ldots-10.{}^{10}C_{10}3^9$ is equal to $\frac{k}{2^8}$, then find k.

SOLUTION

$$\text{Sum} = \sum_{r=1}^{10} r.{}^{10}C_r\left(\frac{1}{2}\right)^{10-r}(-3)^{r-1}\left(\because {}^{10}C_r=\frac{10}{r}\,{}^9C_{r-1}\right)$$

$$=\sum_{r=1}^{10}10.{}^9C_{r-1}\left(\frac{1}{2}\right)^{9-(r-1)}\cdot(-3)^{r-1}=10\left[\frac{1}{2}-3\right]^9$$

$$=10\left[\frac{-5}{2}\right]^9=\frac{-5^{10}}{2^8}=\frac{k}{2^8}\Rightarrow k=-5^{10}$$

ILLUSTRATION 26

Find the value of $\sum_{r+5=0}^{10}(-1)^{r+5}.\,{}^{10}C_{r+5}.2^{5-r}$

SOLUTION

Put $r + 5 = t$; Required term $\sum_{t=0}^{10}(-1)^t\,{}^{10}C_t\,.\,2^{10-t}$

$$=2^{10}\sum_{t=0}^{10}(-1)^t\cdot{}^{10}C_t\cdot\frac{1}{2^t}=2^{10}\left(1-\frac{1}{2}\right)^{10}=1$$

ILLUSTRATION 27

Given $(1-2x+5x^2+10x^3)(1+x)^n=1+a_1x+a_2x^2+$ and that $a_1^2=2a_2$ then find the value of n.

SOLUTION

$$(1-2x+5x^2+10x^3)\,[C_0+C_1x+C_2x^2+\ldots]$$

$$=1+a_1x+a_2x^2+\ldots$$

$$a_1={}^nC_1-2 \text{ and } a_2={}^nC_2-2\cdot{}^nC_1+5=\frac{n(n-1)}{2}-2n+5$$

put $a_1^2=2a_2$

$$(n-2)^2=n(n-1)-4n+10$$

$$n^2-4n+4=n^2-5n+10\Rightarrow n=6$$

ILLUSTRATION 28

Find the number of different terms in the sum $(1+x)^{2009}+(1+x^2)^{2008}+(1+x^3)^{2007}$.

SOLUTION

Number of terms in $(1+x)^{2009}$ = 2010 terms

Additional terms in $(1+x^2)^{2008}$ = terms having $(x^{2010}, x^{2012}, \ldots, x^{4016})$ = 1004 terms

Additional terms in $(1+x^3)^{2007}$ = terms having $(x^{2010}, x^{2013}, \ldots, x^{4014}, \ldots, x^{6021})$ = 1338 terms

(common to 2 and 3) = terms having $(x^{2010}+x^{2016}+\ldots+x^{4014})$ = 335 terms

Hence total = 2010 + 1004 + 1338 – 335= 4352 – 335 = 4017

Alternate

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C))$$

$$=(2010+2009+2008)-(1005+670+670)+335$$

$$=4017.$$

ILLUSTRATION 29

Find the coefficient of x^4 in the expansion of

$$(1 + x + x^2 + x^3)^{11}$$

SOLUTION

$1 + x + x^2 + x^3 = (1 + x) + x^2 (1 + x) = (1 + x) (1 + x^2)$

$\therefore (1 + x + x^2 + x^3)^{11} = (1 + x)^{11} (1 + x^2)^{11}$

$= (1 + {}^{11}C_1 x + {}^{11}C_2 x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots)(1 + {}^{11}C_1 x^2 + {}^{11}C_2 x^4 + \ldots\ldots)$

To find term in x^4 from the product of two brackets on the right-hand-side, consider the following products of terms as

$= 1 \times {}^{11}C_2 x^4 + {}^{11}C_2 x^2 \times {}^{11}C_1 x^2 + {}^{11}C_4 x^4$

$= [{}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4]\, x^4$

$= [55 + 605 + 330]x^4 = 990x^4$

ILLUSTRATION 30

Find the coefficient of t^4 in $(1 - t)^6 (1 + t^2)^5$.

SOLUTION

The coefficient of t^4 in $(1 - t)^6 (1 + t^2)^5$

$$= \sum_{r=0}^{6} {}^6C_r (-1)^r t^r \sum_{P=0}^{5} {}^5C_P\, t^{2P}$$

$= {}^6C_0\, {}^5C_2 + {}^6C_2 \cdot {}^5C_1 + {}^6C_4 \cdot {}^5C_0$

$= (1)(10) + (15)(5) + (15)(1) = 10 + 75 + 15 = 100$

ILLUSTRATION 31

Find the coefficient of x^5 in the expansion of

$$(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$$

SOLUTION

$(1 + x)^{21} + (1 + x)^{22} + \ldots\ldots + (1 + x)^{30}$

$$= (1+x)^{21}\left(\frac{(1+x)^{10}-1}{(1+x)-1}\right) = \frac{1}{x}\left[(1+x)^{31}-(1+x)^{21}\right]$$

$\therefore$ Coefficient of x^5 in the given expression

$$= \text{Coefficient of } x^5 \text{ in} \left[\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}\right]$$

$$= \text{Coefficient of } x^6 \text{ in} \left\{(1+x)^{31}-(1+x)^{21}\right\} = {}^{31}C_6 - {}^{21}C_6$$

ILLUSTRATION 32

If the coefficient x^2 and x^3 in the expansion of

$(1 + 8x + bx^2)(1 - 3x)^9$ in the power of x are equal, find b.

SOLUTION

$$(1 + 8x + bx^2)(1 - 3x)^9 = (1 + 8x + bx^2) \sum_{r=0}^{9} {}^9C_r (-3x)^r$$

$\therefore$ Coefficient of $x^2 = {}^9C_2 . 9 - 8 . {}^9C_1 . 3 + b$

$\therefore$ Coefficient of $x^3 = -\, {}^9C_3 \cdot 27 + 8 \cdot {}^9C_2 \cdot 9 - b \cdot {}^9C_1 \cdot 3$

Both are equal $324 - 216 + b = -2268 + 2592 - 27b$

$$\Rightarrow \quad b = \frac{54}{7}$$

ILLUSTRATION 33

If $2a + 2b + 3c = \frac{1}{5}$ and $a, b, c > 0$ then find maximum value of term independent of x in the expansion of $\left(abx^{\frac{1}{2}} + cx^{\frac{-1}{3}}\right)^{25}$

SOLUTION

$$T_{r+1} = {}^{25}C_r (ab)^{(n-r)} .\, x^{\left(\frac{n-r}{2}\right)} . c^r . x^{\left(\frac{-r}{3}\right)}$$

$n = 25$, Independent of $x = \dfrac{25-r}{2} - \dfrac{r}{3} = 0 \quad \Rightarrow \quad r = 15$

$= {}^{25}C_{15}\, a^{10} . b^{10} . c^{15}$

$10a + 10b + 15c = 1$

AM $\geq$ GM

$$\frac{(a+a+\ldots+a)+(b+b+\ldots+b)+(c+c+\ldots+c)}{35}$$

$$\geq (a^{10}b^{10}c^{15})^{\frac{1}{35}}$$

$$\Rightarrow \quad \left(\frac{1}{35}\right)^{35} \geq a^{10}b^{10}c^{15}.$$

So the maximum value of required term ${}^{25}C_{15}\left(\dfrac{1}{35}\right)^{35}$

ILLUSTRATION 34

Find coefficient of x^5 in the expansion of

$$\left(x + \frac{1}{x}\right)^{10}\left(x - \frac{1}{x}\right)^9$$

SOLUTION

$$\left(x+\frac{1}{x}\right)^{10}\left(x-\frac{1}{x}\right)^{9} = \left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^{9}$$

$$= \left(x+\frac{1}{x}\right)\sum_{r=0}^{9} {}^9C_r (-1)^r x^{18-4r}$$

$$= \sum_{r=0}^{9} {}^9C_r (-1)^r x^{19-4r} + \sum_{r=0}^{9} {}^9C_r (-1)^r x^{17-4r}$$

Coefficient of x^5 is given by $r = 3$ in second sum only.

Coefficient of $x^5 = {}^9C_3(-1)^3 = -84$

ILLUSTRATION 35

If $(1 - x^2 + x^3)\,(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots\ a_i$'s $\in R$, then find the value of $a_1^2 - a_1 - 2a_2$.

SOLUTION

$$(1 - x^2 + x^3)\,({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots)$$

$$= a_0 + a_1x + a_2x^2 + \ldots$$

$$a_0 = 1,\ a_1 = n,\ a_2 = {}^nC_2 - {}^nC_0$$

$$= \frac{n^2 - n - 2}{2} = \frac{a_1^2 - a_1 - 2}{2}$$

$$\Rightarrow \quad a_1^2 - a_1 - 2a_2 = 2$$

ILLUSTRATION 36

Find the value of $\displaystyle\sum_{k=0}^{21} k \cdot {}^{21}C_k \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{21-k}$

SOLUTION

$$\sum_{k=0}^{21} k \cdot {}^{21}C_k \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{21-k}$$

$$= \sum_{k=0}^{21} k \cdot \frac{21}{k}\ {}^{20}C_{k-1} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{21-k}$$

$$= \frac{21}{3} \sum_{k=0}^{21} {}^{20}C_{k-1} \left(\frac{1}{3}\right)^{k-1} \left(\frac{2}{3}\right)^{20-(k-1)}$$

$$= 7\left(\frac{2}{3} + \frac{1}{3}\right)^{20} = 7(1)^{20} = 7$$

ILLUSTRATION 37

Find the coefficient of x^3 in $(1 + x)^{25} \cdot (1 - x)^5$.

SOLUTION

Coefficient x^3 in $(1 - x^2)^5 \cdot (1 + x)^{20}$

Coefficient x^3 in $(1 - 5x^2)(1 + x)^{20}$

$=$ coefficient x^3 in $(1 + x)^{20} - 5$ coefficient x in $(1 + x)^{20}$

$= {}^{20}C_3 - 5 \cdot {}^{20}C_1 = 1140 - 100 = 1040$

ILLUSTRATION 38

Let coefficient of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal. If a lies between the roots of the equation $49x^2 + 7px - 9 = 0$, then the range of p is $(-\infty, -\lambda)$. Find the value of λ.

SOLUTION

Coefficient of x^2 = coefficient of x^3

${}^9C_2 3^7 a^2 = {}^9C_3 3^6 a^3 \qquad \therefore\ a = \dfrac{9}{7}$

a lies between roots of given equation

$$49\left(\frac{81}{49}\right) + 7p\left(\frac{9}{7}\right) - 9 < 0$$

Now, $f(a) < 0 \quad \Rightarrow \quad 81 + 9p - 9 < 0$

$9p < -72 \Rightarrow p < -8 \quad$ Hence $\lambda = 8$

ILLUSTRATION 39

Find the coefficient of x^{14} in $(1 + 2x + 3x^2 + \ldots + 16x^{15})^2$

SOLUTION

$(1 + 2x + \ldots + 16x^{15})(1 + 2x + \ldots + 16x^{15})$

Coefficient of $x^{14} = (1 \times 15) + (2 \times 14) + (3 \times 13) + \ldots + (15 \times 1)$

$$= \sum_{r=1}^{15} r(16 - r) = \sum_{r=1}^{15} (16r - r^2)$$

$$= 16\sum_{r=1}^{15} r - \sum_{r=1}^{15} r^2$$

Using $\displaystyle\sum_{r=1}^{n} r = \frac{n(n+1)}{2};\ \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

$$= 16(15)8 - (5 \cdot 8 \cdot 31) = 680$$

ILLUSTRATION 40

If term independent of x in the expansion of $\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right)(1 + x)^n$, $n \in$ N is 29, then find n.

SOLUTION

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)(1 + x)^n = \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)\left(\sum_{r=0}^{n} {}^nC_r x^r\right)$$

Term independent of $x = {}^nC_0 + {}^nC_1 + {}^nC_2 = 29$

$$\Rightarrow \quad 1 + n + \frac{n(n-1)}{2} = 29$$

$$\Rightarrow \quad n^2 + n - 56 = 0 \quad \Rightarrow \quad (n + 8)(n - 7) = 0$$

$$\Rightarrow \quad n = 7$$

ILLUSTRATION 41

If $\displaystyle\sum_{r=1}^{10} {}^{10}C_r (2)^{20-r} (3)^{r-1} = \frac{a^{10} - b^{10}}{3}$, then find $(a - b)$.

SOLUTION

$$\sum_{r=1}^{10} {}^{10}C_r (2)^{20-r} (3)^{r-1}$$

$$= \frac{2^{10}}{3} \sum_{r=1}^{10} {}^{10}C_r . 2^{10-r} . 3^r = \frac{2^{10}}{3}\left[(2+3)^{10} - 2^{10}\right]$$

$$= \frac{2^{10}}{3}\{5^{10} - 2^{10}\} = \frac{10^{10} - 4^{10}}{3}$$

$$\Rightarrow a - b = 10 - 4 = 6$$

ILLUSTRATION 42

Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \dots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)^{51}C_3$ for some positive integer n then find the value of n.

SOLUTION

Coefficient of x^2 in the expansion of

$(1 + x)^2 + (1 + x)^3 + \dots (1 + x)^{49} + (1 + mx)^{50}$ is

$${}^2C_2 + {}^3C_2 + \dots {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)^{51}C_3$$

$$\Rightarrow {}^3C_3 + {}^3C_2 + \dots {}^{49}C_2 + {}^{50}C_2m^2 = (3n + 1)^{51}C_3$$

$$\Rightarrow {}^{50}C_3 + {}^{50}C_2m^2 = (3n + 1)^{51}C_3 \qquad (\text{As, } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$$

$$\therefore \frac{50.49.48}{6} + \frac{50.49}{2} . m^2 = (3n + 1)\frac{51.50.49}{6}$$

$$8 + \frac{m^2}{2} = (3n+1)\frac{17}{2} \quad \Rightarrow \quad m^2 = 51n + 1$$

Must be a perfect square hence $n = 5$ and $m = 16$

$\therefore n = 5$

ILLUSTRATION 43

Find the coefficient of α^t in the expansion of,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + \dots + (\alpha + q)^{m-1}$ where $\alpha \neq q$ and $p \neq q$.

SOLUTION

$E = (\alpha + p)^{m-1}$

$$\left[1+\frac{\alpha+q}{\alpha+p}+\left(\frac{\alpha+q}{\alpha+p}\right)^2+\dots+\left(\frac{\alpha+q}{\alpha+p}\right)^{m-1}\right]$$

$$= (\alpha+p)^{m-1}\left[\frac{\left(\frac{\alpha+q}{\alpha+p}\right)^m-1}{\left(\frac{\alpha+q}{\alpha+p}\right)-1}\right]$$

$$= \frac{[(\alpha+p)^m-(\alpha+q)^m]}{p-q}$$

Hence coefficient of α^t in $\dfrac{(\alpha+p)^m-(\alpha+q)^m}{p-q}$

$$= \frac{{}^mC_t(p^{m-t}-q^{m-t})}{p-q}$$

ILLUSTRATION 44

Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \dots$ If A_0, A_1, A_2 are in A.P. Then find the value of n.

SOLUTION

$(1 + x^2)^2 (1 + x)^n = (1 + 2x^2 + x^4)({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots)$

$= {}^nC_0 + {}^nC_1x + ({}^nC_2 + 2 \cdot {}^nC_0)x^2 + \dots$

Hence $A_0 = 1$; $A_1 = {}^nC_1$; $A_2 = {}^nC_2 + 2$ which are in A.P.

$$2A_1 = A_0 + A_2$$

$$\Rightarrow \quad 2n = 1 + {}^nC_2 + 2$$

$$\Rightarrow \quad 2n = 3+\frac{n(n-1)}{2}$$

$$\Rightarrow \quad n^2 - 5n + 6 = 0 \Rightarrow (n - 2)(n - 3) = 0 \Rightarrow n = 2, 3$$

ILLUSTRATION 45

Let a and b are two positive real numbers such that $a^2 + b = 2$, then find the maximum value of term independent of x in the expansion of $(ax^{1/6} + bx^{-1/3})^9$

SOLUTION

$$T_{r+1} = {}^9C_r\left(ax^{1/6}\right)^{9-r}\left(bx^{-1/3}\right)^r = {}^9C_ra^{9-r}b^rx^{\frac{9-r}{6}-\frac{r}{3}}$$

$${}^9C_ra^{9-r}b^r\,x^{\frac{3-r}{2}} \Rightarrow r = 3 \text{ (for term independent of } x)$$

$\therefore$ Term independent at x is equal to ${}^9C_3a^6b^3$ …(1)

To get the maximum value of a^6b^3 use AM ≥ GM

$$\text{i.e., } \frac{3\left(\frac{a^2}{3}\right)+3\left(\frac{b}{3}\right)}{6} \geq \left[\frac{a^6b^3}{3^6}\right]^{\frac{1}{6}}$$

$$\Rightarrow \left(\frac{1}{3}\right)^6 \geq \frac{a^6b^3}{3^6}$$

$\therefore (a^6b^3)_{\max} = 1$

$\therefore$ Maximum value of term independent of $x = {}^9C_3$

ILLUSTRATION 46

If in the expansion of $\left(x^3-\dfrac{2}{\sqrt{x}}\right)^n$, a term with the power x^2 exists and 'n' is a double digit number, then find the least value of 'n'.

SOLUTION

$$T_{r+1} = {}^nC_rx^{3n-3r}(-1)^r 2^r x^{-r/2} = {}^nC_rx^{3n-7r/2}(-1)^r \cdot 2^r$$

$$x^2 \text{ exists } 3n-\frac{7^r}{2} = 2 \Rightarrow 3n = \frac{7^r}{2}+2$$

Least value of r for which n is double digit number is $r = 8$.

$\therefore n = 10$

ILLUSTRATION 47

Find the number of rational terms in the expansion of $\left(2^{\frac{1}{7}}+3^{\frac{1}{4}}\right)^{198}$

SOLUTION

$$T_{r+1} = {}^{198}C_r (2)^{\frac{198-r}{7}} (3)^{\frac{r}{4}}$$

$3^{\frac{r}{4}}$ is integral $r \in \{0, 4, 8, 12, 16, \ldots\}$

$2^{28+\frac{2-r}{7}}$ is integral $r \in \{2, 9, 16, 23, \ldots\}$

$\therefore$ For $r = 16, 44, 72, 100, 128, 156, 184$, terms in the expansion are rational. Seven terms are rational.

ILLUSTRATION 48

Find the number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$

SOLUTION

The general term in the expansion of $(y^{1/5} + x^{1/10})^{55}$ is

$$T_{r+1} = {}^{55}C_r (y^{1/5})^{55-r} (x^{1/10})^r$$

$$^{55}C_r \; y^{11-\frac{r}{5}} x^{\frac{r}{10}}$$

Clearly, T_{r+1} will be independent of radical signs if $r/5$ and $r/10$ are integers, where

$0 \le r \le 55 \; r = 0, 10, 20, 30, 40, 50$

Hence, total terms are 6 which are free from radical sign.

ILLUSTRATION 49

Find the number of rational terms in the expansion of $\left(2^{\frac{1}{3}} + 5^{\frac{1}{9}}\right)^{6551}$

SOLUTION

$$T_{r+1} = {}^{6561}C_r (2^{1/3})^{6561-r} (5^{1/9})^r = {}^{6561}C_r 2^{2187-r/3} . 5^{r/9}$$

$\frac{r}{3}$ and $\frac{r}{9}$ must be integers.

$\therefore \; 0 \le r \le 6561 \Rightarrow 0 \le \frac{r}{9} \le 729$

$\therefore \; \frac{r}{9} = 0, 1, 2 \ldots 729$

$\therefore$ Number of rational terms = 730

ILLUSTRATION 50

Find the coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)\ldots\left(x-\frac{1}{2^{49}}\right)$

SOLUTION

Coefficient of x^{49} in this series

$$= -\left[1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{49}}\right] = -\left[\frac{1-\frac{1}{2^{50}}}{1-\frac{1}{2}}\right] = -2\cdot\left[1-\frac{1}{2^{50}}\right]$$

ILLUSTRATION 51

Find the greatest value of the terms independent of x in the expansion of $\left(x\sin\theta + \frac{\cos\theta}{x}\right)^{10}$

SOLUTION

$$\left(x\sin\theta + \frac{\cos\theta}{x}\right)^{10}$$

$$T_{r+1} = {}^{10}C_r (x\sin\theta)^{10-r} \cdot \left(\frac{\cos\theta}{x}\right)^r$$

$$= {}^{10}C_r (\sin\theta)^{10-r} (\cos\theta)^r \cdot x^{10-2r}$$

Hence for independent of x, $r = 5$

$$T_6 = {}^{10}C_5 (\sin\theta)^5 \cdot (\cos\theta)^5 = \frac{{}^{10}C_5 (\sin 2\theta)^5}{2^5}$$

Greatest value of required term $= \frac{{}^{10}C_5}{2^5}$

ILLUSTRATION 52

Find the positive integer which is just greater than $(1 + 0.0001)^{1000}$.

SOLUTION

Expression on expansion of $(1 + 10^{-4})^{1000}$ gives

$$1 + 1000\times 10^{-4} + \frac{1000\times 999}{2} 10^{-8} + {}^{1000}C_3 10^{-12} + \ldots$$

$$< 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots\ldots = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$$

So integer just greater than the given expression must be 2.

ILLUSTRATION 53

If $\left(x + \frac{1}{x} + 1\right)^6 = a_0 + \left(a_1 x + \frac{b_1}{x}\right) + \left(a_2 x^2 + \frac{b_2}{x^2}\right) + \ldots. + \left(a_6 x^6 + \frac{b_6}{x^6}\right)$, then find the value of a_0.

SOLUTION

$$\left(x + \frac{1}{x} + 1\right)^6 = \sum_{r=0}^{6} {}^6C_r \left(x + \frac{1}{x}\right)^r$$

Constant term, we have in ${}^6C_0\left(x+\frac{1}{x}\right)^0 + {}^6C_2\left(x+\frac{1}{x}\right)^2 + {}^6C_4\left(x+\frac{1}{x}\right)^4 + {}^6C_6\left(x+\frac{1}{x}\right)^6$

$$= {}^6C_0 + {}^6C_2 \cdot {}^2C_1 + {}^6C_4 \cdot {}^4C_2 + {}^6C_6 {}^6C_3$$

$$= 1 + 30 + 90 + 20 = 141$$

Test Your Concept-I

(Solutions on page H.147)

1. Find the 7th term of $\left(3x^2-\frac{1}{3}\right)^{10}$
2. Find the term independent of x in the expansion $\left(2x^2-\frac{3}{x^3}\right)^{25}$.
3. Find the term independent of *x* in
   i. $\left(2x-\frac{1}{\sqrt{x}}\right)^{15}$ ii. $\left(\frac{4x^2}{3}-\frac{3}{2x}\right)^{9}$
4. Find the term independent of x in $(1+x)^m\left(1+\frac{1}{x}\right)^n$, is
5. Find the coefficient of x^{20} in the expansion of $(1+x^2)^{40}\cdot\left(x^2+2+\frac{1}{x^2}\right)^{-5}$
6. Find the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ and the term independent of *x* in $\left(x-\frac{2}{x}\right)^{10}$
7. If in the expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, the ratio of the seventh term from the beginning to the seventh term from end is equal to 1/6, then find *n*.
8. Find the number of terms which are free from radical sings in the expansion of $(y^{1/5}+x^{1/10})^{55}$
9. Find the largest real value of *x* such that $\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3}$
10. The coefficients of r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms in the binomial expansion of $(1+y)^m$ are in AP, then find relation between *m* and *r*.
11. In the expansion of $\left(x^{\frac{2}{3}}-\frac{1}{\sqrt{x}}\right)^{30}$, find the term containing x^{13}
12. If coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{13}$ is equal to coefficient of x^{-6} in $\left(ax+\frac{1}{bx^2}\right)^{13}$, then find *a* and *b*
13. If the coefficient of x^8 in $(x-1)(x-2)(x-3)\ldots(x-10)$ is N, then sum of digits of N is

Answer Keys

1. $\frac{70}{3}x^8$ **2.** $\frac{25!}{10!15!}(2)^{15}(3)^{10}$ **3.** (i) 96096 (ii) 2268 **4.** ${}^{m+n}C_n$ or ${}^{m+n}C_m$.

5. ${}^{30}C_5$ or ${}^{30}C_{25}$ **6.** 1 : 32 **7.** 9 **8.** 6 **9.** $2\sqrt{2}-5$.

10. 0 **11.** 29.7.9.13.25 **12.** $b \neq 0$. **13.** 6

MIDDLE TERMS

Since the binomial expansion $(X+Y)^n$ contains $(n+1)$ terms so middle term depends upon the value of n

Nature of n	Number of middle term	Middle Term
Even	1	$\left(\frac{n}{2}+1\right)^{th}$ Term
Odd	2	$\left(\frac{n+1}{2}\right)^{th}$ term & $\left(\frac{n+3}{2}\right)^{th}$ term

Middle terms have greatest binomial coefficient

(a) If n is even, then greatest binomial coefficient $={}^nC_{n/2}$

(b) If n is odd, then greatest binomial coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

ILLUSTRATION 54

Find the sum of middle term in the expansion of $\left(2a-\frac{a^2}{4}\right)^9$

SOLUTION

n is odd, so there exist two middle terms and the middle terms are 5th and 6th term.

$$\text{So, } S={}^9C_4.(2a)^5\left(\frac{-a^2}{4}\right)^4+{}^9C_5(2a)^4.\left(\frac{-a^2}{4}\right)^5$$

$$=\frac{9!}{4!.5!}(2a)^4\left(\frac{-a^2}{4}\right)^4\left[2a-\frac{a^2}{4}\right]=\frac{63}{32}a^{13}(8-a)$$

ILLUSTRATION 55

Find the middle term in the expansion of

a. $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ b. $\left(2x^2-\frac{1}{x}\right)^7$

SOLUTION

(a) Given, $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$

Here, $n = 6$ (even)

$\therefore$ Only one middle term is possible and $\left(\frac{n}{2}+1\right)$th term will be middle term.

$\frac{n}{2}+1=\frac{6}{2}+1=4^{\text{th}}$ term

$$T_4 = T_{3+1} = {}^6C_3\left(\frac{2x}{3}\right)^3\left(-\frac{3}{2x}\right)^3$$

$$= -{}^6C_3\left(\frac{2}{3}\right)^3\left(\frac{3}{2}\right)^3 = -\frac{6!}{3!3!} = -20$$

(b) Given, $\left(2x^2-\frac{1}{x}\right)^7$

Here, $n = 7$ (odd)

$\therefore$ There are two middle terms.

First middle term $= \left(\frac{n+1}{2}\right)^{\text{th}} = \left(\frac{7+1}{2}\right)^{\text{th}}$

$= 4^{\text{th}}$ term

Second middle term $= \left(\frac{n+3}{2}\right)^{\text{th}} = \left(\frac{7+3}{2}\right)^{\text{th}}$

$= 5^{\text{th}}$ term

$$\therefore T_4 = T_{3+1} = {}^7C_3\,(2x^2)^4\cdot\left(-\frac{1}{x}\right)^3$$

$$= -{}^7C_3\,(2)^4\,x^5 = -\frac{7!}{3!4!}\,16x^5$$

$$= -35\times 16x^5 = -560\,x^5$$

$$\text{and } T_5 = T_{4+1} = {}^7C_4\,(2x^2)^3\left(-\frac{1}{x}\right)^4$$

$$= {}^7C_4\,(2)^3\,x^2 = \frac{7!}{4!3!}(2)^3\,x^2$$

$$= 35\times 8\times x^2 = 280\,x^2$$

APPLICATION OF BINOMIAL THEOREM

(i) Divisibility/Remainder problems

ILLUSTRATION 56

Show that $3^{512} - 1$ is divisible by 2^{11}.

SOLUTION

$$3^{512} - 1 = (1 + 8)^{256} - 1$$

$$= (1 + {}^{256}C_1 8 + {}^{256}C_2 8^2 \ldots + {}^{256}C_{256}\, 8^{256}) - 1$$

$$= 8\,[{}^{256}C_1 + {}^{256}C_2\, 8 + \ldots + {}^{256}C_{256}\, 8^{255}]$$

$$= 8\cdot 256[1 + \ldots]$$

$3^{512} - 1 = 2^{11}\times$ (some integer)

ILLUSTRATION 57

Find the remainder, if $1 + 2 + 2^2 + 2^3 + \ldots + 2^{1999}$ is divided by 5.

SOLUTION

$1 + 2 + 2^2 + 2^3 + \ldots + 2^{1999} = \frac{1(2^{2000}-1)}{2-1} = 2^{2000} - 1 = (2^2)^{1000} - 1 = 4^{1000} - 1$

$= (5-1)^{1000} - 1 = (1-5)^{1000} - 1$

$1 - {}^{1000}C_1\cdot 5 + {}^{1000}C_2\cdot 5^2 + \ldots. + {}^{1000}C_{1000}\cdot 5^{1000} - 1.$

$= 5(-{}^{1000}C_1 + {}^{1000}C_2\, 5 + \ldots + {}^{1000}C_{1000}\, 5^{999})$

which is divisible by 5.

$\therefore$ Remainder is zero.

ILLUSTRATION 58

If $6^{83} + 8^{83}$ is divided by 49, then find the remainder.

SOLUTION

$8^{83} + 6^{83} = (1 + 7)^{83} + (7 - 1)^{83} = (1 + 7)^{83} - (1 - 7)8^3$

$= 2[{}^{83}C_1\,{}^{\cdot}{}_7 + {}^{83}C_3\,{}^{\cdot}\,7^{3 + \ldots + 83}C_{83}\cdot 7^{83}]$

$= 2[83\cdot 7 + 7^2\,[{}^{83}C_3\cdot 7 + \ldots + {}^{83}C_{83}\, 7^{81}]]$

$= (2\cdot 7\cdot 83) + 49$I where I is an integer

$\because$ $14\times 83 = 1162 \Rightarrow \frac{1162}{49} = 23\frac{35}{49}$

$\therefore$ Required remainder is 35.

ILLUSTRATION 59

If 7 divides $32^{32^{32}}$ then find the remainder.

SOLUTION

We have $32 = 2^5$

$\therefore$ $(32)^{32} = (2^5)^{32} = 2^{160}$

$(32)^{32} = (3 - 1)^{160}$

$$= {}^{160}C_0\, 3^{160} - {}^{160}C_1\, 3^{159} + \ldots - {}^{160}C_{159}\, 3 + {}^{160}C_{160}\, 1.$$

$$= 3\,(3^{159} - {}^{160}C_1\, 3^{158} + \ldots - {}^{160}C_{159}) + 1$$

$$= 3m + 1,\ m \in \mathrm{I}_+$$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

$$\begin{aligned}\therefore \quad 32^{32^{32}} &= 2^{3(5m+1)}.2^2\\ &= 4.(8)^{5m+1}\\ &= 4.(7+1)^{5m+1}\\ &= 4.({}^{5m+1}C_0(7)^{5m+1} + {}^{5m+1}C_1(7)^{5m} + {}^{5m+1}C_2\\ &\quad (7)^{5m-1} + \ldots + {}^{5m+1}C_{5m}7 + {}^{5m+1}C_{5m+1})\\ &= 4[7\{{}^{5m+1}C_0 7^{5m} + {}^{5m+1}C_1 7^{5m-1} +\\ &\quad {}^{5m+1}C_2 7^{5m-2} + \ldots + {}^{5m+1}C_{5m}\} + 1]\\ &= 4\,[7n+1], \quad n \in I_+\\ &= 28\,n + 4\end{aligned}$$

This show that where $32^{32^{32}}$ is divided by 7, then remainder is 4.

ILLUSTRATION 60

Find the remainder, when $(15^{23} + 23^{23})$ is divided by 19.

SOLUTION

$E = (19 - 4)^{23} + (19 + 4)^{23}$

$= 2[19^{23} + {}^{23}C_2 \cdot 19^{21} \cdot 4^2 + \ldots. + {}^{23}C_{22} \cdot 19 \cdot 4^{22}]$

$= 2 \cdot 19[19^{22} + {}^{23}C_2 \cdot 19^{20} \cdot 4^2 + \ldots. + {}^{23}C_{22} \cdot 4^{22}]$

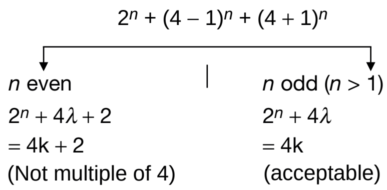
$\Rightarrow$ E is divisible by 19

Hence, required remainder = 0

ILLUSTRATION 61

Let $n \in$ N and $1 \le n \le 20$. If the number $2^n + 3^n + 5^n$ is divisible by 4, then find sum of all possible values of 'n'.

SOLUTION



$\therefore$ Required sum = 3 + 5 + 7 + … + 19 = 99

ILLUSTRATION 62

If a is the remainder when 5^{40} is divided by 11 and b is the remainder when 2^{2003} is divided by 17, then find the value of $b - a$.

SOLUTION

$$\begin{aligned}5^{40} &= (11-6)^{40} = 11k + 6^{40} = 11k + (33+3)^{20}\\ &= 11k + 11\lambda + (11-2)^{10} = 11k + 11\lambda + 11\mu + 2^{10}\\ &= 11t + 1024 = 11(t + 93) + 1\end{aligned}$$

$\Rightarrow \quad a = 1$

Also, $2^{2003} = 8(17-1)^{500} = 8(17\lambda + 1) \Rightarrow b = 8$

$\therefore\ b - a = 7$

ILLUSTRATION 63

Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

SOLUTION

We have $2222^{5555} + 5555^{2222} = (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$ …(1)

The number $2222^{5555} + 4^{5555}$ is divisible by 2222 + 4 = 2226 = 7.318 which is divisible by 7. (because a sum of two odd powers is always divisible by the sum of the bases of the powers),

The difference $5555^{2222} - 4^{2222}$ is also divisible by 7 since it is divisible by 5555 – 4 = 5551 = 7.793 (because the difference of any integral powers with equal exponent is divisible by the difference of the bases) as to the difference $4^{5555} - 4^{2222}$, it can be rewritten as

$4^{2222}(4^{3333} - 1) = 4^{2222}(2^{6666} - 1) = 4^{2222}(64^{1111} - 1)$,

This expression is divisible by 64 – 1 = 63 = 7.9. Hence $4^{5555} - 4^{2222}$ is also divisible by 7. Hence each term written in bracket in Eq. (1) are divisible by 7. Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

ILLUSTRATION 64

If n is any positive integer, show that $2^{3n+3} - 7n - 8$ is divisible by 49.

SOLUTION

The given expression

$= 2^{3n+3} - 7n - 8 = 2^{3n}\,.\,2^3 - 7n - 8$

$= 8^n\,.\,8 - 7n - 8 = 8\,(1 + 7)^n - 7n - 8$

$= 8\,\{1 + {}^nC_1 7 + {}^nC_2 7^2 + \ldots + {}^nC_n 7^n\} - 7n - 8$

$= 8 + 56n + 8\,({}^nC_2 7^2 + \ldots + {}^nC_n 7^n\} - 7n - 8$

$= 49n + 8\,({}^nC_2\,.\,7^2 + \ldots + {}^nC_n\,7^n)$

$= 49\,\{n + 8\,({}^nC_2 + \ldots + {}^nC_n 7^{n-2})\}$

Hence $2^{3n+3} - 7n - 8$ is divisible by 49.

Working Rule

Lets consider a number N = 56464137 (any random number)

Last digit in given number N is 7 = remainder when N is divided by 10

Last 2 digits in given number N is 37 = remainder when N is divided by 100

Last 3 digits in given number N is 137 = remainder when N is divided by 1000

Last 4 digits in given number N is 4137 = remainder when N is divided by 10000

It means whenever last *k* digits are asked in number N, we should look for remainder when N is divided by 10^k

ILLUSTRATION 65

Find the sum of last 3 digits of 3^{100}.

SOLUTION

We have

$$3^{100} = (3^4)^{25} = (81)^{25}$$

$$= (80+1)^{25} = {}^{25}C_0.(80)^{25} + {}^{25}C_1.(80)^{24} + \ldots. +$$

$${}^{25}C_{22}(80)^3 + {}^{25}C_{23}(80)^2 + {}^{25}C_{24}(80) + {}^{25}C_{25}$$

$$= 10^3\Big[{}^{25}C_0\, 8^{25} \times 10^{22} + {}^{25}C_1 \times 8^{24} \times 10^{21} + \ldots.. +$$

$${}^{25}C_{22} \times 8^3\Big] + \frac{25 \times 24}{2} \times (80)^2 + 25 \times 80 + 1$$

$$= 10^3 m + 1920000 + 2000 + 1, \text{ where } m \in \mathrm{N}$$

$$= 10^3(m + 1920 + 2) + 1$$

$$\Rightarrow 3^{100} - 1 = 10^3(m + 1922)$$

$$\Rightarrow 3^{100} - 1 \text{ is divisible by } 1000$$

Thus, last three digits of 3^{100} are 001, last two digits of 3^{100} are 01 and the last digit of 3^{100} is 1.

ILLUSTRATION 66

Find the

(i) last digit **(ii)** last two digits

(iii) last three digits of 17^{256}.

SOLUTION

Since $17^{256} = (17^2)^{128} = (289)^{128} = (290 - 1)^{128}$

$$\therefore\ 17^{256} = {}^{128}C_0\,(290)^{128} - {}^{128}C_1\,(290)^{127} + {}^{128}C_2(290)^{126} - \ldots$$

$$- {}^{128}C_{125}(290)^3 + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= [{}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126}$$

$$- \ldots - {}^{128}C_{125}(290)^3]$$

$$+ {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000\, m + {}^{128}C_2\,(290)^2 - {}^{128}C_1\,(290) + 1 \qquad (m \in I^+)$$

$$= 1000\, m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$$

$$= 1000\, m + (128)\,(127)\,(290)\,(145) - 128 \times 290 + 1$$

$$= 1000\, m + (128)\,(290)\,(127 \times 145 - 1) + 1$$

$$= 1000\, m + 683527680 + 1$$

$$= 1000\, m + 683527000 + 680 + 1$$

$$= 1000\,(m + 683527) + 681$$

Hence last three digits of 17^{256} must be 681, as a result last two digits of 17^{256} is 81 and last digit of 17^{256} is 1.

ILLUSTRATION 67

Find the digit as unit place in the number

$$17^{1995} + 11^{1995} - 7^{1995}$$

SOLUTION

We have

$$17^{1995} + 11^{1995} - 7^{1995} = (7 + 10)^{1995} + (1 + 10)^{1995} - 7^{1995}$$

$$= [7^{1995} + {}^{1995}C_1 7^{1994}\,.\,10^1 + {}^{1995}C_2 7^{1993}\,.\,10^2 + \ldots\ {}^{1995}C_{1995}\,.\,10^{1995}] +$$

$$[{}^{1995}C_0 + {}^{1995}C_1 10^1 + {}^{1995}C_2 10^2 + \ldots + {}^{1995}C_{1995} 10^{1995}] - 7^{1995}$$

$$= [{}^{1995}C_1 7^{1994}\,.\,10^1 + \ldots + 10^{1995}] + [{}^{1995}C_1 10^1 + \ldots + {}^{1995}C_{1995}\,.\,10^{1995}] + {}^{1995}C_0$$

= (a multiple of 10) + 1

Thus, the units place digits is 1.

ILLUSTRATION 68

Find last two digits of 21 to the $(100)^{\text{th}}$ power.

SOLUTION

$$(21)^{100} = (1 + 20)^{100} = 1 + {}^{100}C_1\, 20 + {}^{100}C_2\, 20^2 + \ldots + {}^{100}C_{100}\, 20^{100}$$

$$= 2001 + 10^2[{}^{100}C_2 \cdot 2^2 + \ldots + {}^{100}C_{100}\, 2^{100} \cdot 10^{98}]$$

Hence last two digits are 01.

PROBLEMS BASED ON (*I* + *f*) FORMAT

$$\text{If } \left(\sqrt{p} + q\right)^n = I + f \qquad \ldots(1)$$

where I and n are positive integers, p is not perfect square and q is rational number, and $0 < f < 1$,

$$\text{If } 0 < \sqrt{p} - q < 1 \quad \Rightarrow \quad 0 < (\sqrt{p} - q)^n < 1$$

$$\text{Now let } \left(\sqrt{p} - q\right)^n = f' \qquad \ldots(2)$$

where $0 < f' < 1$

Now we should look for an approach by which we can remove irrationality from expansion by addition or subtraction of equation (1) and (2)

Case (I): n is an odd integer

$$I + f - f' = (\sqrt{p} + q)^n - (\sqrt{p} - q)^n$$

RHS contains even powers of $\sqrt{p}$ because n is odd. Hence RHS and I are integers so $f - f'$ is also integer

$$\Rightarrow f - f' = 0 \text{ since } -1 < f - f' < 1$$

$$\Rightarrow f = f'$$

$(I+f)f = (I+f)f' = \left(\sqrt{p}+q\right)^n \left(\sqrt{p}-q\right)^n = (p-q^2)^n$

Case (II)

If n is even integer $I+f+f' = \left(\sqrt{p}+q\right)^n + \left(\sqrt{p}-q\right)^n$

Here LHS and I are integers $f+f'$ is also integer

$\Rightarrow f+f' = 1$ since $0 < f+f' < 2$

$\Rightarrow (I+f)\,(1-f) = (I+f)f' = \left(\sqrt{p}+q\right)^n \left(\sqrt{p}-q\right)^n$

$= (p-q^2)^n$

ILLUSTRATION 69

Find the greatest integer less than or equal to $(\sqrt{2}+1)^6$.

SOLUTION

Let $(\sqrt{2}+1)^6 = I+f$, where $I \in N$ and $0 < f < 1$ clearly, I is the greatest integer less than or equal to $(\sqrt{2}+1)^6$, Let $f' = (\sqrt{2}-1)^6$ then

$$I+f+f' = \left\{2^6 + {}^6C_2\left(\sqrt{2}\right)^{6-2} + {}^6C_4\left(\sqrt{2}\right)^{6-4} + {}^6C_6\left(\sqrt{2}\right)^{6-6}\right\} \; \{f+f'=1\}$$

$I+1 = 198$

$I = 197$

ILLUSTRATION 70

Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$, where [.] denotes the greatest integer function.

Prove that $Rf = 4^{2n+1}$.

SOLUTION

Greatest integer function is defined as follow:

$[x]$ = integral part of x

$\therefore f = R - [R]$ implies that f is the fraction part of R.

$\therefore 0 < f < 1$

Since $144 > 125 > 121$, $\sqrt{125} = 5\sqrt{5}$ lies between 11 and 12.

$\therefore 0 < 5\sqrt{5} - 11 < 1$ and hence $(5\sqrt{5}-11)^{2n+1}$ will also be a proper fraction

Let $g = (5\sqrt{5}-11)^{2n+1}$

Now $[R] + f - g = R - g$

$= (5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$= 2\left\{{}^{(2n+1)}C_1 (5\sqrt{5})^{2n} . 11^1 + {}^{(2n+1)}C_3 (5\sqrt{5})^{2n-2} \cdot 11^3 + \ldots.\right\}$

= an even integer

Since $[R]$ is an integer, the above implies $f - g = 0$

Hence $Rf = Rg = (5\sqrt{5}+11)^{2n+1} . (5\sqrt{5}-11)^{2n+1}$

$= (125-121)^{2n+1} = 4^{2n+1}$

Hence Proved.

ILLUSTRATION 71

Show that $\left[\left(\sqrt{3}+1\right)^{2n}\right] + 1$ is divisible by 2^{n+2} for all $n \in \mathrm{N}$ where [.] denotes the greatest integer function.

SOLUTION

Let $x = (\sqrt{3}+1)^{2n} = [x] + f$ ...(1)

where $0 < f < 1$

Now let $(\sqrt{3}-1)^{2n} = f'$ ...(2)

where $0 < f' < 1$

Adding (1) and (2) we get

$\Rightarrow [x] + f + f' = (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}$

$= (4+2\sqrt{3})^n + (4-2\sqrt{3})^n$

$= 2^n \{(2+\sqrt{3})^n + (2-\sqrt{3})^n\}$

$= 2^n . 2\{{}^nC_0 (2)^n + {}^nC_2 (2)^{n-2} (\sqrt{3})^2 + {}^nC_4 (2)^{n-4} (\sqrt{3})^4 + \ldots\}$

$\Rightarrow [x] + f + f' = 2^{n+1} k$, ...(3)

(where k is an integer)

Hence $f + f'$ is an integer.

$\Rightarrow$ i.e., $f + f' = 1$ $\{\because 0 < f + f' < 2\}$

From (3),

$[x] + 1 = 2^{n+1} k$

$\left[\left(\sqrt{3}+1\right)^{2n}\right] + 1 = 2^{n+1} k$ [from (1)]

This shows that $[(\sqrt{3}+1)^{2n}] + 1$ divisible by 2^{n+1} for all $n \in \mathrm{N}$.

ILLUSTRATION 72

If n is any positive integer, show that $\left[\left(7+4\sqrt{3}\right)^n\right]$, where [ ] denotes the greatest integer function, is an odd number. Also, if $f = \left(7+4\sqrt{3}\right)^n - \left[\left(7+4\sqrt{3}\right)^n\right]$ then $(7+4\sqrt{3})^n (1-f) = 1$.

SOLUTION

Let $\mathrm{I} = \left[\left(7+4\sqrt{3}\right)^n\right]$

We have $(7+4\sqrt{3})^n = \mathrm{I} + f$, where $0 \le f < 1$.

Also, $7 - 4\sqrt{3} = 1/(7+4\sqrt{3})$

Therefore $0 < (7-4\sqrt{3}) < 1$. This implies that $0 < (7-4\sqrt{3})^n < 1$. Let $f' = (7-4\sqrt{3})^n$

We have $1 + f + f' = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$

$$= 2\left[{}^{n}C_0 .7^{n} + {}^{n}C_2 .7^{n-2}\left(4\sqrt{3}\right)^2 + \ldots\ldots\ldots\right] = 2k$$

where k is some integer. Hence $f + f' = 2k - \text{I}$ is an integer. Also, $0 < f < 1$ and $0 < f' < 1$ implies $0 < f + f' < 2$. But, since $f + f'$ is an integer, we get $f + f' = 1$. Therefore, $2k - \text{I} = 1$ or $\text{I} = 2k - 1$. This shows that I is an odd integer. Also,

$$(7 + 4\sqrt{3})^n (1 - f) = (7 + 4\sqrt{3})^n f'$$

$$= (7 + 4\sqrt{3})^n (7 - 4\sqrt{3})^n = (49 - 48)^n = 1$$

Test Your Concepts-II

(Solutions on page H.149)

1. Consider $E = \left(\sqrt[8]{x} + \sqrt[5]{y}\right)^z = \text{I} + f, 0 \le f < 1$.

(i) If $x = 5, y = 2, z = 100$, then find number of irrational terms in expansion of E

(ii) If $x = 16, y = 1$ and $z = 6$, then prove that $\text{I} = 197$

2. Find the remainder when 2^{100} is divided by 7.

3. If $(7 + 4\sqrt{3})^n = \alpha + \beta$, where α is a positive integer and β is a proper fraction, then find $(1 - \beta)(\alpha + \beta)$.

4. Find the last digit of $(3^P + 2)$ is, where $P = 3^{4n}$ and $n \in \text{N}$.

5. If $6^{83} + 8^{32}$ is divisible by 49, then find the remainder.

6. Find remainder when $(11)^{27} + (21)^{27}$ is divided by 16.

7. Find the remainder, when $(15^{23} + 23^{23})$ is divided by 19.

8. Let $(5 + 2\sqrt{6})^n = p + f$, where $n \in \text{N}$, $p \in \text{N}$ and $0 < f < 1$, then find the value of $f^2 - f + pf - p$.

9. Find the remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9.

10. If the unit digit of $13^n + 7^n - 3^n$, $n \in \text{N}$, is 7, then find the value of n.

11. Prove that the number $(2010)^n - 2(1010)^n - 4(510)^n + 5(10)^n$, $n \in \text{N}$ is always divisible by 10^3.

12. Find last three digits of the number $N = 7^{100} - 3^{100}$.

Answer Keys

1. (i) $1 - (\sqrt{2} - 1)^6$	**2.** 2	**3.** 1	**4.** 5	**5.** 21
6. 0	**7.** 0	**8.** −1	**9.** 2	**10.** $4k + 1$
12. 000				

Numerically Greatest Term

As we know that in binomial expansion each term has three sub-terms binomial coefficient, X^{n-r} and Y^r, so here we are looking for a term who has greatest value numerically

If T_r and T_{r+1} be the r^{th} and $(r + 1)^{\text{th}}$ terms in the expansion of $(X + Y)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^{n}C_r\, X^{n-r}Y^r}{{}^{n}C_{r-1}X^{n-r+1}Y^{r-1}} = \frac{n-r+1}{r}\cdot\frac{Y}{X}$$

Let numerically, T_{r+1} be the greatest term in the above expansion then $T_{r+1} \ge T_r$

$$\text{or } \frac{T_{r+1}}{T_r} \ge 1$$

$$\therefore \frac{n-r+1}{r}\left|\frac{Y}{X}\right| \ge 1 \quad \text{or} \quad r \le \frac{n+1}{1+\left|\dfrac{X}{Y}\right|} \qquad \ldots(1)$$

Now substituting values of n and x in (1), we get

$$r \le m + f \quad \text{or} \quad r \le m$$

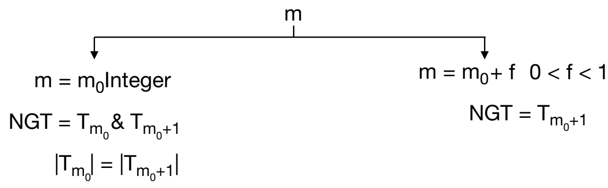
where m is a positive integer and f is a fraction such that $0 < f < 1$.

In the first case T_{m+1} is the greatest term, while in the second case T_m and T_{m+1} are the greatest terms and both are equal.

Working Rule

To find the greatest term (numerically) in the expansion of $(X + Y)^n$

$$\text{Calculate } m = \frac{n+1}{1+\left|\dfrac{X}{Y}\right|} = \frac{n+1}{1+\left|\dfrac{\text{first term}}{\text{last term}}\right|}$$


m
$m = m_0$ Integer
$\text{NGT} = T_{m_0}$ & T_{m_0+1}
$|T_{m_0}| = |T_{m_0+1}|$
$m = m_0 + f \quad 0 < f < 1$
$\text{NGT} = T_{m_0+1}$

For numerically greatest coefficient we should take all variables as 1, in above formula

ILLUSTRATION 73

Find numerically the greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \dfrac{1}{5}$.

SOLUTION

Now, calculate $m = \dfrac{n+1}{1+\left|\dfrac{\text{first term}}{\text{last term}}\right|} = \dfrac{11+1}{1+\left|\dfrac{3}{5\frac{1}{5}}\right|} = 3$

$\therefore$ The numerically greatest terms in the expansion are T_3 and T_4

$$T_3 = {}^{11}C_2 3^9\left(5\left(-\frac{1}{5}\right)\right)^2 = 55\times 3^9$$

$$T_4 = {}^{11}C_3 3^8\left(5\left(-\frac{1}{5}\right)\right)^3 = -55\times 3^9$$

$$|T_3| = |T_4|$$

From above we say that the values of both greatest terms are equal numerically

But if we talk about **algebraically greatest term is** T_3 and **algebraically least term** is T_4.

ILLUSTRATION 74

Find greatest term in the expansion of $(7-4x)^{10}$ when $x = 1$.

SOLUTION

$$m = \frac{n+1}{1+\left|\frac{I}{II}\right|} = \frac{10+1}{1+\left|\frac{7}{-4}\right|} = 4$$

$$T_4 = {}^{10}C_3 7^7(-4)^3 = -120\cdot 4^3\cdot 7^7$$

$$T_5 = {}^{10}C_4 7^6\cdot(-4)^4 = 120\cdot 4^3\cdot 7^7$$

T_4 and T_5 are numerically greatest but T_4 is negative so greatest term is T_5.

ILLUSTRATION 75

Find the index n of the binomial $\left(\dfrac{x}{5}+\dfrac{2}{5}\right)^n$ if the 9th term of the expansion has numerically the greatest coefficient $(n \in \mathrm{N})$.

SOLUTION

Here 9th term has numerically greatest coefficient which does not mean greatest binomial coefficient.

For greatest coefficient, put $x = 1$ and then calculate numerically greatest term.

Here, clearly

$$m = \frac{n+1}{1+\left|\frac{I}{II}\right|} = \frac{n+1}{1+\left|\frac{\frac{1}{5}}{\frac{2}{5}}\right|} = \frac{2(n+1)}{3}$$

As T_9 is numerically greatest term, so $8 < m < 9$

$$\Rightarrow 8 < \frac{2(n+1)}{3} < 9$$

$$\Rightarrow 12 < n+1 < 13.5 \quad \Rightarrow \quad 11 < n < 12.5 \quad \Rightarrow \quad n = 12$$

ILLUSTRATION 76

Find which positive values of x is the fourth term in the expansion of $(5+3x)^{10}$ is greatest.

SOLUTION

Here $x > 0$, Also

$$m = \frac{n+1}{1+\left|\frac{I}{II}\right|} = \frac{11}{1+\left|\frac{5}{3x}\right|} = \frac{11|3x|}{|3x|+5} = \frac{33x}{3x+5} \qquad (\text{As } x > 0)$$

Since, T_4 is greatest. Hence, $3 < m < 4 \Rightarrow 3 < \dfrac{33x}{3x+5} < 4$

$$\Rightarrow 9x + 15 < 33x < 12x + 20$$

$$\Rightarrow \frac{5}{8} < x < \frac{20}{21}$$

ILLUSTRATION 77

Find greatest term in the binomial expansion of $(a+2x)^9$ when $a = 1$ and $x = \dfrac{1}{3}$.

SOLUTION

$$r = \frac{n+1}{1+\left|\frac{I}{II}\right|} = \frac{9+1}{1+\left|\frac{1}{2/3}\right|} = 4$$

$$T_4 = {}^9C_3\left(\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right)^3\frac{9\cdot 8\cdot 7}{6}$$

$$T_5 = {}^9C_4\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^3\frac{2}{3}\cdot\frac{9\cdot 8\cdot 7\cdot 6}{24}$$

$$= \left(\frac{2}{3}\right)^3\frac{9\cdot 8\cdot 7}{6} = T_4$$

Hence for values of 'r' upto 3, $T_{r+1} < T_r$.

When $r = 4$, $T_{r+1} = T_r$ i.e., $T_4 = T_5$ and both of them are the greatest terms.

ILLUSTRATION 78

Show that if the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient then x lies between $\dfrac{n}{n+1}$ and $\dfrac{n+1}{n}$.

SOLUTION

In the expansion of $(1+x)^{2n}$, the middle term is $(n+1)$th term, we know that from binomial expansion, middle term has greatest binomial coefficients.

Now this middle term should be greatest term as mentioned in question

m should not be an integer but it should lie between n and $n+1$

$$m = \frac{2n+1}{1+\left|\frac{1}{x}\right|} \text{ taking } x > 0$$

$$n < \frac{2n+1}{1+\frac{1}{x}} < n+1 \Rightarrow \frac{2n+1}{n} > 1+\frac{1}{x} > \frac{2n+1}{n+1}$$

$$\frac{n+1}{n} > \frac{1}{x} > \frac{n}{n+1} \Rightarrow \frac{n}{n+1} < x < \frac{n+1}{n}$$

MULTINOMIAL THEOREM (FOR A POSITIVE INTEGRAL INDEX)

If n is a positive integer and $x_1, x_2, x_3, \ldots, x_k \in C$ then

$$(x_1+x_2+x_3+\ldots+x_k)^n = \sum \frac{n!}{r_1!r_2!r_3!\ldots r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} \ldots x_k^{r_k}$$

$$0 \le r_1, r_2, r_3, \ldots r_k \le n$$

where $r_1, r_2, r_3, \ldots r_k$ are all non-negative integers subject to the condition

$$r_1 + r_2 + r_3 + \ldots + r_k = n$$

Remark(s)

(i) The number of terms (without adding similar terms) in the multinomial expansion $(x_1+x_2+x_3+\ldots+x_k)^n$ is $^{n+k-1}C_{k-1}$.

(ii) For finding sum of the coefficients in expansion of $(\alpha_1 x + \alpha_2 y + \alpha_3 z + \ldots \alpha_k t)^n$

Put $x = y = \ldots = z = t = 1$ (all variable as 1)

Sum of all the coefficients in given expansion

$= (\alpha_1 + \alpha_2 + \alpha_3 + \ldots \alpha_k)^n$

ILLUSTRATION 79

Find the coefficient of $a^4b^3c^2d$ in the expansion of $(a-b+c-d)^{10}$.

SOLUTION

$$(a-b+c-d)^{10} = \sum \frac{10!(a)^{r_1}(-b)^{r_2}(c)^{r_3}(-d)^{r_4}}{r_1!r_2!r_3!r_4!}$$

$$r_1+r_2+r_3+r_4 = 10$$

For coefficient of $a^4b^3c^2d$ $\quad r_1 = 4, r_2 = 3, r_3 = 2, r_4 = 1$

The coefficient of $a^4b^3c^2d$ in the expansion of

$(a-b+c-d)^{10}$ is $= (-1)^4 \dfrac{10!}{4!3!2!1!} = 12600$

ILLUSTRATION 80

Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$.

SOLUTION

$$(ab+bc+ca)^6 = \sum \frac{6!}{x!y!z!}(ab)^x(bc)^y(ca)^z$$

In this case write $a^3b^4c^5 = (ab)^x (bc)^y (ca)^z$ say

$\therefore a^3b^4c^5 = a^{z+x} \,.\, b^{x+y} \,.\, c^{y+z}$

$\Rightarrow z + x = 3, x + y = 4, y + z = 5$

adding all $2(x+y+z) = 12$

$\therefore x + y + z = 6$

Then $x = 1, y = 3, z = 2$ therefore

The coefficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ or the coefficient of $(ab)^1 (bc)^3 (ca)^2$ in the expansion of $(bc+ca+ab)^6$ is $\dfrac{6!}{1!3!2!} = 60$

ILLUSTRATION 81

Find the number of terms with rational coefficients in the expansion of $\left(\sqrt[3]{5}x+\sqrt{3}y+z\right)^6$

SOLUTION

Any term of the expansion is of the form $= 5^{\frac{p}{3}} \,.\, 3^{\frac{q}{2}}.\, x^p y^q z^r$ where $p, q, r \in I$ and $0 < p, q, r \le 6$ such that $p+q+r = 6$ for rational coefficient

$\therefore p = 0, 3, 6$; $q = 0, 2, 4, 6$ possibilities are (p, q, r)

$= (0, 0, 6), (0, 2, 4), (0, 4, 2), (0, 6, 0), (3, 0, 3) (3, 2, 1) (6, 0, 0)$

$\therefore$ Seven possibilities, hence 7 terms have rational coefficients.

ILLUSTRATION 82

If the number of terms in $\left(x+1+\dfrac{1}{x}\right)^n$ $(n \in I^+)$ is 401, then find n.

SOLUTION

$$\left(x+1+\frac{1}{x}\right)^n = \frac{(1+x+x^2)^n}{x^n} \qquad \ldots(1)$$

$(1+x+x^2)^n$ is of the from $a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$, which contains $(2n+1)$ terms.

So, $\left(x+1+\dfrac{1}{x}\right)^n$ contains $(2n+1)$ terms

$\therefore 2n + 1 = 401 \quad \Rightarrow n = 200$

Alternate

Number of terms in $\left(x+1+\frac{1}{x}\right)^n$ = the number of terms in $\left(x+2+\frac{1}{x}\right)^n$ *hence* $\left(x+2+\frac{1}{x}\right)^n = \left(\sqrt{x}+1/\sqrt{x}\right)^{2n}$ so number of terms = $2n + 1 = 401 \Rightarrow n = 200$

ILLUSTRATION 83

If number of distinct terms in the expansion of $\left(x+\frac{1}{x}+y+\frac{1}{y}\right)^{14}$ is αβγ, where αβγ is a three digits number, then find (α + β + γ).

SOLUTION

$$\left(x+\frac{1}{x}+y+\frac{1}{y}\right)^{14}$$

$$\Rightarrow \quad \left((x+y)+\left(\frac{x+y}{xy}\right)\right)^{14}$$

$$\Rightarrow \quad \frac{(x+y)^{14}(1+xy)^{14}}{(xy)^{14}}$$

Both factors in numerator have 15 independent terms

$\Rightarrow 15 \times 15 = 225$ terms $\quad \Rightarrow \alpha+\beta+\gamma = 2+2+5 = 9$

ILLUSTRATION 84

Find the coefficient of x^7 in the expansion of

$(1 + 3x - 2x^3)^{10}$

SOLUTION

Coefficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$

$$= \sum \frac{10!}{n!n_2!n_3!}(1)^{n_1}(3x)^{n_2}(-2x^3)^{n_3}$$

where $n_1+n_2+n_3 = 10$ and $n_2+3n_3 = 7$, the possible values of n_1, n_2 and n_3 are shown

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

∴ The coefficient of x^7.

$$= \frac{10!}{3!7!0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1$$

$$+\frac{10!}{7!1!2!}(1)^7(3)^1(-2)^2$$

$= 262440 - 204120 + 4320 = 62640.$

ILLUSTRATION 85

Find the coefficient of t^8 in the expansion of $(1 + 2t^2 - t^3)^9$.

SOLUTION

$T_{r+1} = {}^9C_r\, t^{2r}(2-t)^r$

Now coefficient of t^8 appears in $r = 3$, $r = 4$

$r = 3 \Rightarrow {}^9C_3\, t^6(2-t)^3 = {}^9C_3 \times t^6 \times {}^3C_p\, 2^{3-p}(-t)^p$

Put $p = 2 \Rightarrow {}^9C_3 \times {}^3C_2 \times 2^1 \times t^8 = 84 \times 6 \times t^8$

Similarly when $r = 4$.

${}^9C_4\, t^8(2-t)^4 = {}^9C_4 \times t^8 \times 126 \times 16 \times t^8$

Put $p = 0 \Rightarrow {}^9C_4 \times {}^4C_0 \times 2^4 \times t^8 = 126 \times 16 \times t^8$

Hence coefficient of t^8 in the expansion of $(1 + 2t^2 - t^3)^9$ is $84 \times 6 + 126 \times 16 = 504 + 2016 = 2520$

ILLUSTRATION 86

Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

SOLUTION

The number of terms in the expansion of $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$

$$= \frac{(n+3)(n+2)(n+1)}{6}$$

ILLUSTRATION 87

What is the sum of the coefficients of the polynomial after expansion of $(25 - 60x + 22x^2 + 18x^3 - 6x^8)^{2013}$?

SOLUTION

Whenever we are looking for sum all coefficients

Put $x = 1 \Rightarrow$ sum $= (25 - 60 + 22 + 18 - 6)^{2013} = -1$

ILLUSTRATION 88

Find the sum of the coefficients of the expansion

$(ax + by + 3z)^9$

SOLUTION

Put $x = y = z = 1$

⇒ sum of coefficients

$= (a\,.\,1 + b\,.\,1 + 3.\,1)^9 = (a + b + 3)^9$

ILLUSTRATION 89

What is the sum of the coefficients of the polynomial after expansion of $(3 - 4x + 5x^2)^{10}$?

SOLUTION

Here we need sum of all coefficients

Put $x = 1$, sum of coefficients $= (3 - 4 + 5)^{10} = 4^{10}$

BINOMIAL THEOREM FOR NEGATIVE AND FRACTIONAL INDEX

If $n \in R$ then,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \infty$$

Remark(s)

(i) The above expansion is valid for any rational number other than a whole number if $|x| < 1$.

(ii) When the index is a negative integer or a fraction then number of terms in the expansion of $(1 + x)^n$ is infinite, and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) The first terms must be unity in the expansion, when index '*n*' is a negative integer or fraction

$$(x+y)^n = \left[\begin{array}{l} x^n\left(1+\frac{y}{x}\right)^n = x^n\left\{1+n.\frac{y}{x}+\frac{n(n-1)}{2!}\left(\frac{y}{x}\right)^2 \right. \\ \qquad\qquad\qquad\qquad \left. +\ldots\ldots\right\} if\left|\frac{y}{x}\right|<1 \\ y^n\left(1+\frac{x}{y}\right)^n = y^n\left\{1+n.\frac{x}{y}+\frac{n(n-1)}{2!}\left(\frac{x}{y}\right)^2 \right. \\ \qquad\qquad\qquad\qquad \left. +\ldots\ldots\right\} if\left|\frac{x}{y}\right|<1 \end{array}\right.$$

(iv) The general term in the expansion of $(1 + x)^n$ is $T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}x^r$

(v) $\boxed{\text{Coefficient of } x^r \text{ in } (1-x)^{-n} = {}^{n+r-1}C_r \text{, where } n \in N}$

Proof: $T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{r!}(-x)^r$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}(-1)^r(-x)^r$$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}x^r$$

$$= \frac{(n-1)!\,n(n+1)\ldots\ldots(n+r-1)}{(n-1)!\,r!}x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$

(vi) When '*n*' is any rational number other than whole number then approximate value of $(1 + x)^n$ is $1 + nx$ (x^2 and higher powers of x can be neglected)

(vii) Expansions to be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r(r + 1)x^r + \ldots \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r + 1)x^r + \ldots \infty$

ILLUSTRATION 90

The expansion of $\frac{1}{(4-3x)^{1/2}}$ by binomial theorem will be valid, find condition on x.

SOLUTION

The expansion of $\frac{1}{(4-3x)^{1/2}} = (4-3x)^{-1/2}$

$= \frac{1}{2}\left(1-\frac{3x}{4}\right)^{-1/2}$ is valid if $\left|\frac{3x}{4}\right| < 1$

$\Rightarrow |3x| < 4$

$\Rightarrow |x| < \frac{4}{3} \quad \Rightarrow \quad -\frac{4}{3} < x < \frac{4}{3}$

ILLUSTRATION 91

In the expansion of $(1 + x + x^2 + \ldots + x^{27})(1 + x + x^2 + \ldots. + x^{14})^2$, find the coefficient of x^{28}.

SOLUTION

As we know coefficient of x^r in $(1 - x)^{-n}$, $n \in N$ is ${}^{n+r-1}C_r$

Now given product is $= \frac{1-x^{28}}{1-x}\left(\frac{1-x^{15}}{1-x}\right)^2$

$= \frac{(1-x^{28})(1-x^{15})^2}{(1-x)^3} = \frac{(1-x^{28})(1-2x^{15})}{(1-x)^3}$

$= (1 - 2x^{15} - x^{28})(1 - x)^{-3}$

Hence coefficient of x^{28} in expansion of $(1 - 2x^{15} - x^{28})(1 - x)^{-3}$

Coefficient of x^{28} in $(1 - x)^{-3} - 2x^{15}(1 - x)^{-3} - x^{28}(1 - x)^{-3}$

$= {}^{30}C_2 - 2 \cdot {}^{15}C_2 - 1 = 435 - 210 - 1 = 224$

ILLUSTRATION 92

If x is so small such that its square and higher powers may be neglected then find the value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$.

SOLUTION

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right) = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)$$

$$= 1 - \frac{41}{24}x$$

Test Your Concepts-III

(Solutions on page H.150)

1. If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, then find greatest term in the expansion for $x = \frac{1}{2}$.

2. The 4th term in the expansion of $\left(2+\frac{3x}{8}\right)^{10}$ has the maximum numerical value, then find the suitable value of x.

3. In the expansion of $(x + y + z)^{10}$ then show that following

i. The expansion will contains 66 terms

ii. The coefficient of $x^3y^2z^5$ will be $\frac{(10)!}{3!2!5!}$

iii. There cannot be term of $x^3y^2z^4$

iv. $\sum\limits_{\alpha+\beta+\gamma=10} \frac{(10)!}{\alpha!\beta!\gamma!} = 3^{10}$

4. Find the coefficient of $a^3b^4c^7$ in the expansion of $(ab + bc + ca)^8$.

5. Find the number of terms in the expansion of $(a + b + c + d + e + f)^n$.

6. In the expansion of $\left(4x^2 - \frac{3}{x}\right)^{15}$ find

(i) The sum of coefficients

(ii) The sum of binomial coefficients

(iii) The numerically greatest coefficient

7. If $|x| < 1$, then in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \ldots)^{1/2}$, then find the coefficient of x^n.

8. Find the $(r + 1)^{\text{th}}$ term in the expansion of $(1 - x)^{-4}$

9. In the expansion of $\left(\frac{1+x}{1-x}\right)^2$, then find coefficient of x^n.

10. If x is very small and $\dfrac{\left(1+\frac{3x}{4}\right)^{-4}\sqrt{16-3x}}{(8+x)^{2/3}} = P + Qx$, then find P and Q.

11. Find the coefficient of x^r in the expansion of $(1 + 3x + 6x^2 + 10x^3 + \ldots)^2$.

12. Find the coefficient of term that contains $x^{10}y^3z^2$ in the expansion of the binomial $(x^2 - 2y + z)^{10}$.

Answer Keys

1. 5th term

2. $x \in \left[\frac{-64}{21}, -2\right] \cup \left[2, \frac{64}{21}\right]$

4. 280

5. ${}^{n+5}C_n$

6. (i) 1 (ii) 2^{15} (iii) ${}^{15}C_6 2^{18}.3^6$

7. 1

8. $\frac{(r+3)(r+2)(r+1)}{6} x^r$

9. $4n$

10. $P = 1$ and $Q = -\frac{305}{96}$

11. $\frac{(r+5)(r+4)(r+3)(r+2)(r+1)}{5!}$

12. $\frac{10!}{5!3!2!}(-2)^3$

SUM OF THE COEFFICIENT OF A BINOMIAL EXPANSION

If $(x + y)^n = {}^nC_0x^ny^0 + {}^nC_1x^{n-1}y + \ldots. + {}^nC_nx^0y^n$ …(1)

To get sum of binomial coefficients ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$,

we put $x = y = 1$ in eq (1) to get

$(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots.. + {}^nC_n$

Sum of all binomial coefficients = 2^n

$$\boxed{\sum_{r=0}^{n} {}^nC_r = {}^nC_0 + {}^nC_1 + {}^{n-}C_2 + \ldots\ldots\ldots + {}^nC_n = 2^n}$$

Put $x = 1; y = -1$

${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots = 0$

$$\boxed{{}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = 2^{n-1}}$$

Sum of binomial coefficients present at odd places = Sum of binomial coefficients present at even places = 2^{n-1}

ILLUSTRATION 93

If n is an odd natural number, then find $\sum\limits_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$.

SOLUTION

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r} = \sum_{r=0}^{\frac{n+1}{2}} \left\{\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}}\right\}$$

$$= \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left\{ \frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_r} \right\} = \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left[\frac{1}{{}^nC_r} - \frac{1}{{}^nC_r} \right]$$

$$= 0$$

ILLUSTRATION 94

If $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$, then find the value of $\sum\limits_{r=0}^{n}\sum\limits_{s=0}^{n}({}^nC_r + {}^nC_s)$.

SOLUTION

Given, $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x + \ldots + {}^nC_nx^n$ …(1)

Put $x = 1$ in Eq. (1), we get

$$\sum_{r=0}^{n} {}^nC_r = 2^n \qquad \ldots(2)$$

We have,

$$\sum_{r=0}^{n}\sum_{s=0}^{n}({}^nC_r + {}^nC_s) = \sum_{r=0}^{n}\sum_{s=0}^{n}{}^nC_r + \sum_{r=0}^{n}\sum_{s=0}^{n}{}^nC_s$$

$$= \sum_{s=0}^{n}\left(\sum_{r=0}^{n}{}^nC_r\right) + \sum_{r=0}^{n}\left(\sum_{s=0}^{n}{}^nC_s\right)$$

$$= \sum_{s=0}^{n} 2^n + \sum_{r=0}^{n} 2^n \qquad \text{[using Eq. (2)]}$$

$= (n + 1)2^n + (n + 1)2^n$

$= (2n + 2)2^n = (n + 1)2^{n+1}$

ILLUSTRATION 95

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots + C_nx^n$ then prove that $(1.2)\, C_2 + (2.3)C_3 + \ldots ((n - 1) . n)\, C_n = n\,(n - 1)\, 2^{n-2}$

SOLUTION

$$(1.2)\ C_2 + (2.3)C_3 + \ldots ((n-1).n)\ C_n = \sum_{r=2}^{n} r(r-1)\,{}^nC_r$$

$$\left(\because {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2}\right)$$

$$= \sum_{r=2}^{n} r\cdot(r-1)\cdot\frac{n\cdot(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2}$$

$$= n(n-1)\sum_{r=2}^{n}{}^{n-2}C_{r-2}$$

$= n(n - 1)\, ({}^{n-2}C_0 + {}^{n-2}C_1 + \ldots.. + {}^{n-2}C_{n-2}) = n(n - 1)\, 2^{n-2}$

Alternate

The given series is $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$

Differentiating both sides w.r. to x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

Again differentiating both sides w.r t. x, we get

$n\ (n - 1)\ (1 + x)^{n-2} = 0 + 0 + (1.2)\ C_2 + (2.3)\ C_3x + \ldots + ((n - 1).\ n)\ C_nx^{n-2}$.

Putting $x = 1$, then $n\ (n - 1)\ (1 + 1)^{n-2} = (1.2)\ C_2 + (2.3)\ C_3 + \ldots. + (n - 1)\ nC_n$

or $(1.2)\ C_2 + (2.3)\ C_3 + \ldots + (n - 1)\ n\ .\ C_n = n\ (n - 1)\ 2^{n-2}$.

ILLUSTRATION 96

Find the sum of the series

$$\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \ldots\ldots \text{ upto } m \text{ terms} \right)$$

SOLUTION

$$\sum_{r=0}^{n} (-1)^r\ {}^nC_r \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \ldots\ldots \text{ upto } m \text{ terms} \right)$$

$$= \sum_{r=0}^{n} {}^nC_r \left(-\frac{1}{2}\right)^r + \sum_{r=0}^{n} {}^nC_r \left(-\frac{3}{4}\right)^r + \sum_{r=0}^{n} {}^nC_r \left(-\frac{7}{8}\right)^r + \ldots$$

$$= \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \left(1 - \frac{7}{8}\right)^n + \ldots\ldots.. \text{ upto } m \text{ terms}$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \ldots\ldots. \text{ upto } m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left(1 - \left(\frac{1}{2^n}\right)^m\right)}{\left(1 - \frac{1}{2^n}\right)} = \frac{2^{mn} - 1}{2^{mn}\left(2^n - 1\right)}$$

ILLUSTRATION 97

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots + C_nx^n$ then prove that $1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots + n^2.C_n = n\ (n + 1)\ .\ 2^{n-2}$

SOLUTION

$$1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots + n^2.C_n = \sum_{r=1}^{n} r^2\ {}^nC_r$$

$$= \sum_{r=1}^{n} r^2 \cdot \frac{n}{r}\,{}^{n-1}C_{r-1} = n\sum_{r=1}^{n}(r - 1 + 1)^{n-1}C_{r-1}$$

$$= n\left(\sum_{r=1}^{n}(r-1)^{n-1}C_{r-1} + \sum_{r=1}^{n}{}^{n-1}C_{r-1}\right)$$

$$= n\left(\sum_{r=1}^{n}(r-1)\frac{(n-1)}{r-1}\,{}^{n-2}C_{r-2} + \left({}^{n-1}C_0 + {}^{n-1}C_1 + \ldots.. + {}^{n-1}C_{n-1}\right)\right.$$

$$= n\left((n-1)\sum_{r=1}^{n} {}^{n-2}C_{r-2} + 2^{n-1} \right)$$

$= n((n - 1)\, 2^{n-2} + 2^{n-1})$ (neglecting first term of summation)

$= n(2^{n-2}(n - 1 + 2)) = n(n + 1)\, 2^{n-2}$

Alternate

Given $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots. + C_nx^n$

Differentiating both sides w.r. to x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_n\, x^{n-1}$ …(1)

Now multiplying both sides by x in (1) then

$nx(1 + x)^{n-1} = C_1\, x + 2C_2x^2 + 3C_3x^3 + \ldots + nC_nx^n$

Differentiating both sides by w.r t. x, we get

$n\{x.\ (n - 1)\ (1 + x)^{n-2} + (1 + x)^{n-1}.\ 1\} = C_1.1 + 2^2\, C_2x + 3^2\, C_3x^2 + \ldots + n^2C_nx^{n-1}$

putting $x = 1$, then

$n\{1.\ (n - 1)\ .\ 2^{n-2} + 2^{n-1}\} = 1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots + n^2.C_n$

or $1^2.C_1 + 2^2.C_2 + 3^2.C^3 + \ldots. + n^2.C_n = n(n + 1)2^{n-2}$.

ILLUSTRATION 98

If the sum $S = {}^{20}C_2 + 2 \cdot {}^{20}C_3 + 3 \cdot {}^{20}C_4 + \ldots + 19.\ {}^{20}C_{20}$ can be expressed as $1 + p{\cdot}2^q$, then find the least value of $(p + q)$.

SOLUTION

Let $S = {}^{20}C_2 + 2 \cdot {}^{20}C_3 + 3 \cdot {}^{20}C_4 + \ldots + 19 \cdot {}^{20}C_{20}$ …(1)

add ${}^{20}C_0 + {}^{20}C_1 + \ldots + {}^{20}C_{20}$ ($= 2^{20}$) on both sides of equation (1)

$$\begin{array}{l} S + 2^{20} = {}^{20}C_2 + 2\cdot {}^{20}C_3 + 3\cdot {}^{20}C_4 + \ldots + 19\cdot {}^{20}C_{20} \\ \quad + {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + {}^{20}C_3 + {}^{20}C_4 + \ldots + {}^{20}C_{20} \\ \hline S + 2^{20} = 1 + {}^{20}C_1 + 2\cdot {}^{20}C_2 + 3\cdot {}^{20}C_3 + 4\cdot {}^{20}C_4 \\ \qquad + \ldots + 20\cdot {}^{20}C_{20} \end{array} \quad \ldots(2)$$

Now

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

$n(1 + x)^{n-1} = C_1 + 2 \cdot C_2x + 3 \cdot C_3x^2 + \ldots + n \cdot C_nx^{n-1}$

Put $x = 1$ and $n = 20$

$20 \cdot 2^{19} = {}^{20}C_1 + 2\, {}^{20}C_2 + 3\, {}^{20}C_3 + \ldots + 20\, {}^{20}C_{20}$.

Hence from (2), $S + 2^{20} = 1 + 20 \cdot 2^{19}$

$S = 1 + 20 \cdot 2^{19} - 2^{20} = 1 + 10 \cdot 2^{20} - 2^{20} = 1 + 9 \cdot 2^{20}$

$= 1 + p \cdot 2^q$; So, $p = 9$, $q = 20$.

Hence, $(p + q)_{\text{least}} = 9 + 20 = 29$.

ILLUSTRATION 99

If $(1 + x)^n = C_0 + C_1\, x + C_2x^2 + \ldots + C_nx^n$ then prove that

$C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)2^n$.

SOLUTION

$$\text{LHS} = \sum_{r=0}^{n} (2r+1)\,{}^nC_r \;= 2\sum_{r=0}^{n} r\cdot {}^nC_r + \sum_{r=0}^{n} {}^nC_r$$

$$= 2\sum_{r=0}^{n} r\cdot\frac{n}{r}\,{}^{n-1}C_{r-1} + ({}^nC_0 + {}^nC_1 + \ldots.. + {}^nC_n)$$

$$= 2n\sum_{r=1}^{n} {}^{n-1}C_{r-1} + 2^n \;= 2n\cdot 2^{n-1} + 2^n$$

$$= 2^n(n + 1)$$

Alternate

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Now replacing x by x^2 then $(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots + C_nx^{2n}$

multiplying both sides by x^1, we get $x\,(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots. + C_nx^{2n+1}$

then differentiating both sides w.r. to x, we get

$x.n(1 + x^2)^{n-1}.2x + (1 + x^2)^n.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots + (2n + 1)\, C_nx^{2n}$

Putting $x = 1$, then we get

$n\ .\ 2^{n-1}\ .\ 2 + 2^n = C_0 + 3C_1 + 5C_2 + \ldots + (2n + 1)\, C_n$

or $C_0 + 3C_1 + 5C_2 + \ldots. + (2n + 1)\, C_n = (n + 1)\, 2^n$.

ILLUSTRATION 100

If $(1 + x)^n = C_0 + C_1\, x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$ then prove that $C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1)\, C_n = 0$.

SOLUTION

$$\text{LHS} = \sum_{r=0}^{n} (2r+1)\,{}^nC_r(-1)^r \;= \sum_{r=0}^{n} (2r\,{}^nC_r(-1)^r + {}^nC_r(-1)^r)$$

$$= 2\sum_{r=0}^{n} r\times\frac{n}{r}\,{}^{n-1}C_{r-1}(-1)^r + \sum_{r=0}^{n} {}^nC_r(-1)^r$$

$$= 2n\sum_{r=0}^{n} {}^{n-1}C_{r-1}(-1)^r + ({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots)$$

$$= 2n(-{}^{n-1}C_0 + {}^{n-1}C_1 - {}^{n-1}C_2 + \ldots) + 0 = 2n(0) + 0 = 0$$

Alternate

The given series is $(1 + x)^n = C_0 + C_1\, x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$ replacing x by x^2 then $(1+x^2)^n = C_0 + C_1x^2 + C_2x^4 + \ldots + C_nx^{2n}$

Multiplying both sides by x, then

$x\,(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \ldots. + C_nx^{2n+1}$

Differentiating both sides w.r. to x, we get

$x\ .\ n(1 + x^2)^{n-1}\, 2x + (1 + x^2)^n.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \ldots + (2n + 1)\, C_nx^{2n}$

Putting $x = i$ in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \dots + (2n+1)(-1)^n C_n$$

or $\quad C_0 - 3C_1 + 5C_2 - \dots + (-1)n(2n+1)C_n = 0$

ILLUSTRATION 101

If $C_r = {}^nC_r$, then

prove that $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dots\dots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

SOLUTION

$$\text{Sum} = \sum_{r=0}^{n} \frac{{}^nC_r}{r+1} = \sum_{r=0}^{n} \left(\frac{n+1}{r+1}\right) {}^nC_r \left(\frac{1}{n+1}\right)$$

$$= \frac{1}{n+1} \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1} \left({}^{n+1}C_0 + {}^{n+1}C_1 + \dots\dots + {}^{n+1}C_{r+1} - {}^{n+1}C_0\right) = \frac{1}{n+1}\left(2^{n+1} - 1\right)$$

Alternate

Consider the expansion $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ …(1)

Integrating both sides of (1) within limits 0 to 1, we get

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + C_2x^2 + \dots + C_nx^n)\,dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left. C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow \quad \frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

Hence $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

ILLUSTRATION 102

Prove that $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dots + (-1)^n \dfrac{C_n}{n+1} = \dfrac{1}{n+1}$.

SOLUTION

$$\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{(r+1)} = \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r \left(\frac{n+1}{r+1}\right) {}^nC_r$$

$$= \frac{1}{n+1} \left(\sum_{r=0}^{n} {}^{n+1}C_{r+1} (-1)^r\right)$$

$$= \frac{-1}{n+1} \left(\sum_{r=0}^{n} {}^{n+1}C_{r+1} (-1)^{r+1}\right) \qquad \dots(1)$$

$$(1 + x)^{n+1} = \sum_{r=0}^{n+1} {}^{n+1}C_r x^r$$

$$= {}^{n+1}C_0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \dots + {}^{n+1}C_{n+1}x^{n+1}$$

put $x = -1$

$$0 = {}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 + \dots\dots + {}^{n+1}C_{n+1}(-1)^{n+1}$$

$$\Rightarrow -{}^{n+1}C_1 + {}^{n+1}C_2 + \dots\dots + {}^{n+1}C_{n+1}(-1)^{n+1} = -{}^{n+1}C_0 = -1 \qquad \dots(2)$$

so required expression (using (2))

$$= -\frac{1}{n+1}(-1) = \frac{1}{n+1}$$

Alternate

Consider the expansion

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots C_nx^n$ …(1)

Integrating both sides of (1) within limits –1 to 0, we get

$$\int_{-1}^{0} (1+x)^n\,dx = \int_{-1}^{0} (C_0 + C_1x + C_2x^2 + \dots C_nx^n)\,dx$$

$$\Rightarrow \left.\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0} = \left. C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\Rightarrow \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1}$$

Hence proved.

ILLUSTRATION 103

Prove that $\dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \dots = \dfrac{2^n - 1}{n+1}$

SOLUTION

We know that from last two illustrations

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \frac{C_5}{6} + \dots = \frac{2^{n+1}-1}{n+1} \qquad \dots(1)$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_5}{6} + \dots. = \frac{1}{n+1} \qquad \dots(2)$$

Subtracting (2) from (1), we have

$$2\left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots\right) = \frac{2^{n+1}-2}{n+1}$$

Dividing each sides by 2, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots = \frac{2^n - 1}{n+1}.$$

ILLUSTRATION 104

Find the sum $\dfrac{\binom{11}{0}}{1} + \dfrac{\binom{11}{1}}{2} + \dfrac{\binom{11}{2}}{3} + \ldots + \dfrac{\binom{11}{11}}{12}$

$\left(\text{where} \binom{n}{r} \text{denotes } {}^nC_r\right)$

SOLUTION

$$\sum_{r=0}^{11} \frac{{}^{11}C_r}{r+1} = \left(\sum_{r=0}^{11} \frac{12}{r+1} {}^{11}C_r\right)\frac{1}{12} = \frac{1}{12}\left(\sum_{r=0}^{11} {}^{12}C_{r+1}\right)$$

$$= \frac{1}{12}\left[{}^{12}C_1 + {}^{12}C_2 + \ldots. + {}^{12}C_{12} \right]$$

$$= \frac{1}{12}(2^{12} - {}^{12}C_0) = \frac{2^{12}-1}{12}$$

Alternate

Consider the expansion of

$$(1 + x)^{11} = {}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x^2 + \ldots + {}^{11}C_{11}\, x^{11}$$

integrating both sides between 0 and 1, we get

$$\int_0^1 {}^{11}C_0 + {}^{11}C_1 x + {}^{11}C_2 x + \ldots + {}^{11}C_{11} x^{11} dx = \int_0^1 (1+x)^{11}\, dx$$

$$\Rightarrow {}^{11}C_0 x + {}^{11}C_1 \frac{x^2}{2} + {}^{11}C_2 \frac{x^3}{3} \ldots + {}^{11}C_{11} \frac{x^{12}}{12}\Bigg|_0^1 = \frac{(1+x)^{12}}{12}\Bigg|_0^1$$

$$\frac{\binom{11}{0}}{1} + \frac{\binom{11}{1}}{2} + \frac{\binom{11}{2}}{3} + \ldots. + \frac{\binom{11}{11}}{12} = \frac{2^{12}-1}{12}$$

ILLUSTRATION 105

Show that:

$$2\,.\,C_0 + 2^2\,.\,\frac{C_1}{2} + 2^3.\frac{C_2}{3} + \ldots + 2^{n+1}\frac{{}^nC_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

SOLUTION

$$\sum_{r=0}^{n} \frac{{}^nC_r 2^{r+1}}{r+1} = \frac{1}{n+1}\sum_{r=0}^{n} 2^{r+1}\, {}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}((1 + 2)^{n+1} - {}^{n+1}C_0) = \frac{3^{n+1}-1}{n+1}$$

Alternate

Integrating the expansion of $(1 + x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n\, dx = \int_0^2 (C_0 + (C_1 x + \ldots + C_n x^n)\, dx$$

$$\Rightarrow \frac{(1+x)^{n+1}}{n+1}\Bigg|_0^2 = C_0 x + C_1\frac{x^2}{2} + \ldots + C_n\frac{x^{n+1}}{n+1}\Bigg|_0^2$$

$$\Rightarrow 2\,.\,C_0 + 2^2.\frac{C_1}{2} + 2^3.\,\frac{C_2}{3} + \ldots + 2^{n+1}\,\frac{C_n}{n+1} = \frac{3^{(n+1)}-1}{n+1}$$

ILLUSTRATION 106

Find the sum to $(n+1)$ terms of the series $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \dfrac{C_3}{5} + \ldots$

SOLUTION

We have

$$(1-x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \ldots$$

$$\Rightarrow x(1-x)^n = C_0 x - C_1 x^2 + C_2 x^3 - C_3 x^4 \ldots$$

Integrating both side with respect to x from 0 to 1

$$\int_0^1 x(1-x)^n\, dx = \left(\frac{C_0 x^2}{2} - \frac{C_1 x^3}{3} + \frac{C_2 x^4}{4} - \ldots\right)_0^1$$

[replace x by 1-t in LHS]

$$= \int_0^1 \left(t^n - t^{n+1}\right)dt = \left|\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right|_0^1 = \frac{1}{n+1} - \frac{1}{n+2} =$$

$$\frac{1}{(n+1)(n+2)}$$

Thus $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \ldots = \dfrac{1}{(n+1)(n+2)}$

ILLUSTRATION 107

Find the value of $\displaystyle\sum_{r=0}^{50} (-1)^r \frac{{}^{50}C_r}{(r+1)}$.

SOLUTION

$$T_r = (-1)^r \frac{51.{}^{50}C_r}{(r+1)51} = (-1)^r \frac{{}^{51}C_{r+1}}{51}$$

$$\sum_{r=0}^{50} (-1)^r \frac{{}^{51}C_{r+1}}{51} = -\frac{1}{51}(-{}^{51}C_1 + {}^{51}C_2 - \ldots. + {}^{51}C_{51})$$

$$= \frac{-1}{51}(-1) = \frac{1}{51}$$

ILLUSTRATION 108

Prove that $\dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \ldots = \dfrac{2^n}{n+1}$ for $\forall\, n \in N$.

SOLUTION

Consider the expansion $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots. + C_n x^n$ …(1)

Integrating both sides of (1) within limits –1 to 1, we get

$$\int_{-1}^{1}(1+x)^n\,dx = \int_{-1}^{1}(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots + C_nx^n)dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_{-1}^{1}$$

$$= \left. C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \frac{C_3x^4}{4} + \frac{C_4x^5}{5} + \ldots + \frac{C_nx^{n+1}}{n+1}\right|_{-1}^{1}$$

$$\Rightarrow \frac{2^{n+1}-0}{n+1} = \left(C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \ldots + \frac{C_n}{n+1}\right) - \left(-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \frac{C_3}{4} - \frac{C_4}{5} + \ldots + \frac{C_n}{n+1}(-1)^{n+1}\right)$$

$$\frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots.\right)$$

Hence $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \ldots = \frac{2^n}{n+1}$

ILLUSTRATION 109

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, show that

$$\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4}{3.4}C_2 + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

SOLUTION

$$\text{L.H.S} = \sum_{r=0}^{n}\frac{{}^nC_r 2^{r+2}}{(r+1)(r+2)}$$

$$= \frac{1}{(n+2)(n+1)}\sum_{r=0}^{n}\frac{(n+2)(n+1)}{(r+2)(r+1)}\,{}^nC_r\,2^{r+2}$$

$$= \frac{1}{(n+2)(n+1)}\sum_{r=0}^{n}{}^{n+2}C_{r+2}2^{r+2}$$

$$= \frac{1}{(n+2)(n+1)}\left((1+2)^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1 2^1\right]$$

$$= \frac{3^{n+2}-1-2(n+2)}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

Alternate

Given $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ …(1)

Integration both sides of (1) within limits 0 to x we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)dx$$

$$\left.\frac{(1+x)^{n+1}}{(n+1)}\right]_0^x = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \left.\frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \frac{(1+x)^{n+1}-1}{(n+1)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{(n+1)} \quad \ldots(2)$$

Integrating again both sides of (2) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1}-1}{(n+1)}dx = \int_0^2\left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots + \frac{C_nx^{n+1}}{n+1}\right)dx$$

$$\Rightarrow \frac{1}{(n+1)}\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\Bigg]_0^2 = \frac{C_0x^2}{1.2} + \frac{C_1x^3}{2.3} + \frac{C_2x^4}{3.4} + \ldots + \left.\frac{C_nx^{n+2}}{(n+1)(n+2)}\right]_0^2$$

$$\frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\} = \frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4C_2}{3.4} + \ldots + \frac{2^{n+2}C_n}{(n+1)\,(n+2)}$$

Hence,

$$\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4C_2}{3.4} + \ldots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2}-2n-5}{(n+1)\,(n+2)}$$

PRODUCT SERIES OF BINOMIAL COEFFICIENT

(i) Upper index of both binomial coefficients present in series sum are fixed and lower indexes are changing

- **(a) Difference of lower indexes is constant:** Write one expansion in ascending power of x and other expansion as descending power of x, then look for the suitable coefficient of x which represents given series
- **(b) Sum of lower indexes is constant:** Write both the expansion in ascending order of x then look for the suitable coefficient of x which represents given series

ILLUSTRATION 110

Show that $\sum_{r=0}^{n}{}^nC_r^{\,2} = {}^{2n}C_n$

SOLUTION

$$\sum_{r=0}^{n}{}^nC_r^2 = {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots + {}^nC_n^2$$
$$= {}^nC_0\,.\,{}^nC_0 + {}^nC_1\,.\,{}^nC_1 + {}^nC_2\,.\,{}^nC_2 + \ldots + {}^nC_n\,.\,{}^nC_n$$

Hence difference of lower index is constant and zero.

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_{n-r}x^{n-r} + \ldots + {}^nC_nx^n$

$(x + 1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \ldots + {}^nC_n$

${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots + {}^nC_n^2 = {}^nC_0 \,.\, {}^nC_0 + {}^nC_1 \,.\, {}^nC_1 + {}^nC_2 \,.\, {}^nC_2 + \ldots + {}^nC_n \,.\, {}^nC_n$

= coefficient of x^n in $(1 + x)^n (x + 1)^n$

= coefficient of x^n in $(1 + x)^{2n} = {}^{2n}C_n$

$$\sum_{r=0}^{n} {}^nC_r = {}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \ldots + {}^nC_n^2 = {}^{2n}C_n$$

It is very frequently used result.

ILLUSTRATION 111

Find the value of

$$\binom{50}{1}^2 + 2\binom{50}{2}^2 + 3\binom{50}{3}^2 + \ldots + 50\binom{50}{50}^2$$

$$\left(\text{where}\binom{n}{r} = {}^nC_r\right)$$

SOLUTION

Let $S = 0\,({}^{50}C_0)^2 + 1({}^{50}C_1)^2 + 2({}^{50}C_2)^2 + \ldots + 50({}^{50}C_{50})^2$ …(1)

Also $S = 50({}^{50}C_{50})^2 + 49({}^{50}C_{49})^2 + \ldots + 0({}^{50}C_0)^2$ …(2)

(writing in reverse order)

∴ On adding (1) and (2), we get

$2S = 50[({}^{50}C_0)^2 + ({}^{50}C_1)^2 + \ldots + ({}^{50}C_{50})^2]$

$\Rightarrow 2S = 50 \cdot {}^{100}C_{50}$

$[\text{As } ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n]$

$\Rightarrow S = 25 \cdot {}^{100}C_{50}$

ILLUSTRATION 112

Show that:

$$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{(2n)!}{(n-r)!\,(n+r)!}$$

SOLUTION

Here differences of lower suffixes of binomial coefficients in each term is r, i.e.,

$r - 0 = r + 1 - 1 = r + 2 - 2 = \ldots\ldots = \ldots\ldots = n - (n - r) = r$

Given $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-r}\,x^{n-r} + \ldots + C_nx^n$ …(1)

Now $(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_rx^{n-r} + C_{r+1}\,x^{r-r-2} + \ldots + C_n$ …(2)

Multiplying (1) and (2), we get

$(1 + x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots + C_{n-r}x^{n-r} + \ldots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_rx^{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \ldots + C_n)$

Now term $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}\,C_n$ is coefficient of x_{n-r} on L.H.S.

$$= {}^{2n}C_{n-r} = \frac{2n\,!}{(n-r)!(n+r)!}$$

$$\Rightarrow C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n = \frac{2n\,!}{(n-r)!(n+r)!}$$

Alternate

Consider $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_rx^r + C_{r+1}\,x^{r+1} + \ldots + C_nx^n$ and

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_r}{x^r} + \frac{C_{r+1}}{x^{r+1}} + \ldots + \frac{C_n}{x^n}$$

In the product of these two expansions, collecting the coefficient of x^r

$C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n =$ coefficient of x^r in $\dfrac{(1+x)^{2n}}{x^n}$

= coefficient of x^{n+r} in $(1 + x)^{2n}$

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n+r)!(n-r)!}$$

ILLUSTRATION 113

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$ then prove that

$$C_1^{\,2} + 2C_2^{\,2} + 3C_3^{\,2} + \ldots + nC_n^{\,2} = \frac{(2n-1)!}{\left((n-1)!\right)^2}$$

SOLUTION

∵ General term $= nC_n^{\,2}$

Difference of lower indexes is constant

Given $(1 + x)^n = C_0 + C_1x + \ldots\ldots + C_nx^n$

differentiate with respect to x

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_n.x^{n-1}$ …(1)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + \ldots + C_n$ …(2)

Multiplying (1) and (2),

$$n(1+x)^{n-1}.(x+1)^n = (C_1 + 2C_2x + 3C_3x^2 + \ldots + {}^nC_n.x^{n-1})\,(C_0x^n + C_1x^{n-1} + \ldots + C_n)$$

And equate coefficient of x^{n-1} on both sides, we get

$$\Rightarrow \quad C_1^2+2C_2^2+3C_3^2+\ldots+nC_n^2=n\,.\,{}^{2n-1}C_{n-1}$$

ILLUSTRATION 114

Show that $\sum\limits_{r=0}^{n} r(n-r)\,C_r^{\,2}=n^2.\,{}^{2n-2}C_n$.

SOLUTION

As difference of lower indexes is constant

We have

$(1+x)^n=C_0+C^1x+C_2x^2+\ldots+C_nx^n$

Differentiating both sides w.r.t. x, we get

$n(1+x)^{n-1}=C_1+2C_2x+3C_3x^2+\ldots+{}^nC_nx^{n-1}$ …(1)

Now

$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\ldots+C_{n-1}\,x+C_n$

Differentiating both sides w.r.t.x, we get

$n(1+x)^{n-1}=nC_0x^{n-1}+(n-1)C_1\;x^{n-2}+(n-2)C_2x^{n-3}+\ldots+C_{n-1}$ …(2)

Multiplying (1) and (2), we have

$[C_1+2C_2x+3C_3x^2+\ldots+{}^nC_nx^{n-1}][nC_0x^{n-1}+(n-1)C_1x^{n-2}+(n-2)C_2x^{n-3}+\ldots+C_{n-2}x+C_{n-1}]$ …(3)

$=n^2(1+x)^{n-1}\,(x+1)^{n-1}=n^2(1+x)^{2n-2}$

The coefficient of x^{n-2} on the LHS of (3) is

$n^2\,.\,{}^{2n-2}C_{n-2}=n^2\,.\,{}^{2n-2}C_n$

The coefficient of $x^{n-}2$ on the RHS of (5) is

$1.\;(n-1)\,C_1^{\,2}+2.(n-2)\,C_2^{\,2}+\ldots.(n-1)\,.\,1C^2_{\,n-1}$

$$=\sum_{r=0}^{n-1} r(n-r)\,C_r^{\,2}=\sum_{r=0}^{n} r(n-r)\,C_r^{\,2}$$

Hence $\sum\limits_{r=0}^{n} r(n-r)\,C_r^{\,2}=n^2({}^{2n-2}C_n)$

ILLUSTRATION 115

Find $({}^{10}C_1)^2+({}^{10}C_3)^2+({}^{10}C_5)^2+\ldots+({}^{10}C_9)^2$.

SOLUTION

We know that

$({}^{10}C_0)^2+({}^{10}C_1)^2+\ldots+({}^{10}C_{10})^2={}^{20}C_{10}$ …(1)

and $({}^{10}C_0)^2-({}^{10}C_1)^2+\ldots+({}^{10}C_{10})^2={}^{10}C_5\,(-1)^5$ …(2)

(1) – (2)

$\Rightarrow 2[({}^{10}C_1)^2+({}^{10}C_3)^2+\ldots+({}^{10}C_9)^2]={}^{20}C_{10}+{}^{10}C_5$

$$\Rightarrow ({}^{10}C_1)^2+({}^{10}C_3)^2+\ldots+({}^{10}C_9)^2=\frac{{}^{20}C_{10}+{}^{10}C_5}{2}$$

ILLUSTRATION 116

Find $\sum\limits_{r=0}^{15}\left({}^{15}C_r\,{}^{40}C_{15}\,{}^{20}C_r-{}^{35}C_{15}\,{}^{15}C_r\,{}^{25}C_r\right)$

SOLUTION

$${}^{40}C_{15}\sum_{r=0}^{15}{}^{15}C_r\,{}^{20}C_r-{}^{35}C_{15}\sum_{r=0}^{15}{}^{15}C_r\,{}^{25}C_r$$

$$\sum_{r=0}^{15}{}^{15}C_r\cdot{}^{20}C_r=\text{Coefficient of }x^{15}\text{ in expansion of }(1+x)^{35}$$

$$={}^{35}C_{15}$$

$$\sum_{r=0}^{15}{}^{15}C_r\cdot{}^{25}C_r=\text{Coefficient of }x^{25}\text{ in expansion of }(1+x)^{40}$$

$$={}^{40}C_{25}$$

Required sum $={}^{40}C_{15}\,{}^{35}C_{15}-{}^{35}C_{15}\,{}^{40}C_{25}=0$

ILLUSTRATION 117

Find the sum of the series $({}^{100}C_1)^2+2({}^{100}C_2)^2+3({}^{100}C_3)^2+\ldots+100({}^{100}C_{100})^2$

SOLUTION

$S=0\cdot({}^{100}C_0)^2+1\cdot({}^{100}C_1)^2+2\cdot({}^{100}C_2)^2+3\cdot({}^{100}C_3)^2+\ldots+100\cdot({}^{100}C_{100})^2$

$$\begin{array}{l} S=100\cdot({}^{100}C_0)^2+99\cdot({}^{100}C_1)^2+\ldots+0\cdot({}^{100}C_{100})^2\\ \hline 2S=100[({}^{100}C_0)^2+({}^{100}C_1)^2+.({}^{100}C_2)^2+\ldots+({}^{100}C_{100})^2]\end{array}$$

$$2S=100\cdot{}^{200}C_{100}=100\cdot\frac{(200)!}{100!\cdot100!}$$

$$\Rightarrow 2S=\frac{100\cdot200\cdot(199)!}{100\cdot(99)!\cdot(100)!}\quad\Rightarrow S=100\cdot{}^{199}C_{99}$$

ILLUSTRATION 118

Let $S_n=\binom{n}{0}\binom{n}{1}+\binom{n}{1}\binom{n}{2}+\ldots+\binom{n}{n-1}\binom{n}{n}$, where $n\in N$.

If $\dfrac{S_{n+1}}{S_n}=\dfrac{15}{4}$, then find the sum of all possible values of 'n'.

SOLUTION

$$(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_{n-1}x^{n-1}+{}^nC_nx^n$$

$$(x+1)^n={}^nC_0x^n+{}^nC_1x^{n-1}+{}^nC_2x^{n-2}+\ldots+{}^nC_{n-1}x+{}^nC_n$$

$$S_n={}^nC_n.{}^nC_1+{}^nC_{n-1}.{}^nC_2+\ldots+{}^nC_1.{}^nC_n={}^{2n}C_{n+1}$$

$$\text{Now, }\frac{S_{n+1}}{S_n}=\frac{{}^{2n+2}C_{n+2}}{{}^{2n}C_{n+1}}=\left(\frac{2n+2}{n+2}\right)\left(\frac{2n+1}{n+1}\right)\frac{{}^{2n}C_n}{{}^{2n}C_{n+1}}$$

$$=\frac{2(2n+1)}{(n+2)}\left(\frac{n+1}{2n-n-1+1}\right)$$

$$=\frac{2(2n+1)(n+1)}{n(n+2)}=\frac{15}{4}\Rightarrow 8\left(2n^2+3n+1\right)=15\left(n^2+2n\right)$$

$\Rightarrow n^2-6n+8=0 \quad \Rightarrow n=2, 4$

$\therefore$ Sum of all possible $n = 6$.

ILLUSTRATION 119

Evaluate $C_0^2+\frac{C_1^2}{2}+\frac{C_2^2}{3}+....+\frac{C_n^2}{n+1}$

SOLUTION

Given $(1+x)^n = C_0 + C_1x + \ldots + C_nx^n$

Integrating both sides w.r.t x from 0 to x.

So, $\frac{(1+x)^{n+1}-1}{(1+n)}=C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+...+\frac{C_nx^{n+1}}{n+1}$ ...(1)

and $(x+1)^n=C_0x^n+C_1x^{n-1}+...+C_n$ ...(2)

Multiplying (1) and (2), we get $\frac{1}{(n+1)}\left\{(1+x)^{2n+1}-(1+x)^n\right\}$

$$=\left(C_0x+\frac{C_1x^2}{2}+....+\frac{C_nx^{n+1}}{n+1}\right)(C_0x^n+C_1x^{n-1}+....+C_n)$$

Now equate coefficient of x^{n+1} on both sides

$\Rightarrow$ required sum $=\frac{1}{n+1}\left\{{}^{2n+1}C_{n+1}-0\right\}={}^{2n+1}C_{n+1}\cdot\frac{1}{n+1}$

$$=\frac{(2n+1)!}{((n+1)!)^2}$$

ILLUSTRATION 120

Find the sum of the following series

$S=C_1^2+\frac{1+2}{2}C_2^2+\frac{1+2+3}{3}C_3^2+\ldots$ up to n terms.

SOLUTION

The r^{th} term of the given series

$$=\frac{1+2+......+r}{r}C_r^2=\frac{r(r+1)}{2r}C_r^2=\frac{1}{2}(r+1)C_r^2$$

$$\therefore S=\frac{1}{2}(1+1)C_1^2+\frac{1}{2}(2+1)C_2^2+\frac{1}{2}(3+1)C_3^2+\ldots+\frac{1}{2}(n+1)C_n^2$$

We know that

$C_0+C_1x+C_2x^2+\ldots+C_nx^n=(1+x)^n$

$\Rightarrow C_0x+C_1x^2+C_2x^3+\ldots+C_nx^{n+1}=x(1+x)^n$

Differentiating both sides w.r.t. x we get

$C_0+2C_1x+3C_2x^2+4C_3x^3+\ldots+(n+1)C_nx^n$

$=(1+x)^n+nx(1+x)^{n-1}$ ...(1)

Also $C_0+C_1\left(\frac{1}{x}\right)+C_2\left(\frac{1}{x}\right)^2+C_3\left(\frac{1}{x}\right)^3+\ldots+C_n\left(\frac{1}{x}\right)^n$

$=\left(1+\frac{1}{x}\right)^n$ ...(2)

Now, $C_0^2+2C_1^2+3C_2^2+4C_3^2+\ldots+(n+1)C_n^2$ = coefficient of constant term in

$[C_0+2C_1x+3C_2x^2+4C_3x^3+\ldots+(n+1)C_nx^n]\times$

$$\left[C_0+C_1\left(\frac{1}{x}\right)+C_2\left(\frac{1}{x}\right)^2+....+C_n\left(\frac{1}{x}\right)^n\right]$$

= coefficient of constant term in

$[(1+x)^n+nx(1+x)^{n-1}](1+1/x)^n$

= coefficient of x^n in

$[(1+x)^n+nx(1+x)^{n-1}](x+1)^n$

= coefficient of x^n in $[(1+x)^{2n}+nx(1+x)^{2n-1})]$

$$={}^{2n}C_n+n\,.\,{}^{2n-1}C_{n-1}=\frac{(2n)!}{n!\,n!}\left(1+\frac{n}{2}\right)$$

$$={}^{2n}C_n\left(1+\frac{n}{2}\right)$$

$$\Rightarrow 2C_1^2+3C_2^2+4C_3^2+\ldots+(n+1)C_n^2={}^{2n}C_n\left(1+\frac{n}{2}\right)-1$$

$$\Rightarrow S=\frac{1}{2}\left[{}^{2n}C_n\left(1+\frac{n}{2}\right)-1\right]$$

ILLUSTRATION 121

Evaluate ${}^mC_r+{}^mC_{r-1}.{}^nC_1+{}^mC_{r-2}.{}^nC_2+\ldots+{}^nC_r$.

SOLUTION

Sum of lower indexes = r

Consider $(1+x)^m={}^mC_0+{}^mC_1x+\ldots+{}^mC_rx^r+\ldots+{}^mC_m.x^m$ ...(1)

$(1+x)^n={}^nC_0+{}^nC_1x+\ldots{}^nC_rx^r+\ldots+{}^nC_n.x^n$ ...(2)

Multiply (1) and (2)

$(1+x)^m(1+x)^n=({}^mC_0+\ldots+{}^mC_rx^r+\ldots+{}^mC_mx^m)$

$({}^nC_0+\ldots+{}^nC_r.x^r+....+{}^nC_n.x^n)$ ...(a)

Now equate coefficients of x^r on sides in equation (a)

$\Rightarrow \quad {}^mC_r.{}^nC_0+{}^mC_{r-1}{}^nC_1+\ldots+{}^mC_0.{}^nC_r$

= coefficient of x^r in $(1+x)^{m+n}$

Required sum = ${}^{m+n}C_r$.

(II) WHEN UPPER INDEX IS VARIABLE

ILLUSTRATION 122

Find value of

${}^nC_0\,.\,{}^{2n}C_r-{}^nC_1\,.\,{}^{2n-2}C_r+\ldots=2^{2n-r}\,{}^nC_{r-n}$

SOLUTION

${}^nC_0\,.\,{}^{2n}C_r-{}^nC_1\,.\,{}^{2n-2}C_r+\ldots$

= coefficient of x^r in ${}^nC_0.(1+x)^{2n}-{}^nC_1.(1+x)^{2n-2}+\ldots$

= coefficient of x^r in ${}^nC_0.((1+x)^2)^n-{}^nC_1.((1+x)^2)^{n-1}+\ldots$

= coefficient of x^r in $[(1+x)^2-1]^n$

= coefficient of x^r in $[2x+x^2]^n = x^n\,(2+x)^n$

If $r > n$ then required expression = coefficient of x^{r-n} in $(2+x)^n$

$= {}^{n}C_{r-n}\, 2^{2n-r}$ if $r > n$

If $r < n$ then required expression = 0

ILLUSTRATION 123

Prove that ${}^{n}C_0 . {}^{2n}C_n - {}^{n}C_1\, {}^{2n-2}C_n + {}^{n}C_2 . {}^{2n-4}C_n - \ldots = 2^n$.

SOLUTION

LHS

$$= \text{co-efficient of } x^n \text{ in}\left[\,{}^{n}C_0(1+x)^{2n} - {}^{n}C_1(1+x)^{2n-2} + {}^{n}C_2(1+x)^{2n-4} - \ldots\right]$$

$$= \text{co-efficient of } x^n \text{ in}\left[\,{}^{n}C_0\left((1+x)^2\right)^n - {}^{n}C_1\left((1+x)^2\right)^{n-1} + {}^{n}C_2\left((1+x)^n\right)^{n-2} - \ldots\right]$$

$$= \text{co-efficient of } x^n \text{ in}\left[(1+x)^2-1\right]^n$$

$$= \text{co-efficient of } x^n \text{ in}\left(2x+x^2\right)^n$$

$$= \text{co-efficient of } x^n \text{ in } x^n(2+x)^n = 2^n$$

ILLUSTRATION 124

Find the value of the expression ${}^{n}C_0\, {}^{4n}C_n - {}^{n}C_1\, {}^{4n-3}C_n + \ldots$ to $n + 1$ terms.

SOLUTION

${}^{n}C_0\, {}^{4n}C_n - {}^{n}C_1\, {}^{4n-3}C_n + {}^{n}C_2\, {}^{4n-6}C_n - \ldots\ldots + (-1)^n\, {}^{n}C_n\, {}^{4n-3n}C_n$

= coefficient of x^n in $[{}^{n}C_0\,(1+x)^{4n} - {}^{n}C_1\,(1+x)^{4n-3} + {}^{n}C_2\,(1+x)^{4n-6} - \ldots(-1)^n\,{}^{n}C_n(1+x)^n]$

= coefficient of x^n in $(1+x)^n\,[{}^{n}C_0\,(1+x)^{3n} - {}^{n}C_1\,(1+x)^{3n-3} + {}^{n}C_2\,(1+x)^{3n-6} - \ldots.(-1)^n\,{}^{n}C_n]$

= coefficient of x^n in $(1+x)^n\,[(1+x)^3-1]^n$

= coefficient of x^n in $(1+x)^n\,[x^3+3x^2+3x]^n = 3^n$

ILLUSTRATION 125

Find the value of ${}^{30}C_0 . {}^{20}C_{10} + {}^{31}C_1 . {}^{19}C_{10} + {}^{32}C_2 . {}^{18}C_{10} + \ldots - + {}^{40}C_{10} . {}^{10}C_{10}$.

SOLUTION

${}^{30}C_0 . {}^{20}C_{10} + {}^{31}C_1 . {}^{19}C_{10} + {}^{32}C_2 . {}^{18}C_{10} + \ldots + {}^{40}C_{10} . {}^{10}C_{10}$

$= {}^{30}C_{30} . {}^{20}C_{10} + {}^{31}C_{30} . {}^{19}C_{10} + {}^{32}C_{30} . {}^{18}C_{10} + \ldots + {}^{40}C_{30} . {}^{10}C_{10}$

Coefficient of x^{30} in $(1+x)^{30} . {}^{20}C_{10} + (1+x)^{31} . {}^{19}C_{10} + (1+x)^{32} . {}^{18}C_{10} + \ldots + (1+x)^{40} . {}^{10}C_{10}$

Coefficient of x^{40} in $(1+x)^{30} . (1+x)^{20} + (1+x)^{31} . (1+x)^{19} + (1+x)^{32} . (1+x)^{18} + \ldots + (1+x)^{40} . (1+x)^{10}$

Required expression = coefficient of x^{40} in $11.(1+x)^{50} = 11.{}^{50}C_{40}$

ILLUSTRATION 126

If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ then find

(i) $a_0 - a_2 + a_4 - a_6 + \ldots\ldots.$ (ii) $a_1 - a_3 + a_5 - a_7 + \ldots\ldots.$

SOLUTION

$(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$

Substitute $x = i$

$$(1+i)^n = a_0 + a_1i + a_2i^2 + a_3i^3 + a_4i^4 + a_5i^5 + a_6i^6 \ldots\ldots + a_ni^n$$

$$\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n = a_0 + a_1i - a_2 - a_3i + a_4 + a_5i - a_6 \ldots\ldots + a_n$$

$$\Rightarrow 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) = (a_0 - a_2 + a_4 - a_6 + \ldots..) + i\,(a_1 - a_3 + a_5 - a_7 + \ldots..)$$

Now compare real and imaginary parts

$$a_0 - a_2 + a_4 - a_6 + \ldots.. = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

$$a_1 - a_3 + a_5 - \ldots\ldots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

ILLUSTRATION 127

Given that $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n$, where x is complex number and $C_0, C_1, \ldots, C_n$ are constants then find

(i) $C_0 + C_3 + C_6 + \ldots$ **(ii)** $C_1 + C_4 + C_7 + \ldots$

(iii) $C_2 + C_5 + C_8 + \ldots$

SOLUTION

(i) $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \ldots$

Put $x = 1, \omega, \omega^2$ and adding we get,

$$3(C_0 + C_3 + C_6 + \ldots.) = 2^n + (1+\omega)^n + (1+\omega^2)^n$$

$$= 2^n + (-\omega^2)^n + (-\omega)^n$$

$$[\because 1 + \omega + \omega^2 = 0]$$

$$-\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

$$-\omega^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\therefore C_0 + C_3 + C_6 + \ldots . = \frac{1}{3}\left[2^n + \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right] = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

(ii) $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots .$

Multiply both sides by x^2

$$x^2(1+x)^n = C_0x^2 + C_1x^3 + C_2x^4 + C_3x^5 + C_4x^6 + C_5x^7 + C_6x^8 + C_7x^9 + \ldots \quad \ldots(1)$$

Now put $x = 1, \omega, \omega^2$ in (1) we get,

$$1.2^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + \ldots$$

$$\omega^2(1+\omega)^n = C_0\omega^2 + C_1 + C_2\omega + C_3\omega^2 + C_4 + C_5\omega + C_6\omega^2 + C_7 + \ldots$$

$$\omega\left(1+\omega^2\right)^n = C_0\omega + C_1 + C_2\omega^2 + C_3\omega + C_4 + C_5\omega^2 + C_6\omega + C_7 + \ldots$$

Then adding we get,

$$2^n + \omega^2(1+\omega)^n + \omega\left(1+\omega^2\right)^n = 3(C_1 + C_4 + C_7 + \ldots)$$

$$\therefore 3(C_1 + C_4 + C_7 + \ldots .) = 2^n + e^{\frac{4\pi i}{3}}.e^{\frac{n\pi i}{3}} + e^{\frac{4\pi i}{3}}.e^{\frac{-n\pi i}{3}}$$

$$\left[\because\ \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}; \quad \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{\frac{4\pi i}{3}}\right]$$

$$= 2^n + \omega^2\left(-\omega^2\right)^n + \omega(-\omega)^n$$

$$= 2^n + e^{-i2\pi/3} \cdot e^{in\pi/3} + e^{i2\pi/3}e^{-in\pi/3}$$

$$= 2^n + e^{i(n-2)\pi/3} + e^{-i(n-2)\pi/3}$$

$$\therefore C_1 + C_4 + C_7 + \ldots = \frac{1}{3}\left[2^n + 2\cos\frac{n-2}{3}\pi\right]$$

(iii) $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + \ldots$

Multiply by x

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + C_4x^5 + C_5x^6 + C_6x^7 + C_7x^8 + C_8x^9 + \ldots$$

Put x = 1, ω, ω^2 we get

$$1.2^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + \ldots$$

$$\omega(1+\omega)^n = C_0\omega + C_1\omega^2 + C_2 + C_3\omega + C_4\omega^2 + C_5 + C_6\omega + C_7\omega^2 + C_8 + \ldots .$$

$$\omega^2(1+\omega^2)^n = C_0\omega^2 + C_1\omega + C_2 + C_3\omega^2 + C_4\omega + C_5 + C_6\omega^2 + C_7\omega + C_8 + \ldots .$$

Adding these we get,

$$3.(C_2 + C_5 + C_8 + \ldots) = 2^n + \omega(1+\omega)^n + \omega^2\left(1+\omega^2\right)^n$$

$$= 2^n + \omega\left(-\omega^2\right)^n + \omega^2(-\omega)^n$$

$$= 2^n + e^{i2\pi/3}e^{in\pi/3} + e^{-i2\pi/3}e^{-in\pi/3}$$

$$= 2^n + e^{\frac{(n+2)\pi i}{3}} + e^{-\frac{(n+2)\pi i}{3}}$$

$$= 2^n + 2\cos\frac{(n+2)\pi}{3}$$

$$\therefore C_2 + C_5 + C_8 + \ldots = \frac{1}{3}\left[2^n + 2\cos(n+2)\frac{\pi}{3}\right]$$

ILLUSTRATION 128

If n is a positive integer and if $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r\, x^r$ then, prove that

(i) $a_r = a_{2n-r}$ for $0 \le r \le 2n$.

(ii) $a_0 + a_1 + \ldots\ldots\ldots + a_{n-1} = \frac{1}{2}(3^n - a_n)$

(iii) $(r + 1)a_{r+1} = (n - r)a_r + (2n - r + 1)a_{r-1}$ $(0 < r < 2n)$

SOLUTION

(i) We have

$$\sum_{r=0}^{2n} a_r\left(\frac{1}{x}\right)^r = \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \frac{1}{x^{2n}}(x^2 + x + 1)^n$$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^{2n-r} = (x^2 + x + 1)^n = \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_{2n-r} x^{2n-r}.$$

On comparison we have $a_r = a_{2n-r}$

Equating the coefficients of x^{2n-r} on both sides, we get $a_r = a_{2n-r}$ for $0 \le r \le 2n$.

(ii) Putting $x = 1$ in (1), we get

$a_0 + a_1 + a_2 + \ldots + a_{2n} = (1 + 1 + 1)^n = 3^n$

But $a_r = a_{2n-r}$ for $0 \le r \le n - 1$.

$\therefore 2(a_0 + a_1 + \ldots\ldots\ldots\ldots a_{n-1}) + a_n = 3^n$

$\Rightarrow a_0 + a_1 + \ldots\ldots\ldots\ldots\ldots a_{n-1} = \frac{1}{2}(3^n - a_n)$

(iii) Differentiating (1) w.r.t. x, we get

$$n(1 + 2x)(1 + x + x^2)^{n-1} = \sum_{r=0}^{2n} ra_r x^{r-1}$$

$$\Rightarrow n(1 + 2x)(1 + x + x^2)^n = (1 + x + x^2)\sum_{r=0}^{2n} ra_r x^{r-1}$$

$$\Rightarrow n(1 + 2x)\sum_{r=0}^{2n} a_r x^r = (1 + x + x^2)\sum_{r=0}^{2n} ra_r x^{r-1}$$

Equating the coefficients of x^r $(0 < r < 2n)$, we get $na_r + 2na_{r-1} = (r+1)a_{r+1} + ra_r + (r-1)a_{r-1}$

$\Rightarrow (r+1)a_{r+1} = (n-r)\,a_r + (2n-r+1)\,a_{r-1}$

ILLUSTRATION 129

If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 \ldots + a_{2n}\,x^{2n}$ then show that $a_0 + a_3 + a_6 + \ldots = a_1 + a_4 + a_7 + \ldots = a_2 + a_5 + a_8 + \ldots = 3^{n-1}$.

SOLUTION

Putting $x = 1, \omega, \omega^2$, where ω is a non real cube root of unity,

$3^n = a_0 + a_1 + a_2 + a_3 + a_4 + \ldots$ ...(1)

$0 = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 +$ ...(2)

because $1 + \omega + \omega^2 = 0$

$0 = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + \ldots$ ...(3)

Adding these

$3^n = 3(a_0) + a_1\,(1 + \omega + \omega^2) + a_2\,(1 + \omega^2 + \omega^4) + a_3\,(1 + \omega^3 + \omega^6) + \ldots = 3\,(a_0 + a_3 + a_6 + \ldots)$

$\therefore a_0 + a_3 + a_6 + \ldots = 3^{n-1}$.

From (1) + (2) × ω^2 + (3) × ω, we get

$3^n + 0 \times \omega^2 + 0 \times \omega$

$= a_0\,(1 + \omega^2 + \omega) + a_1\,(1 + \omega^3 + \omega^3) + a_2\,(1 + \omega^4 + \omega^5) + a_3\,(1 + \omega^5 + \omega^7) + a_4\,(1 + \omega^6 + \omega^9) + \ldots$

$\therefore 3^n = 3\,(a_1 + a_4 + a_7 + \ldots)$, (using $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$)

$\therefore a_1 + a_4 + a_7 + \ldots = 3^{n-1}$

Again, from (1) + (2) × ω + (3) × ω^2, we get

$3^n = a_0\,(1 + \omega + \omega^2) + a_1\,(1 + \omega^2 + \omega^4) + a_2\,(1 + \omega^3 + \omega^6) + \ldots$

$= 3\,(a_2 + a_5 + a_8 + \ldots)$

$(a_2 + a_5 + a_8 + \ldots) = 3^{n-1}$

Test Your Concepts-IV

(Solutions on page H.151)

1. Find the sum of all coefficients in $(2x^2 - 3x + 1)^{11}$

2. Find the sum of the coefficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$.

3. If $\sum_{r=1}^{10} r(r-1)\,{}^{10}C_r = k \cdot 2^9$, then find k.

4. If ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are binomial coefficients of different terms in the expansion of $(1 + x)^n$, $(n \in N)$, then find

$$\left(1+\frac{{}^nC_1}{{}^nC_0}\right)\left(1+\frac{{}^nC_2}{{}^nC_1}\right)\left(1+\frac{{}^nC_3}{{}^nC_2}\right)\cdots\left(1+\frac{{}^nC_n}{{}^nC_{n-1}}\right)$$

5. If the sum of the coefficients in the expansion of $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ vanishes, then find the value of α

6. If $(1 + x - 3x^2)^{2145} = a_0 + a_1x + a_2x^2 + \ldots$, then last digit of $a_0 - a_1 + a_2 - a_3 + \ldots$

7. Find the coefficient of x^{49} in the polynomial

$$\left(x-\frac{{}^{50}C_1}{{}^{50}C_0}\right)\left(x-2^2\cdot\frac{{}^{50}C_2}{{}^{50}C_1}\right)\left(x-3^2\cdot\frac{{}^{50}C_3}{{}^{50}C_2}\right)\cdots\left(x-50^2\cdot\frac{{}^{50}C_{50}}{{}^{50}C_{49}}\right)$$

8. If $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$, then find the value of $\sum_{r=0}^{n}\sum_{s=0}^{n} {}^nC_r \cdot {}^nC_s$

9. Find the value of $\sum_{r=0}^{s}\sum_{s=1}^{n} {}^nC_s \cdot {}^sC_r$

10. If $1 + (1 + x) + (1 + x)^2 + \ldots. + (1 + x)^n = \sum_{k=0}^{n} a_kx^k$, prove that

(i) $\sum_{k=0}^{n} a_k = 2^{n+1} - 1$ (ii) $a_{n-2} = \frac{n(n+1)}{2}$

11. Find the value of ${}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \ldots + {}^{950}C_0$

12. Find the value of $\sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \frac{1 + r\log_e 10}{(1 + \log_e 10^n)^r}$

13. Find the coefficient of x^9 in the polynomial given by, $(x + 1)(x + 2) \ldots (x + 10) + (x + 2)(x + 3) \ldots (x + 11) + \ldots + (x + 11)(x + 12) \ldots (x + 20)$.

14. In the expansion of $(1 + x)^n (1 + y)^n (1 + z)^n$, find sum of the coefficients of the terms of degree r.

15. If $\sum_{k=0}^{100}\left(\frac{k}{k+1}\right){}^{100}C_k = \frac{a(2^{100}) + b}{c}$, where a, b and $c \in N$, then find the number of digits in least value of $(a + b + c)$.

16. Find the value of $S = \sum_{r=0}^{n-1} \frac{1}{(n-r)^2}\left(\frac{{}^nC_{r+1}}{{}^nC_r}\right)^2$.

17. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1 + a_2x^2 + \ldots + a_{20}x^{20}$, then find a_1.

18. If $(1 + x - 2x^2)^6 = a_0 + a_1x + a_2x^2 + \ldots + a_{12}x^{12}$, then find value of $a_2 + a_4 + a_6 + \ldots + a_{12}$.

19. If $T_0, T_1, T_2, \ldots, T_n$ represents the terms in the expansion of $(x + a)^n$, then find the value of $(T_0 - T_2 + T_4 - T_6 \ldots)^2 + (T_1 - T_3 + T_5 \ldots)^{20}$.

Answer Keys

1. 0

2. 3×6^{10}

3. 45

4. $\dfrac{(n+1)^n}{n!}$

5. 1

6. 3

7. −22100

8. $2^n \times 2^n$

9. $3^n - 1$

11. $^{1001}C_{50}$

12. 0

13. 1155

14. $\sum\limits_{p,q,s \geq n} (^nC_p)(^nC_q)(^nC_s)$

15. 3

16. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2}$

17. 20

18. 31

19. $(x^2 + a^2)^n$

SOLVED PROBLEMS

PROBLEM 1

(i) If $a_n = \left(1+\frac{1}{n}\right)^n$ then prove that $2 < a_n < 3\ n\ \forall\ N$.

(ii) Using binomial theorem, prove the inequality $n^{n+1} > (n+1)^n;\ n \ge 3,\ n \in N$

(iii) Using binomial theorem, prove the inequality $n^n > (n+1)^{n-1};\ n \ge 3,\ n \in N$.

SOLUTION

(i) We have $a_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{1.2}.\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}.\frac{1}{n^3} + \ldots$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \ldots$$

= 2 + positive quantity $(\because\ n \in N)$

$\therefore\ a_n > 2$ …(1)

Also

$$a_n = \left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right)$$

$$a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} \ldots + \frac{1}{1.2.3.\ \ldots.n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}$$

$$\left(\because \frac{1}{2.3} < \frac{1}{2.2}; \frac{1}{2.3.4} < \frac{1}{2.2.2}, \text{etc.}\right)$$

$$< 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\left(1-\frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}} < 3 \text{ when } n \ge 2$$

$a_n < 3$ …(2)

From (1) and (2), we get

$2 < a_n < 3$

(ii) From part (i), $\left(1+\frac{1}{n}\right)^n < 3$

$\Rightarrow \left(1+\frac{1}{n}\right)^n < n \quad (\because\ n \ge 3)$

$\Rightarrow \frac{(n+1)^n}{n^n} < n$

$\Rightarrow (n+1)^n < n^{n+1}$

(iii) $\frac{(n+1)^n}{n^n} < n$

$\Rightarrow n^n > \frac{(n+1)^n}{n} > \frac{(n+1)^n}{n+1} \left(\text{since } \frac{1}{n} > \frac{1}{n+1}\right)$

$\Rightarrow n^n > \frac{(n+1)^n}{(n+1)}$

$\Rightarrow n^n > (n+1)^{n-1}$

PROBLEM 2

Evaluate $\sum_{r=1}^{i} \frac{{}^nC_{2r-1}}{r}$ where $i = \left[\frac{n+1}{2}\right]$,

i.e., greatest integer $\le \frac{n+1}{2}$

SOLUTION

Here $t_r = \frac{{}^nC_{2r-1}}{r} = 2\ .\ \frac{{}^nC_{2r-1}}{2r}$

$= 2.\frac{1}{n+1}\,{}^{n+1}C_{2r}$ $\left\{\because \frac{1}{r+1}\,{}^nC_r = \frac{1}{n+1}.\,{}^{n+1}C_{r+1}\right\}$

$\therefore$ the required sum $= \frac{2}{n+1}\sum_{r=1}^{i}\ {}^{n+1}C_{2r}$

$$= \frac{2}{n+1}\{{}^{n+1}C_2 + {}^{n+1}C_4 + {}^{n+1}C_6 + \ldots + {}^{n+1}C_{2i}\}$$

$$= \frac{2}{n+1}\{({}^{n+1}C_0 + {}^{n+1}C_2 + {}^{n+1}C_4 + \ldots) - {}^{n+1}C_0\}$$

$$= \frac{2}{n+1}\{2^{(n+1)-1} - 1\} = \frac{2\left(2^n - 1\right)}{n+1}$$

Alternate

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n$

$$\therefore \int_0^x (1+x)^n\,dx = \int_0^x ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n)$$

dx or $\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^x$

$$= \left[{}^nC_0x + \frac{{}^nC_1x^2}{2} + \frac{{}^nC_2x^3}{3} + \frac{{}^nC_3x^4}{4} + \ldots\ldots + \frac{{}^nC_nx^{n+1}}{n+1}\right]_0^x$$

$$\therefore \frac{(1+x)^{n-1}}{n+1}-\frac{1}{n+1}={}^{n}C_0x+\frac{{}^{n}C_1x^2}{2}+\frac{{}^{n}C_2x^3}{3}+\frac{{}^{n}C_3x^4}{4}+\ldots$$

Putting $x = 1, -1$, we get $\frac{2^{n+1}}{n+1}-\frac{1}{n+1}$

$$={}^{n}C_0+\frac{{}^{n}C_1}{2}+\frac{{}^{n}C_2}{3}+\frac{{}^{n}C_3}{4}+\frac{{}^{n}C_4}{5} \qquad \ldots(1)$$

$$0-\frac{1}{n+1}=-{}^{n}C_0+\frac{{}^{n}C_1}{2}-\frac{{}^{n}C_2}{3}+\frac{{}^{n}C_3}{4}-\frac{{}^{n}C_4}{5}+ \qquad \ldots(2)$$

Adding these, $\frac{2^{n+1}}{n+1}-\frac{2}{n+1}$

$$=2\left(\frac{{}^{n}C_1}{2}+\frac{{}^{n}C_3}{4}+\frac{{}^{n}C_5}{6}+\ldots\ldots\right)$$

$$\therefore \frac{{}^{n}C_1}{1}+\frac{{}^{n}C_3}{2}+\frac{{}^{n}C_5}{3}+\ldots\ldots=\frac{2\left(2^n-1\right)}{n+1}$$

PROBLEM 3

Show that

$$C_0-C_1\frac{1+x}{1+nx}+C_2\frac{1+2x}{(1+nx)^2}-C_3\frac{1+3x}{(1+nx)^3}+\ldots=0$$

SOLUTION

We have

$$C_0-C_1\frac{1+x}{1+nx}+C_2\frac{1+2x}{(1+nx)^2}-C_3\frac{1+3x}{(1+nx)^3}+\ldots$$

$$=\left[C_0-C_1\frac{1}{1+nx}+C_2\frac{1}{(1+nx)^2}-C_3\frac{1}{(1+nx)^3}+\ldots..\right]+\frac{x}{1+nx}$$
$$\left[-C_1+2C_2\frac{1}{1+nx}-3C_3\frac{1}{(1+nx)^2}+\ldots.\right] \qquad \ldots(1)$$

As we know

$(1-y)^n = C_0 - C_1y + C_2y^2 - C_3y^3 + \ldots$

On differentiating both sides w.r.t. y,

$-n(1-y)^{n-1} = -C_1 + 2C_2y - 3C_3y^2 + \ldots$

Thus $C_0-C_1\frac{1}{1+nx}+C_2\frac{1}{(1+nx)^2}$

$$-C_3\frac{1}{(1+nx)^3}+\ldots=\left(1-\frac{1}{1+nx}\right)^n=\left(\frac{nx}{1+nx}\right)^n \qquad \ldots(2)$$

and $-C_1+2C_2\frac{1}{1+nx}-3C_3\frac{1}{1+nx}+\ldots$

$$=-n\left(1-\frac{1}{1+nx}\right)^{n-1}=-n\left(\frac{nx}{1+nx}\right)^{n-1} \qquad \ldots(3)$$

Using these values in equation (1),

$$C_0-C_1\frac{1+x}{1+nx}+C_2\frac{1+2x}{(1+nx)^2}-C_3\frac{1+3x}{(1+nx)^3}+\ldots=$$

$$\left(\frac{nx}{1+nx}\right)^n-\frac{x}{1+nx}.n\left(\frac{nx}{1+nx}\right)^{n-1}=\left(\frac{nx}{1+nx}\right)^n-\left(\frac{nx}{1+nx}\right)^n=0$$

PROBLEM 4

Prove that

$$\frac{1}{m!}C_0+\frac{n}{(m+1)!}C_1+\frac{n(n-1)}{(m+2)!}C_2+\ldots+\frac{n(n-1)\ldots\ldots.2.1}{(m+n)!}C_n$$
$$=\frac{(m+n+1)(m+n+2)\ldots\ldots..(m+2n)}{(m+n)!}$$

SOLUTION

Since $(1+x)^{m+n} . (1+x)^n = (1+x)^{m+2n}$

$\Rightarrow ({}^{m+n}C_0 + {}^{m+n}C_1x + {}^{m+n}C_2x^2 + \ldots\ldots\ldots + {}^{m+n}C_{m+n}x^{m+n}) \times$

$(C_0 + C_1x + C_2x^2 + \ldots + C_nx^n) = (1+x)^{m+2n}$

Equating the coefficients of x^n on both sides, we find

$${}^{m+n}C_n . C_0 + {}^{m+n}C_{n-1} . C_1 + {}^{m+n}C_{n-2} . C_2 + \ldots\ldots\ldots + {}^{m+n}C_0 . C_n$$
$$= {}^{m+2n}C_n$$

$$\Rightarrow \frac{(m+n)!}{m!\ n!}C_0+\frac{(m+n)!}{(n-1)!(m+1)!}C_1+\frac{(m+n)!}{(n-2)!(m+2)!}C_2$$

$$+\ldots+\frac{(m+n)!}{(m+n)!}C_n=\frac{(m+2n)!}{(m+n)!\ n!}$$

Dividing both sides by $(m+n)!/n!$, we find

$$\frac{1}{m!}C_0+\frac{n}{(m+1)!}C_1+\frac{n(n-1)}{(m+2)!}C_2+\ldots+\frac{n(n-1)\ldots\ldots..2.1}{(m+n)!}C_n$$

$$=\frac{(m+2n)!}{(m+n)!\,(m+n)!}=\frac{(m+n+1)(m+n+2)\ldots\ldots(m+2n)}{(m+n)!}$$

PROBLEM 5

Let $\sum_{r=0}^{100}\sum_{s=0}^{100}(C_r^2+C_s^2+C_r.C_s) = m({}^{2n}C_n) + 2^p$ where m, n and p are even natural numbers and C_r represents the coefficient of x^r in the expansion of $(1+x)^{100}$. Find the value of $(m+n+p)$.

SOLUTION

Consider $\sum_{r=0}^{100}\sum_{s=0}^{100}(C_r^2+C_s^2+C_r.C_s)$

$$= \sum_{r=0}^{100}\sum_{s=0}^{100}(1\times C_r^2 + 1\times C_s^2 + C_r.C_s)$$

$$= \left(\sum_{s=0}^{100} 1\right)\left(\sum_{r=0}^{100} C_r^2\right) + \left(\sum_{r=0}^{100} 1\right)\left(\sum_{s=0}^{100} C_s^2\right) + \left(\sum_{r=0}^{100} C_r\right)\left(\sum_{s=0}^{100} C_s\right)$$

$= (101)\,(C_0^2 + C_1^2 + \ldots + C_{100}^2) + (101)\,(C_0^2 + C_1^2 + C_2^2 + \ldots + C_{100}^2)$

$\quad + (C_0 + C_1 + C_2 + \ldots + C_{100})\,(C_0 + C_1 + \ldots + C_{100})$

$= 2 \times (101) \times {}^{200}C_{100} + (2^{100})\,(2^{100}) = m({}^{2n}C_n) + 2^p$

So, $m = 202$; $n = 100$; $P = 200$

T = 202 + 100 + 200 = 502

PROBLEM 6

If $(1 + 2x + 2x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$

Show that $a_0\,a_{2n} - a_1a_{2n-1} + \ldots +$

$$a_{2n}a_0 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^n\left({}^nC_{n/2}\right) & \text{if } n \text{ is even} \end{cases}$$

SOLUTION

We have

$(1 + 2x + 2x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$ ...(1)

Replacing x by $-x$ in (1), we get

$(1 - 2x + 2x^2)^n = a_0 - a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$ ...(2)

Now, $a_0\,a_{2n} - a_1a_{2n-1} + a_2\,a_{2n-2} - \ldots + a_{2n}a_0$

= coefficient of x^{2n} in $[a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}] \times [a_0 - a_1x + a_2x^2 - \ldots + a_{2n}x^{2n}]$

= coefficient of x^{2n} in $(1 + 2x + 2x^2)^n\,(1 - 2x + 2x^2)^n$

= coefficient of x^{2n} in $[(1 + 2x^2)^2 - (2x)^2]^n = (1 + 4x^4)^n$

= coefficient of y^n in $(1 + 4y^2)^n$

If n is odd, y^n will not occur in the expansion of $(1 + 4y^2)^n$ so coefficient of y^n in $(1 + 4y^2)^n$ is 0.

If n is even, as $2m$, then coefficient of y^{2m} in $(1+ 4y^2)^{2m}$ is ${}^{2m}C_m\,4^m = {}^{2m}C_m 2^{2m} = (2^n)\,({}^nC_{n/2})$

PROBLEM 7

Let $f(x) = 1 - x + x^2 - x^3 + \ldots + x^{16} - x^{17} = a_0 + a_1(1 + x) + a_2(1 + x)^2 + \ldots + a_{17}(1 + x)^{17}$. Find the value of a_2.

SOLUTION

We have,

$f(x) = a_0 + a_1(1 + x) + a_2(1 + x)^2 + \ldots$

$f'(x) = a_1 + 2a_2(1 + x) + 3a_3(1 + x)^2 + \ldots$

$f''(x) = 2a_2 + 6a_3\,(1 + x) + \ldots$

$f''(-1) = 2a_2$

Now $f(x) = 1 - x + x^2 - x^3 + \ldots + x^{16} - x^{17}$

$f'(x) = -1 + 2x - 3x^2 + \ldots + 16x^{15} - 17x^{16}$

$f''(x)]_{x=-1} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + 16 \cdot 17$

$$= \sum_{r=1}^{16} r(r+1) = \sum_{r=1}^{16} r^2 + \sum_{r=1}^{16} r$$

$$= \frac{16\cdot 17\cdot 33}{6} + \frac{16\cdot 17}{2} = \frac{16\cdot 17}{2}[11 + 1]$$

$f''(-1) = 16 \cdot 17 \cdot 6 \quad \Rightarrow 2a_2 = 16 \cdot 17 \cdot 6$

Hence, $a_2 = 16 \cdot 51 = 816$.

PROBLEM 8

If $(1+x)^n = \sum_{r=0}^{n} C_r\ x^r$, find the value of

(i) $\sum\limits_{0 \le r < s \le n} (C_r \cdot C_s)$ **(ii)** $\sum\sum\limits_{o \le r < s \le n} (C_r - C_s)^2$

(iii) $\sum\sum\limits_{0 \le r < s \le n} (C_r + C_s)^2$

SOLUTION

(i) We have

$(C_0 + C_1 + C_2 + \ldots\ldots C_n)^2$

$$= \sum_{r=0}^{n} C_r^2 + 2\sum\sum_{0 \le r < s \le n} C_rC_s$$

$$\Rightarrow 2\sum\sum_{0 \le r < s \le n} C_rC_s = (2^n)^2 - {}^{2n}C_n$$

$$\Rightarrow \sum\sum_{o \le r < s \le n} C_rC_s = 2^{2n-1} - \frac{1}{2}\,{}^{2n}C_n$$

(ii) $\sum\sum\limits_{o \le r < s \le n} (C_r - C_s)^2$

$= (C_0 - C_1)^2 + (C_0 - C_2)^2 + \ldots\ldots(C_0 - C_n)^2 \quad +$

$\quad (C_1 - C_2)^2 + (C_1 - C_3)^2 + \ldots\ldots(C_1 - C_n)^2 + \ldots\ldots$

$$= n\left(C_0^2 + C_1^2 + \ldots\ldots C_n{}^2\right) - 2\sum\sum_{0 \le r < s \le n} C_rC_s$$

$= n.\ {}^{2n}C_n - 2^{2n} + {}^{2n}C_n$ (using result of (i))

$= (n+1)\ {}^{2n}C_n - 2^{2n}$

(iii) $$\sum\sum_{0 \le r < s \le n} (C_r + C_s)^2 = n\left(\sum_{i=o}^{n} C_i^2\right) + 2\sum\sum_{0 \le r < s \le n} C_rC_s$$

$= (n-1)\ {}^{2n}C_n + 2^{2n}$

PROBLEM 9

Let $\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 .... + a_{2n}x^{2n}$, where n is odd integer, if

$S_1 = a_0 + a_4 + a_8 + ...$

$S_2 = a_1 + a_5 + a_9 + ...$

$S_3 = a_2 + a_6 + a_{10} + ...$

$S_4 = a_3 + a_7 + a_{11} + ...$ then find relation between them.

SOLUTION

$\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Put $x = i, -i$

$$i^n = a_0 + a_1 i - a_2 - a_3 i + a_4 + a_5 i + \ldots \qquad \ldots(1)$$

$$(-i)^n = a_0 - a_1 i - a_2 - a_3 i + a_4 - a_5 i + \ldots \qquad \ldots(2)$$

$$(1) + (2) \Rightarrow 2(a_0 - a_2 + a_4 - a_6 + \ldots) = 0 \qquad (\therefore n \text{ is odd})$$

$\Rightarrow a_0 + a_4 + a_8 + \ldots = a_2 + a_6 + \ldots \Rightarrow S_1 = S_3$

(1) – (2) and equating real parts

$a_1 - a_3 + a_5 - a_7 + \ldots = 0$

$a_1 + a_5 + a_9 + \ldots = a_3 + a_7 + \ldots \Rightarrow S_2 = S_4$

PROBLEM 10

Show that

(i) If $x^2 < 1$, the coefficient of x^n in $\dfrac{(1+2x)^n + (1+3x)^n}{(1-x)}$ is $3^n + 4^n$.

(ii) The coefficient of x^2 in $(1 + x + x^2)^{10}$ is ${}^{11}C_2$.

SOLUTION

(i) We have, $\dfrac{(1+2x)^n + (1+3x)^n}{1-x}$

$= (1 + 2x)^n (1 - x)^{-1} + (1 + 3x)^n (1 - x)^{-1}$

$= [1 + {}^nC_1(2x) + {}^nC_2(2x)^2 + \ldots][1+x+x^2 + \ldots] + [1+{}^nC_1(3x) + {}^nC_2(3x)^2 + \ldots][1 + x + x^2 + \ldots]$

Coefficient of x^n is

$(1 + {}^nC_1 2 + {}^nC_2 2^2 + {}^nC_3 2^3 + \ldots + {}^nC_n 2^n) + (1 + {}^nC_1 3 + {}^nC_2 3^2 + \ldots + {}^nC_n 3^n)$

$\Rightarrow (1 + 2)^n + (1 + 3)^n = 3^n + 4^n$

(ii) We have, $(1 + x + x^2)^{10} \Rightarrow \left(\dfrac{1-x^3}{1-x}\right)^{10}$

$= (1 - x^3)^{10} (1 - x)^{-10}$

$= (1 - {}^{10}C_1x^3 + {}^{10}C_2x^6 + \ldots) (1 - x)^{-10}$

Coefficient of x^2 is depend on $(1 - x)^{-10}$

$\because$ Coefficient of x^2 of $(1 - x)^{-10} = {}^{2+10-1}C_2 = {}^{11}C_2$

($\because$ Coefficient of x^r in $(1 - x)^{-r} = {}^{n+r-1}C_r$)

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If the coefficient of $b^{12}c^6d^{15}\ e^{4n}$ in the expansion of $\left(a^7+b^3+2c^2+d^5+e^3\right)^{26}$ is non-zero, then its value is

   (A) $\frac{26!}{64.20!}$ (B) $\frac{26!}{108.16!}$

   (C) $\frac{26!}{96.16!}$ (D) $\frac{26!}{36.20!}$

2. If $\{x\}$ represents the fractional part of x, then $\left\{\frac{5^{200}}{8}\right\}$ is

   (A) $\frac{1}{4}$ (B) $\frac{1}{8}$

   (C) $\frac{3}{8}$ (D) $\frac{5}{8}$

3. The value of $\sum_{r=1}^{12}{}^{24}C_r$ is equal to

   (A) $2^{23}-\frac{1}{2}\left({}^{24}C_{12}\right)+1$ (B) $2^{24}-{}^{24}C_{12}-1$

   (C) $2^{23}+\frac{1}{2}\left({}^{24}C_{12}\right)-1$ (D) $2^{23}-1$

4. The coefficient of x^{50} in the expansion of
   $(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots..+1001\,x^{1000}$

   (A) ${}^{1000}C_{50}$
   (B) ${}^{1001}C_{50}$

   (C) ${}^{1002}C_{50}$

   (D) 2^{1001}

5. If P_n denotes the product of all the coefficients of $(1+x)^n$ and $9!P_{n+1}=10\ {}^{9}P_n$ then n is equal to

   (A) 10 (B) 9
   (C) 19 (D) none of these

6. Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum 99^n+1 is

   (A) 3 (B) 4
   (C) 2 (D) None of these

7. If $C_0, C_1, C_2, \ldots\ldots\ldots C_n$ are the Binomial coefficients in the expansion $(1+x)^n$. 'n' being even, then $C_0+(C_0+C_1)+(C_0+C_1+C_2)+\ldots\ldots(C_0+C_1+C_2+\ldots.+C_{n-1})$ is equal to

   (A) $n2^n$ (B) $n.2^{n-1}$

   (C) $n.2^{n-3}$ (D) $n.2^{n-2}$

8. If $\sum_{r=0}^{n}\left\{\frac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}\right\}^3=\frac{25}{24}$, then n is equal to

   (A) 3 (B) 4
   (C) 5 (D) 6

9. The value of ${}^{12}C_2+{}^{13}C_3+{}^{14}C_4+\ldots\ldots\ldots\ldots+{}^{999}C_{989}$ is

   (A) ${}^{1000}C_{11}-12$ (B) ${}^{1000}C_{11}+12$

   (C) ${}^{999}C_{11}-12$ (D) ${}^{1000}C_{989}$

10. The sum $S_n=\sum_{k=0}^{n}(-1)^K.{}^{3n}C_k$, where $n=1,2,\ldots\ldots\ldots$ is

    (A) $(-1)^n.{}^{3n-1}C_{n-1}$ (B) $(-1)^n.{}^{3n-1}C_n$

    (C) $(-1)^n.{}^{3n-1}C_{n+1}$ (D) None of these

11. The coefficient of x^9 in $\left(x-{}^{21}C_0\right)\left(x-{}^{21}C_1\right)\left(x-{}^{21}C_2\right)\ldots\ldots\left(x-{}^{21}C_{10}\right)$ is

    (A) $2^{40}-\frac{1}{2}\,{}^{42}C_{20}$ (B) $2^{39}-\frac{1}{2}\,{}^{42}C_{21}$

    (C) $2^{40}-{}^{42}C_{20}$ (D) $2^{39}-\frac{1}{4}\,{}^{42}C_{21}$

12. $\sum_{k=1}^{10}\frac{(-1)^{k-1}}{k}.\left({}^{10}C_k\right)=$

    (A) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+----+\frac{1}{11}$

    (B) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+----+\frac{1}{10}$

    (C) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+----+\frac{1}{9}$

    (D) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+----+\frac{1}{12}$

13. The remainder when 27^{40} is divided by 12 is

    (A) 3 (B) 7
    (C) 9 (D) 11

14. If $\sum_{r=0}^{2n}a_r(x-1)^r=\sum_{r=0}^{2n}b_r(x-2)^r$ and $b_r=(-1)^{r-n}$ for all $r\geq n$, then $a_n=$

    (A) ${}^{2n+1}C_{n-1}$ (B) ${}^{3n}C_n$

    (C) ${}^{2n+1}C_n$ (D) 0

15. $\sum_{r=0}^{n} \frac{n-3r+1}{n-r+1} \frac{{}^nC_r}{2^r}$ is equal to

(A) $\frac{1}{2^n}$ (B) $\frac{1}{3^n}$

(C) $\frac{1}{4^n}$ (D) $\frac{1}{2^n}+1$

16. The greatest integer less than or equal to $\left(\sqrt{2}+1\right)^6$ is

(A) 194 (B) 195

(C) 196 (D) 197

17. The coefficient of x^6 in the expansion of $\left(1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\frac{x^5}{\underline{|5}}\right)^2$ is

(A) $\frac{2}{15}$ (B) $\frac{4}{15}$

(C) $\frac{31}{360}$ (D) $\frac{2}{45}$

18. Coefficient of x^{2009} in $(1 + x + x^2 + x^3 + x^4)^{1001}$ $(1 – x)^{1002}$ is

(A) 0 (B) $4.\ {}^{1001}C_{501}$

(C) –2009 (D) none of these

19. Coefficient of x^6 in $\left((1+x)\left(1+x^2\right)^2\left(1+x^3\right)^3 ..... \left(1+x^n\right)^n\right)$ is

(A) 26 (B) 28

(C) 30 (D) 35

20. The coefficient of x^2 in $(1+x)(1+2x)(1+4x)..... \left(1+2^n x\right)$ is

(A) $\frac{\left(2^n+1\right)\left(2^n+2\right)}{3}$ (B) $\frac{\left(2^{n+1}-1\right)\left(2^{n+1}-2\right)}{3}$

(C) $\frac{\left(2^{n+1}+1\right)\left(2^{n+1}+2\right)}{3}$ (D) $\frac{\left(2^{n+1}+1\right)\left(2^{n+1}-2\right)}{5}$

21. Let $n \in N$ *and* $n<\left(5\sqrt{3}+8\right)^4$. Then the greatest value of n is

(A) 77040 (B) 77042

(C) 77041 (D) none of these

22. The co-efficient of x^{100} in the expansion of $(1+x)(1-2x)(1+3x)(1-4x)....(1+101x)$ is

(A) $101!\sum_{r=1}^{101}\frac{(-1)^r}{r}$ (B) $101!\sum_{r=1}^{101}\frac{(-1)^{r-1}}{r}$

(C) $101!\sum_{r=0}^{101}\frac{1}{r+1}$ (D) $\sum_{r=1}^{101}\frac{(-1)^r.r}{r+1}$

23. If x = 111 .... 1 (20 digits), y = 333 ...... 3 (10 digits) and z = 222 ..... 2 (10 digits) then $x – z$ equal to

(A) y^2 (B) $2y^2$

(C) y^3 (D) $3y^3$

24. The coefficient of abc^3de^2 in the expansion of $(a+b+c+d+e)^8$ is equal to

(A) 3630 (B) 3600

(C) 3360 (D) none of these

25. If each coefficient in the expansion of the expression $x(1+x)^n\ (n \in N)$ in powers of 'x' is divided by the exponent of corresponding power, then the sum of the values thus obtained is equal to

(A) $\frac{2^n}{n+1}$ (B) $\frac{2^n-1}{n+1}$

(C) $\frac{2^n+1}{n+1}$ (D) $\frac{2^{n+1}-1}{n+1}$

26. If $x^{15}-x^{13}+x^{11}-x^9+x^7-x^5+x^3-x=7$ then

(A) x^{16} is 15

(B) x^{16} is less than 15

(C) x^{16} greater than 15

(D) Nothing can be said about x^{16}

27. If $n > 3$ and $a, b \in R$, then the value of

$ab-n(a-1)(b-1)+\frac{n(n-1)}{1.2}(a-2)$
$(b-2)-.....+(-1)^n(a-n)(b-n)$ is equal to

(A) a^n+b^n (B) $\frac{a^n-b^n}{a-b}$

(C) $(ab)^n$ (D) 0

28. The expression $\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7-\left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$ is a polynomial in x of degree

(A) 7 (B) 5

(C) 4 (D) 3

29. Co-efficient of a^t in the expansion of,
$(a + p)^{m-1} + (a + p)^{m-2} (a + q) + (a + p)^{m-3} (a + q)^2 + ... (a + q)^{m-1}$

(A) $\frac{{}^mC_t\left(p^t-q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t}-q^{m-t}\right)}{p-q}$

(C) $\frac{{}^mC_t\left(p^t+q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t}+q^{m-t}\right)}{p-q}$

30. The number of distinct terms in the expansion of $\left(x+y^2\right)^{13}+\left(x^2+y\right)^{14}$ is

(A) 27 (B) 29

(C) 28 (D) 25

31. $3\ {}^n\mathrm{C}_0+10\ {}^n\mathrm{C}_1+28\ {}^n\mathrm{C}_2+72\ {}^n\mathrm{C}_3+......(n+1)$ terms =

(A) $n(14n – 4) + 3$

(B) $2^{n-2}+4n+14n(n-1)-1$

(C) $3^{n-1}(4n+9)$

(D) $3^{n+1}+(n-1)2^n$

32. $\sum_{m=1}^{n}\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m} {}^{n}C_m \cdot {}^{m}C_p \cdot {}^{p}C_k\right)\right) =$

(A) $3^n - 2^n$
(B) $4^n - 3^n$
(C) $3^n + 2^n$
(D) $4^n - 1$

33. The value of ${}^{2000}C_2 + {}^{2000}C_5 + {}^{2000}C_8 + \dots\dots + {}^{2000}C_{2000} = ?$

(A) $\frac{2^{1999}-1}{3}$
(B) $\frac{2^{1999}+1}{3}$
(C) $\frac{2^{2000}+1}{3}$
(D) $\frac{2^{2000}-1}{3}$

34. $3^{10}\ {}^{20}C_0\ {}^{20}C_{10} - 3^9 \cdot {}^{20}C_1\ {}^{19}C_9 + \dots. + {}^{20}C_{10}\ {}^{10}C_0$ equals (${}^{n}C_r$ denote coefficient of x^r in $(1 + x)^n$.)

(A) ${}^{10}C_{10}.3^{10}$
(B) ${}^{20}C_{10}.3^{10}$
(C) ${}^{20}C_{10}.2^{10}$
(D) ${}^{20}C_{10}\ {}^{10}C_8$

35. $\sum_{r=1}^{n}\sum_{p=0}^{r-1} {}^{n}C_r . {}^{r}C_p . 2^p$ is equal to

(A) $4^n - 3^n + 1$
(B) $4^n - 3^n - 1$
(C) $4^n - 3^n + 2$
(D) $4^n - 3^n$

36. The value of $\binom{50}{6} - \binom{5}{1}\binom{40}{6} + \binom{5}{2}\binom{30}{6} - \binom{5}{3}\binom{20}{6} + \binom{5}{4}\binom{10}{6}$ where $\binom{n}{r}$ denotes ${}^{n}C_r$ is

(A) 15625
(B) 0
(C) 1000000
(D) 2250000

37. The sum of the rational terms in the expansion of $\left(\sqrt{2} + \sqrt[5]{3}\right)^{10}$ is:

(A) 41
(B) 42
(C) 39
(D) 45

38. Value of $\sum_{K=1}^{\infty}\sum_{r=0}^{K} \frac{1}{3^K}\left({}^{K}C_r\right)$ is

(A) 4/3
(B) 2
(C) 1
(D) 2/3

39. Let $f(x) = a_0 + a_1x + a_2x^2 + \dots.. + a_nx^n + \dots..$ and $\frac{f(x)}{1-x} = b_0 + b_1x + b_2x^2 + \dots.. + b_nx^n + \dots..$ then

(A) $b_n + b_{n-1} = a_n$
(B) $b_n - b_{n-1} = a_n$
(C) $\frac{b_n}{b_{n-1}} = a_n$
(D) None

40. The coefficient of x^4 of the expansion $(1 + 5x + 9x^2 + \dots\dots \infty)(1 + x^2)^{11}$ is

(A) ${}^{11}C_2 + 4.{}^{11}C_1 + 3$
(B) ${}^{11}C_2 + 3.{}^{11}C_1 + 4$
(C) $3.{}^{11}C_2 + 4.{}^{11}C_1 + 3$
(D) 171

41. If the expression of $\left(1 + x + x^2\right)^n$ be written as $a_0 + a_1x + a_2x^2 + \dots\dots.. + a_{2n}x^{2n}$, then the value of $\frac{a_0 + a_1 + a_3 + a_4 + a_6 + a_7 + \dots..}{a_2 + a_5 + a_8 + \dots\dots.}$

(A) 1
(B) 4
(C) 0
(D) 2

42. If $C_0, C_1, C_2 \dots..$ are binomial coefficients in the expansion $\sum_{r=0}^{n} C_r x^r$, then value of the expression (series) $\frac{2C_0}{1} + \frac{3C_1}{2} + \frac{4C_2}{3} + \frac{5C_3}{4} + \dots\dots +$ is

(A) $\frac{2^n + 1}{n+1}$
(B) $\frac{2^n - 1}{n+1}$
(C) $\frac{2^n(n+3) - 1}{n+1}$
(D) $\frac{2^n(n+2) - 1}{n+1}$

43. If $\left(3 + x^{2008} + x^{2009}\right)^{2010} = a_0 + a_1x + \dots + a_nx^n$ then the value of $a_0 - \frac{1}{2}a_1 - \frac{1}{2}a_2 + a_3 - \frac{1}{2}a_4 - \frac{1}{2}a_5 + a_6 \dots\dots$ is_____

(A) 0
(B) 3^{2010}
(C) 2^{2010}
(D) 1

44. If $C_0, C_1, C_2, \dots., C_n$ are Binomial Coefficients, such that $S_n = \sum_{r=0}^{n} \frac{1}{{}^{n}C_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{{}^{n}C_r}$ then $\frac{t_n}{S_n}$ equals

(A) $\frac{n}{2}$
(B) $\frac{n(n+1)}{2}$
(C) $\frac{n+1}{2}$
(D) None of these

45. If $(1 + x)^n = \sum_{r=0}^{n} C_r x^r$, then the value of $3C_1 + 7C_2 + 11C_3 + \dots. + (4n - 1)C_n$ is

(A) $(4n - 1)2^n$
(B) $(2n - 1)2^n$
(C) $(2n - 1)2^n + 1$
(D) $(4n - 1)2^n - 1$

46. Let $\left(1+x^2\right)^2(1+x)^n = a_0 + a_1x + a_2x^2 + \dots$ *if* a_0, a_1, a_2 are in A.P., then the largest value of n is

(A) 3
(B) 5
(C) 4
(D) 2

47. The term independent of x in $\left(1 + x + x^{-2} + x^{-3}\right)^{10}$ is n then the last digit of n is

(A) 1
(B) 3
(C) 7
(D) 9

48. If the number of terms in $\left(1 + x^{-1} + x^{-2}\right)^n$ is 53, then the largest prime p so that $n!$ is divisible by 5^{p-1} is

(A) 3
(B) 5
(C) 7
(D) 11

49. If $\sum_{r=0}^{n}\left\{a_r(x-y+2)^r - b_r(y-x-1)^r\right\} = 0 \ \forall\ n \in N$, then b_n is equal to

(A) a_n (B) $(-1)^n a_n$

(C) $(-1)^n 2^n a_n$ (D) $2^n a_n$

50. The value of $\sum_{r=0}^{15} {}^{15}C_r\left(r-\frac{15}{2}\right)^2$ is

(A) $2^{10}.15$ (B) $2^{12}.15$

(C) $2^{13}.15$ (D) $2^{15}.15$

51. If period of $f(x) = |\sec 4\pi x| + |\operatorname{cosec} 6\pi x|$ is P, then coefficient of x^{-24P} in expansion of $\left(x+\frac{b}{x^3}\right)^{32P}$ is

(A) ${}^{12}C_3 b^3$ (B) ${}^{12}C_9 b^9$

(C) ${}^{16}C_7 b^7$ (D) None of these

52. The coefficient of x^{20} in $(1 + x)(1 + 2x)(1 + 4x)(1 + 8x)$ .... $(1 + 20^{20}x)$ is

(A) $2^{211} - 2^{190}$

(B) $2^{191} - 2^{171}$

(C) $2^{231} - 2^{209}$

(D) $2^{161} - 2^{142}$

53. If a_1, a_2, a_3, a_4 are any 4 consecutive binomial coefficients of the expansion of $(1+x)^n$ respectively, then $\frac{a_1}{a_1+a_2}, \frac{a_2}{a_2+a_3}, \frac{a_3}{a_3+a_4}$ are in

(A) A.G.P (B) G.P

(C) H.P (D) A.P

54. If the sum of the coefficients in the expansion $(x - 2y + 3z)^n$ is 128, then the greatest coefficient in $(1+x)^n$ is

(A) 25 (B) 35

(C) 45 (D) 31

55. Geometric mean of all the divisors of ${}^{2011}C_1 + 2.{}^{2011}C_2 + 3.{}^{2011}C_3 + 4.{}^{2011}C_4 + ... + 2011.{}^{2011}C_{2011}$ is

(A) $\sqrt{2011}$ (B) $2^{1005}\sqrt{2011}$

(C) 2^{1005} (D) 2011

56. If $C_0, C_1 \ldots C_{10}$ are the binomial coefficient in the expansion of $(1+x)^{10}$, then $2.C_0 + \frac{2^2}{2}.C_1 + \frac{2^3}{3}.C_2 + \ldots + \frac{2^{11}}{11}.C_{10}$

(A) $\frac{2^{11}}{11}$ (B) $\frac{2^{11}-1}{11}$

(C) $\frac{3^{11}}{11}$ (D) $\frac{3^{11}-1}{11}$

57. If $(1+px+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then $\sum_{r=0}^{2n}(2r+1)a_r =$

(A) $(p+2)^n$ (B) $(2p+1)(p+2)^2$

(C) $(2n+1)(p+2)^n$ (D) $(p+2)^{n+1}$

58. If $x+y=1$ then $\sum_{r=0}^{n} r.{}^nC_r x^r y^{n-r}$

(A) $2nx$ (B) nx

(C) ny (D) nx^2

59. If $x = \left(2+\sqrt{3}\right)^n$, then the value of $x - x^2 + x[x]$ where [.] denotes the greatest integer function, is equal to

(A) 1 (B) 2

(C) 2^{2n} (D) 2^n

60. If $\left(1+x+x^2\right)^{25} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, then $a_0 + a_2 + a_4 + \ldots + a_{50}$ is

(A) even

(B) odd and of the form $3n$

(C) odd and of the form $(3n - 1)$

(D) odd and of the form $(3n + 1)$

61. The value of $\underset{n\to\infty}{Lt}\sum_{r=1}^{n}\left(\sum_{t=0}^{r-1}\frac{1}{7^n}{}^nC_r.{}^rC_t 5^t\right)$ is equal to

(A) 0 (B) 5

(C) 1 (D) 3

62. The coefficient of x^{1274} in the expansion of $(x+1)(x-2)^2(x+3)^3(x-4)^4.....(x+49)^{49}(x-50)^{50}$ is

(A) 1275 (B) –1275

(C) $-\sum_{i=1}^{50} i^2$ (D) $\sum_{i=1}^{50} i^2$

63. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then the value of $\sum\sum_{0\le i<j\le n}(C_i - C_j)^2$

(A) $(n+1)^{2n}C_n - 2^{2n}$ (B) $(n+1)^{2n}C_n + 2^{2n}$

(C) $(n-1)^{2n}C_n + 2^{2n}$ (D) None of these

64. The value of $\sum_{r=1}^{n}(-1)^{r-1}\left(\frac{r}{r+1}\right).{}^nC_r$ is

(A) $\frac{1}{n+1}$ (B) $\frac{1}{n}$

(C) $\frac{1}{n-1}$ (D) 0

65. The last two digits of the number $(23)^{14}$ are

(A) 01 (B) 03

(C) 09 (D) None of these

66. $\sum_{0\le i<j\le n}\sum\left(C_i+C_j\right)^2$

(A) $(n-1).{}^{2n}C_n+2^{2n}$ (B) ${}^{2n}C_n+2^{2n}$

(C) ${}^{2n}C_n-(n+1)2^n$ (D) None

67. If $C_r={}^nC_r$, then $\left(C_0-C_2+C_4-C_6+\ldots\right)^2+\left(C_1-C_3+C_5-C_7+\ldots\right)^2$ is

(A) 2^{2n} (B) 2^n

(C) 2^{n^2} (D) $2^{\frac{n+1}{2}}$

68. In the expansion of $\left(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{\sqrt{a}}}\right)^{21}$ the term containing same powers of a and b is

(A) 11th (B) 13th

(C) 12th (D) 6th

69. The coefficient of x^{20} in $\sum_{m=0}^{18}{}^{18}C_m(x-2)^{18-m}3^m+\sum_{m=0}^{18}{}^{19}C_m(x-2)^{19-m}3^m+\ldots\ldots+\sum_{m=0}^{30}{}^{30}C_m(x-2)^{30-m}3^m$

(A) ${}^{31}C_{20}$ (B) ${}^{31}C_{21}$

(C) ${}^{30}C_{21}$ (D) none of these

70. If the middle term in the expansion $(1+x)^{2n}$ is the greatest term, then the value of $[x]$ (where [ ] denotes the greatest integer function) should be

(A) $\{0,2\}$ (B) $\{1,2\}$

(C) $\{0,1\}$ (D) none of these

71. If the greatest value of the term independent of x is expansion of $\left(x\sin p+x^{-1}\cos p\right)^{10}$ is achieved at P = θ. Then the locus of point from which pair of tangents be drawn to $x^2+y^2=4$ including an angle θ is

(A) $x^2+y^2=4\left(4+2\sqrt{2}\right)$ (B) $x^2+y^2=4\left(4+\sqrt{2}\right)$

(C) $2x^2+y^2=4\left(3+2\sqrt{2}\right)$ (D) $x^2+y^2=4\left(1+2\sqrt{2}\right)$

72. The sum of the series $(n)C_0{}^2+(n-1)C_1^2+(n-2)C_2^2+(n-3)C_3^2+\ldots\ldots\, C_{n-1}^2$ is

(A) $n\,{}^{2n-1}C_n$ (B) $n\,{}^{2n}C_{n-1}$

(C) $(n-1)^{2n-1}C_n$ (D) $(n-1)^{2n}C_{n+1}$

73. Let $a_n=\underbrace{111\ldots..1}_{n\text{ times}}$. The remainder when a_{124} is divided by 271 is

(A) 23 (B) 25

(C) 27 (D) 29

74. Let $n=100^{50}-43$, the sum of the digits in the decimal notation of n is

(A) 890 (B) 893

(C) 894 (D) 897

75. If the sum of the coefficients in the expansion of $(q+r)^{20}(1+(p-2)x)^{20}$ is equal to square of the sum of the coefficients in the expansion of $[2r\,q\,x-(r+q).\,y]^{10}$, where p, r, q are positive constants, then

(A) $\sqrt{rq}\le p$

(B) $\frac{r+q}{2}\ge p$

(C) r, p and q are in G.P.

(D) $1/r, 1/p$ an $1/q$ are in H.P.

76. If $f(x)$ is periodic with period 't' such that $f(2x+3)+f(2x+7)=2$, then the coefficient of m^{-3t} in the expansion of $\left(m+\frac{b}{m^3}\right)^{4t}$ is

(A) ${}^{16}C_7\,b^7$ (B) ${}^{16}C_7\,b^9$

(C) ${}^{16}C_7\,b^8$ (D) ${}^{16}C_9\,b^9$

77. The sum of the coefficients of terms in the expansion of $(ax+by+cz)^{102}$ of the type $x^py^qz^r$ so that p,q,r are in arithmetic progression is

(A) ${}^{102}C_{34}2^{68}$ (B) ${}^{102}C_{34}2^{34}$

(C) ${}^{102}C_{51}2^{51}$ (D) 2^{102}

78. If $n\in N$, then value of $S=\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{{}^{r+2}C_r}$ is

(A) $\frac{1}{n+2}$ (B) $\frac{2}{n+2}$

(C) $\frac{3}{n+2}$ (D) $\frac{4}{n+2}$

79. The coefficient of x^{50} in the expression $(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots.+(1001)x^{1000}$ is

(A) ${}^{(1000)}C_{53}$ (B) ${}^{(1002)}C_{52}$

(C) ${}^{(1002)}C_{50}$ (D) ${}^{(1002)}C_{51}$

80. $2.{}^{10}C_2+3.2.{}^{10}C_3+4.3.{}^{10}C_4+\ldots+10.9.{}^{10}C_{10}=$

(A) 35.2^9 (B) 35.2^8

(C) 45.2^{10} (D) 45.2^9

81. The value of $\left(\frac{{}^{50}C_0}{1}+\frac{{}^{50}C_2}{3}+\frac{{}^{50}C_4}{5}+\ldots.+\frac{{}^{50}C_{50}}{51}\right)$ is

(A) $\frac{2^{50}}{51}$ (B) $\frac{2^{50}-1}{51}$

(C) $\frac{2^{50}-1}{50}$ (D) $\frac{2^{50}+1}{51}$

82. $\sum_{i=0}^{m}\left({}^{10}C_i\right)\left({}^{20}C_{m-i}\right)$ when ${}^{p}C_q = 0$ if $p < q$ is maximum when $m =$

(A) 10 (B) 15

(C) 5 (D) 20

83. If $C_r = {}^{(2n+1)}C_r$, then $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^{2n-1}C_{2n-1}^2$ is equal to

(A) $-4n^2 - 4n$ (B) ${}^{2n}C_n$

(C) $\frac{1}{n}\,{}^{(2n)}C_n$ (D) 0

84. If n is even positive integer, then the condition that the greatest term in the expansion of $(1+x)^n$ may have the greatest coefficient also is

(A) $\frac{n}{n+2} < x < \frac{n+2}{n}$ (B) $\frac{n+1}{n} < x < \frac{n}{n+1}$

(C) $\frac{n}{n+4} < x < \frac{n+4}{n}$ (D) None

85. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots.. + a_nx^n$ then $a_0 + a_3 + a_6 + a_9 + \ldots.. =$

(A) $\frac{1}{3}\left[2^n + 2\sin\frac{n\pi}{3}\right]$ (B) $\frac{1}{3}\left[2^n + \cos\frac{n\pi}{3}\right]$

(C) $\frac{1}{3}\left[2^n + \sin\frac{n\pi}{3}\right]$ (D) $\frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If the expansion of $\left(x^2 + \frac{1}{x}\right)^p$, ($p \in$ N) contains a term of x^{10}, then

(A) p must be greater than 4

(B) $2p$–10 must be divisible by 5

(C) $2p$ – 10 must be divisible by 3

(D) p must be odd

2. The number $\frac{5^k + 3}{4}(k \in N)$, when divided by 10, may leave remainder

(A) 2 (B) 6

(C) 7 (D) 8

3. The value of $C_0^2 + 3.C_1^2 + 5.C_2^2 + ----$ to $(n+1)$ terms where $C_r = {}^{n}C_r$, is

(A) ${}^{2n-1}C_{n-1}$

(B) $2n+1.{}^{2n-1}C_n$

(C) $2(n+1).{}^{2n-1}C_n$

(D) ${}^{2n-1}C_n + (2n+1)^{2n-1}C_{n-1}$

4. If n is a positive integer and $\left(3\sqrt{3}+5\right)^{2n+1} = \alpha + \beta$ where α is an integer and $0 < \beta < 1$ then

(A) α is an even integer

(B) $(\alpha+\beta)^2$ is divisible by 2^{2n+1}

(C) the integer just below $\left(3\sqrt{3}+5\right)^{2n+1}$ is divisible by 3

(D) α is divisible by 10

5. Which of the following statements are true?

(A) The digit at unit place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is 1.

(B) $(106)^{85} - (85)^{106}$ is divisible by 7.

(C) The positive integer which is just greater than $(1 + 0.0001)^{1000}$ is 2.

(D) If $(1 + 2x - 3x^2)^{2010} = a_0 + a_1x + a_2x^2 + \ldots + a_{4020}x^{4020}$, then $a_0 + a_2 + a_4 + a_6 + \ldots. + a_{4020}$ is an even integer.

6. If the term independent of x in the expansion of $\left(\sqrt{x} + \frac{k}{x^2}\right)^{10}$ is 405, then k may be

(A) 3 (B) –3

(C) 2 (D) –2

7. In the expansion of binomial $\left(2^{\frac{1}{5}} + 3^{\frac{1}{2}}\right)^{20}$

(A) No.of rational terms is 3

(B) No. of Irrational terms is 18

(C) greatest rational term is 11th term

(D) greatest rational term is ${}^{20}C_{10}.\ 2^2.\ 3^5$

8. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then

(A) $(C_0 - C_2 + C_4 - \ldots\ldots.)^2 + (C_1 - C_3 + C_5 - \ldots\ldots)^2 = 2^n$

(B) $C_0 + C_2 + C_4 + \ldots\ldots\ldots. = 2^n$

(C) $C_0 - C_2 + C_4 + \ldots\ldots\ldots. = 2^{n/2}\sin\frac{n\pi}{8}$

(D) $C_1 - C_3 + C_5 + \ldots\ldots\ldots. = 2^{n/2}\sin\frac{n\pi}{4}$

9. If the expansion of $\left(x-\frac{a}{x}\right)^n$ and $\left(x+\frac{b}{x^2}\right)^n$ in power of x, both have one term independent of x then n must be divisible by

(A) 6
(B) 2
(C) 3
(D) 4

10. $(a\alpha x+a^2y+1)^{2009}$ is a polynomial in x and y if the sum of the co-efficients vanishes for some real value of a. Then possible value of α is / are

(A) –2
(B) – 3
(C) 1
(D) 2

11. If $\left(1+2x+3x^2\right)^{10}=a_0+a_1x+a_2x^2+\ldots+a_{20}x^{20}$, then

(A) $a_1=20$
(B) $a_2=210$
(C) $a_{19}=20.3^9$
(D) $a_{20}=2^2.3^7.7$

12. In the expansion of $\left(\sqrt[4]{3}+\sqrt[3]{4}\right)^{136}$

(A) Number of rational terms is 12
(B) Number of irrational terms is 125
(C) Number of total terms is 137
(D) Number of rational terms is 0

13. Let S = $C_0^2+1.\frac{C_1^2}{C_0}+2.\frac{C_2^2}{C_1}+3.\frac{C_3^2}{C_2}+\ldots\ldots..+n.\frac{C_n^2}{C_{n-1}}$ where $c_r={}^nC_r$ then

(A) 2^n divides 'S'
(B) n+2 divides S + n
(C) 2^{n-1} divides S + n
(D) S is a positive integer

14. If $x+y=1, 0<x<1, n\in N$ then

(A) $x^n\sum_{r=0}^{n}{}^nC_r\left(\frac{x}{y}\right)^{-r}=1$
(B) $\sum_{r=0}^{n}r\,{}^nC_rx^{n-r}y^r=ny$
(C) $\sum_{r=0}^{n}r^2\,{}^nC_rx^ry^{n-r}=ny^2+x^2$
(D) None

15. For $n\in N$ if $\left(\frac{5}{x^2}+x^4\right)^n$ has a term independent of 'x' then 'n' can be ....

(A) 18
(B) 21
(C) 27
(D) 99

16. $N=144^{255}+192^{255}$, then

(A) N is divisible by 7
(B) N is divisible by 35
(C) N is divisible by 4
(D) N is divisible by 28

17. In the expansion of $\left(\frac{1}{x}+x\tan x\right)^5$; $T_4:T_2=\frac{2}{27}\pi^4$ then x can be

(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{3}$
(C) $\frac{-\pi}{3}$
(D) $\frac{-\pi}{6}$

18. If the coefficients of 3 consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 7 : 35 then

(A) 'n' is divisible by 5
(B) 'n' is divisible by 35
(C) 'n' is divisible by 23
(D) 'n' is not divisible by any number other than itself

19. Which of the following is true?

(A) If $x^2<1$, the coefficient of x^n in $\frac{(1+2x)^n+(1+3x)^n}{1-x}$ is 3^n+4^n
(B) The value of $\sum_{r=0}^{n}(-1)^r\,{}^n\text{C}_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots..\infty\right)$ is equal to $\frac{1}{2^n-1}$
(C) The remainder obtained when $1!+2!+3!+\ldots.+95$ is divisible by 15 is 3
(D) The coefficient of x^2 in $\left(1+x+x^2\right)^{10}$ is ${}^{12}C_2$.

20. The 4th term in the expansion of $\left(2+\frac{3x}{8}\right)^{10}$ has the maximum numerical value then the value of x can be in the interval

(A) $\left[\frac{-64}{21},-2\right]$
(B) $\left[2,\frac{64}{21}\right]$
(C) $\left(-\infty,\frac{-64}{21}\right)$
(D) $\left(\frac{64}{21},\infty\right)$

21. The value of ${}^{1000}\text{C}_{50}+{}^{999}\text{C}_{49}+{}^{998}\text{C}_{48}+\ldots\ldots+{}^{950}\text{C}_0$ is

(A) ${}^{1001}\text{C}_{50}$
(B) ${}^{1002}\text{C}_{951}-{}^{1001}\text{C}_{51}$
(C) ${}^{1001}\text{C}_{951}$
(D) ${}^{1002}\text{C}_{51}-{}^{1001}\text{C}_{950}$

22. $\left(1^2-a_1\right)+\left(2^2-a_2\right)+\left(3^2-a_3\right)+\ldots..\left(n^2-a_n\right)=\frac{n(n^2-1)}{3}$

(A) $a_n=n$
(B) a_1,a_2,a_3 .... are in A.P.
(C) a_1,a_2,a_3 .... are in G.P.
(D) $\sum_{i=1}^{n}a_i=\frac{n(n+1)}{2}$

23. The largest coefficient in the expansion of $(4+3x)^{25}$ is

(A) ${}^{25}C_{11}\,3^{25}\left(\frac{4}{3}\right)^{14}$
(B) 6!
(C) ${}^{25}C_{14}4^{14}3^{11}$
(D) ${}^{25}C_{14}4^{11}3^{14}$

24. Let $\text{R}=\left(8+3\sqrt{7}\right)^{20}$ and $[\text{R}]=$ the greater integer less than or equal to R. Then

(A) [R] is even
(B) [R] is odd
(C) $R-[R]=1-\frac{1}{\left(8+3\sqrt{7}\right)^{20}}$
(D) $R+R[R]=1+R^2$

25. A value of r satisfying the equation ${}^{69}\text{C}_{3r-1}-{}^{69}\text{C}_{r^2}$ ${}^{69}\text{C}_{3r-1}-{}^{69}\text{C}_{r^2}={}^{69}\text{C}_{r^2-1}-{}^{69}\text{C}_{3r}$ is

(A) 1
(B) 2
(C) 3
(D) 7

26. The value of ${}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+k}C_k$ is equal to

(A) ${}^{n+k+1}C_k$ (B) ${}^{n+k+1}C_{n+1}$

(C) ${}^{n+k}C_{n+1}$ (D) none of these

27. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$, then

(A) $a_0 - a_2 + a_4 - a_6 + \ldots\ldots = 0$, if n is odd

(B) $a_1 - a_3 + a_5 - a_7 + \ldots\ldots = 0$, if n is even

(C) $a_0 - a_2 + a_4 - a_6 + \ldots\ldots = 0$, if $n = 4p$, $p \in I^+$

(D) $a_1 - a_3 + a_5 - a_7 + \ldots\ldots = 0$, if $n = 4p + 1$, $p \in I^+$

28. If $(1+x)^{2n} = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$ then

(A) $a_0 + a_2 + a_4 + ... = 1/2(a_0 + a_1 + a_2 + a_3 + ...)$

(B) $a_{n+1} < a_n$

(C) $a_{n-3} = a_{n+3}$

(D) $a_{n+1} > a_n$

29. If $\left(4+\sqrt{15}\right)^n = I + f$ where n is odd natural number, I is an integer and $0 < f < 1$ then

(A) I is a natural number (B) I is an even integer

(C) $(I+f)(1-f) = 1$ (D) $(I+f)(I-f) = 1$

30. $\sum\limits_{k=0}^{n} {}^{n}C_k\left({}^{n+1}C_{k+1} + {}^{n+1}C_{k+2} + - - - + {}^{n+1}C_{n+1}\right)$ is equal to

(A) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + - - - + {}^{2n+1}C_n$

(B) 4^n

(C) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + - - - + {}^{2n+1}C_{2n+1}$

(D) 2^{2n+1}

31. If $\left(1+ax+bx^2\right)^4 = a_0 + a_1x + a_2x^2 + ... + a_8x^8$ where $a, b, a_0, a_1, a_2 ..... a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$ then

(A) $a_0 = a_1 \neq a_2$ (B) $a_1 = a_2 = 1$

(C) $a_0 = a_1 = a_2 = 4a$ (D) $b = \dfrac{5}{32}$

32. The value of $\sum\limits_{k=0}^{7}\left[\dfrac{\binom{7}{k}}{\binom{14}{k}}\sum\limits_{r=k}^{14}\binom{r}{k}\binom{14}{r}\right]$, where $\binom{n}{r}$ denotes ${}^{n}C_r$, is

(A) 6^7 (B) greater than 7^6

(C) 8^7 (D) greater than 7^8

33. The coefficient of x^{324} in the expansion of $(x + 1)(x - 2)^2 (x + 3)^3 (x - 4)^4 .... (x - 24)^{24} (x - 25)^{25}$ is

(A) 325

(B) $\left(\sum\limits_{k=0}^{8} {}^{8}C_k + 4\cdot\sum\limits_{s=0}^{4} {}^{4}C_s + \sum\limits_{r=0}^{2} {}^{2}C_r + {}^{5}C_0\right)$

(C) $1^2 - 2^2 + 3^2 - 4^2 + \ldots - 24^2 + 25^2$

(D) $1^1 + 2^2 + 3^3 + 4^4 + \ldots + 25^{25}$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

To find coefficient of $x^r (0 \le r \le n-1)$ in the expansion of $(x+a)^{n-1} + (x+a)^{n-2}(x+b) + ... + (x+a)(x+b)^{n-2} + (x+b)^{n-1}$ we first sum up the series and then find the coefficient of x^r in it

1. The coefficient of $x^r (0 \le r \le n-1)$ in the expansion of $(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + ... + (x+1)^{n-1}$

(A) ${}^{n}C_r$ (B) ${}^{n}C_r(2^{n-r} - 1)$

(C) ${}^{n}C_r 2^{n-r}$ (D) ${}^{n}C_r(2^r - 1)$

2. Coefficient of x^r in $2^n + 2^{n-1}(x+2) + 2^{n-2}(x+2)^2 + ... + (x+2)^n$

(A) $2^{n-r}({}^{n+1}C_{r+1})$ (B) $2({}^{n}C_r)^2$

(C) $2^r({}^{n}C_r) - n$ (D) $2\left({}^{n+1}C_{r+1}\right)$

3. Coefficient of x^{50} in the expression $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + ... + 1001x^{1000}$

(A) ${}^{1002}C_{50}$ (B) ${}^{1001}C_{50}$

(C) ${}^{1000}C_{50}$ (D) ${}^{999}C_{50}$

Comprehension 2

If $\left(1 + x + x^2\right)^{100} = \sum\limits_{r=0}^{200} a_r x^r$

4. Which of the following is true

(A) $a_{28} = a_{72}$ (B) $a_{56} = a_{144}$

(C) $a_{200} = a_{300}$ (D) none of these

5. $a_0 + a_1 + ... + a_{99}$ is equal to

(A) $\dfrac{3^{99} - a_{99}}{2}$ (B) $\dfrac{3^{101} - a_{99}}{2}$

(C) $\dfrac{3^{100} - a_{100}}{2}$ (D) none of these

6. $37a_{37}$ is equal to

(A) $64a_{36}+165a_{35}$ (B) $64a_{35}+148a_{36}$

(C) $56a_{32}+168a_{22}$ (D) none of these

Comprehension 3

Let $\sum_{r=0}^{20} a_r x^r=\left(1+x+x^2\right)^{10}$. Then

7. $a_1+a_3+a_5+a_7+a_9+a_{11}+a_{15}+a_{17}+a_{19}=$

(A) $\frac{1}{2}$ (B) 3^{10}

(C) $\frac{1}{2}\left(3^{10}-1\right)$ (D) $\frac{1}{2}\left(3^{10}+1\right)$

8. $a_0+a_3+a_6+a_9+a_{12}+a_{15}+a_{18}=$

(A) 3^9 (B) 3^9-1

(C) 3^9+1 (D) $3^{10}-1$

9. $a_5-\binom{10}{1}a_4+\binom{10}{2}a_3-\binom{10}{3}a_2+\binom{10}{4}a_1-\binom{10}{5}a_0=$

(A) 0 (B) $\binom{10}{4}$

(C) $-\binom{10}{5}$ (D) $\binom{10}{2}$

Comprehension 4

Consider a G.P. with first term $(1+x)^n, (|x|<1)$, common ratio $\left(\frac{1+x}{2}\right)$ and number of terms (n + 1). Let 'S' be sum of all the terms of the G.P., then

10. 'S' equals

(A) $\frac{1}{2^n}\left[2^{n+1}(1+x)^n-(1+x)^{2n+1}\right].\lim_{m\to\infty}\sum_{k=0}^{m}x^k$

(B) $\frac{1}{2^n}\left[2^{n+1}(1+x)^{n+1}-(1+x)^{2n}\right].\lim_{m\to\infty}\sum_{k=0}^{m}x^k$

(C) $\frac{1}{2^n}\left[2^{n}(1+x)^{n+1}-(1+x)^{2n+1}\right].\lim_{m\to\infty}\sum_{k=0}^{m}x^k$

(D) $\frac{1}{2^n}\left[2^{n}(1+x)^{n}-(1+x)^{2n}\right].\lim_{m\to\infty}\sum_{k=0}^{m}x^k$

11. The coefficient of x^n in 'S' is

(A) 2^n (B) 2^{n+1}

(C) 2^{2n} (D) 2^{2n+1}

Comprehension 5

Integration plays a vital role in proving identities involving binomial coefficients. If we apply integration techniques, several elegant binomial identities can be proved. For example

$$C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots\ldots+\frac{(-1)^n C_n}{n+1}=\int_0^1(1-x)^n\,dx=\frac{1}{n+1}$$

12. The value of $\frac{C_0}{1}-\frac{C_1}{3}+\frac{C_2}{5}-\ldots\ldots+(-1)^n\frac{C_n}{2n+1}$ is

(A) $\frac{n!}{(2n+1)!}$ (B) $\frac{2^{2n}.n!}{(2n+1)!}$

(C) $\frac{2^{2n}.(n!)^2}{(2n+1)!}$ (D) $\frac{2^{2n}.(2n+1)!}{n!}$

13. $\frac{C_0}{n(n+1)}-\frac{C_1}{(n+1)(n+2)}+\frac{C_2}{(n+2)(n+3)}-\ldots\ldots(n+1)$ terms is

(A) $\int_0^1 x^{n+1}(1-x)^{n+1}dx$ (B) $\int_0^1 x^{n}(1-x)^{n+1}dx$

(C) $\int_0^1 x^{n-1}(1-x)^{n+1}dx$ (D) $\int_0^1 x^{n}(1-x)^{n}dx$

14. The value of ${}^nC_1-\left(1+\frac{1}{2}\right){}^nC_2+\left(1+\frac{1}{2}+\frac{1}{3}\right){}^nC_3-\ldots.$ $+(-1)^{n-1}\left(1+\frac{1}{2}+\ldots.+\frac{1}{n}\right){}^nC_n$ is____

(A) 0 (B) 1

(C) n (D) $\frac{1}{n}$

15. The value of $\left(C_1-3.C_3+5.C_5-\ldots\ldots..\right)^2+\left(2.C_2-4.C_4+6.C_6-\ldots\ldots..\right)^2$ is

(A) $n2^{n-1}$ (B) $n^2.2^{n-1}$

(C) $n(n-1)2^{n-1}$ (D) $n^2.2^n$

16. If $\sum_{r=0}^{n}\frac{1}{C_r}=a$ then $\sum\sum_{0\le i<j\le n}\left(\frac{1}{C_i}+\frac{1}{C_j}\right)=$

(A) na (B) n^2a

(C) $\frac{na}{2}$ (D) $\frac{n^2a}{2}$

Comprehension 6

If m, n, r are positive integers and if $r<m$, $r<n$ then ${}^mC_r+{}^mC_{r-1}{}^nC_1+{}^mC_{r-2}{}^nC_2+\ldots\ldots\ldots..+{}^nC_r$ = coefficient of x^r in $(1+x)^m(1+x)^n$ = coefficient of x^r in $(1+x)^{m+n}={}^{m+n}C_r$

Based on above information answer the following

17. The value of $r\,(0\le r\le 30)$ for which ${}^{20}C_r.{}^{10}C_0+{}^{20}C_{r-1}.{}^{10}C_1+\ldots\ldots..+{}^{20}C_0.{}^{10}C_r$ is minimum, is

(A) 0 (B) 1

(C) 5 (D) 15

18. If $S_n={}^nC_0.{}^nC_1+{}^nC_1.{}^nC_2+\ldots\ldots.+{}^nC_{n-1}.{}^nC_n$ and if $\frac{S_{n+1}}{S_n}=\frac{15}{4}$ then $n=$

(A) 4, 6 (B) 8, 10

(C) 2, 4 (D) 6, 8

19. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ and n is odd then the value of $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots + (-1)^n C_n^2 =$

(A) $(-1)^n\, {}^{2n}C_{n-1}$ (B) 0

(C) ${}^{2n}C_{n-1}$ (D) $(-1)^{n/2}\, {}^{n}C_{n/2}$

Comprehension 7

If n is any rational number

$$(1+x)^n = 1 + \frac{n}{\underline{|1}}x + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots.. + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r + \ldots\infty \text{ where } -1 < x < 1$$

20. Coefficient of x^2 in the expansion $\dfrac{4+x}{\sqrt{1+4x}}$ is

(A) 22 (B) 11

(C) – 7 (D) – 74

21. The $(r+1)^{\text{th}}$ term in the expansion of $(1-2x)^{-2/3}$

(A) $\dfrac{2.5.8\ldots(3r-1)}{\underline{|r}}\left(\dfrac{2x}{3}\right)^r$ (B) $\dfrac{2.5.8\ldots(3r-1)}{\underline{|r}}x^r$

(C) $\dfrac{2.5.8\ldots(3r-1)}{\underline{|r}}\left(\dfrac{3x}{2}\right)^r$ (D) $\dfrac{2.5.8\ldots(3r-1)}{\underline{|r}}(2x)^r$

22. If $y = 2x + 3x^2 + 4x^3 + \ldots\ldots \infty$ then $x =$

(A) $1 + \dfrac{1}{\sqrt{1+y}}$ (B) $1 - \dfrac{1}{\sqrt{1+y}}$

(C) $1 + \sqrt{1+y}$ (D) $\sqrt{1+y} - 1$

Comprehension 8

If $(1+x)^n = C_0x^0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then

23. Value of $\dfrac{(C_0+C_1)(C_1+C_2)(C_2+C_3)+\ldots\ldots\ldots\ldots(C_{n-1}+C_n)}{C_1C_2\ldots\ldots\ldots C_{n-1}C_n}$ is

(A) $\dfrac{(n+1)^n}{n!}$ (B) $\dfrac{n+1}{n!}$

(C) $\dfrac{(n+1)^{n-1}}{n!}$ (D) $\dfrac{(n+1)^n}{n}$

24. Value of $C_0 + (C_0 + C_1) + \ldots\ldots\ldots + (C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_{n-1})$ is

(A) $(n+1).2^n$ (B) $n.2^{n-1}$

(C) $(n.+1)2^{n-1}$ (D) $n.2^n$

Comprehension 9

If $\forall n \in \mathrm{N} \left(\sum\limits_{r=0}^{2} x^r\right)^n = a_0 + a_1x + a_2x^2 + \ldots.. + a_{2n}x^{2n}$ where $a_1, a_2, \ldots\ldots, a_{2n} \in \mathrm{Z}$

25. $\sum\limits_{r=0}^{2n} \dfrac{r}{a_r} =$

(A) $n\sum\limits_{r=0}^{2n}\dfrac{1}{a_r}$ (B) $2n\sum\limits_{r=0}^{2n}\dfrac{1}{a_r}$

(C) $\dfrac{3n}{2}\sum\limits_{r=0}^{2n}\dfrac{1}{a_r}$ (D) none of these

26. If $a = \sum\limits_{k=1}^{[n/2]} {}^nC_{2k}\,{}^{2k}C_k$ where [.] denotes greatest integer function then $(a_n - a)$ is equal to

(A) 0 (B) 1

(C) – 1 (D) 2

27. If $a_0{}^2 - a_1{}^2 + a_2{}^2 - a_3{}^2 + \ldots\ldots\ldots + a_{2n}{}^2 = kn_n$ then $k =$

(A) 1 (B) 2

(C) 1/2 (D) 0

28. The value of $a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} - \ldots.. + a_{2n-2r}a_{2n}$ is

(A) a_{r+1} (B) a_{r+2}

(C) a_{n+r+1} (D) a_{n+r}

Comprehension 10

Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots. + C_nx^n$ when $C_r = {}^nC_r$, then

29. $\sum\limits_{r=1}^{n}(-1)^{r-1}\dfrac{r}{r+1}C_r =$

(A) $\dfrac{1}{n}$ (B) 0

(C) $\dfrac{1}{n-1}$ (D) $\dfrac{1}{n+1}$

30. $\sum\limits_{r=0}^{n}\dfrac{C_r}{r+1} =$

(A) $\dfrac{2^{n+1}}{n}$ (B) $\dfrac{2^{n+1}+1}{n+1}$

(C) $\dfrac{2^{n+1}-1}{n+1}$ (D) $\dfrac{2^{n+1}-1}{n}$

31. $\sum\limits_{r=0}^{n} {}^nC_r\cos(r+1)x =$

(A) $2^n\cos^n\dfrac{x}{2}\cos\dfrac{nx}{2}$ (B) $2^n\cos^n\dfrac{x}{2}\cos\dfrac{(n+1)}{2}x$

(C) $2^n\cos^n\dfrac{x}{2}\cos\dfrac{(n+2)}{2}x$ (D) $2^n\cos^n\dfrac{x}{2}\cos\dfrac{(n-1)}{2}x$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match the following:

COLUMN-I	COLUMN-II
(A) $\sum\limits_{0\le i<j\le n} (i+j)(C_i.C_j)$ is	(p) $n.2^n$
(B) $\sum\limits_{0\le i<j\le n} \sum ({}^nC_i + {}^nC_j)$ is equal to	(q) $n.2^{n-1}$
(C) $\sum\limits_{0\le i<j\le n} \sum i\,{}^nC_j$ is equal to	(r) $n(n-1)2^{n-3}$
(D) $\sum\limits_{r=0}^{n} r\,{}^nC_r$ is equal to	(s) $\frac{n}{2}.\left[2^{2n} - {}^{2n}C_n\right]$

2. Match the items of Column I with the items of Column II

COLUMN-I	COLUMN-II
(A) If the sum of the coefficients in the expansion of $(a^2x^2 - 2ax + 1)^{50}$ is zero, then the value of $a + 2a + 3a + ... + 10a$ is	(p) 1
(B) If $x = (3 + 2\sqrt{2})^6 = [x] + f$ where $[x]$ is the integral part of x and $f = x - [x]$, then $x(1 - f)$ is	(q) 15
(C) If the sum of the coefficients of x^5 and x^{10} in the expansion of $[x^3 - (1/x^2)]^n$ is zero, then n equals	(r) 129
(D) Number of integral terms in the expansion of $(\sqrt{5} + \sqrt[8]{7})^{1024}$ is	(s) 55

3. Match the following:

COLUMN-I	COLUMN-II
(A) Number of distinct terms in the expansion of $(x + y - z)^{16}$ is	(p) 2^{12}
(B) Number of terms in the expansion of $\left(x + \sqrt{x^2 - 1}\right)^6 + \left(x - \sqrt{x^2 - 1}\right)^6$ is	(q) 97
(C) The number of irrational terms in $\left(\sqrt[8]{5} + \sqrt[6]{2}\right)^{100}$ is	(r) 4
(D) The sum of numerical coefficients in the expansion of $\left(1 + \frac{x}{3} + \frac{2y}{3}\right)^{12}$ is	(s) 153

4. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then

COLUMN-I	COLUMN-II
(A) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$ is equals	(p) $\frac{1}{2}(3^n + a_n)$
(B) $a_0a_2 - a_1a_3 + a_2a_4 ...... + a_{2n-2}a_{2n}$ is equals	(q) $\frac{1}{2}(3^n - 1)$
(C) $a_0 + a_1 + a_2 + ... + a_n$ is equals	(r) a_{n+1}
(D) $a_2 + a_4 + ... + a_{2n}$ is equals	(s) a_n

5. Consider the binomial expansion of $(1+2x)^n$ (n is a positive integer) in which the sum of the coefficients is 6561. Let $(1 + 2x)^n = \mathrm{R} = \mathrm{I} + f$ where I is the largest integer not exceeding R and $0 < f < 1$.

COLUMN-I	COLUMN-II
(A) If r^{th} term in the expansion is the greatest term, when $x = \frac{1}{2}$ then r cannot exceed	(p) 3
(B) If i^{th} term is having the greatest coefficient, then i can be	(q) 4
(C) The number of integral terms in the expansion when $x = \sqrt[3]{3}$ is less than	(r) 5
(D) For $x = \frac{1}{\sqrt{2}}$, the value of $R(1-f)$ is less than	(s) 6
	(t) 7

6. Match the following: -

COLUMN-I	COLUMN-II
(A) ${}^{m}C_1\ {}^{n}C_m - {}^{m}C_2\ {}^{2n}C_m + {}^{m}C_3\ {}^{3m}C_m + \ldots$ is	(p) The coefficient of x^m in the expansion of $\left(1-(1+x)^n\right)^m$
(B) ${}^{n}C_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^{m}C_m$ is	(q) The coefficient of x^m in $\frac{(1+x)^{n+1}}{x}$
(C) $C_0C_n + C_1C_{n-1} + \ldots + C_nC_0$ is	(r) The coefficient of x^{n+1} in $(1+x)^{2n}$
(D) $2^m\ {}^{n}C_0 - 2^{m-1}\ {}^{n}C_1\ {}^{n-1}C_{m-1} + \ldots + (-1)^m\ {}^{n}C_m\ {}^{n-m}C_0$ is	(s) The coefficient of x^m in the expansion of $(1+x)^n$
	(t) The coefficient of x^n in $(1+x)^{2n}$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. When the terms in the binomial expansion of $\left(\sqrt{x} + \frac{1}{2x^{1/4}}\right)^n$ are arranged in decreasing powers of 'x', the coefficient of the first three terms are in arithmetic progression. The number of terms in the expansion with integer powers of 'x' is

2. If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ are equal if $b = \frac{1}{9}$ then the value of a is

3. If ${}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - {}^{n}C_3 + \ldots + (-1)^r . {}^{n}C_r = 28$, then n is equal to

4. The expression of $(1+x+x^2)^n$ be written as $a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{2n}x^{2n}$ then the value of $\frac{a_0 + a_1 + a_3 + a_4 + a_6 + a_7 + \ldots\ldots\ldots\ldots\ldots}{a_2 + a_5 + a_8 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$ is

5. If $f(n) = \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \ldots\ldots + (-1)^n . \frac{C_n}{n+2}$ then $\lim\limits_{n\to\infty}\left(3n^2 + 4n + 2\right)f(n) =$

6. If it is given that coefficient of 2nd, 3rd and 4th term in expansion of $(a + b)^n$ are in A.P. and the value of 3rd term is 84 $\left(a, b \in (1, \infty)\right)$, then the maximum value of $\left(\log_{2^{1/5}} a^2\right)\left(\log_{2^{1/2}} b^2\right)$ is equal to _____________

7. If $\frac{{}^{n}C_1 + 3^2\ {}^{n}C_3 + 5^2\ {}^{n}C_5 + \ldots..}{n(n+1)} = 2^{n-\alpha}\ \forall n \geq 3,\ n \in \mathrm{N}$ then α is equal to _____________

8. If the coefficient of x^{24} in $\left(\frac{C_1}{C_0} - x\right)\left(x - 2^2.\frac{C_2}{C_1}\right)\left(x - 3^2.\frac{C_3}{C_2}\right) \ldots\ldots\ldots \left(x - 25^2.\frac{C_{25}}{C_{24}}\right)$ where C_r stands for 25_{C_r} is $3000 - 25\,\mathrm{K}$ then K is

9. The digit in the unit place of the number is $29^{3487} + 31^{2005} + 19^{2005}$ __________

10. In the binomial expansion $(a-b)^n; n \geq 5$ the sum of the 5th and 6th terms is zero then $\frac{nb - 5a}{b}$ is equal to

11. Let $f(n) = \left(\sqrt{2}+1\right)^n$, '$n$' being an odd positive integer [.] the greatest integer function then the value of 'n' for which $\left[f(x)\right] = 82$ is

12. If the sum of all the coefficients of the terms in the expansion of $(x + y + z + w)^6$ which contain x but not y, is $95t$ then the value of t is _____.

13. If $(1+ax)^n = 1 + 8x + 24x^2 + \ldots.$, then $a + n =$

14. If the coefficient of x^{10} in $\left(x + {}^{10}C_0\right)\left(x + {}^{10}C_1\right) \ldots\ldots\ldots\ldots \left(x + {}^{10}C_{10}\right)$ is K then $\left[\frac{K}{256}\right]$ (Where [.] denotes greatest integer function) is

15. If P and Q are, respectively, the sum of even and odd terms in the expansion of $(x+a)^{10}$, then $(x+a)^{20}-(x-a)^{20}= k$ PQ where k is _______

16. If the sixth term in the expansion of $\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{(1/5)\log_2(3^{x-1}+1)}}\right]^7$ is 84, then the sum of the values of x is

17. If α and β are the roots of equation $x^2+4x+p=0$ where $p = \sum_{r=0}^{n} {}^nC_r \frac{1+rx}{(1+nx)^r}(-1)^r$ then the value of $|\alpha-\beta|$ is

18. ${}^nC_o\,{}^{4n}C_n - {}^nC_1\,{}^{(4n-3)}C_n + {}^nC_2\,{}^{4n-6}C_n - {}^nC_3\,{}^{4n-9}C_n + \ldots +$ to n terms is equal to $(1+k)^n$ then the value of k is

19. If the sum of rational coefficients in expansion of $\left(\sqrt[3]{5}x+\sqrt{3}y+z\right)^6$ is k then last digit in k is

20. The value of $99^{50}-99.98^{50}+\frac{99.98}{1.2}(97)^{50}-\ldots\ldots+99$ is

21. The coefficient of x^{203} in the expression $(x-1)(x^2-2)(x^3-3)\ldots..(x^{20}-20)$ is K then $\frac{K}{13}$ equal to

22. If $53^{53}-33^3$ is divided by 10, then the remainder is ___

23. The coefficient of x^{20} in $(1+x)(1+2x)(1+4x)(1+8x)\ldots\ldots.$ $(1+2^{20}x)$ is $2^{190}(2^{17+k}-2^{k-4})$ then $k =$ _______

24. The polynomial $p(x)=1-x+x^2-x^3+\ldots..+x^{16}-x^{17}$ can be written as a polynomial in y where $y = x + 1$, then let coefficient of y^2 be k then $\left[\frac{k}{400}\right]$ (where [.] denote greatest integer function) is

25. The digit in units place of $(1!+2!+3!+\ldots\ldots\ldots.+2011!)^{2011}$ is ____________

26. The value of $\frac{{}^{47}C_4}{{}^{57}C_4}+\sum_{J=0}^{3}\frac{{}^{50-J}C_3}{{}^{57}C_{53}}+\sum_{K=0}^{5}\frac{{}^{56-K}C_{53-K}}{{}^{57}C_4}$ is equal to

27. Let $R_n=\left(5\sqrt{5}+11\right)^{2n+1}$ and $f_n=R_n-[R_n]$ where [.] denotes G.I.F, then value of $\left[\frac{1}{2^{14}}(R_3f_3)\right]$ is

28. The arithemetic mean of the numbers ${}^{25}C_0, \frac{{}^{25}C_2}{3}, \frac{{}^{25}C_4}{5}, \frac{{}^{25}C_6}{7}, \ldots\ldots, \frac{{}^{25}C_{24}}{25}$ is equal to $\frac{2^n}{n^2}$ where $n\in N$ then sum of the digits of n equals.

29. If the co-efficient of $x^{\frac{n^2+n-14}{2}}$ in $(x-1)(x^2-2)(x^3-3)(x^4-4)$ $\ldots.(x^n-n), (n\geq 8)$ is K than the value of $K-10$ is ____

30. If in the expansion of $(2^{1/3}+3^{-1/3})^n$, the ratio of the 7th term from the beginning to the 7th term from the end is 1: 6, then n is equal to

31. Let $a=3^{\frac{1}{223}}+1$ and for all $n > 3$, let $f(n)={}^nC_0.a^{n-1}-{}^nC_1.a^{n-2}+{}^nC_2.a^{n-3}-\ldots+(-1)^{n-1}.{}^nC_{n-1}.a^0$. If the value of $f(2007)+f(2008)=9.3^k$ where $k\in N$, then find ‘k’

32. When 32^{33} is divided by 34, it leaves the remainder $3k + 5$ then the value of k is _____.

33. The remainder obtained when $(2109)^{360}-(1396)^{333}$ is divided by 7 is _______

34. If $n\in N$ and $C_k={}^nC_k$, and
$$\sum_{k=1}^{n}k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2=\frac{n(n+1)^2(n+2)}{3p}$$
then p is

35. If $C_0, C_1, C_2 \ldots. C_n$ denote binomial coefficient in the expansion of $(1+x)^n$ and given $a = b = 2$ and $n = 10$, then $aC_0+(a+b)C_1+(a+2b)C_2+\ldots..+(a+nb)C_n = 4k.2^9$ where k is ____

36. Given ${}^8C_1x(1-x)^7+2.{}^8C_2x^2(1-x)^6+3.{}^8C_3x^3(1-x)^5+\ldots..+8.x^8=ax+b$, Find $a+b$

37. If $\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{{}^{r+2}C_r}=\frac{k}{n+2}$ find k

38. $s=a+(a+d)+(a+2d)+\ldots\ldots+(a+nd)$ and $A=a+(a+d)^nC_1+(a+2d)^nC_2+\ldots..+(a+nd)^nC_n$ then $(n+1)A=k^nS$ where $k =$ _________

39. Consider two polynomials $f(x)$ and $g(x)$ as $g(x)=\sum_{r=0}^{200}\alpha_r x^r$ and $f(x)=\sum_{r=0}^{200}\beta_r x^r$. Given (i) $\beta_r=1\ \forall\ r\geq 100$, (ii) $f(x+1)=g(x)$. Let $A=\sum_{r=100}^{200}\alpha_r$. Find the remainder when A is divided by 15.

40. $(1+x)(1+x+x^2)(1+x+x^2+x^3)\ \ldots\ (1+x+x^2+\ldots..+x^{100})$ when written in the ascending power of x then (the highest exponent of x) – 5045 is

41. Find the coefficient of x^{103} in $(1+x+x^2+x^3+x^4)^{199}(x-1)^{201}$.

42. The sum of the series $3.{}^{2007}C_0-8.{}^{2007}C_1+13\,{}^{2007}C_2-18.{}^{2007}C_3+\ldots.$ upto 2008 terms is K, then K is

43. Let $(1+x^3)^n=\sum_{r=0}^{n}a_rx^r(1-x)^{3n-2r}$, $n > 2$, then find the value of n so that a_1, a_2, a_3 are in arithmetic geometric progression

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1. [JEE Main 2020]
If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$ is 61, then n is equal to _______.

2. [JEE Main 2020]
The coefficient of x^4 in the expansion of $(1 + x + x^2)^{10}$ is ________.

3. [JEE Main 2020]
If $C_r \equiv {}^{25}C_r$ and $C_0 + 5 \cdot C_1 + 9 \cdot C_2 + \dots + (101) \cdot C_{25} = 2^{25} \cdot k$, then k is equal to _________.

4. [JEE Main 2020]
For a positive integer n, $\left(1+\frac{1}{x}\right)^n$ is expanded in increasing powers of x. If three consecutive coefficients in this expansion are in the ratio, 2 : 5 : 12, then n is equal to ....

5. [JEE Main 2020]
Let $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{13}}$ is equal to ______.

6. [JEE Main 2020]
The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^6$ in powers of x, is _____.

7. [JEE Main 2020]
The natural m, for which the coefficient of x in the binomial expansion of $\left(x^m + \frac{1}{x^2}\right)^{22}$ is 1540, is ______.

8. [JEE Main 2020]
The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + \dots + x^{10}$ is:
(A) 120
(B) 330
(C) 210
(D) 420

9. [JEE Main 2020]
If a, b and c are the greatest values of ${}^{19}C_p$, ${}^{20}C_q$ and ${}^{21}C_r$ respectively, then:
(A) $\frac{a}{11} = \frac{b}{22} = \frac{c}{21}$
(B) $\frac{a}{10} = \frac{b}{11} = \frac{c}{21}$
(C) $\frac{a}{11} = \frac{b}{22} = \frac{c}{42}$
(D) $\frac{a}{10} = \frac{b}{11} = \frac{c}{42}$

10. [JEE Main 2020]
If α and β be the coefficients of x^4 and x^2 respectively in the expansion of $\left(x+\sqrt{x^2-1}\right)^6 + \left(x-\sqrt{x^2-1}\right)^6$, then:
(A) $\alpha + \beta = -30$
(B) $\alpha + \beta = 60$
(C) $\alpha - \beta = -132$
(D) $\alpha - \beta = 60$

11. [JEE Main 2020]
In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if l_1 is the least value of the term independent of x when $\frac{\pi}{8} \le \theta \le \frac{\pi}{4}$ and l_2 is the least value of the term independent of x when $\frac{\pi}{16} \le \theta \le \frac{\pi}{8}$, then the ratio $l_2 : l_1$ is equal to:
(A) 8 : 1
(B) 1 : 8
(C) 16 : 1
(D) 1 : 16

12. [JEE Main 2020]
Let $\alpha > 0, \beta > 0$ be such that $\alpha^3 + \beta^2 = 4$. If the maximum value of the term independent of x in the binomial expansion of $\left(\alpha x^{\frac{1}{9}} + \beta x^{-\frac{1}{6}}\right)^{10}$ is $10k$, then k is equal to:
(A) 176
(B) 336
(C) 352
(D) 84

13. [JEE Main 2020]
If the number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^n$ is exactly 33, then the least value of n is:
(A) 264
(B) 256
(C) 128
(D) 248

14. [JEE Main 2020]
If for some positive integer n, the coefficients of three consecutive terms in the binomial expansion of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14, then the largest coefficient in this expansion is
(A) 792
(B) 252
(C) 462
(D) 330

15. [JEE Main 2020]
If the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is k, then $18k$ is equal to :
(A) 9
(B) 11
(C) 5
(D) 7

16. [JEE Main 2020]
The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to:
(A) ${}^{51}C_7 + {}^{30}C_7$
(B) ${}^{51}C_7 - {}^{30}C_7$
(C) ${}^{50}C_7 - {}^{30}C_7$
(D) ${}^{50}C_6 - {}^{30}C_6$

17. [JEE Main 2020]
If the constant term in the binomial expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then $|k|$ equals
(A) 2
(B) 1
(C) 3
(D) 9

18. [JEE Main 2020]
If $\{p\}$ denotes the fractional part of the number p, then $\left\{\frac{3^{200}}{8}\right\}$, is equal to
(A) $\frac{1}{8}$
(B) $\frac{5}{8}$
(C) $\frac{3}{8}$
(D) $\frac{7}{8}$

19. **[JEE Main 2020]**

The number of ordered pairs (r, k) for which $6 \cdot {}^{35}C_r = (k^2 - 3) \cdot {}^{36}C_{r+1}$, where k is an integer, is

(A) 4

(B) 3

(C) 2

(D) 6

20. **[JEE Main 2019]**

If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to:

(A) 8

(B) 14

(C) 6

(D) 4

21. **[JEE Main 2019]**

The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is

(A) 12

(B) 15

(C) 10

(D) 14

22. **[JEE Main 2019]**

The sum of the co-efficient of all even degree terms in x in the expansion of

$\left(x+\sqrt{x^3-1}\right)^6 + \left(x-\sqrt{x^3-1}\right)^6$, $(x > 1)$ is equal to:

(A) 26

(B) 24

(C) 32

(D) 29

23. **[JEE Main 2019]**

The coefficient of x^{18} in the product $(1 + x)(1 - x)^{10}(1 + x + x^2)^9$ is:

(A) 84

(B) 126

(C) –126

(D) –84

24. **[JEE Main 2019]**

If ${}^{20}C_1 + (2^2)\,{}^{20}C_3 + (3^2)\,{}^{20}C_3 + \ldots\ldots + (20^2)\,{}^{20}C_{20} = A(2^{\beta})$, then the ordered pair (A, β) is equal to:

(A) (420, 18)

(B) (380, 18)

(C) (420, 19)

(D) (380, 19)

25. **[JEE Main 2019]**

Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, for all $x \in R$, then $\frac{a_2}{a_0}$ is equal to

(A) 12.25

(B) 12.75

(C) 12.00

(D) 12.50

26. **[JEE Main 2019]**

Let $S_n = 1 + q + q^2 + \ldots + q^n$ and

$T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n$ where

q is real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2 \,.\, S_1 + \ldots + {}^{101}C_{101} \,.\, S_{100} = \alpha T_{100}$ then α is equal to

(A) 202

(B) 200

(C) 2^{100}

(D) 2^{99}

27. **[JEE Main 2019]**

The value of r for which ${}^{20}C_r\,{}^{20}C_0 + {}^{20}C_{r-1}\,{}^{20}C_1 + {}^{20}C_{r-2}\,{}^{20}C_2 + \ldots + {}^{20}C_r$ is maximum, is

(A) 20

(B) 15

(C) 10

(D) 11

28. **[JEE Main 2019]**

The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is:

(A) 0

(B) 8

(C) 6

(D) 4

29. **[JEE Main 2019]**

If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then n can be:

(A) 11

(B) 12

(C) 9

(D) 14

30. **[JEE Main 2019]**

The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is:

(A) 54

(B) 55

(C) 49

(D) 48

31. **[JEE Main 2019]**

A ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $\left(2^{1/3}+\frac{1}{2(3)^{1/3}}\right)^{10}$ is:

(A) $1 : 2(6)^{1/3}$

(B) $1 : 4(6)^{1/3}$

(C) $2(36)^{1/3} : 1$

(D) $4(36)^{1/3} : 1$

32. **[JEE Main 2019]**

If $\sum_{r=0}^{25}\{{}^{50}C_r \,.\, {}^{50-r}C_{25-r}\} = K({}^{50}C_{25})$ then K is equal to

(A) 2^{24}

(B) $2^{25} - 1$

(C) 2^{25}

(D) $(25)^2$

33. **[JEE Main 2019]**

If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i + {}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$ then k is equal to

(A) 100

(B) 200

(C) 50

(D) 400

34. **[JEE Main 2019]**

If the third term in the binomial expansion of $\left(1+x^{\log_2 x}\right)^5$ equals 2560, then a possible value of x is

(A) $2\sqrt{2}$

(B) $4\sqrt{2}$

(C) $\frac{1}{8}$

(D) $\frac{1}{4}$

35. **[JEE Main 2019]**

The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right) \cdot \left(2x^2-\frac{3}{x^2}\right)^6$ is equal to:

(A) –36

(B) –108

(C) 36

(D) –72

36. [JEE Main 2019]

The positive value of λ for which the coefficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is

(A) 4
(B) $2\sqrt{2}$
(C) 3
(D) $\sqrt{5}$

37. [JEE Main 2019]

If some three consecutive coefficients in the binomial expansion of $(x + 1)^n$ in powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficients is:

(A) 232
(B) 964
(C) 625
(D) 227

38. [JEE Main 2019]

The sum of the series $2.{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + \ldots + 62.{}^{20}C_{20}$ is equal to

(A) 2^{23}
(B) 2^{25}
(C) 2^{26}
(D) 2^{24}

39. [JEE Main 2019]

If the fourth term in the binomial expansion of $\left(\sqrt{x^{\frac{1}{1+\log_{10}x}}}+x^{\frac{1}{12}}\right)^6$ is equal to 200, and $x > 1$, then the value of x is

(A) 10
(B) 10^3
(C) 100
(D) 10^4

40. [JEE Main 2019]

If the fourth term in the Binomial expansion of $\left(\frac{2}{x}+x^{\log_8 x}\right)^6$ $(x > 0)$ is 20×8^7, then a value of x is:

(A) 8^2
(B) 8^3
(C) 8
(D) 8^4

41. [JEE Main 2019]

If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1 + ax + bx^2)(1 - 3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to:

(A) (28, 861)
(B) (28, 315)
(C) (–21, 714)
(D) (–54, 315)

42. [JEE Main 2019]

The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is:

(A) 23
(B) 58
(C) 38
(D) 35

43. [JEE Main 2018]

The sum of the co-efficient of all odd degree terms in the expansion of $(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$, $(x > 1)$ is:

(A) 0
(B) 1
(C) 2
(D) –1

44. [JEE Main 2017]

The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is:

(A) $2^{20} - 2^{10}$
(B) $2^{21} - 2^{11}$
(C) $2^{21} - 2^{10}$
(D) $2^{20} - 2^{9}$

45. [JEE Main 2016]

If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion is:

(A) 243
(B) 729
(C) 64
(D) 2187

46. [JEE Main 2015]

The sum of coefficients of integral power of x in the binomial expansion $\left(1-2\sqrt{x}\right)^{50}$ is:

(A) $\frac{3^{50}}{2}$
(B) $\frac{3^{50}-1}{2}$
(C) $\frac{3^{50}+1}{2}$
(D) $\frac{2^{50}+1}{2}$

47. [JEE Main 2014]

If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to:

(A) $\left(14, \frac{272}{3}\right)$
(B) $\left(16, \frac{272}{3}\right)$
(C) $\left(16, \frac{251}{3}\right)$
(D) $\left(14, \frac{251}{3}\right)$

48. [JEE Main 2013]

The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is

(A) 4
(B) 120
(C) 210
(D) 310

49. [AIEEE 2012]

If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$ is:

(A) an irrational number
(B) an odd positive integer
(C) an even positive integer
(D) a rational number other than positive integers

50. [AIEEE 2011]

The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is

(A) –132
(B) –144
(C) 132
(D) 144

51. [AIEEE 2010]

Let $S_1 = \sum_{j=1}^{10} j(j-1).{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j.{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2.{}^{10}C_j$

Statement-1: $S_3 = 55 \times 2^9$.

Statement-2: $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(A) Statement-1 is false, Statement-2 is true

(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(D) Statement-1 is true, Statement-2 is false

52. **[AIEEE-2009]**

The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is

(A) 2	(B) 7
(C) 8	(D) 0

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Single Correct Choice Type Questions

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. **[JEE Adv. 2014]**

Coefficient of x^{11} in the expansion of $(1 + x^2)^4\,(1 + x^3)^7\,(1 + x^4)^{12}$ is

(A) 1051	(B) 1106
(C) 1113	(D) 1120

2. **[JEE Adv. 2010]**

For $r = 0, 1, \ldots, 10$, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

(A) $B_{10} - C_{10}$

(B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$

(C) 0

(D) $C_{10} - B_{10}$

Multiple Correct Choice Type Questions

1. **[JEE Adv. 2020]**

For non-negative integers s and r,

$$\text{let } \binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$$

For positive integers m and n, let $g(m, n) = \sum_{p=0}^{m+n} \dfrac{f(m,n,p)}{\binom{n+p}{p}}$ where for any non-negative integer p,

$$f(m, n, p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}.$$

Then which of the following statements is/are TRUE?

(A) $g(m, n) = g(n, m)$ for all positive integers m, n

(B) $g(m, n + 1) = g(m + 1, n)$ for all positive integers m, n

(C) $g(2m, 2n) = 2g(m, n)$ for all positive integers m, n

(D) $g(2m, 2n) = (g(m, n))^2$ for all positive integers m, n

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. **[JEE Adv. 2019]**

$$\text{Suppose det}\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$

Holds for some positive integer n. Then $\sum_{k=0}^{n} \dfrac{{}^nC_k}{k+1}$ equals ___________

2. **[JEE Adv. 2018]**

Let $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r$, $r \in \{1,2,\ldots,10\}$ denote binomial coefficients. Then, the value of $\dfrac{1}{1430}X$ is ___________

3. **[JEE Adv. 2016]**

Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is

4. **[JEE Adv. 2013]**

The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** B	**3.** C	**4.** C	**5.** B	**6.** C	**7.** B	**8.** C	**9.** A	**10.** B
11. D	**12.** B	**13.** C	**14.** C	**15.** A	**16.** D	**17.** C	**18.** A	**19.** B	**20.** B
21. C	**22.** B	**23.** A	**24.** C	**25.** D	**26.** C	**27.** D	**28.** D	**29.** B	**30.** C
31. C	**32.** B	**33.** D	**34.** C	**35.** D	**36.** D	**37.** A	**38.** B	**39.** B	**40.** D
41. D	**42.** C	**43.** C	**44.** A	**45.** C	**46.** A	**47.** A	**48.** C	**49.** B	**50.** C
51. C	**52.** A	**53.** D	**54.** B	**55.** B	**56.** D	**57.** C	**58.** B	**59.** A	**60.** A
61. C	**62.** B	**63.** A	**64.** A	**65.** C	**66.** A	**67.** B	**68.** B	**69.** B	**70.** C
71. A	**72.** A	**73.** C	**74.** C	**75.** B	**76.** A	**77.** A	**78.** B	**79.** C	**80.** D
81. A	**82.** B	**83.** A	**84.** A	**85.** D					

Multiple Correct Choice Type Questions

1. A, C	**2.** A, C	**3.** C, D	**4.** A, B, D	**5.** A, B, C, D
6. A, B	**7.** A, B, C, D	**8.** A, D	**9.** A, B, C	**10.** A, B, D
11. A, B, C	**12.** A, B, C	**13.** B, C, D	**14.** A, B	**15.** A, B, C, D
16. A, C, D	**17.** B, C	**18.** C, D	**19.** A, B, C	**20.** A, B
21. A, B, C, D	**22.** A, B, D	**23.** A, C	**24.** B, C, D	**25.** C, D
26. A, B	**27.** A, B	**28.** A, B, C	**29.** A, C	**30.** A, B
31. B, C, D	**32.** A, B	**33.** A, B, C		

Linked Comprehension Type Questions

1. B	**2.** A	**3.** A	**4.** B	**5.** C	**6.** A	**7.** C	**8.** A	**9.** A	**10.** A
11. A	**12.** C	**13.** C	**14.** D	**15.** B	**16.** A	**17.** A	**18.** C	**19.** B	**20.** A
21. A	**22.** B	**23.** A	**24.** B	**25.** A	**26.** B	**27.** A	**28.** D	**29.** D	**30.** C
31. C									

Matrix Match/Column Match Type Questions

1. A → (s)	B → (p)	C → (r)	D → (q)
2. A → (s)	B → (p)	C → (q)	D → (r)
3. A → (s)	B → (r)	C → (q)	D → (p)
4. A → (s)	B → (r)	C → (p)	D → (q)
5. A → (r, s, t)	B → (s, t)	C → (q, r, s, t)	D → (p, q, r, s, t)
6. A → (p)	B → (q)	C → (t)	D → (s)

Integer/Numerical Answer Type Questions

1. 3	**2.** 9	**3.** 9	**4.** 2	**5.** 3	**6.** 4	**7.** 3	**8.** 3	**9.** 9	**10.** 4
11. 5	**12.** 7	**13.** 6	**14.** 4	**15.** 4	**16.** 3	**17.** 4	**18.** 2	**19.** 3	**20.** 0
21. 1	**22.** 6	**23.** 4	**24.** 2	**25.** 7	**26.** 1	**27.** 1	**28.** 8	**29.** 3	**30.** 9
31. 7	**32.** 9	**33.** 2	**34.** 4	**35.** 6	**36.** 8	**37.** 2	**38.** 2	**39.** 1	**40.** 5
41. 0	**42.** 0	**43.** 7							

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1. 30	**2.** 615.00	**3.** 51	**4.** 118	**5.** 8	**6.** 120.00	**7.** 13	**8.** B	**9.** C	**10.** C
11. C	**12.** B	**13.** B	**14.** C	**15.** D	**16.** B	**17.** C	**18.** A	**19.** A	**20.** A
21. B	**22.** B	**23.** A	**24.** A	**25.** A	**26.** C	**27.** A	**28.** A	**29.** D	**30.** A
31. D	**32.** C	**33.** A	**34.** D	**35.** A	**36.** A	**37.** A	**38.** B	**39.** Bonus	**40.** A
41. B	**42.** C	**43.** C	**44.** A	**45.** Bonus	**46.** C	**47.** B	**48.** C	**49.** A	**50.** B
51. D	**52.** A								

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Single Correct Choice Type Questions

1. C	**2.** D

Multiple Correct Choice Type Questions

1. A, B, D

Integer/Numerical Answer Type Questions

1. 6.20	**2.** 646	**3.** 5	**4.** 6

From the Author's Desk

“Dear Learners,

Permutation and Combination is a concept-based chapter of Algebra consisting of numerous concepts. From this chapter questions are always asked in JEE Mains and Advanced. Word problems, Circular arrangement, one or more than on selection, restricted permutation, gap method, restricted selection, Beggar method, dearrangement problems are key concepts to focus on.

It is a conceptual chapter so each question requires thought process to attempt, hence practice is the key to handle it. Students are advised to solve adequate questions covering various concepts in given time frame. Do not forget to attempt PYQs from JEE Mains and Advanced.

Good Luck!!!”

CHAPTER

6 Permutation and Combination

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Permutation based problems
(b) Combination based problems
(c) Geometry based P and C
(d) Circular permutation
(e) Multinomial theorem for selection/arrangement
(f) One or more than one selection
(g) Dearrangement

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

By the term **permutation** we mean arrangement of distinct or non-distinct things whereas by **combination** we mean selection of some or all things from distinct or non-distinct things.

FUNDAMENTAL PRINCIPLE OF COUNTING

Fundamental Principle of Addition

Suppose a work A can occur in m ways and B can occur in n ways and both cannot occur simultaneously. Then A or B can occur in $(m + n)$ ways. This rule is also applicable for two or more exclusive works.

For example: If we have to find the number of two digit numbers which either ends in 7 or are even numbers. The number of two digit numbers ending in 7 is 9 and number of two digit even numbers is 45. Hence required number is $9 + 45 = 54$

Fundamental Principle of Multiplication

Suppose there are two works A and B. Let A can occur in m ways and for each way of occurrence of A, B can occur in n ways, then both A and B can occur in mn ways.

For example: Let there be two questions A and B which can be solved by 2 and 3 methods respectively. Then both A and B can be solved in $2 \times 3 = 6$ ways.

ILLUSTRATION 1

There are 20 IITs and 25 NITs who offer engineering course

(i) find number of ways a student can opt for an engineering course

(ii) find number of ways by which a student can choose two colleges each from IITs and NITs

SOLUTION

(i) A student can opt for any college as he has to take admission in only one college so he has total ways of opting them as

$= 20 + 25 = 45$ ways (principle of addition)

(ii) student needs to choose one college from IITs and NITs separately so no. of ways = (any one college from 20 IITs)(Any one college from 25 NITs)

$= 20.25 = 500$ (Principle of multiplication)

ILLUSTRATION 2

There are 6 roads between A and B and 4 roads between B and C.

(i) In how many ways can one drive from A to C by way of B?

(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips?

(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

SOLUTION

(i) In order to drive from A to C, one can drive A to B in 6 ways and B to C in 4 ways. Thus, the number of ways to drive from A to C is $6 \times 4 = 24$ ways.

(ii) Clearly, there are 24 ways to drive from A to C. Now, in order to drive from C to A, one can drive C to B in 4 ways and B to A in 6 ways, i.e. one can drive from C to A in 24 ways.

Thus, the number of ways to drive from A to C and back to A

$= 24 \times 24 = 576$ ways

(iii) Clearly, one can drive from A to B in 6 ways and B to C in 4 ways, i.e. one can drive from A to B in 24 ways. But to come back one can drive from C to B in 3 ways, as we cannot choose the way from which we came and one can drive from B to A in 5 ways, i.e. one can drive from C to A in 15 ways.

Thus, the number of ways to drive the circular trip without using the same road

$= 24 \times 15 = 360$

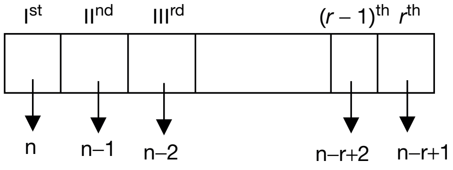
PERMUTATION

Each of the arrangements in a definite order which can be made by taking some or all of the things is called permutation.

(i) Let r and n be positive integers such that $1 \le r \le n$ then, the number of all linear permutations of n distinct objects taken r at a time is given by

$$n(n-1)(n-2).....(n-(r-1)) = \frac{n!}{(n-r)!} = {}^{n}P_r$$

Proof:



No. of ways

$= n(n-1)\,(n-2)\,\ldots\ldots\,(n-r+2)\,(n-r+1)$

$$= \frac{n(n-1)(n-2)......(n-r+2)(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!} = {}^{n}P_r$$

(ii) The number of permutations of n distinct things taken all at a time is given by ${}^{n}\mathrm{P}_n = n!$

ILLUSTRATION 3

If ${}^{2n}\mathrm{P}_4 = 396 \times {}^{n}\mathrm{P}_2$, then the values of n, is

SOLUTION

We have ${}^{2n}\mathrm{P}_4 = 396 \times {}^{n}\mathrm{P}_2$

$$\Rightarrow \quad \frac{(2n)!}{(2n-4)!} = 396 \times \frac{(n)!}{(n-2)!}$$

$$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)!}{(2n-4)!}$$

$$= 396 \times \left(\frac{n(n-1)(n-2)!}{(n-2)!}\right)$$

$\Rightarrow \quad 2^2 \cdot n(n-1)\,(2n-1)\,(2n-3) = 396 \times n(n-1)$

$\Rightarrow \quad (2n-1)\,(2n-3) = 99$

$\Rightarrow \quad 4n^2 - 8n - 96 = 0$

$\Rightarrow \quad n^2 - 2n - 24 = 0 \quad \Rightarrow \quad (n-6)\,(n+4) = 0$

$\Rightarrow \quad n = 6$ [since, n cannot be negative]

ILLUSTRATION 4

If ${}^{\lambda+m}\mathrm{P}_2 = 56$, and ${}^{\lambda-m}\mathrm{P}_2 = 12$, then $\lambda + 2m$, is

SOLUTION

We have, ${}^{\lambda+m}\mathrm{P}_2 = 56$

$$\Rightarrow \quad \frac{(\lambda+m)!}{(\lambda+m-2)!} = 56$$

$$\Rightarrow \quad \frac{(\lambda+m)(\lambda+m-1)(\lambda+m-2)!}{(\lambda+m-2)!} = 56$$

$\Rightarrow \quad (\lambda+m)(\lambda+m-1) = 56 = 8 \times 7$

$\therefore \quad \lambda + m = 8$ ...(1)

Now, consider ${}^{\lambda-m}\mathrm{P}_2 = 12$

$$\Rightarrow \quad \frac{(\lambda-m)!}{(\lambda-m-2)!} = 12$$

$$\frac{(\lambda-m)(\lambda-m-1)(\lambda-m-2)!}{(\lambda-m-2)!} = 12$$

$\Rightarrow \quad (\lambda-m)(\lambda-m-1) = 12 = 4 \times 3$

$\therefore \quad \lambda - m = 4$ ...(2)

On solving Eqs. (1) and (2), we get

$\lambda = 6$ and $m = 2$

$\therefore \quad \lambda + 2m = 10$

ILLUSTRATION 5

The value of

$1 + 1 \cdot {}^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \dots + n \cdot {}^nP_n$, is

SOLUTION

Consider, the given series

$$1 + 1 \cdot {}^1P_1 + 2 \cdot {}^2P_2 + 3 \cdot {}^3P_3 + \dots + n \cdot {}^nP_n$$

$$= 1 + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! \quad [\because {}^nP_n = n!, \forall\, n \in N]$$

$$= 1! + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! \quad [\therefore 1! = 1]$$

$$= 2! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! \quad [\therefore n! \cdot (n+1) = (n+1)!]$$

$$= 2!\,(1 + 2) + 3 \cdot 3! + \dots + n \cdot n!$$

$$= 3! + 3 \cdot 3! + \dots + n \cdot n! = n! + n \cdot n!$$

$$= n!\,(1 + n) = (n + 1)!$$

Alternate:

$1 + 1.1! + 2.2! + 3.3! + \dots + n.n!$

$$= 1 + \sum_{r=1}^{n} r.r! = 1 + \sum_{r=1}^{n} (r+1-1)r!$$

$$= 1 + \sum_{r=1}^{n} [(r+1)r! - r!] = 1 + \sum_{r=1}^{n} \left((r+1)! - r!\right)$$

$$= 1 + \left[(2! - 1!) + (3! - 2!) + (4! - 3!) + \dots + (n+1)! - n!\right]$$

$$= 1 + (n+1)! - 1 = (n+1)!$$

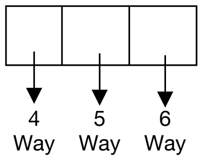
PERMUTATION PROBLEMS BASED ON NUMBERS/DIGITS

ILLUSTRATION 6

If repetitions of digits are not permitted,

(i) How many 3-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9.

SOLUTION



Clearly, units's place can be filled in 6 ways. Now, tenth place can be filled with any of remaining 5 digits, i.e. in 5 ways. Now, the hundred's place can be filled with in any of the remaining 4 digits, i.e. in 4 ways. Thus, the total number of required numbers $= 6 \times 5 \times 4 = 120$

(ii) How many of these are less than 400.

SOLUTION

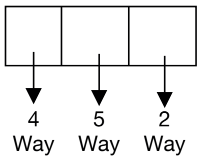
For number to be less than 400, hundred's place cannot be greater than 3. So, hundred's place can be filled with any of 2 digits 2, 3 i.e. 2 ways. Now tenth place can be filled with any of the remaining 5 digits. i.e. in 5 ways and unit place can be filled with any of the remaining 4 digit, i.e. in 4 ways.

$$= 2 \times 5 \times 4 = 40$$

(iii) How many are even.

SOLUTION

For number to be even, unit's place is 2 or 6 i.e. unit's place can be filled in 2 ways. Now, tenth place can be filled with any of the remaining 5 digits, i.e. in 5 ways and hundred's place can be filled with any of the remaining 4 digits, i.e. in 4 ways.



Thus, the number of required numbers $= 4 \times 5 \times 2 = 40$

(iv) How many are odd.

SOLUTION

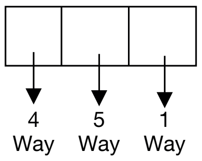
Since, total numbers are 120. and even numbers are 40.

$\therefore$ Number of required numbers $= 120 - 40 = 80$

(v) How many are multiples of 5.

SOLUTION

For number to be multiple of 5, unit's place is 5, i.e. unit's place can be filled in one way. Now, tenth place can be filled with any of the remaining 5 digits, i.e. in 5 ways and hundred's place can be filled with any of the remaining 4 digits, i.e. in 4 ways.



Thus, the number of required numbers $= 4 \times 5 \times 1 = 20$

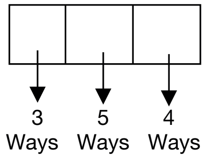
ILLUSTRATION 7

How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6, if no digit is repeated in the same number?

SOLUTION

Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in

3 ways. Now, ten's and unit's places can be filled by 5 and 4 ways respectively.



Hence, the required number of numbers

$$= 3 \cdot 5 \cdot 4 = 60$$

ILLUSTRATION 8

How many positive integers less than 10^5 can be written with only the digits 7 and 9?

SOLUTION

The number of 1 digit numbers is 2

The number of 2 digit number is 2^2

The number of 3 digit numbers is 2^3

The number of 4 digit numbers is 2^4

The number 5 digit numbers is 2^5

Total = $2 + 2^2 + 2^3 + 2^4 + 2^5 = 62$

ILLUSTRATION 9

If a seven-digit number made up of all distinct digits 8, 7, 6, 4, 2, x, and y is divisible by 3, then find

(i) maximum value of $x - y$
(ii) maximum value of $x + y$
(iii) minimum value of xy
(iv) minimum value of $x + y$

SOLUTION

8, 7, 6, 4, 2, x and y

Any number is divisible by 3 if sum of digits is divisible by 3, i.e., $x + y + 27$ is divisible by 3. x and y can take values from 0, 1, 3, 5, 9. Possible pairs are (5, 1), (3, 0), (9, 0), (9, 3) and (1, 5), (0, 3) (0, 9), (3, 9).

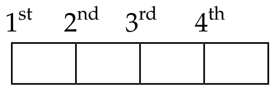
(i) maximum value of $x - y$ is 9
(ii) maximum value of $x + y$ is 12
(iii) minimum value of xy is 0
(iv) minimum value of $x + y$ is 3

ILLUSTRATION 10

Find number of four digits numbers in which repetition of digits are not allowed, even digit appear at even places, odd digit appear at odd places and the number is divisible by 4 is equal to

SOLUTION

1st, 3rd place must be filled with odd number and 2nd, 4th place must be filled with even number and last two digits must be divisible by 4 hence possibilities for 3rd and 4th place



12, 32, 52, 72, 92, 16, 36, 56, 76, 96

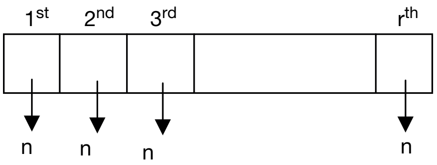
Total 10 possibilities for 3rd and 4th place

For 1st and 2nd place 4 and 4 ways respectively.
Hence total number of ways are $4 \cdot 4 \cdot 10 = 160$

PERMUTATIONS WHEN OBJECT CAN REPEAT

Number of permutations of n different objects taken r at a time when thing could be repeated any number of times in n^r.

Proof:



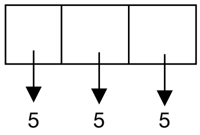
Number of ways = $n.n \ldots n = n^r$

ILLUSTRATION 11

How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

SOLUTION

The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways.



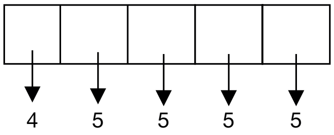
Therefore, the required number of three digits numbers = $5 \times 5 \times 5 = 125$.

ILLUSTRATION 12

Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.

SOLUTION

In five digits number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place = 4.
Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.



So, the required number of numbers = $4 \times 5 \times 5 \times 5 \times 5 = 2500$.

ILLUSTRATION 13

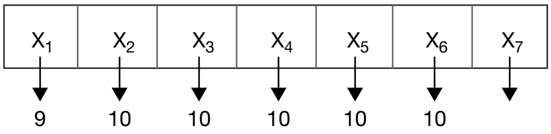
When repetition is allowed find the number of 7 digits numbers using 0 to 9

(i) such that sum of digits is even

(ii) which are palindrome

SOLUTION

(i)



if $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

$$= \begin{cases} \text{even}, x_7 = 0,2,4,6,8 \\ \text{odd}, x_7 = 1,3,5,7,9 \end{cases}$$

$\therefore$ Total number of ways $= 45 \times 10^5$

Alternate:

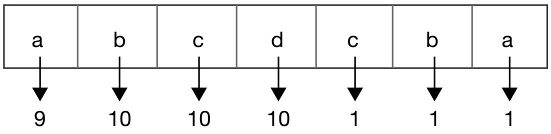
Sum of digits can either be even or odd

$$\therefore \text{ Required number } = \frac{1}{2}\begin{pmatrix}\text{Total Possible} \\ \text{Numbers}\end{pmatrix}$$

$$= \frac{1}{2}(9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10)$$

$$= 45 \cdot 10^5$$

(ii) A string (Consisting digits, alphabets etc) is called a palindrome when read from left to right and right to left, it remains same. For instance (1435341 which is a prime palindrome number)



Total palindromes = $9 \times 10^3 = 9000$

ILLUSTRATION 14

Consider a determinant $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ where $a, b, c, d \in \{0, 1\}$. If n denote the total number of determinants whose entries are 0 or 1 only and m denotes the number of determinants with non-zero values, then find the ratio m/n.

SOLUTION

Value of determinant $D = ad - bc$; $n = 2^4 = 16$; D can be non zero if

Case-I: $ad = 1$ and $bc = 0$ i.e.,

$b = 1; c = 0$

$b = 0; c = 1 \qquad \Rightarrow 3$ cases

$b = 0; c = 0$

Case-II: $bc = 1$ and $ad = 0$ i.e.,

$a = 1 ; d = 0$

$a = 0 ; d = 1 \qquad \Rightarrow 3$ cases

$a = 0, d = 0$

Hence total determinant with non-zero value = 6.

$$\therefore \quad \frac{m}{n} = \frac{6}{16}$$

ILLUSTRATION 15

Find the number of 10-digit numbers such that the product of any two consecutive digits in the number is a prime number.

SOLUTION

As per the given condition, digit 1 should occur at alternate places of the number and at the remaining 5 places either 2, 3, 5 or 7 should appear. Now when the number starts with 1, number of numbers = 4^5 and when the number starts with either 2, 3, 5 or 7, number of numbers = 4^5. So, total number = $2 \times 4^5 = 2048$

ILLUSTRATION 16

A three digits number abc is such that

(i) its hundreds digit is equal to the sum of the other two digits, and

(ii) $b(c + 1) = 52 - 4a$

Find number of such numbers.

SOLUTION

$N = abc$

$a = b + c$

$b(c + 1) = 52 - 4a$

$b(c + 1) = 52 - 4(b + c)$

$bc + 5b + 4c + 20 = 52 + 20$ (add 20 on both sides)

$(b + 4)\,(c + 5) = 72$

$$\text{Hence} \left\{\begin{matrix} b = 8 & c = 1 \\ b = 5 & c = 3 \\ b = 4 & c = 4 \\ b = 2 & c = 7 \end{matrix}\right\} \Rightarrow 4 \text{ numbers.}$$

ILLUSTRATION 17

Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Find the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$.

SOLUTION

Possible solutions are

(1, 2, 3, 4, 10), (1, 2, 3, 5, 9), (1, 2, 3, 6, 8), (1, 2, 4, 5, 8), (1, 2, 4, 6, 7), (1, 3, 4, 5, 7), (2, 3, 4, 5, 6).

Hence, 7 solutions are there.

ILLUSTRATION 18

(a) How many new words can be formed by using all the letters of the word 'MIRACLE'?

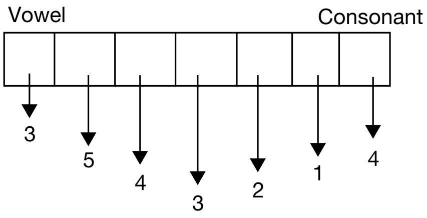
SOLUTION

Number of new words = total number of words –1

= 7! – 1

(b) How many words starts with a vowel and ending with a consonant.

SOLUTION



Vowel → IAE

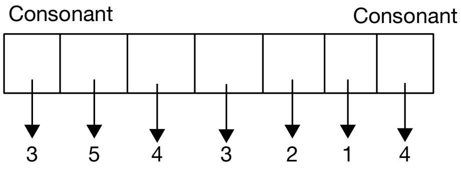
Consonant → MRCL

Number of words = 3(5.4.3.2.1)4 = 12. 5!

= 12.120 = 1440

(c) How many words starts with a consonant and ends with a consonant.

SOLUTION

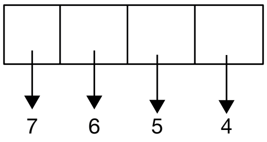


Number of words = 4.(5!) 3 = 1440

ILLUSTRATION 19

(a) How many 4 lettered word can be formed by using all the letters of the word 'HISTORY'?

SOLUTION

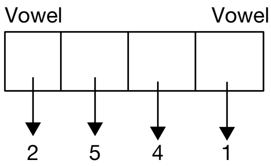


Number of words = 7.6.5.4 = 840

(b) Find number of 4-letter words which start and end with a vowel.

SOLUTION

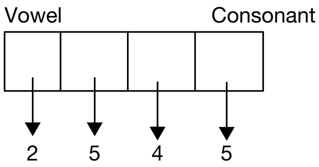
Vowels → IO Consonant → HSTRY



Number of required words = 2.5.4.1 = 40

(c) Find number of 4-letter word starts with vowel and ends with a consonant.

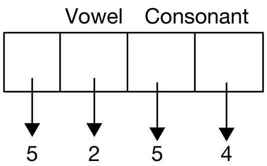
SOLUTION



Number of required words = 2.5.4.5 = 200

(d) Find number of 4-letter word whose 3rd letter is consonant and 2nd letter is vowel.

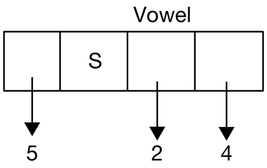
SOLUTION



Number of words = 5.2.5.4 = 10(20) = 200

(e) Find number of 4-letter word whose 2nd letter is 'S' and 3rd letter is vowel.

SOLUTION



Number of words = 5.2.4 = 40

ILLUSTRATION 20

Find the number of different words ending and beginning with a consonant which can be made out of letters of the word EQUATION.

SOLUTION

Total 8 – letters, 3 – consonants, 5 – vowels

The consonants are to occupy 1st and last place and it can be done is ${}^3P_2$ ways we will now be left with 5 vowels and 1 consonant i.e 6 letters which can be arranged in 6! Ways. Hence the number of words under given conditions is ${}^{3}P_{2}\times 6! = 4320$.

PERMUTATION WHEN NOT ALL OBJECTS ARE DISTINCT

Out of $(p+n+q)$ things in which p things are of type I and n things are of type II and q things are of type III then number of ways of arranging them in a row is $=\dfrac{(p+n+q)!}{p!n!q!}$

Where same type of things are alike (identical)

ILLUSTRATION 21

(a) How many different words can be formed by using all the letters of the word 'ALLAHABAD'?

SOLUTION

There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $=\dfrac{9!}{4!\,2!}=7560$

(b) In how many of them vowels occupy the even positions?

SOLUTION

There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in $\dfrac{4!}{4!}=1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\dfrac{5!}{2!}$ ways. = 60 ways

(c) In how many of them both L do not come together?

SOLUTION

Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\dfrac{8!}{4!}$ ways.

So, the number of words in which both L come together $=\dfrac{8!}{4!}=1680$. Hence, the number of words in which both L do not come together = Total no. of words – No. of words in which both L come together = 7560 – 1680 = 5880.

ILLUSTRATION 22

How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

SOLUTION

Any number of greater than a million will contain all the seven digits. Now, we have to arrange these seven digits, out of which 2 occurs twice, 3 occurs three times and the rest are distinct.

The number of such arrangements $=\dfrac{7!}{2!\times 3!}=420$

These arrangements also include those numbers which contain 0 at the million's place.

Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

These 6 digits can be arranged in $\dfrac{6!}{2!\times 3!}=60$ ways

Hence, the number of required numbers = 420 – 60 = 360.

Dictionary Problems

In this section we will discuss questions based on their rank according to their alphabetical preference in English, as words are mentioned in dictionary but we will not consider what is the meaning of that word so our main focus is to keep in all possibilities of arrangement of those letters and their rank in pseudo dictionary (As we don't consider the meaning of word formed that's why we are calling it pseudo).

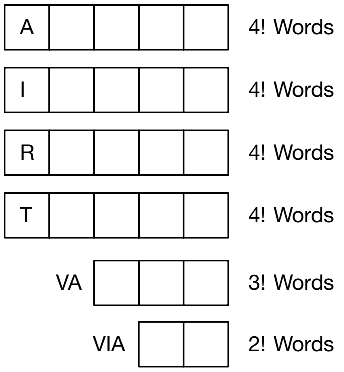
Profile (a) When all letters are distinct.

ILLUSTRATION 23

(a) Find the rank of word VIRAT.

SOLUTION

A I R T V is alphabetic order of given letters

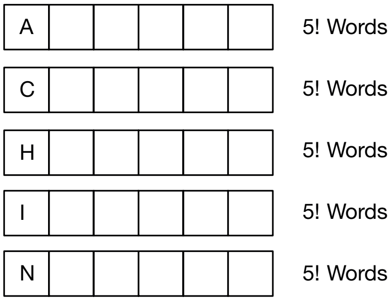


VIRAT........105th words.

(b) Find the rank of word SACHIN.

SOLUTION

A C H I N S its alphabetic order of given letters


A
5! Words
C
5! Words
H
5! Words
I
5! Words
N
5! Words

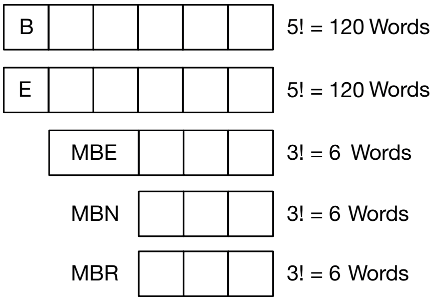
SACHIN → 601^{th} word

ILLUSTRATION 24

Find the 260^{th} word when all the letters of word 'NUMBER' are arranged in alphabetic order.

SOLUTION

BEMNRU its alphabetic order.


B
5! = 120 Words
E
5! = 120 Words
MBE
3! = 6 Words
MBN
3! = 6 Words
MBR
3! = 6 Words

MBUENR → 259^{th} word

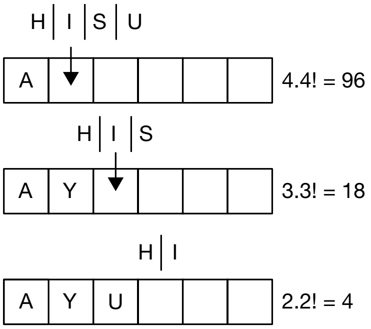
MBUERN → 260^{th} word.

ILLUSTRATION 25

(a) Find the rank of word AYUSHI.

SOLUTION

AHISUY its alphabetic order.


H | I | S | U
A
4.4! = 96
H | I | S
A Y
3.3! = 18
H | I
A Y U
2.2! = 4

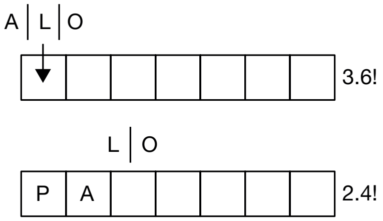
AYUSHI → 119^{th} word

(b) Find the rank of the word 'PARLOUS'.

SOLUTION

The letters of the word in alphabetical order is A, L, O, P, R, S, U

The number of words begin with


A | L | O
3.6!
L | O
P A
2.4!

PARLOSU – 1

PARLOUS – 1

∴ Rank of the word = 3.6! + 2.4! + 1 + 1 = 2210

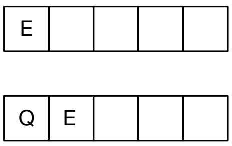
Profile (b) When few letters are identical.

ILLUSTRATION 26

Find the rank of word QUEUE.

SOLUTION

EEQUU its alphabetic order



$\frac{4!}{2!}$ = 12 (remaining letters Q, U, U, E)

$\frac{3!}{2!}$ = 3 (remaining letters E, U, U)

QUEEU is 16^{th} word

QUEUE → 17^{th} word.

ILLUSTRATION 27

Find the rank of word RAMAYAN.

SOLUTION

AAAMNRY. Its alphabetic order

A………$\frac{6!}{2!} = 360$ (remaining letters A→ 2, MNRY)

M………$\frac{6!}{3!} = 120$ (remaining letters A→ 3, NRY)

N………$\frac{6!}{3!} = 120$ (remaining letters A→ 3, MRY)

RAA……… 4! = 24 (remaining letters A, M, N, Y)

RAMAA……… 2! = 2 (remaining letters N, Y)

RAMAN………2! = 2

RAMAYAN → 629^{th}

ILLUSTRATION 28

Find the rank of word SUCCESS.

SOLUTION

CCESSSU its alphabetic order

C……… $\frac{6!}{3!} = 120$ (remaining letters $S \to 3$, C, E, U)

E……… $\frac{6!}{3!2!} = 60$ (remaining letters $S \to 3$, $C \to 2$, U)

SC……. $\frac{5!}{2!} = 60$ (remaining letters $S \to 2$, C, E, U)

SE……. $\frac{5!}{2!2!} = 30$ (remaining letters $S \to 2$, $C \to 2$, U)

SS……. $\frac{5!}{2!} = 60$ (remaining letters $C \to 2$, S, U, E)

SUCCESS → 331th word

ILLUSTRATION 29

Find the number of words that come before COCHIN when all words formed by the letters arranged in dictionary order. What is 80th word?

SOLUTION

The letters of the word in alphabetical order is C, C, H, I, N, O

C C _ _ _ _ 4!

C H _ _ _ _ 4!

C I _ _ _ _ 4!

CN _ _ _ _ 4!

C O C H I N

Hence there are 96 words coming before COCHIN

C C _ _ _ _ 4! = 24

C H _ _ _ _ 4! = 24

C I _ _ _ _ 4! = 24

C N C _ _ _ _ 3! = 6

C N H C I O = 1

C N H C O I = 1

Total Ways = 80

∴ 80th word is CNHCOI

Test Your Concepts-I

(Solutions on page H.182)

1. If ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, then find $m + n$.
2. Find the value of ${}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.
3. The number of n-digit numbers which consists of the digits 1 and 2 only if each digit is to be used atleast once, is equal to 512, then find *n*.
4. How many of the 900 three-digit numbers have atleast one even digit?
5. Find the number of natural numbers are their from 1 to 1000 which have none of their digits repeated.
6. Find the number of natural numbers between 100 and 1000 such that atleast one of their digits is 7.
7. Find number of ways in which 7 different colours in a rainbow can be arranged, if green is always in the middle.
8. Find the number of 5 lettered palindromes which can be formed using the letters from the English alphabets.
9. Find the number of ways in which the letters of the word 'MIRACLE' can be arranged, if vowels always occupy the odd place.
10. Find the number of words which can be formed using all the letters of word 'AKSHI', if each word begins with vowels or terminates in vowel.
11. The 120 permutations of MAHES are arranged in dictionary order, as if each were an ordinary five letter word, then find the last letter of the 86th word in the list.
12. Find the rank of word SALONI.
13. Find the rank of the word FABLE.
14. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits, so that odd digits occupy even positions.
15. Find the number of permutations of n different objects taken r at a time in which two specified objects always occur together.

Answer keys

1. 10 **2.** ${}^{n}P_r$ **3.** 9 **4.** 775 **5.** 738 **6.** 252

7. 6! **8.** 26^3 **9.** 576 **10.** 84 **11.** E **12.** 612

13. 74 **14.** 60 **15.** $\frac{(n-2)!}{(n-r)!} \cdot 2(r-1)$

COMBINATIONS (SELECTION)

The number of all combinations of n distinct things taken 'r' at a time is given by ${}^nC_r = \dfrac{n!}{(n-r)!r!}$

Properties of ${}^nC_r$

(i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) Let n and r be non-negative integers such that $r \le n$. Then, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) Let n and r be non-negative integers such that $1 \le r \le n$ then ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1}$

ILLUSTRATION 30

Find the number of positive integers satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$.

SOLUTION

$${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$$

$$\frac{(n+1)(n)(n-1)}{6} - \frac{(n+1)n}{2} \le 100;\ n \ge 2$$

$$\frac{n(n+1)}{6}[n-1-3] \le 100$$

$$n(n+1)\,(n-4) \le 600$$

$n = 2, 3, 4, \ldots, 9$ eight positive numbers are accepted.

ILLUSTRATION 31

Let P_n denotes the number of permutations of n distinct things taken all at a time and

$$x_n = {}^{n+5}C_4 - \left(\frac{143}{96}\right)\left(\frac{P_{n+5}}{P_{n+3}}\right) \text{(where } n \in N).$$

Find the possible value of n for which x_n is negative.

SOLUTION

$$x_n = {}^{n+5}C_4 - \left(\frac{143}{96}\right)\left(\frac{P_{n+5}}{P_{n+3}}\right)$$

given $x_n < 0$

hence ${}^{n+5}C_4 < \dfrac{143}{96} \times \dfrac{(n+5)!}{(n+3)!}$

$$\frac{(n+5)!}{4!(n+1)!} < \frac{143}{96}\,\frac{(n+5)!}{(n+3)(n+2)(n+1)!}$$

$$\frac{1}{24} < \frac{143}{96} \cdot \frac{1}{(n+3)(n+2)}$$

$$\Rightarrow 4(n+3)(n+2) < 143 \Rightarrow 4(n^2 + 5n + 6) - 143 < 0$$

$$\Rightarrow 4n^2 + 20n - 119 < 0 \Rightarrow 4n^2 + 34n - 14n - 119 < 0$$

$$\Rightarrow 2n(2n + 17) - 7(2n + 17) < 0$$

$$\Rightarrow (2n + 17)\,(2n - 7) < 0 \Rightarrow n \in \left(-\frac{17}{2}, \frac{7}{2}\right)$$

$\therefore n \in \{1, 2, 3\}$ are accepted values.

ILLUSTRATION 32

(a) In how many ways can a team of 3 boys and 4 girls can be formed among 5 boys and 6 girls

SOLUTION

Required number of ways $= {}^5C_3 \,.\, {}^6C_4 = 10.15 = 150$

(b) A term of 6 person is to be formed such that boys are in majority

SOLUTION

There are two possibilities

(i) 4 Boys, 2 Girls → ${}^5C_4 .\, {}^6C_2 = 5.15 = 75$

(ii) 5 Boys, 1 Girls → ${}^5C_5 .\, {}^6C_1 = 6$

Required number of ways = 75 + 6 = 81 ways

(c) A team of 4 boys and 2 girls is to be formed such that, a particular boy and a particular girl refuse to team up together

SOLUTION

Required ways = Total – (both are in team)

$$= {}^5C_4 . {}^6C_2 - ({}^4C_3 . {}^5C_1)$$

$$= 5.15 - (20) = 75 - 20 = 55 \text{ ways}$$

ILLUSTRATION 33

16 players are available for a cricket match, find no. of ways of selecting team if

(a) 3 particular players are always in team and 2 players are injured,

SOLUTION

3 Player are always in team, its means only 8 to be selected. And 2 are injured so available player are 11 number of ways $= {}^{11}C_8$

(b) 3 particular players are always in team and 2 other particular players do not play the same match

SOLUTION

Required ways

= (Total ways when 3 particular players are in team) – (Other particular 2 player are always in team)

$= {}^{13}C_8 - {}^{11}C_6$ (6 players are to be taken from remaining 11)

$= {}^{13}C_8 - {}^{11}C_6$

ILLUSTRATION 34

ARTICLE

(a) How many 4-lettered words can be formed using given word?

SOLUTION

No. of 4-lettered words can the formed using given word $= {}^7C_4 . 4! = 840$

(b) How many 4-lettered words containing 2 vowels and 2 consonants?

SOLUTION

No. of required ways

$$= \underbrace{{}^3C_2}_{\text{Vowel selection}} . \underbrace{{}^4C_2}_{\text{Consonant selection}} . \underbrace{4!}_{\text{arrangement}} = 432$$

(c) How many words can be formed using all the letters when all vowels together.

SOLUTION

$\underbrace{\boxed{\text{AIE}}}_{\text{Consider it as one}}$ RTCL

Number of words $= 5! \times (3!) \rightarrow$ Internal arrangement of vowels

$= 720$

(d) Number of words using all the letters when all consonants are not together.

SOLUTION

A,I,E, $\boxed{\text{RTCL}}$

Required number of words = total words – (All consonants are together)

$= 7! - 4! \times 4! \rightarrow$ Internal arrangement of consonants

$= 5040 - 576 = 4464$

ILLUSTRATION 35

A committee of 5 is to be formed out of 6 gentlemen and 4 ladies. In how many ways this can be done, when

(i) at least two ladies are included

(ii) at most two ladies are included?

SOLUTION

(i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways :

Case I

Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

Case II

Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

Case III

Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, therefore, by the fundamental principle of addition, the total number of ways of forming the committee

$$= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$

$$= 120 + 60 + 6 = 186$$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways:

Case I

Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

Case II

Selecting 4 gents out of 6 and 1 lady out of 4 then number of ways $= {}^6C_4 \cdot {}^4C_1$

Case III

Selecting 3 gents only out of 6 and two ladies out of 4. This can be done in ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so, the total number of ways of forming the committee

$$= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186$$

ILLUSTRATION 36

Find total number of ways of selecting two numbers from the set {1, 2, 3, 4, ….., 90} so that their sum is divisible by 3.

SOLUTION

Given numbers can be rearranged as

$3k + 1$ type	1	4	7……..88	$\rightarrow$ 30
$3k + 2$ type	2	5	8……..89	$\rightarrow$ 30
$3k$ type	3	6	9……..90	$\rightarrow$ 30

That means we must take two numbers from last row or one number each from first and second row.

$$\text{Total ways} = \underbrace{{}^{30}C_2}_{\text{Both } 3k \text{ type}} + \underbrace{{}^{30}C_1 \cdot {}^{30}C_1}_{(3k+1 \text{ and } 3k+2 \text{ type})} = 435 + 900 = 1335$$

ILLUSTRATION 37

All possible two-factor products are formed from the numbers 1, 2, 3 …. 100. How many multiples out of the total obtained are multiples of 3?

SOLUTION

The total number of two-factor products $= {}^{100}C_2$

Out of the numbers 1, 2, 3, ….100; the multiples of 3 are 3, 6, 9, …99; i.e. there are 33 multiples of 3, and therefore there are 67 non-multiples of 3.

$\therefore$ The number of two-factor products which are not multiples of $3 = {}^{67}C_2$

$\therefore$ The required number $= {}^{100}C_2 - {}^{67}C_2$

$$= 4950 - 2211 = 2739$$

Alternate

The number of two-factor products formed when both factors are multiples of $3 = {}^{33}C_2$ and the number of two-factors products formed when one is a multiple of 3 and the other a non-multiple of $3 = {}^{33}C_1 \times {}^{67}C_1$ one is a

In either case the product is a multiple of 3.

$\therefore$ The required number $= {}^{33}C_2 + 33 \times 67$

$$= 528 + 2211 = 2739$$

PERMUTATION AND COMBINATION WITH RESTRICTION PROBLEMS

ILLUSTRATION 38

Using all the letters of word 'GANAPATI' find

(a) Total number of new words

SOLUTION

G N P T I A A A

Total new words $= \dfrac{8!}{3!} - 1$ (given word is one arrangement)

$= 6719$

(b) Number of words which starts with G and ends with I

SOLUTION

G	..	.	..	.	..	.	I

(Remaining letters A, A, A, N, P, T)

Required number of words $= \dfrac{6!}{3!} = 120$

(c) Number of words either start with G or ends with I

SOLUTION

Required number of words

= number of words starts with G + number of words ends with I – number of words starts with G and ends with I (They are present in both cases)

$$= \frac{7!}{3!} + \frac{7!}{3!} - \frac{6!}{3!} = 1560$$

(d) Number of words either start with G or ends with I but do not starts with G and ends with I together

SOLUTION

Required number of words $= \dfrac{7!}{3!} + \dfrac{7!}{3!} - 2\left(\dfrac{6!}{3!}\right)$

$$= 1440$$

(e) Number of words neither starts with G nor ends with I

SOLUTION

Number of words neither starts with G nor ends with I

= Total number of words – either starts with G or ends with I

$$= \frac{8!}{3!} - \left(\frac{7!}{3!} + \frac{7!}{3!} - \frac{6!}{3!}\right) = 6720 - 1560 = 5160$$

(f) Number of words when all A's are together.

SOLUTION

$\boxed{\text{AAA}}$ G N P I T

Internal arrangement of AAA is 1

Number of required words $= 6! \times 1 = 6! = 720$

(g) Number of words when G, N, P are together.

SOLUTION

$\boxed{\text{GNP}}$ AAATI

Number of words $= \dfrac{6!}{3!} \times (3!) \rightarrow$ Internal arrangement of G, N, P

$$= 6! = 720$$

(h) Number of words when G, N, P are together and in given order

SOLUTION

$\boxed{\text{GNP}}$ AAATI

As G N P are together and remain in given order hence we should not consider other internal arrangement of G N P

So required number of words $= \dfrac{6!}{3!} \times 1$

$$= 120.$$

RELATIVE ORDER PROBLEMS

(i) Number of words when all A come before I

SOLUTION

Here relative order of A A A and I is given that all A's appear before I, so first we should satisfy the constraint given in question, choose 4 places out of 8 places and then satisfy their sequence of relative order

–A – A –A – I –

Number of required words

Arrangement of G, N, P, T

$$= {}^8C_4 \times 1 \times 4!$$

One ways of preference

Selection of 4 places for 3A, I

$= 1680$

(j) Number of words when T comes before G, N and P letters

SOLUTION

$$- \boxed{T} - G - N - P -$$

Number of required words

G, N, P can shuffle their position among them

$$= {}^8C_4 \times 3! \times \left(\frac{4!}{3!}\right) \rightarrow \text{Arrangement of (AAA, I)}$$

Selection of 4 places for T, G, N, P

$= 1680$

(k) Number of words when P comes before G and N

SOLUTION

$$-P- -G- N-$$

Number of required words

arrangement of G & N

$$= {}^8C_3 \times 2! \times \left(\frac{5!}{3!}\right) \rightarrow \text{Arrangement of remaining letters T I A A A}$$

Selection of 3 places for P, G, N

$= 2240$

GAP METHOD

(l) Number of words where G, N, P are separated from each other

SOLUTION

Here we want G, N, P to be separated from each other so first we should arrange remaining alphabets then arrange these alphabets in gaps among them

$$\times A \times A \times T \times I \times A \times$$

Number of words

$$= \left(\frac{5!}{3!}\right) \times {}^6C_3 \times 3! \rightarrow \text{Arrangement for G, N, P}$$

Gap selection for G, N, P

Arrangement of remaining letters (A → 3, T, I)

$= 2400$

(m) Number of words when vowels are separated from each other.

SOLUTION

$$\times G \times N \times P \times T \times$$

required number of ways

$$= 4! \times {}^5C_4 \times \left(\frac{4!}{3!}\right) \rightarrow \text{Arrangement of vowels AAAI}$$

Gap selection

Arrangement of remaiming letters (GNPT)

$= 480$

(n) Number of words when two A's are together but separated from 3rd A

SOLUTION

Let's consider both AA together $\boxed{AA}$ and separated from third A

$$\times G \times N \times P \times T \times I \times$$

Number of required words

Arrangement of $\boxed{AA}$ & A

$$= (5!) \times {}^6C_2 \times 2!$$

Gap selection

Arrangement of remaining letters (GNPTI)

$= 1800$

ILLUSTRATION 39

Using all the letters of word 'INTERMEDIATE' find

(a) Number of total words

SOLUTION

$$I \rightarrow 2,\ T \rightarrow 2,\ E \rightarrow 3,\ M, N, R, D, A$$

$$\text{Total words} = \frac{12!}{2!2!3!}$$

(b) Number of total words where no two vowels are together

SOLUTION

Vowels → E, E, E, I, I, A

Reaming letters → M, N, R, D, T, T

$$\times M \times N \times R \times D \times T \times T \times$$

Number of required words

$$= \underbrace{\left(\frac{6!}{2!}\right)}_{\substack{\text{Arrangement of}\\ \text{remaining letters}\\ \text{(MNRDT,T)}}} \times \underbrace{{}^7C_6}_{\substack{\text{Gap selection}\\ \text{for vowels.}}} \times \overbrace{\left(\frac{6!}{3!2!}\right)}^{\substack{\text{Arrangement}\\ \text{of vowels}}}$$

$= 151200$

(c) Number of total words where vowels and consonants are alternate

SOLUTION

$\times \text{M} \bullet \text{N} \bullet \text{R} \bullet \text{D} \bullet \text{T} \bullet \text{T} \times$

Number of required words

$$= \underbrace{\left(\frac{6!}{2!}\right)}_{\substack{\text{Arrangement of}\\ \text{Consonants}}} \times \underbrace{\left({}^2C_1\right)}_{\substack{\text{Middle gaps must be filled}\\ \text{to have them in alternate, from 2}\\ \text{corners we choose one place.}}} \times \overbrace{\left(\frac{6!}{3!\,2!}\right)}^{\substack{\text{Arrangement}\\ \text{of vowels}}}$$

$= 43200$

ILLUSTRATION 40

10 persons are sitting in a row find number of ways of selecting 3 persons such that no two of them are consecutive.

SOLUTION

After selecting three persons there are going to be 7 persons left. For selecting 3 persons not to be consecutive they should come from the gaps among 7 seated persons

$$\times P_1 \times P_2 \times P_3 \times P_4 \times P_5 \times P_6 \times P_7 \times$$

Hence required number of ways $= {}^8C_3 = 56$.

ILLUSTRATION 41

In a row, ten chairs are placed, find number of ways of sitting 4 persons, such that no two of them are sitting consecutive.

SOLUTION

Here we need 4 persons to be seated on chairs such that no two of them are consecutive it means there are going to be 6 vacant chairs so there are 7 gaps and 4 persons to be seated there

$$\times C_1 \times C_2 \times C_3 \times C_4 \times C_5 \times C_6 \times$$

Number of required ways $= {}^7C_4.(4!) \rightarrow$ Arrangement of persons

$= 1680$

ILLUSTRATION 42

Find number of words from using all the letters of given word ARRANGE such that relative order of vowels and consonants remains unchanged.

SOLUTION

We want that relative order of vowels and consonants remains unchanged, it means vowels and consonant should be rearranged among themselves.

ARRANGE Their sequence of order V C C V C C V

Vowels(V) → A, A, E Consonant(C) → R, R, N, G

Number of required words

$$= \underbrace{\left(\frac{3!}{2!}\right)}_{\substack{\text{Arrangement of}\\ \text{vowels}}} \underbrace{\left(\frac{4!}{2!}\right)}_{\substack{\text{Arrangement of}\\ \text{consonants}}}$$

$= 36$

ILLUSTRATION 43

15 persons, amongst whom are A, B and C are to speak at a function. Find in how many ways can the speech be done if A wants to speak before B and B is to speak before C?

SOLUTION

It is a relative order problem. We choose 3 place for A, B and C by ${}^{15}C_3$ and arranging remaining 12 persons by 12! so required number of ways $= {}^{15}C_3 \cdot 12!$

ILLUSTRATION 44

Number of different words that can be formed by using all letters of word APPLICATION such that no two vowels are together is

SOLUTION

Vowels A A I I O
Consonant P P L C T N
First arrange consonants

$$\times P \times P \times L \times C \times T \times N \times$$

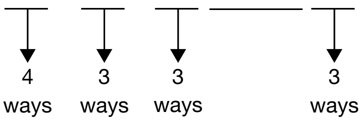
$$\text{No. of ways} = \underbrace{\left(\frac{6!}{2!}\right)}_{\substack{\text{Arrangement of}\\ \text{Consonants}}} \times \underbrace{{}^7C_5}_{\text{Gap selection}} \times \overbrace{\left(\frac{5!}{2!2!}\right)}^{\substack{\text{Arrangement}\\ \text{of vowels}}} = (45)7!$$

ILLUSTRATION 45

If P_n denotes the number of n-letter words which can be formed by using all the letter of the word 'GOOGLE' under the condition that no two consecutive letters are identical, then the value of $\frac{P_n}{P_{n-1}}$ is (where unlimited supply of each letter of the above word is given)

SOLUTION

There are only 4-different letters G, O, L, E



$$\therefore\ P_n = 4.3^{n-1} \Rightarrow P_{n-1} = 4.3^{n-2}$$

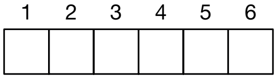
$$\therefore\ \frac{P_n}{P_{n-1}} = 3$$

ILLUSTRATION 46

Find number of ways in which 3 boys and 3 girls can be seated on a line where two particular girls do not want to sit adjacent to a particular boy.

SOLUTION

Let B_1 be the boy and G_1, G_2 be the girls who do not want to sit with B_1



(i) If B_1 occupies first place then G_1 or G_2 cannot sit on second place

Number of ways = ${}^4P_2 \times 3! = 72$ ways

If B_1 occupies 6th place

Similarly number of ways = 72 ways

(ii) Now if B_1 sit at 2nd place,

Number of ways = ${}^3P_2 \times 3! = 36$ ways

Similarly B_1 can sit on 3rd, 4th, 5th place in same number of ways = 36

$\Rightarrow$ Total ways = $72 \times 2 + 36 \times 4 = 288$ ways

ILLUSTRATION 47

In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?

SOLUTION

In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below:

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here E denotes the position of an English book and × denotes gaps between them.

Since there are 21 books on English, the number of gaps are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical. Hence, the required number of ways = $1 \times {}^{22}C_{19} \times 1 = 1540$.

ILLUSTRATION 48

There are 15 seats in the first row of a cinema hall. The torch man has the instruction that the seat number 6 must be occupied. Find the number of ways in which 4 seats of the first row can be allotted so that no two of them are consecutive.

SOLUTION

As seat number 6 must be occupied so leave seat numbers 5, 6 and 7 and now consider seat numbers 1 to 4 and 8 to 15. The number of ways of choosing 3 seats out of 12 seats so that no two are consecutive 9 seats will be vacant and 3 filled and those 3 filled seats are from gaps of 9 vacant seats.

$$= {}^{10}C_3 = 120$$

Also, the number of ways in which seat numbers 4 and 8 are occupied at the same time is $2 + 6 = 8$.

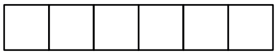
Thus, the required number of ways is 128.

ILLUSTRATION 49

Find number of ways in which 3 men and their wives can be arranged in a line such that none of the 3 men stand in a position that is ahead of his wife.

SOLUTION

It is a relative order problem



$$— W_1 — H_1 —$$
$$— W_2 — H_2 —$$
$$— W_3 — H_3 —$$

${}^6C_2$ (for H_1 W_1) and ${}^4C_2$ (for H_2 W_2) and H_3 W_3 will have only one way

Hence total ways $({}^6C_2 \cdot 1)\ ({}^4C_2 \cdot 1) = \frac{6!}{2!2!2!} = 90$

Test Your Concepts-II

(Solutions on page H.183)

1. Find the value of $\frac{{}^{20}C_n}{{}^{25}C_m}$, when both numerator and denominator have their greatest values.
2. There are 9 children comprising of 3 boys and 6 girls. Let m denotes the number of ways when the 3 boys are in succession and n denotes the corresponding figure when the two end position are occupied by boys. If $m = kn$ then find k.
3. A student has to answer 10 questions, choosing at least 4 from each of part A and part B. If three are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions?
4. There are 20 questions in a question paper. If no two students solve the same combination of questions but solve equal number of questions then find the maximum number of students who appeared in the examination.
5. Find the number of all three elements subsets of the set $\{a_1, a_2, ..., a_n\}$ which contains a_3.
6. How many words can be made with the letters of the word "GENIUS" if each word neither begins with G nor ends in S?
7. Find the sum of all the numbers which can be formed by using 1, 3, 5, 7 all at a time and which have no digit repeated.
8. Find the number of ways in which four persons can be accommodate in 3 different chairs, if each person can occupy only one chair.
9. If m men and n women are to be seated in a row so that no two women sit together. If $m > n$, then find the number of ways in which they can be seated.
10. A new flag is to be designed with six vertical strips using some or all of the colours yellow, green, blue and red. Then, find the number of ways this can be done such that no two adjacent strips have same colour.
11. Number of 9-digit numbers divisible by nine digits from 0 to 9, if each digit is used at most once is $k\cdot(8!)$, then find k.
12. The tamer of wild animals has to bring one by one 5 lions and 4 tigers to the circus area. Find the number of ways this can be done, if no two tigers immediately follow each other.
13. Find the number of different seven-digit numbers that can be written using only three digits 1, 2 and 3 under the condition that the digit 2 occurs exactly twice in each number.
14. A letter lock consists of three rings each marked with 10 different letters. Then find the number of ways in which it is possible to make an unsuccessful attempt to open the lock.
15. It is required to seat 5 men and 4 women in a row, so that the women occupy the even places, find the number of such arrangements.
16. Find the number of arrangements of the letters of the word BANANA in which the two N's do not appear adjacently.

Answer keys

1. $\frac{143}{4025}$
2. 1
3. 266
4. ${}^{20}C_{10}$
5. ${}^{n-1}C_2$
6. 504
7. $16 \times 1111 \times 3!$
8. 24
9. $\frac{m!(m+1)!}{(m-n+1)!}$
10. 12×81
11. 17
12. $5!\,{}^6P_4$
13. 672
14. 999
15. 2880
16. 40

P AND C IN GEOMETRY-BASED PROBLEMS

ILLUSTRATION 50

12 points are in a plane such that 5 of them are collinear then find total number of

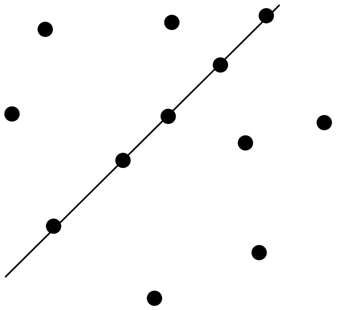
(a) straight lines **(b)** triangles

(c) quadrilaterals **(d)** Pentagon

SOLUTION

(a)

$$\text{No. of Straight lines} = {}^{12}C_2 - \underset{\substack{\downarrow \\ \text{Possibilities from} \\ \text{collinear point}}}{{}^{5}C_2} + \overset{\substack{\text{Collinear line} \\ \uparrow}}{1} = 57$$



(b) No. of Triangles = Total Possibilities − possibilities from Collinear Points

$= {}^{12}C_3 - {}^{5}C_3 = 210$

(c) Quadrilateral is not formed when

Case I:

All four point from collinear points $= {}^{5}C_4$

Case II:

3 from collinear point and one from non-collinear Points $= {}^{5}C_3 \cdot {}^{7}C_1$

No. of Quadrilaterals

= Total Possibilities – Possibilities from Collinear Points

$= {}^{12}C_4 - ({}^{5}C_4 + {}^{5}C_3 . {}^{7}C_1) = 420$

(d) Pentagon is not formed

Case I:

All five points are from collinear points $= {}^{5}C_5$

Case II:

Four points are from collinear points and one from non-collinear points $= {}^{5}C_4 \cdot {}^{7}C_1$

Case III:

Three points are from collinear points and two from non-collinear points $= {}^{5}C_3 \cdot {}^{7}C_2$

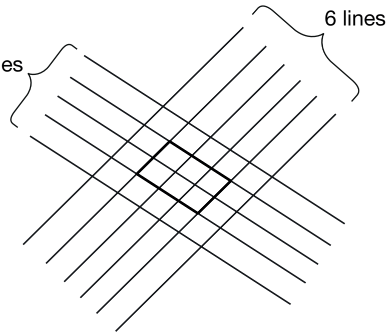
Required number of pentagon formed $= {}^{12}C_5 - ({}^{5}C_5 + {}^{5}C_4 \cdot {}^{7}C_1 + {}^{5}C_3 \cdot {}^{7}C_2)$

$= 792 - (1 + 35 + 210) = 546$

ILLUSTRATION 51

Five parallel lines are drawn in one direction and 6 parallel lines are drawn in other direction. Find number of parallelograms formed due to their intersection

SOLUTION



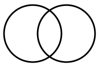
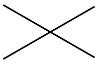
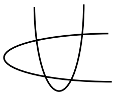
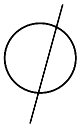
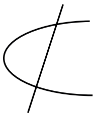
Parallelogram is made of two pairs of parallel lines so no. of parallelogram $= {}^{5}C_2 . {}^{6}C_2 = 10.15 = 150$

ILLUSTRATION 52

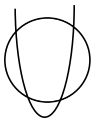
4 distinct circles, 3 distinct parabolas and 5 distinct straight lines are present in plane. Find maximum number of points of intersections.

SOLUTION

For maximum point of intersection, following cases are formed.

Circle-Circle		No. of point of intersection $= {}^{4}C_2 . 2 = 12$
Straight line-Straight line		No. of Point of intersection $= {}^{5}C_2 . 1 = 10$
Parabola – Parabola		No. of Point of intersection $= {}^{3}C_2 . 4 = 12$
Circle – Straight Line		No. of Point of intersection $= {}^{4}C_1 . {}^{5}C_1 . 2 = 40$
Straight Line – Parabola		No. of Point of intersection $= {}^{5}C_1 . {}^{3}C_1 . 2 = 30$

Circle – Parabola



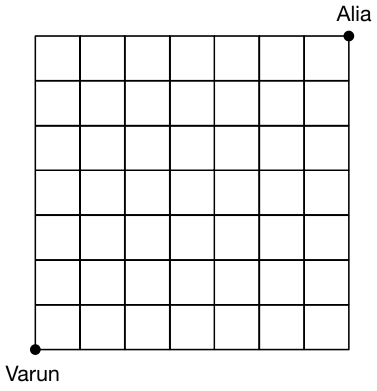
No. of Point of intersection = ${}^4C_{1.}{}^3C_{1.}4$ = 48

Total maximum point of intersections = 152

ILLUSTRATION 53

Varun and Alia are shooting for a film in a city and then accidently lost their way. Varun ask Alia to share her location and ask her to not move from her location. Given picture tells their current position so now Varun walks only right side and upwards and wants to meet Alia in shortest path then find

(i) Number of ways so that they can meet
(ii) Number of rectangles
(iii) Number of squares
(iv) Number of rectangular which are not square



SOLUTION

(i) For varun to meet allia varun has to move 7 horizontal(H) and 8 vertical(V) blocks. So number of ways = $\frac{15!}{8!(7!)}$ = (number of ways of arrangements of 7H, 8V)

(ii) Number of rectangles = ${}^8C_2.\,{}^9C_2 = 1008$

(iii) Number of squares

$$= \underbrace{7\times8}_{\text{Size}1\times1} + \underbrace{6\times7}_{\text{Size}2\times2} + \underbrace{5\times6}_{\text{Size}3\times3} + \underbrace{4\times5}_{\text{Size}4\times4} + \overbrace{3\times4}^{\text{Size }5\times5} + \underbrace{2\times3}_{\text{Size}6\times6} + \underbrace{1\times2}_{\text{Size}7\times7}$$

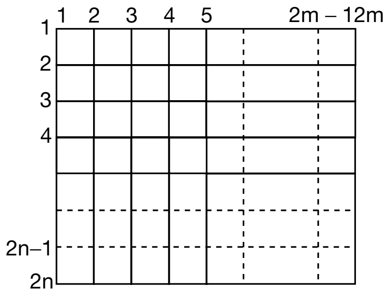
$= 168$

(iv) Number of rectangles which are not square

$= 1008 - 168 = 840.$

ILLUSTRATION 54

A rectangle with sides $2m - 1$ and $2n - 1$ is divided into squares of unit length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is



SOLUTION

If we see the blocks in terms of lines, then there are $2m$ vertical lines and $2n$ horizontal lines.

To from the required rectangle we must select two horizontal lines, one even numbered (out of 2, 4, …, $2n$) and one odd numbered (out of 1, 3, …., $2n - 1$) and similarly two vertical lines.

The number of rectangles is

$${}^mC_1 \times {}^mC_1 \times {}^nC_1 \times {}^nC_1 = m^2n^2$$

ILLUSTRATION 55

In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70 then find number of diagonals of the polygon?

SOLUTION

4 points construct a quadrilateral and one point of intersection of diagonals always lies inside the quadrilateral.

Number of diagonals = ${}^nC_4 = 70$

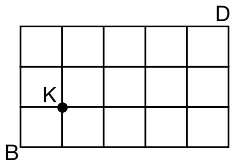
$n(n-1)\,(n-2)(n-3) = 70 \times 24$

$n(n-1)\,(n-2)\,(n-3) = 8 \times 7 \times 6 \times 5 \Rightarrow n = 8$

Now number of diagonals = ${}^nC_2 - n(\text{sides})$
$= {}^8C_2 - 8 = 28 - 8 = 20$

ILLUSTRATION 56

Lets create a scene from Bhahubali movie. Devsena is trapped at position D and Bhahubali is at point B. Bhahubali decides to reach to Devsena but Kattappa is positioned at point K and he cannot move from this position K. Then find number of different routes from B to D in the figure if Bhahubali can move only up or to the right and by avoiding position of Kattappa.



SOLUTION

We can transform above problem as Bhahubali need s to move 5H(horizontal) or 3V(vertical) but he need to avoid position K

= total ways – ways through position K

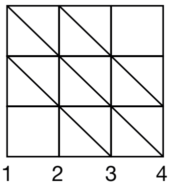
= (arrangement of 5 H and 3 V)–(2 ways from B to K then arrangement of 4H and 2 V)

$$= {}^8C_3 - ({}^2C_1 \cdot {}^6C_2) = 56 - 30 = 26$$

ILLUSTRATION 57

Find number of triangles whose vertices are points in the xy-plane with integer coordinates (x, y) satisfying $1 \le x \le 4$ and $1 \le y \le 4$.

SOLUTION



No. of triangles possible

= Total possibilities – collinear points on diagonal – collinear points on horizontal and vertical lines

$$= {}^{16}C_3 - 2[2 \times {}^3C_3 + {}^4C_3] - 8 \times {}^4C_3 = {}^{16}C_3 - 12 - 32$$

$$= 560 - 44 = 516$$

ILLUSTRATION 58

In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both A and B, and no two are parallel. Then find the number of intersections points the lines have.

SOLUTION

In the general position, 37 straight lines have ${}^{37}C_2$ points of intersection. But 13 straight lines passing through the point A yield one intersection point instead of ${}^{13}C_2$ and 11 straight lines passing through the point B yield one intersection point instead of ${}^{11}C_2$.

∴ Required point of intersections

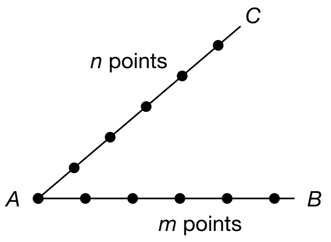
$$= {}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + \underset{\hookrightarrow \text{A \& B}}{2}$$

$$= 666 - 78 - 55 + 2 = 535$$

ILLUSTRATION 59

There are m points on one straight line AB and n points on another straight line AC, none of them being A. How many triangles can be formed with these points as vertices? How many can be formed if point A is also included?

SOLUTION



To get a triangle, we either take two points on AB and one point on AC, or one point on AB and two points on AC. Therefore, the number of triangles we obtain

$$= ({}^mC_2)\,({}^nC_1) + ({}^mC_1)\,({}^nC_2)$$

$$= \frac{m(m-1)}{2}n + m\frac{n(n-1)}{2}$$

$$= \frac{1}{2}mn\,(m - 1 + n - 1) = \frac{1}{2}mn\,(m + n - 2)$$

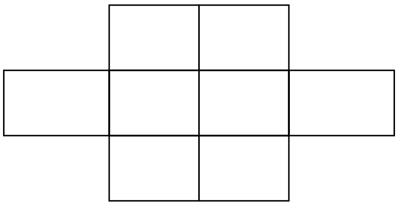
If the point A is included, we get ${}^mC_1\,{}^nC_1$ additional triangles.

Thus, in this case we get $= \dfrac{mn}{2}(m + n - 2) + mn$

$$= \frac{mn(m+n)}{2} \text{ triangles}$$

ILLUSTRATION 60

Number of ways 5X can be placed such that no row remains empty.



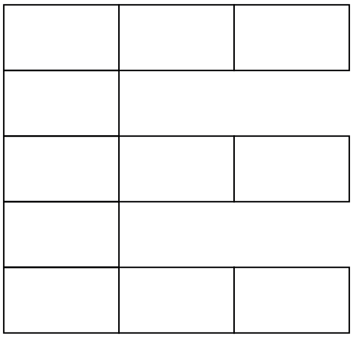
SOLUTION

Required ways = Total – First or third row can remain empty

$$= {}^8C_5 - ({}^6C_5 + {}^6C_5) = \frac{8.7.6}{6} - (12) = 56 - 12 = 44$$

ILLUSTRATION 61

Arrange AAA BBB such that no two rows remain empty.



SOLUTION

2nd and 4th row always get filled. Among 1st, 3rd, 5th row, only one row will have 2 alphabets and remaining two rows have one alphabet each.

So number of ways.

$$= \underbrace{({}^3C_1.{}^3C_2)}_{\text{Selection of row with 2 alphabets.}}\overbrace{{}^3C_1.{}^3C_1}^{\text{Selection of one place from remaining two rows for one alphabet.}} \times \underbrace{\left(\frac{6!}{3!3!}\right)}_{\text{Arrangement of AAABBB}}$$

ILLUSTRATION 62

Let a, b are natural numbers such that their LCM is 166320 and HCF is 792, then find number of possible ordered pairs (a, b).

SOLUTION

$$166320 = 2^4.3^3.7.5.11$$

$$792 = 2^3.3^2.11$$

$$a = 2^{x_1}\ 3^{y_1}\ 7^{z_1}\ 5^{u_1}\ 11^{w_1}$$

$$b = 2^{x_2}\ 3^{y_2}\ 7^{z_2}\ 5^{u_2}\ 11^{w_2}$$

$$\left.\begin{array}{l} (x_1, x_2) \to (3,4)(4,3)\ 2 \text{ ways} \\ (y_1, y_2) \to (2,3)(3,2)\ 2 \text{ ways} \\ (z_1, z_2) \to (1,0)(0,1)\ 2 \text{ ways} \\ (u_1, u_2) \to (1,0)(0,1)\ 2 \text{ ways} \\ (w_1, w_2) \to (1,1)\ 1 \text{ way} \end{array}\right\} 16 \text{ ordered pair}$$

ILLUSTRATION 63

If r, s, t are different prime numbers and a, b, c are the positive integers such that LCM of a, b, c is $r^2\ s^4\ t^2$ and HCF of a, b, c is rs^2t, then find number of ordered triplets (a, b, c).

SOLUTION

$$\begin{array}{lll} a & b & c \end{array}$$

$$\left.\begin{array}{llll} r & r & r^2 & 3\text{ ways} \\ r & r^2 & r^2 & 3\text{ ways} \end{array}\right\} \Rightarrow 6 \text{ ways}$$

$$\left.\begin{array}{llll} t & t & t^2 & 3\text{ ways} \\ t & t^2 & t^2 & 3\text{ ways} \end{array}\right\} \Rightarrow 6 \text{ ways}$$

$$\left.\begin{array}{llll} s^2 & s^2 & s^4 & 3\text{ ways} \\ s^2 & s^3 & s^4 & 6\text{ ways} \\ s^2 & s^4 & s^4 & 3\text{ ways} \end{array}\right\} \Rightarrow 12 \text{ ways}$$

Total ways $= 6 \times 6 \times 12 = 432$

ILLUSTRATION 64

Two boys and six girls are officers of a Math club. When the photographer takes a picture for the school yearbook, she asks the eight officers of the club and one faculty sponsor to sit in a row with the faculty sponsor in the middle and the two boys not next to one another. Find number of different seating arrangements.

SOLUTION

$$\times\times\times\times\boxed{\text{F}}\times\times\times\times$$

∴ Required number of ways = Total – no. of ways in which the two boys are together

$$= 8! - \underset{\substack{\text{two adjacent}\\\text{positions for}\\\text{boys}}}{6} \cdot \underset{\substack{\text{no. of ways}\\\text{in which the}\\\text{boys can sit}}}{2} \cdot \underset{\substack{\text{other 6 girls}\\\text{can sit}}}{6!}$$

$$= 8 \cdot 7 \cdot 6! - 12 \cdot 6!$$

$$= 6!\,[56 - 12] = 44 \cdot 6! = 31680$$

ILLUSTRATION 65

Find number of ways in which all the letters of the word "COCONUT" be arranged such that atleast one 'C' comes at odd place.

SOLUTION

CC; OO ; N; U ; T

∴ Number of ways = total – (Both C, C come at even places) $= \dfrac{7!}{2!2!} - \left({}^3C_2 \times \dfrac{5!}{2!}\right) = 1080$

ILLUSTRATION 66

Find number of words that can be formed using all the letters of the word GARGEE if no two alike letters are together.

SOLUTION

Set A represents number of ways when G's are together
Set B represent number of ways when E's are together

Required number of ways $= \text{Total} - n(\text{A} \cup \text{B})$

$$= \frac{6!}{2!2!} - (n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B}))$$

$$= \frac{6!}{2!2!} - \left(\frac{5!}{2!} + \frac{5!}{2!} - 4!\right) = 180 - 96 = 84$$

Alternate:

GG EE AR

Number of words when G's are separated, using gap method

$$= \frac{4!}{2!} \cdot {}^5C_2 = 120$$

Number of words when G's are separated but E's are together $= 3! \times {}^4C_2 = 36$

∴ Number of ways when no two alike letters are together $= 120 - 36 = 84$

ILLUSTRATION 67

How many three digit numbers are of the form abc, with $a < b$, $c < b$?

SOLUTION

It is a three digit number, $a \neq 0$

$a \geq 1, a < b \Rightarrow b = 2, 3, 4, \ldots 9$

Let $b = r$, where $r = 2, 3, \ldots.9$, then a can take $(r - 1)$ values $1, 2, \ldots, r - 1$ and c can take r values $0, 1, 2, \ldots, r - 1$

Thus for each value of r, bc can take $r(r–1)$ values,

$$\therefore \quad \sum_{r=2}^{9} r(r-1) = \sum_{r=1}^{9} r^2 - \sum_{1}^{9} r = \frac{9.10.19}{6} - 45 = 240$$

ILLUSTRATION 68

How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \leq a + b + c + d + e \leq 10$?

SOLUTION

(i) $a = b = c = d = e = 1$ 1 way

(ii) $a = b = c = d = 1, e \neq 1, e = 2$ to 6 $\frac{5!}{4!} \times 5 = 25$ ways

(iii) $a = b = c = 1$, last two can be

$(2, 2), (3, 3), (2, 4), (2, 3), (2, 5)$

$$\text{Number of ways} = 2.\frac{5!}{2!.3!} + 3.\frac{5!}{3!} = 80$$

(iv) $a = b = 1, c = d = e = 2$

$$\text{Number of ways} = 2\frac{5!}{3!2!} = 10$$

$$\text{Total sequence} = 1 + 25 + 80 + 10$$
$$= 116$$

Test Your Concepts-III

(Solutions on page H.185)

1. Find the number of diagonals that can be drawn by joining the vertices of an octagon.

2. The straight lines l_1, l_2, l_3 are parallel and lie in the same plane. A total number of m points are taken on l_1, n points on l_2, k points on l_3. Find the maximum number of triangles formed with vertices at these points.

3. A polygon has 44 diagonals, then find the number of its sides.

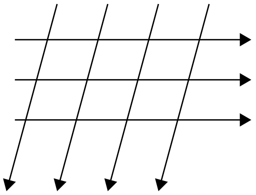
4. There are 8 different lines, 7 different circle and 6 different equilateral triangles, then find maximum number of points of intersection.

5. If $A_1A_2....A_{18}$ is a regular polygon, then find number of ways in which a quadrilateral can be formed with the help of vertices of given polygon, such that no side of quadrilateral is a side of given polygon.

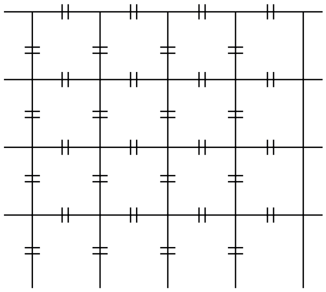
6. There are 10 points in a plane of which no three are collinear and 4 points are concyclic the find number of different circles that can be drawn through atleast 3 of these points.

7. Find the maximum number of points into which 4 circles and 4 straight lines intersect.

8. Find the number of parallelograms in given figure.

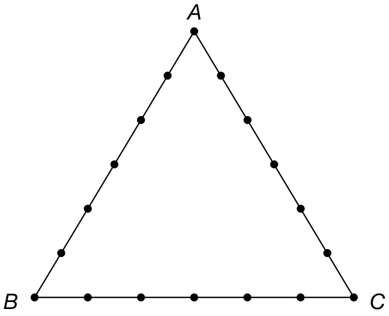


9. Find the number of squares in given figure.

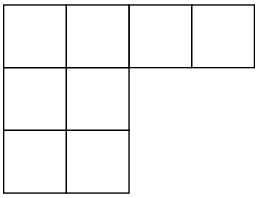


10. There are n concurrent lines and another line parallel to one of them. Find the number of different triangles that will be formed by the $(n + 1)$ lines.

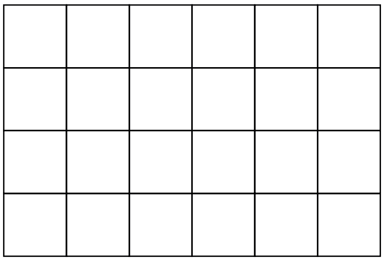
11. 18 points are indicated on the perimeter of a ΔABC (see figure). How many triangles are there with vertices at these points?



12. Find the number of ways in which 5X's can be placed in the squares of the figure so that no row remains empty.



13. Find the number of rectangles in the grid shown which are not squares.



Answer keys

1. 20 **2.** ${}^{m+n+k}C_3 - {}^{m}C_3 - {}^{n}C_3 - {}^{k}C_3$ **3.** 11 **4.** 620 **5.** 1287 **6.** 117 **7.** 50

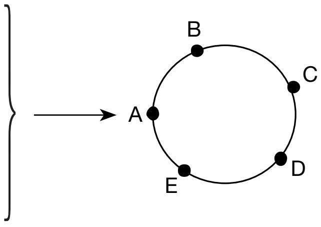
8. 18 **9.** 20 **10.** $\frac{(n-1)(n-2)}{2}$ **11.** 711 **12.** 44 **13.** 160

CIRCULAR PERMUTATION

Arrangement of Distinct Objects Round a Circular Table

Let consider 5 alphabets and their linear as well as circular arrangement. Clearly these 5 linear are different but all these 5 linear arrangements are considered equal as one in a circular table as there is no starting point and no end point.

ABCDE
BCDEA
CDEAB
DEABC
EABCD



5 linear arrangements = 1 circular arrangement

If n persons are arranged in a straight line, there are $n!$ different ways in which this can be done. Whereas n persons sit a round a circular table, each circular arrangement will be $= 1/n($ arrangements in a line)

$$= \frac{1}{n}(n)! = (n-1)!$$

Remark

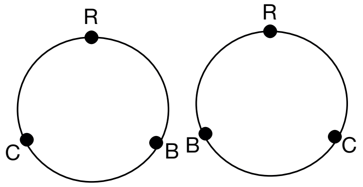
In this case the number of circular permutation m clockwise and anticlockwise are distinct.

∴ the number of circular permutations of n different persons taken all at a time is $(n-1)!$

Arrangement of Beads (All Distinct) Around a Circular Wire

A circular wire differs from a circular table because when we can turn it over. We see that the other side presents an arrangement of different beads different from that on the first side.

If the wire on the left is turned over we obtain the arrangements on the right thus two different arrangements for beads around a circular wire are not distinct. So number of ways in this case $\frac{1}{2}(n-1)!$



Concept:

Whenever clockwise and anticlockwise sense are not distinguishable then number of circular arrangement of n distinct things is $\frac{1}{2}(n-1)!$ distinct.

Example:
forming garland, necklace, circular wire

ILLUSTRATION 69

Number of ways of arrange 4 different flowers in a circle.

SOLUTION

Here we are arranging flowers in circle so anticlockwise and clockwise are two different sense so number of ways of doing it $= (4-1)! = 6$

ILLUSTRATION 70

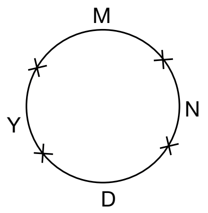
All the letters of MONDAY are arranged in a circle such that vowels are not together.

SOLUTION

Required number of ways = total – vowels are together = (6 –1)! – (5 –1)!2!

= 120 – 48 = 72 $\boxed{OA}$ MNDY

Alternate:



Number of ways

$$= \underbrace{(4-1)!}_{\text{Arrangement of consonant}} \times \underbrace{{}^4C_2}_{\text{Gap selection}} \times \underbrace{2!}_{\text{Vowels}}$$

$= 6 \times 6 \times 2 = 72$

ILLUSTRATION 71

11 distinct pearls are used to form a necklace if 6 particular pearls are always together.

SOLUTION

$\boxed{P_1\,P_2\,P_3\,P_4\,P_5\,P_6}\quad P_7\,P_8\,P_9\,P_{10}\,P_{11}$

$$\text{Number of ways} = \frac{1}{2}(6-1)! \times 6! = \frac{1}{2}5! \times 6! = 43200$$

ILLUSTRATION 72

3 Red, 5 Yellow, 4 Blue coloured flowers are used to form a garland such that same colour followers are together. Find number of different garlands possible if Same colour followers are (a) different (b) identical

SOLUTION

(a) $R_1R_2R_3$ $\boxed{Y_1Y_2Y_3Y_4Y_5}$ $\boxed{B_1B_2B_3B_4}$

$$\text{Number of ways} = \frac{1}{2}.(3-1)!\ .\underbrace{3!}_{\text{Red flowers Internal arrangement}}\ \ \overbrace{5!}^{\text{Yellow flowers internal arrangement}}\ \underbrace{4!}_{\text{Blue flowers internal arrangement}}$$

= 17280

(b) $\boxed{R\,R\,R}$ $\boxed{Y\,Y\,Y\,Y\,Y}$ $\boxed{B\,B\,B\,B}$

$$\text{Number of ways} = \frac{1}{2}.(3-1)! \times 1 \times 1 \times 1 = 1$$

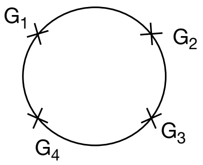
All are identical flowers so internal arrangement is 1

ILLUSTRATION 73

The number of ways in which 3 boys and 4 girls be seated around a circular table if no two boys sit together is

SOLUTION

First make the 4 girls sit around a circular table. This can be done in (4 – 1)! = 3! = 6. After that '3' boys can occupy any three gaps.

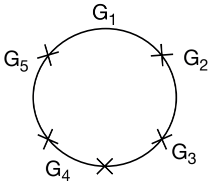


Required number of ways = $6 \cdot {}^4C_3 \cdot 3! = 144$

ILLUSTRATION 74

Find the number of ways of arranging 5 boys and 5 girls around a circular table such that boys and girls are alternate.

SOLUTION



First make the 5 girls sit around a circular table. This can be done in (5 – 1)! = 4! = 24. After that '5' boys can occupy in gap created between girls so they will become in alternate order = $4! \cdot {}^5C_5 \cdot 5! = 24.5! = 2880$ ways.

ILLUSTRATION 75

n different things are arranged around a circle. In how many ways can 3 objects be selected when no two of the selected objects are consecutive.

SOLUTION

Let the object be denoted by $a_1, a_2, a_3 \ldots, a_n$. arranged in a circle we have to select 3 objects so that no two of them are consecutive. For this we first find the number of ways in which 2 or 3 objects are consecutive. Now, number of ways in which 2 or 3 objects are consecutive is obtained as follows with a_1 the number such triples is $a_1\,a_2\,a_3$, $a_1\,a_2\,a_4$, $a_1\,a_2\,a_5$, …, $a_1\,a_2\,a_{n-1}$.

[Note that we have excluded $a_1a_2a_n$ since it will be repeated again for if we start with a_n, then we shall get triples: $a_na_1\,a_2$, $a_na_1\,a_3$]

So number of such triples when we start with a_1 is $(n-3)$. similarly with $a_2, a_3, a_4 \ldots$ we shall get the numbers of triplet that is $(n-3)$. But total number of triplet is ${}^nC_3$. Hence required number of ways

$$= {}^{n}C_3 - n\,(n-3)$$

$$= \frac{n(n-1)(n-2)}{1.2.3} - n\,(n-3)$$

$$= \frac{n}{6}[n^2 - 3n + 2 - 6n + 18]$$

$$= \frac{n}{6}(n^2 - 9n + 20) = \frac{n}{6}(n-4)\,(n-5).$$

ILLUSTRATION 76

3 married couples are to seated around a circular table. Let x denotes the number of ways where atmost two couples sit together and let y denotes the number of ways where only two particular couples sit together, then find $x + y$.

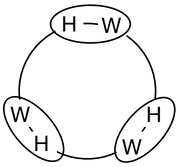
SOLUTION

n(All couples are together)

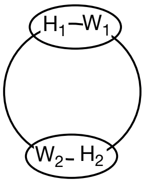
$= 2 \times 2 \times 2 \times 2 \times 2 = 16$ ways

$\therefore\ x = \text{Total} - 16$

$x = 5! - 16 = 104$



To find y, first Let H_1, W_1, H_2, W_2 sit in $2 \times 2 = 4$ ways.



Now, H_3 and W_3 can sit in 2 ways

$\therefore\ y = 4 \times 2 = 8$

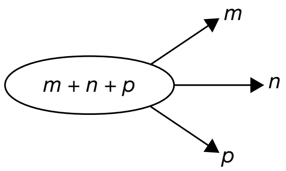
Now $x + y = 112$

DIVISION OF ITEMS INTO GROUPS

Division of Items into Groups of Unequal Size

(i) The number of ways in which $(m+n+p)$ distinct items can be divided into unequal groups containing m, n, p items is

$$^{m+n+p}C_m \cdot {}^{n+p}C_n = \frac{(m+n+p)!}{m!\,n!\,p!} \quad (m \neq n \neq p)$$



(ii) The number of ways to distribute $(m + n + p)$ distinct items amongst three persons in the groups containing m, n and p items is

= (no. of ways to divide) × (distribution)

$$= \frac{(m+n+p)!}{m!\,n!\,p!}.3!$$

Division of Groups of Equal Size

(i) The number of ways in which pq different items can be divided equally into p groups, each containing q items is $= \left(\frac{(pq)!}{(q!)^p}\right).\frac{1}{p!}$

(ii) The number of ways in which pq different items can be divided equally into p groups, among p persons.

Each containing q items $= \left(\frac{(pq)!}{(q!)^p} \times \frac{1}{p!}\right).p! = \frac{(pq)!}{(q!)^p}.$

ILLUSTRATION 77

In how many ways 6 different toys can be distributed between two brothers in 1 : 2 ratio.

SOLUTION

First 6 toys has be divided in two groups of 4 toys and 2 toys = 6!/4!2!
By 2! Ways these toys can be distributed among brothers.
Hence required no. of ways = (6!/4!2!)2! = 30

ILLUSTRATION 78

In how many ways 8 different toys can be distributed among 3 brothers, such that each receives atleast one toy, no two brothers get equal number of toys and younger brother gets maximum number of boys.

SOLUTION

Let there are three brothers B1 B2 B3(youngest)

Group formation	**Distribution among brothers B3 is youngest so he get maximum toys**
1, 2 and 5 toys	$\left(\frac{8!}{1!\,2!\,5!}\right)2! \rightarrow$ B_1, B_2 distribution
1, 3 and 4 toys	$\left(\frac{8!}{1!\,3!\,4!}\right)2! \rightarrow$ B_1, B_2 distribution

Hence required no. of ways

= (8!/1! 2! 5!)2! + (8!/1! 3! 4!)2!

= 896

ILLUSTRATION 79

Sixteen players S_1, S_2, S_3, ……, S_{16} play in a tournament. Find number of ways in which they can be grouped into eight pairs so that S_1 and S_2 are in different groups?

SOLUTION

Required number of ways

= (number of ways in which 16 players can be divided in 8 couples) – (number of ways when S_1 and S_2 are in the same group)

$$= \frac{16!}{2^8 \cdot 8!} - \frac{(14!)}{2^7 \cdot 7!} = \frac{16 \cdot 15 \cdot 14!}{2 \cdot 2^7 \cdot 8 \cdot 7!} - \frac{(14!)}{2^7 \cdot 7!}$$

$$= \frac{(14!)}{2^7 \cdot 7!}\left[\frac{16 \cdot 15}{16} - 1\right] = \frac{(14) \cdot (14)!}{14 \cdot (6!) \cdot 2^6} = \frac{(14)!}{2^6 \cdot 6!}$$

ILLUSTRATION 80

In an eleven-storey building 9 people enter a lift cabin from ground floor. It is known that they will leave the lift in groups of 2, 3 and 4 at different residential storeys. Find number of ways in which they can get down.

SOLUTION

9 people can be divided into groups of 2, 3 and 4 in $\frac{9!}{2!3!4!}$ ways.

Now each group can leave lift at different floor in ${}^{10}C_3 \times 3!$ ways

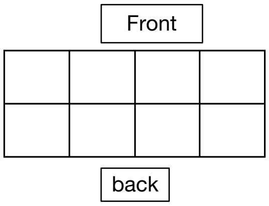
Required ways $= \frac{9!}{2!3!4!} 10 \cdot 9 \cdot 8 = \frac{10!}{4}$

Note that in an eleven storeyed building there will be 10 floors.

ILLUSTRATION 81

A van can accommodate 8 persons, 4 in the front and 4 at the back. If heights of 8 persons are all different, then in how many ways they can be seated in the van so that the person who is sitting in the front is shorter than just at back?

SOLUTION



Select one seat at the back, it can be done in ${}^4C_1$, now tallest person will occupy this seat and infront of him anyone of 7 persons can be seated in ${}^7C_1$ ways. Again select one seat at the back from remaining 3, it can be done in ${}^3C_1$ ways, now tallest person among 6 will occupy this seat and infront of him any one of 5 persons can be seated in ${}^5C_1$ ways and so on.

Thus, total number of required ways

$$= {}^4C_1 \times {}^7C_1 \times {}^3C_1 \times {}^5C_1 \times {}^2C_1 \times {}^3C_1 \times 1$$

$$= 28 \times 90 = 2520$$

Alternate:

We should make 4 pairs of persons out of 8 persons then

Number of ways of making pairs $= 8!/((2!)^4.4!) = 105$

There is only one way of arranging them in front and back seat(taller person takes back seat and shorter will take front seat).

Now these 4 pairs have 4 choices to sit (from left to right) $= 105 * 4! = 2520$

ILLUSTRATION 82

Two cars of capacity 4 each and 3 autos of capacity 3 each are available for ride (internal seating arrangement of vehicles are not considered)

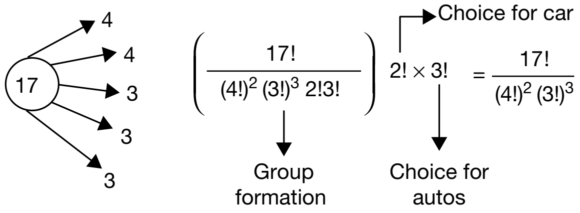
(a) In how many ways 17 persons departs

(b) If two particular persons insist on going by same car, then solve the above problem

(c) 3 Particular persons insist on going by same particular car.

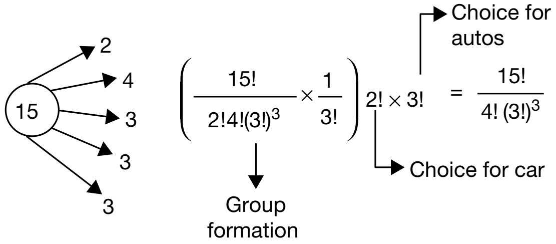
SOLUTION

(a) Required no. of ways =



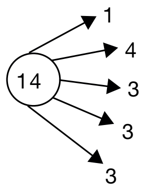
$$\left(\frac{17!}{(4!)^2 (3!)^3 2!3!}\right) 2! \times 3! = \frac{17!}{(4!)^2 (3!)^3}$$

(b) Let P_1P_2 want to go by same car, group containing 2 persons joins P_1 and P_2 then required number of ways =



$$\left(\frac{15!}{2!4!(3!)^3} \times \frac{1}{3!}\right) 2! \times 3! = \frac{15!}{4!\,(3!)^3}$$

(c) Let P_1,P_2,P_3 insist on going by same particular car so we do not need to make choice for cars, group of 1 person joins P_1, P_2, P_3 so required ways

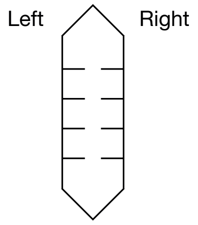


$$\left(\underbrace{\frac{14!}{1!4!(3!)^3}}_{\text{Group formation}} \times \frac{1}{3!}\right)\overbrace{1! \times 3!}^{\text{Choice for autos}} = \frac{14!}{4!\,(3!)^3}$$

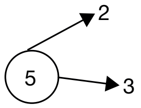
ILLUSTRATION 83

A boat is to be managed by 8 persons, 4 on each side find number of ways if two particular persons can work only on left side and 1 particular person on right side only.

SOLUTION



5 persons have no choices, we need 2 person on left and 3 on right side.



$$\text{Ways} = \underbrace{\left(\frac{5!}{2!\,3!}\right)}_{\text{Group formation}} \overbrace{4!}^{\text{Left side arrangement}} \times \underbrace{4!}_{\text{Right side arrangement}}$$

Test Your Concepts-IV

(Solutions on page H.186)

1. Find number of ways such that 6 persons are arranged on a circular table such that 2 particular persons are not together.
2. 12 guests at a dinner party are to be seated along a circular table. Supposing that the master and mistress of the house have fixed seats opposite one another, and that there are two specified guests who must always, be placed next to one another, the find number of ways in which the company can be placed.
3. In how many ways can 4 men and 4 women be seated at a round table, if
   i. no restriction is imposed.
   ii. each women is to be seated between two men.
   iii. two particular women must sit together.
   iv. two particular women must not sit together.
   v. all women must sit together.
   vi. there is exactly one person between two particular women.
4. How many words can be formed from the letters of the word DAUGHTER, so that
   i. the vowels always come together.
   ii. all vowels never come together.
5. How many necklaces can be made of 18 beads, 8 being blue, 5 red, 4 green and 1 yellow?
6. A man has 3 friends. If N is number of ways he can invite one friend everyday for dinner on 6 successive nights so that no friend is invited more than 3 times then find N.
7. Find the number of ways in which 6 red roses and 3 white roses (all roses different) can form a garland so that all the white roses come together.
8. Six persons A, B, C, D, E and F are to be seated at a circular table. Find the number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right.
9. Find the total number of ways of dividing 15 different things into groups of 8, 4 and 3 respectively.
10. Find the number of ways in which 200 different things can be divided into groups of 100 pairs.
11. Find the number of ways of distributing 50 identical things among 8 persons in which such a way that three of them get 8 things each, two of them get 7 things each and remaining 3 get 4 things each.

Answer keys

1. 72

2. 22.10!

3. (i) 5040 (ii) 144 (iii)1440 (iv) 3600 (v) 576 (vi) 1440

4. (i) 4320 (ii) 36000

5. $\frac{(17)!}{2\cdot 8!\cdot 5!\cdot 4!}$

6. 510

7. 2160

8. 18

9. $\frac{(15)!}{8!\cdot 4!\cdot 3!}$

10. $\frac{(200)!}{(100)!\cdot 2^{100}}$

11. $\frac{8!}{(3!)^2\cdot 2!}$

MORE THAN ONE SELECTION FROM DISTINCT ITEMS

The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

Proof:

Out of n items one item can be selected in ${}^nC_1$ ways two item can be selected in ${}^nC_2$ ways and so on.

Thus, the required number of ways is

$$ {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots. + {}^nC_n $$

$$ = ({}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots. + {}^nC_n) - {}^nC_0 = 2^n - 1 $$

Alternate:

Item		Number of things can be taken		Number of ways
T_1	⟶	0, 1	⟶	2
T_2	⟶	0, 1	⟶	2
T_3	⟶	0, 1	⟶	2
⋮		⋮		⋮
T_n	⟶	0, 1	⟶	2

Total ways = 2.2.2…2

Atleast one selected = Total – None selected

$= 2^n - 1$

MORE THAN ONE SELECTION FROM GROUP OF IDENTICAL AND NON-IDENTICAL ITEMS

The total number of ways of selecting one or more items from m identical items of first kind, n identical items of second kind, p identical items of third kind and rest k distinct items is $= (m + 1)\,(n + 1)\,(p + 1).2^k - 1$

Proof:

Item	Things can be taken	No. of ways
First kind thing	0, 1, 2,,…..m	$m + 1$
Second kind thing	0, 1, 2,,…..n	$n + 1$
Third kind thing	0, 1, 2,,…..p	$p + 1$
T1	0, 1	2
T2	0, 1	2
T3	0, 1	2
.		
.		
Tk	0,1	2

The total number of ways of selecting one or more items

= Total number of selection – none selected

$= (m + 1)\,(n + 1)\,(p + 1).2^k - 1$

ILLUSTRATION 84

A student is allowed to select at most n books from a collection of $(2n + 1)$ distinct books. If the total number of ways in which he can select a book is 63. Find the value of n.

SOLUTION

Given students select at most n books from a collection ($2n + 1$) books. It means that he select One book or two book or three book or… or n books.

$$ \Rightarrow \quad {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots {}^{2n+1}C_n = 63 $$

But we know that,

$$ \Rightarrow \quad {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 \ldots {}^{2n+1}C_{2n+1} = 2^{2n+1} $$

Now, ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$

$$ 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots. {}^{2n+1}C_n) + ({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots. {}^{2n+1}C_{2n}) = 2^{2n+1} $$

$$ \Rightarrow \quad 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots. {}^{2n+1}C_n) = 2^{2n+1} $$

$$ 2 + 2.63 = 2^{2n+1} $$

$$ n = 3 $$

ILLUSTRATION 85

In a library 5 Physics, 4 Chemistry and 3 Maths books are available. Find number of was of selecting atleast one book if same subject books are (a) different (b) alike

SOLUTION

(a) Same subject books are different, we have P_1, P_2, P_3, P_4, P_5 for Physics, C_1, C_2, C_3, C_4 for Chemistry and M_1, M_2, M_3 for Maths

Number of ways of selecting atlast one book $= 2^{12} - 1$

(b) We have 5 alike books for Physics, 4 alike books for Chemistry, 3 alike books for Maths

Number of ways of selections atleast one brock $= (5 + 1)\,(4 + 1)\,(3 + 1) - 1 = 6.5.4 - 1 = 119$

ILLUSTRATION 86

Given five different green dyes, four different blue dyes and three different red dyes, how many combinations of dyes can be chosen taking at least one green, one blue dye?

SOLUTION

Any one dye of a particular colour can be either chosen or not; and, thus there are 2 ways in which each one may be dealt with.

Number of ways of selection so that at least one green dye is included $= 2^5 - 1 = 31$ (l is subtracted to correspond to the case when none of the green dyes is chosen). A similar argument may be advanced in respect of other two colours also.

Number of combinations

$$ = \overset{\text{For green}}{(2^5 - 1)}\ \overset{\text{For blue}}{(2^4 - 1)}\ \underset{\text{For red}}{(2^3)} $$

$$ = 31 \times 15 \times 8 = 3720 $$

ILLUSTRATION 87

Find number of ways of selecting at least one fruit out of 4 Apples, 3 Bananas and 1 each of Mango, Pineapple and Guava. Assume same type of fruits are identical.

SOLUTION

Among '4' apples, we can select either 0, 1, 2, 3, or all the four apples i.e. we have total '5' choices. similarly, for '3' bananas, we have four choices and for 3 other fruits, we have '2' choices each. Hence total ways = $5\times4\times2\times2\times2 = 160$. But this includes a case of selecting '0' apples, '0' of every other fruit Hence total number of ways = 160 – 1 =159 ways

ILLUSTRATION 88

In a basket 6 Apples, 5 Mangoes and 4 Bananas are placed. if same type of fruits are identical then find number of ways of selecting

(i) atleast one fruit

SOLUTION

Required number of ways = $(6 + 1)(5 + 1)(4 + 1) - 1 = 7.6.5 - 1 = 210 - 1 = 209$

(ii) Atleast 2 Mangoes are selected

SOLUTION

	Number of things taken	Number of ways
Apples	→ 0, 1, 2, 3, 4, 5, 6,	→ 7
Mangoes	→ 2, 3, 4, 5	→ 4
Banana	→ 0, 1, 2, 3, 4,	→ 5

Required number of ways = 7.4.5 = 140

(iii) At least 3 Apples are selected

SOLUTION

	Number of things taken	Number of ways
Apples	→ 3, 4, 5, 6	→ 4
Mangoes	→ 0, 1, 2, 3, 4, 5	→ 6
Banana	→ 0, 1, 2, 3, 4,	→ 5

Required number of ways = 4.6. 5 = 120

(iv) At least 2 Apples, 2 Mangoes, 2 Bananas selected

SOLUTION

	Number of things taken	Number of ways
Apples	→ 2, 3, 4, 5, 6	→ 5
Mangoes	→ 2, 3, 4, 5	→ 4
Banana	→ 2, 3, 4	→ 3

Required number of ways = 5.4. 3 = 60

(v) If same type of fruits are distinct then no of ways if selecting at least one fruit.

SOLUTION

Required number of ways = $2^6\,2^5\,2^4 - 1 = 2^{15} - 1$

(vi) If same type of fruits are distinct, then number of ways of selecting atleast two Mangoes

SOLUTION

$$\text{No. of ways} = \underbrace{2^6}_{\text{For apples}}.(2^5 - (\underbrace{1}_{\text{No mangoes selected}} + \underbrace{{}^5C_1}_{\text{One mango selected}})).\overbrace{2^4}^{\text{For bananas}}$$

$$= 2^{10}(2^5 - 6) = 2^{11}.13$$

ILLUSTRATION 89

5 Pen, 4 Pencil, 3 Markers and 3 distinct things are available for selection and considering same type of objects are alike then find number of ways of selecting

(i) at least one object

SOLUTION

No. of required ways = $(5 + 1)(4 + 1)(3 + 1)(2)^3 - 1$

$$= 6.5.4.2^3 - 1 = 959$$

(ii) atleast 2 object are selected

SOLUTION

No of required ways = Total – (none + exactly one thing selected)

$$= 6.5.4.2^3 - (1 + {}^6C_1) = 960 - 7 = 953$$

(iii) Atleast 3 objects are selected

SOLUTION

Exactly two objects are selected

(a) Both identical → ${}^3C_1$

(b) Both different → ${}^6C_2$

Atleast 3 objects are selected = atleast two selected – exactly two selected

$$= 6.5.4.2^3 - 1 - 6 - ({}^3C_1 + {}^6C_2)$$

$$= 6.5.4.2^3 - 7 - 3 - 15 = 935$$

Let $N = p^a q^b r^c$ where p, q, r are prime numbers and a, b, c are whole numbers

(i) Total number of divisors = $(a + 1)(b + 1)(c + 1)$

(ii) Number of proper divisors = Total number of divisors – 2 ($\because$ 1 and N are not considered as proper divisor of number N)

(iii) Sum of all divisors $= \sum_{x=0}^{a}\sum_{y=0}^{b}\sum_{z=0}^{c} p^x q^y r^z$

$$= (1+p+p^2 .... + p^a)(1+q+q^2 .... + q^b)$$
$$(1+r+r^2+..+r^c)$$

(iv) Sum of proper divisors = sum of all divisors −1 − N

(v) Number of ways of splitting N into two factors

$$= \begin{cases} (a+1)(b+1)(c+1), N \text{ is not perfect square} \\ \dfrac{(a+1)(b+1)(c+1)+1}{2}, N \text{ is a perfect square} \end{cases}$$

(vi) Number of ways of splitting N into two co-prime factors = 2^{n-1} where n is number of prime numbers present in N.

ILLUSTRATION 90

N = 21600 find

(i) No. of total divisors of N

SOLUTION

$N = 21600 = 2^5.3^3.5^2$

Here we can consider it as similar to previous question such that there are 5 apples (2s), 3 mangoes (3s) and 2 bananas (5s)

	Number of things taken	Number of ways
2	0, 1, 2, 3, 4, 5	6
3	0, 1, 2, 3	4
5	0, 1, 2	3

Total divisors = 6.4.3 = 72

(ii) No. of proper divisors of N

SOLUTION

1 and number itself are not considered as proper divisors hence no. of proper divisors = 70

(iii) Number of even divisors of N

SOLUTION

	Number of things taken	Number of ways
2	1, 2, 3, 4, 5,	5
3	0, 1, 2, 3	4
5	0, 1, 2,	3

Number of even divisors = 5.4.3 = 60

(iv) Number of odd divisors of N

SOLUTION

	Number of things taken	Number of ways
2	0	1
3	0, 1, 2, 3	4
5	0, 1, 2	3

Number of odd divisors = 1.4.3 = 12

Alternate:

Number of odd divisors = Total number of divisors – Number of even divisor = 72 – 60 = 12

(v) Number of divisors multiple of 12

SOLUTION

$12 = 2^2.3$

	Number of things taken	Number of ways
2	2, 3, 4, 5	4
3	1, 2, 3	3
5	0, 1, 2	3

Number of divisors multiple of 12 = 4.3.3 = 36

(vi) Number of divisors multiple of 60

SOLUTION

$60 = 2^2.3. 5$

	Number of things taken	Number of ways
2	2, 3, 4, 5	4
3	1, 2, 3	3
5	1, 2	2

Number of divisors multiple of 60 = 4.3.2 = 24

(vii) Sum of all divisors

SOLUTION

Any divisor of $2^5.3^3.5^2$ is of the form $2^a.3^b.5^c$ where $0 \le a \le 5, 0 \le b \le 3$ and $0 \le c \le 2$.

Hence, the sum of the divisors

$$\sum_{a=0}^{5}\sum_{b=0}^{4}\sum_{c=0}^{2} 2^a 3^b 5^c = \sum_{a=0}^{5} 2^a \sum_{b=0}^{4} 3^b \sum_{c=0}^{2} 5^c$$

$$= (1+2+\ldots+2^5)(1+3+\ldots+3^3)(1+5+5^2)$$

$$= \left(\frac{2^6-1}{2-1}\right)\left(\frac{3^4-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right) = 63(40)(31)$$

$$= 78120$$

(viii) Sum of proper divisors

SOLUTION

Sum of the proper divisors = total sum – 1 – N

$$= 78120 - 1 - 21600 = 56519$$

ILLUSTRATION 91

Let $f(n) = \sum_{r=0}^{n}\sum_{k=r}^{n}\binom{k}{r}$. Find the total number of divisors of $f(9)$.

SOLUTION

$$\sum_{k=r}^{n} {}^{k}C_r = {}^{r}C_r + {}^{r+1}C_r + {}^{r+2}C_r + \dots + {}^{n}C_r$$

$$= 1 + {}^{r+1}C_1 + {}^{r+2}C_2 + {}^{r+3}C_3 + \dots + {}^{n}C_{n-r}$$

$$= {}^{r+1}C_0 + {}^{r+1}C_1 + {}^{r+2}C_2 + \dots + {}^{n}C_{n-r}$$

$$= {}^{n+1}C_{n-r}$$

Now, ${}^{n+1}C_{n-r} = {}^{n+1}C_{r+1}$

$$\therefore f(n) = \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$

$$= {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots + {}^{n+1}C_{n+1}$$

$$= {}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \dots + {}^{n+1}C_{n+1} - {}^{n+1}C_0$$

$$\Rightarrow f(n) = (2^{n+1}) - 1$$

$$\Rightarrow f(9) = 2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$$

Hence, number of divisors are

$$(1 + 1)(1 + 1)(1 + 1) = 8$$

ILLUSTRATION 92

X, Y, Z are sets of all positive divisors of 10^{60}, 20^{50} and 30^{40} respectively. Then find $n(X \cup Y \cup Z)$.

SOLUTION

$x = 2^{60} \times 5^{60}$; $n(x) = 61 \times 61$

$n(y) = 101 \times 51$, where $y = 2^{100} \times 5^{50}$

$z = 2^{40} \times 3^{40} \times 5^{40}$

$n(z) = 41^3$

$n(x \cap y) = 61 \times 51$

$n(y \cap z) = 41^2 = n(z \cap x)$

$n(x \cap y \cap z) = 41^2$

$$n(x \cup y \cup z) = n(x) + n(y) + n(z) - n(x \cap y) - (y \cap z) - n(z \cap x) + n(x \cap y \cap z)$$

$$= 61^2 + 101 \times 51 + (41)^3 - 61 \times 51 - 41^2 - 41^2 + 41^2$$

$$= 61(61 - 51) + 41^2(41 - 1) + 101 \times 51 = 73001$$

Multinomial Theorem for Selection

(a) If there are ℓ object of one kind, m object of second kind, n object of third kind then the number of ways of choosing r objects out of these $(\ell + m + n)$ is the coefficient of x^r in the expansion of $(1 + x + x^2 + x^3 + \dots + x^\ell)(1 + x + x^2 + \dots + x^m)(1 + x + x^2 + x^3 + \dots + x^n)$

(b) If one object of each kind is to be included, there the number of ways of choosing r objects out of these objects (i. e., $\ell + m + n + \dots$) is the coefficient of x^r in the expansion of $(x + x^2 + x^3 + \dots + x^\ell)(x + x^2 + x^3 + \dots + x^m)(x + x^2 + x^3 + \dots + x^n)$.

(We need following result from binomial theorem frequently. Coefficient of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$)

ILLUSTRATION 93

Find number of ways of selecting 4 letters from word 'MISSISSIPPI'

SOLUTION

SSSS IIII PPM

There are 5 possibilities of selecting 4 letters

(a) all same → ${}^2C_1 = 2$

(b) 3 same, 1 different → ${}^2C_1 . {}^3C_1 = 6$

(c) 2 same, 2 same → ${}^3C_2 = 3$

(d) 2 same, 2 different → ${}^3C_1 . {}^3C_2 = 9$

(e) 4 different → ${}^4C_4 = 1$

Total ways = 21

Alternate (Use Multinomial Theorem)

SSSS IIII PPM

Number of ways of selecting 4 letters from word 'MISSISSIPPI'

= Coefficient of x^4 in

$$\underbrace{(1 + x + x^2 + x^3 + x^4)}_{\text{For S}}\underbrace{(1 + x + x^2 + x^3 + x^4)}_{\text{For I}}\underbrace{(1 + x + x^2)}_{\text{For P}}\underbrace{(1 + x)}_{\text{For M}}$$

$$= \text{coefficient of } x^4 \text{ is } \left(\frac{1 - x^5}{1 - x}\right)^2\left(\frac{1 - x^3}{1 - x}\right)(1 + x)$$

$$= \text{coefficient of } x^4 \text{ in } (1 + x^{10} - 2x^5)(1 + x - x^3 - x^4)(1 - x)^{-3}$$

Ignoring higher power

$$= \text{coefficient of } x^4 \text{ in } 1 \cdot (1 + x - x^3 - x^4)(1 - x)^{-3}$$

$$= \text{coefficient of } x^4 \text{ in } (1 - x)^{-3} + x(1 - x)^{-3} - x^3(1 - x)^{-3} - x^4(1 - x)^{-3}$$

$$(\because \text{coefficient of } x^4 \text{ in } (1 - x)^{-n} = {}^{n+r-1}C_r)$$

$= \text{coefficient of } x^4 \text{ is } {}^{3+4-1}C_4 + {}^{3+3-1}C_3 - {}^{3+1-1}C_1 - {}^{3+0-1}C_0$

$= {}^6C_4 + {}^5C_3 - {}^3C_1 - {}^2C_0$

$= 15 + 10 - 3 - 1 = 21.$

ILLUSTRATION 94

5 Pens, 4 Pencils, 3 Markers are available for selection. Find number of ways of selecting exactly 4 things if same type of things are alike.

SOLUTION

5 Pens 4 Pencils 3 Markers

There are 4 possibilities of selecting 4 things

(a) all alike $\rightarrow {}^2C_1 = 2$

(b) 3 alike, 1 distinct $\rightarrow {}^3C_1.{}^2C_1 = 6$

(c) 2 alike, 2 alike $\rightarrow {}^3C_2 = 3$

(d) 2 alike, 2 distinct $\rightarrow {}^3C_1 \cdot {}^2C_2 = 3$

Number of ways = 14

Alternate

5 Pens 4 Pencils 3 Markers

Number of ways of selecting 4 things

$= \text{Coefficient of } x^4 \text{ in}$

$$\overbrace{(1^0 + x^1 + x^2 + x^3 + x^4 + x^5)}^{\text{Pens}}\ \overbrace{(x^0 + x^1 + x^2 + x^3 + x^4)}^{\text{Pencils}}\ \overbrace{(x^0 + x^1 + x^2 + x^3)}^{\text{Markers}}$$

$= \text{coefficient of } x^4 \text{ in } 1.\dfrac{(1-x^6)}{1-x}.1.\dfrac{(1-x^5)}{1-x}.1.\dfrac{(1-x^4)}{1-x}$

$= \text{coefficient of } x^4 \text{ in } (1 - x^6)\,(1 - x^5)\,(1 - x^4)\,(1 - x)^{-3}$

(higher power are neglected)

$\text{Coefficient of } x^4 \text{ in } \dfrac{1.1.(1-x^4)}{(1-x)^3} = (1-x^4)(1-x)^{-3}$

$= \text{coefficient of } x^4 \text{ in } (1 - x)^{-3} - x^4(1 - x)^{-3}$

$= {}^{3+4-1}C_4 - {}^{3+0-1}C_0 = {}^6C_4 - 1 = 14.$

ILLUSTRATION 95

10 Shirts, 5 T-shirts, 3 caps, 1 pen, 1 pencil and 1 marker are available. Find number of ways of selecting exactly 3 things if same type of things are

(i) distinct

SOLUTION

As all things are distinct so we can select it as $= {}^{21}C_3$

(ii) identical

SOLUTION

10 Shirts, 5 T-shirts, 3 caps, 1 pen, 1 pencil and 1 marker

There are 4 possibilities of selecting 3 things

(a) 3 alike $\rightarrow {}^3C_1 = 3$

(b) 2 alike, 1 distinct $\rightarrow {}^3C_1.{}^5C_1 = 15$

(c) 3 distinct $\rightarrow {}^6C_3 = 20$

Total number of required ways = 38

ILLUSTRATION 96

(a) In JEE-ADVANCED – 2020 maximum marks in each subject PCM is 60. Find number of ways by which Ankita can score exactly 60 marks as total marks (assuming that there is no negative marking and marks are given in whole numbers)

SOLUTION

$P + C + M = 60 \qquad 0 \le P, C, M \le 60$

Number of ways of scoring 60 marks

$$= \text{coefficient of } x^{60} \text{ in } \overbrace{(x^0 + x^1 + x^2 + \ldots\ldots + x^{60})}^{\text{Physics}}\ \underbrace{(x^0 + x^1 + \ldots\ldots + x^{60})}_{\text{Chemistry}}\ \underbrace{(x^0 + x^1 + \ldots\ldots +)}_{\text{Maths}}$$

$= \text{coefficient of } x^{60} \text{ in } \left(1\left(\dfrac{(1-x^6)}{1-x}\right)\right)^3$ (neglecting higher power)

$= \text{coefficient of } x^{60} \text{ in } 1.(1 - x)^{-3}$ [coeff of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$]

$= {}^{3+60-1}C_{60} = {}^{62}C_{60} = {}^{62}C_2 = \dfrac{62.61}{2} = 31(61) = 1891$

(b) Find number of ways of scoring 60, if individual cut-off of P, C, M are 7, 8, 9 marks respectively.

$P + C + M = 60. \qquad P \ge 7\,;\, C \ge 8\,;\, M \ge 9$

$\text{Coefficient of } x^{60} \text{ in}$

$$\overbrace{(x^7 + x^8 + \ldots\ldots + x^{60})}^{\text{Physics}}\ \underbrace{(x^8 + x^9 + \ldots\ldots + x^{60})}_{\text{Chemistry}}\ \overbrace{(x^9 + x^{10} + \ldots\ldots + x^{60})}^{\text{Maths}}$$

$= \text{coefficient of } x^{60} \text{ in}$

$$x^7\left(\frac{1-x^{37}}{1-x}\right)x^8\left(\frac{1-x^{37}}{1-x}\right)x^9\left(\frac{1-x^{37}}{1-x}\right)$$

$= \text{coefficient of } x^{60} \text{ in } x^{24}\,(1 - x^{37})^3\,(1 - x)^{-3}$

$= \text{coefficient of } x^{36} \text{ in } (1 - x^{37})\,(1 - x)^{-3}$

Neglecting higher power terms

$= \text{coefficient of } x^{36} \text{ in } 1.\ (1-x)^{-3} = {}^{3+36-1}C_{36}$

$= {}^{38}C_{36} = 703$

(c) If marks are awarded in multiple of 0.5, then find number of ways of scoring total marks 60

SOLUTION

Now marks are awarded in multiple of 0.5 so

Coefficient of x^{60} in $(x^0 + x^{0.5} + x^1 + x^{1.5} + \ldots + x^{59.5} + x^{60})^3 \rightarrow$ PCM

let $x = t^2$

$= \text{coefficient of } t^{120} \text{ in } (1 + t + t^2 + t^3 + \ldots + t^{119} + t^{120})^3$

$$= \text{coefficient of } t^{120} \text{ in } \left(1.\frac{(1-t^{121})}{1-t}\right)^3$$

$= \text{coefficient } t^{120} \text{ in } (1 - t^{121})^3 (1 - t)^{-3}$

$= \text{coefficient } t^{120} \text{ in } (1)(1 - t)^{-3}$

$$= {}^{3+120-1}C_{120} = {}^{122}C_2 = \frac{122.121}{2} = 61.\,122 = 7381$$

MULTINOMIAL THEOREM FOR ARRANGEMENT

If there are ℓ object of one kind, m object of second kind, n object of third kind then the number of possible arrangements / permutations of r object out of these object $(\ell + m + n)$ is the coefficient of x^r in the expansion of

$$r!\left(1+\frac{x}{1!}+\frac{x}{2!}+\ldots\ldots+\frac{x^{\ell}}{\ell!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^m}{m!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots\ldots+\frac{x^n}{n!}\right)$$

ILLUSTRATION 97

Find number of words formed by 4 letters selected from given word 'EXAMINATION'.

SOLUTION

Alphabets present in words are AA, I, I, N, NE, X, M, T, O
Number of words using 4 letters

= coefficient of x^4 in

$$4!\overset{\rightarrow A}{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}\overset{\rightarrow I}{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}\overset{\rightarrow N}{\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)}$$

$$\overset{\rightarrow E}{\left(1+\frac{x}{1!}\right)}\overset{\rightarrow M}{\left(1+\frac{x}{1!}\right)}\overset{\rightarrow T}{\left(1+\frac{x}{1!}\right)}\overset{\rightarrow X}{\left(1+\frac{x}{1!}\right)}\overset{\rightarrow O}{\left(1+\frac{x}{1!}\right)}$$

$$= \text{coeff of } x^4 \text{ in } 4!\left(1+x+\frac{x^2}{2}\right)^3 (1+x)^5$$

coefficient of x^4 in

$$\left((1+x)^3+\frac{x^6}{8}+\frac{3}{2}(1+x)^2x^2+\frac{3}{4}(1+x)x^4\right)(1+x)^5$$

= coeff of x^4 in 24

$$\left[(1+x)^8+\frac{x^6}{8}(1+x)^5+\frac{3}{2}x^2(1+x)^7+\frac{3}{4}x^4(1+x)^6\right]$$

$$= 24\left[{}^8C_4+0+\frac{3}{2}.{}^7C_2+\frac{3}{4}.1\right] = 2454$$

Alternate:

A A, I I, N N E, X , M , T, O

Three cases are possible for making words of 4 letters

(a) 2 same, 2 distinct → ${}^3C_1\,{}^7C_2 \times 4!/2! = 756$

(b) 2 same 2 same → ${}^3C_2.{}^{4!}/_{2!2!} = 18$

(c) 4 distinct → ${}^8C_4\,.\,4! = 1680$

Number of words using 4 letters = 2454

ILLUSTRATION 98

Find number of 5 lettered words formed with the letters of the word CALCULUS.

SOLUTION

C C L L U U S A

Three cases for making 5 lettered word

(i) 2 alike + 2 alike + 1 different → ${}^3C_2 \cdot {}^3C_1 \cdot \left(\frac{5!}{2!2!1!}\right) = 270$

(ii) 2 alike + 3 different → ${}^3C_1 \cdot {}^4C_3 \cdot \frac{5!}{2!} = 720$

(iii) All 5 different → ${}^5C_5 \cdot 5! = 120$

∴ Total number of word = 1110

ILLUSTRATION 99

Find the number of 5 letter words that can be formed using the letters of the word INDEPENDENCE.

SOLUTION

EEEE NNN DD I P C

Possible cases of 5 letters words:

(i) 4 alike, 1 different → $1 \times {}^5C_1 \times \frac{5!}{4!} = 25$

(ii) 3 alike, 2 alike of other kind → ${}^2C_1 \times {}^2C_1 \times \frac{5!}{3!2!} = 40$

(iii) 3 alike, 2 different → ${}^2C_1 \times {}^5C_2 \times \frac{5!}{3!} = 400$

(iv) 2 alike, 2 alike of other kind, 1 different

→ ${}^3C_2 \times {}^4C_1 \times \frac{5!}{2!2!} = 360$

(v) 2 alike, 3 different → ${}^3C_1 \times {}^5C_3 \times \frac{5!}{2!} = 1800$

(vi) all different→ $^6C_5 \times 5! = 720$

∴ Required number of words

$= 25 + 40 + 400 + 360 + 1800 + 720 = 3345$

Distribution of Distinct Things Without Any Restriction

No. of ways of distributing n distinct things among p persons without any restriction (none, one, two or all or any number of things can be given to one person) = P^n

Distribution of Identical Things Without Any Restriction

No. of ways of distributing n identical things among p persons without any restriction (none, one, two or all or any number of things can be given to one person) $= {}^{n+p-1}C_{p-1}$ **(Beggar Method)**

Proof:

Person	Actual quantity of things given	Things can be given
P_1	r_1	0, 1, 2,,…..n
P_2	r_2	0, 1, 2,,…..n
P_3	r_3	0, 1, 2,,…..n
.	.	.
.	.	.
.	.	.
.	.	.
.	.	.
P_p	r_p	0, 1, 2,,…..n

$$0 \le r_1, r_2, r_3, \ldots\ldots r_p \le n$$

$$r_1 + r_2 + r_3 + \ldots\ldots + r_p = n \qquad \ldots(1)$$

Number of non-negative integral solution of equation (1)

$$= \text{coefficient of } x^n \text{ in } (x^0 + x^1 + \ldots. + x^n)(x^0 + x^1 + \ldots. + x^n) \ldots\ldots.. (x^0 + x^1 + \ldots + x^n)$$

$$= \text{coefficient of } x^n \text{ in } (1 + x + x^2 + \ldots.\, x^n)^P$$

$$= \text{coefficient of } x^n \text{ in } 1.\left(\frac{(1 - x^{n+1})}{1 - x}\right)^P = (1 - x^{n+1})^P$$
$$(1 - x)^{-P}$$

Neglecting terms of higher power

$$= \text{coefficient of } x^n \text{ in } (1 - x)^{-P}$$

$$= {}^{p+n-1}C_n = {}^{n+p-1}C_{p-1}.$$

ILLUSTRATION 100

In how many ways 3 rings can be worn on 4 fingers, if any no. of rings can be worn on any finger **(i)** rings are distinct

SOLUTION

Required number of ways $= 4^3$

(ii) rings are identical

SOLUTION

$x_1 + x_2 + x_3 + x_4 = 3;\ x_i \ge 0;\ i = 1, 2, 3, 4$

Required number of ways $= {}^{3+4-1}C_{4-1} = {}^6C_3 = 20$

ILLUSTRATION 101

Find number of ways of distributing 10 Apples, 5 Mangoes, 4 Oranges among 4 persons if each can receive any number of fruits and same type of fruits are identical.

SOLUTION

Using beggar method: 4 person.

Required number of ways $= {}^{10+4-1}C_{4-1} \cdot {}^{5+4-1}C_{4-1} \cdot {}^{4+4-1}C_{4-1}$

$$= \underset{\substack{\downarrow \\ \text{Apple} \\ \text{distribution}}}{{}^{13}C_3} \cdot \underset{\substack{\downarrow \\ \text{Mangoes} \\ \text{distribution}}}{{}^{8}C_3} \cdot \underset{\substack{\downarrow \\ \text{Oranges} \\ \text{distribution}}}{{}^{7}C_3}$$

$$= {}^{13}C_3 \cdot {}^8C_3 \cdot {}^7C_3$$

ILLUSTRATION 102

Number of ways in which 30 identical things are distributed among 6 persons is if each gets (i) odd number of things

SOLUTION

Let person P_i get x_i number of things such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$$

(i) If x_i is odd or $x_i = 2\lambda_i + 1$, where $\lambda_i \ge 0$, then

$2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) + 6 = 30$

or $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 12$

Then number of solutions is ${}^{12+6-1}C_{6-1} = {}^{17}C_5$.

(ii) even number of things (excluding 0)

SOLUTION

If x_i is even or $x_i = 2\lambda_i$, where $\lambda_i \ge 1$, then

$2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) = 30$

or $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 15$ where $\lambda_i \ge 1$,

let $\lambda_i = P_i + 1$ then

$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 = 9$ where $P_i \ge 1$

Therefore, required number of ways is ${}^{9+6-1}C_{6-1} = {}^{14}C_5$.

ILLUSTRATION 103

If 5 apple, 6 bananas and 10 oranges are to be distributed among 5 persons such that everybody gets atleast one fruit of each species (fruits of the same species are alike) then find number of ways

SOLUTION

Distribution of alike objects.

5 apples are distributed in 1 way (give one apple to each person)

For 6 bananas, first give one to each person then 1 banana is left, there are 5 ways for it to distribute

For 10 oranges, first give one to each person then 5 oranges are left, now use beggar method = ${}^{5+5-1}C_{5-1} = {}^{9}C_4$

∴ Total number of ways = $1 \cdot 5 \cdot {}^{9}C_4 = 630$

ILLUSTRATION 104

If the number of ways in which a selection of 100 balls can be made out of 100 identical red balls, 100 identical blue balls and 100 identical white balls is *abcd* (where *abcd* is four digit number), then find $a + c - b - d$

SOLUTION

x = number of red balls, y = number of blue balls, z = number of white balls

$x + y + z = 100$

∴ Number of ways = ${}^{102}C_2 = 5151$ (using beggar's method)

$\Rightarrow a = 5;\ b = 1;\ c = 5;\ d = 1$

$\Rightarrow (a + c - b - d) = 5 + 5 - 1 - 1 = 8$

ILLUSTRATION 105

Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls.

SOLUTION

First give one ball to each Box. Now only 2 balls are remaining and it can be given to any Box = ${}^{2+3-1}C_{3-1} = {}^{4}C_2 = 6$

ILLUSTRATION 106

$xy = 24$, find number of positive integral solution.

SOLUTION

1, 24; 2, 12; 3, 8; 4, 6; 6, 4; 8, 3; 12, 2; 24, 1 ⇒ 8 ways

Alternate:

$xy = 24 = 2^3 \cdot 3^1$

No. of positive integral solution = distribution of 2, 2, 2 and 3 in two persons x and y.

$= ({}^{3+2-1}C_{2-1}) \cdot ({}^{1+2-1}C_{2-1})$

$= {}^{4}C_1 \cdot {}^{2}C_1 = 8$

ILLUSTRATION 107

If $xyz = 432$, find number of positive integral solution.

SOLUTION

$xyz = 432 = 2^4 \cdot 3^3$

No. of positive integral solutions

= distribution of 2, 2, 2, 2 and 3, 3, 3 in two persons x and y.

$= {}^{4+3-1}C_{3-1} \cdot {}^{3+3-1}C_{3-1}$

$= {}^{4+2}C_2 \cdot {}^{3+2}C_2 = {}^{6}C_2 \cdot {}^{5}C_2$

$= 150$

ILLUSTRATION 108

$xyzw = 17640$, find

(i) Number of positive integral solution

(ii) Number of integral solution

SOLUTION

x, y, z, w can be considered as 4 persons.

(i) $xyzw = 17640 = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^2$

Number of positive integral solutions

${}^{3+4-1}C_{4-1} \cdot {}^{2+4-1}C_{4-1} \cdot {}^{1+4-1}C_{4-1} \cdot {}^{2+4-1}C_{4-1}$

$= {}^{6}C_3 \cdot {}^{5}C_3 \cdot {}^{4}C_3 \cdot {}^{5}C_3$

(ii) $xyzw = 17640$

(a) all four Positive → 8000

(b) all four Negative → 8000

(c) 2 are positive, 2 are negative → ${}^{4}C_2 \cdot {}^{2}C_2 \cdot 8000 = 48000$

Total number of integral solution = 64000

ILLUSTRATION 109

Find number of integral solution of equation

$xyzw = -17640$

SOLUTION

$xyzw = -17640$

(a) one variable positive, 3 variable negative

$\rightarrow {}^{4}C_3 \cdot {}^{1}C_1 \cdot 8000 = 32000$

(b) one variable negative, 3 variable positive

$\rightarrow {}^{4}C_3 \cdot {}^{1}C_1 \cdot 8000 = 32000$

Number of integral solution = 64000

Working Rule

Equation $x_1 + x_2 + x_3 + \cdots + x_p = n$ where $0 \le x_1, x_2, x_3, \ldots\ldots, x_p \le n$

No. of non-negative integral solution

= No. of ways of distributing n identical things among P persons without any restriction

$= {}^{n+P-1}C_{P-1}$

ILLUSTRATION 110

Find the positive number of solutions of $x + y + z + w = 20$ under the following conditions.

(i) zero values of x, y, z, w are included

SOLUTION

Since $x + y + z + w = 20$

Here $x \geq 0, y \geq 0, z \geq 0, w \geq 0$

The number of solution of the given equation is same as the number of ways of distributing 20 things among 4 person.

Hence total number of solutions $= {}^{20+4-1}C_{4-1}$,

$$= {}^{23}C_3 = \frac{23.22.21}{1.2.3} = 1771$$

(ii) zero values are excluded

SOLUTION

Since $\quad x + y + z + w = 20 \qquad \ldots(1)$

Here $\quad x \geq 1, y \geq 1, z \geq 1, w \geq 1$

or $\quad x - 1 \geq 0, y - 1 \geq 0, z - 1 \geq 0, w - 1 \geq 0$

Let $x_1 = x - 1 \Rightarrow x = x_1 + 1$

$y_1 = y - 1 \Rightarrow y = y_1 + 1$

$z_1 = z - 1 \Rightarrow z = z_1 + 1$

$w_1 = w - 1 \Rightarrow w = w_1 + 1$

Then from (1), $x_1 + 1 + y_1 + 1 + z_1 + 1 + w_1 + 1 = 20$

$\Rightarrow x_1 + y_1 + z_1 + w_1 = 16$

and $x_1 \geq 0, y_1 \geq 0, z_1 \geq 0, w_1 \geq 0$,

Hence total number of solution $= {}^{16+4-1}C_{4-1}$,

$$= {}^{19}C_3 = \frac{19.18.17}{1.2.3} = 57.17 = 969$$

ILLUSTRATION 111

If $x + y + z = 30$ where $x \geq -4$, $y > 1$, $z \geq 0$ find number of integral solution?

SOLUTION

Here $x \geq -4, y > 1, y \geq 2, z \geq 0, y \geq 2$

$x + 4 \geq 0; y - 2 \geq 0; z \geq 0$

Let $\quad t_1 = x + 4; t_2 = y - 2; t_3 = z$

$x + y + z = 30$

Now $\quad t_1 - 4 + t_2 + 2 + t_3 = 30$

$t_1 + t_2 + t_3 = 32$

Number of required solution $= {}^{32+3-1}C_{3-1} = {}^{34}C_2 = 561$

ILLUSTRATION 112

If $x + y + z = -12$, find the number of negative integral solution.

SOLUTION

Let $\quad x = -t_1; y = -t_2; z = -t_3$

Now $\quad t_1 + t_2 + t_3 = 12$

$t_1, t_2, t_3 \geq 1$

$t'_1 + t'_2 + t'_3 = 9 \quad$ where $t_i' = t_i + 1$

Number of required solution $= {}^{9+3-1}C_{3-1} = {}^{11}C_2$

ILLUSTRATION 113

If $x + y + z \leq 20$, find the number of non-negative integral solution.

SOLUTION

$x + y + z + w = 20 \quad$ where $x, y, z \geq 0$ and

$0 \leq w \leq 20$

Number of non-negative integral solution $= {}^{24+4-1}C_{4-1} = {}^{23}C_3$

ILLUSTRATION 114

If $x + y + z + w < 25$ where $x > -2$, $y > 1$, $z \geq 2$, $w \geq -2$, find the number of integral solution.

SOLUTION

Here $x \geq -1, y \geq 2, z \geq 2, w \geq -2$

$x + 1 \geq 0; y - 2 \geq 0; z - 2 \geq 0; w + 2 \geq 0$

Let, $x + 1 = t_1 ; y - 2 = t_2 ; z - 2 = t_3 ; w + 2 = t_4$

Given, $x + y + z + w < 25$

$x + y + z + w \leq 24$

$t_1 - 1 + t_2 - 2 + t_3 - 2 + t_4 - 2 \leq 24$

$t_1 + t_2 + t_3 + t_4 \leq 23 \quad$ where $\quad t_1, t_2, t_3, t_4 \geq 0$

$t_1 + t_2 + t_3 + t_4 + t_5 = 23 \quad$ where $\quad t_5 \geq 0$

Number of required solution $= {}^{23+5-1}C_{5-1} = {}^{27}C_4$

ILLUSTRATION 115

If $17 \leq x + y + z \leq 20$, find the number of positive integral solution.

SOLUTION

Here $x \geq 1; y \geq 1; z \geq 1: x = x' + 1, y = y' + 1, z = z' + 1$

$17 \leq x + y + z + w \leq 20 \Rightarrow 14 \leq x' + y' + z' \leq 17$

$x' + y' + z' = 14 \qquad$ No. of required solution $= {}^{16}C_2$

$x' + y' + z' = 15 \qquad$ No. of required solution $= {}^{17}C_2$

$x' + y' + z' = 16$ No. of required solution $= {}^{18}C_2$

$x' + y' + z' = 17$ No. of required solution $= {}^{19}C_2$

Total no. of required solution $= {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2$

$= 8.15 + 17.8 + 9.17 + 19.9$

$= 580$

ILLUSTRATION 116

If $x_1 + x_2 + x_3 + 4x_4 = 20$, find the number of non-negative integral solution.

SOLUTION

Let $x_4 = 0$ then $x_1 + x_2 + x_3 = 20$	No. of non integral solution $= {}^{22}C_2$
Let $x_4 = 1$ then $x_1 + x_2 + x_3 = 16$	No. of non integral solution $= {}^{18}C_2$
Let $x_4 = 2$ then $x_1 + x_2 + x_3 = 12$	No. of non integral solution $= {}^{14}C_2$
Let $x_4 = 3$ then $x_1 + x_2 + x_3 = 8$	No. of non integral solution $= {}^{10}C_2$
Let $x_4 = 4$ then $x_1 + x_2 + x_3 = 4$	No. of non integral solution $= {}^{6}C_2$
Let $x_4 = 5$ then $x_1 + x_2 + x_3 = 0$	No. of non integral solution $= 1$

Required solution $= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{6}C_2 + 1$

$= 536$

Alternate:

The total number of non-negative integral solutions of the given equation

$$= \text{Coefficient of } x^{20} \text{ in } \underbrace{(x^0 + x^1 + x^2 + \ldots)^3}_{\text{for } x_1 | x_2 | x_3} \underbrace{(x^0 + x^4 + x^8 + x^{16} + \ldots)}_{\text{for } x_4}$$

$$= \text{Coefficient of } x^{20} \text{ in } \left(\frac{1}{1-x}\right)^3 \left(x^0 + x^4 + x^8 + x^{12} + x^{16} + x^{20}\right)$$

$$= \text{Coefficient of } x^{20} \text{ in } [(1-x)^{-3}(1 + x^4 + x^8 + x^{12} + x^{16} + x^{20} + \ldots)]$$

$= \text{Coefficient of } x^{20}$ in $(1-x)^{-3}$ + Coeff. x^{16} in $(1-x)^{-3}$ + Coeff. of x^{12} in $(1-x)^{-3} + \ldots +$ Coeff. of x^0 in $(1-x)^{-3}$

$\because$ Coefficient of x^r in $(1-x)^{-n} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$

$$= {}^{20+3-1}C_{3-1} + {}^{16+3-1}C_{3-1} + {}^{12+3-1}C_{3-1} + {}^{8+3-1}C_{3-1} + {}^{4+3-1}C_{3-1} + 1$$

$= {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{6}C_2 + 1$

$= 231 + 153 + 91 + 45 + 15 + 1$

$= 536.$

ILLUSTRATION 117

Find number of ways in which is 16 identical toys are distributed among 3 students such that each receives not less than 3 toys.

SOLUTION

As atleast 3 toys should be given to each student so let give 3–3 toys to all students so 9 toys are distributed Now remaining 7 toys are to be distributed among three students without any restriction so we can use beggar's method so required number of ways

$= {}^{7+3-1}C_{3-1} = {}^{9}C_2 = 36$

ILLUSTRATION 118

In a bakery, 4 type of biscuits are available. How many ways Nittin can buy 10 biscuits if he decides to buy atleast one from each variety

SOLUTION

Let Nittin buys x_1, x_2, x_3 and x_4 biscuits of 4 types of biscuits respectively

We want $x_1 + x_2 + x_3 + x_4 = 10$

He wants to take atleast one from each variety so 4 biscuits are bought of each variety now only 6 biscuits should be bought, we have 4 varieties are available so its similar to beggars method. So required ways $= {}^{6+4-1}C_{4-1}$ $= {}^{9}C_3 = 84$

ILLUSTRATION 119

If x is an odd integer and y, z are even integers, satisfying $x + y + z = 47$ and $x, y, z \geq 0$ find number of solutions

SOLUTION

$x + y + z = 47$

$\Rightarrow$ The problem is similar to put 47 identical things in 3 diff. boxes, Let $x' = x - 1$

$\Rightarrow x' + y + z = 46$

Now x'_1, y, z are all even numbers

$x' = 2t_1$, $y = 2t_2$, $z = 2t_3$

$2(t_1 + t_2 + t_3) = 46$

$\Rightarrow$ $t_1 + t_2 + t_3 = 23$ where $t_1, t_2, t_3 \geq 0$

Number of required solution $= {}^{23+3-1}C_{3-1} = {}^{25}C_2 = 300$

Test Your Concepts-V

(Solutions on page H.188)

1. Find the number of integers which lie between 1 and 10^6 and which have sum of digits equal to 12.
2. Find the number of ways in which a score of 11 can be made from a throw by three persons, each throwing a single die once.
3. Find the number of ways in which any four letters can be selected from the word CORGOO.
4. Find the number of different words that can be formed out of the letters of the word MORADABAD taken four at a time.
5. The number of words which can be formed by taking two same and two different letters from the letters of the word 'COMBINATION'.
6. Find the number of positive integral solution of the equation $xyz = 3000$.
7. Find the number of positive integral solution of $abc = 30$.
8. Find number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$.
9. Find number of positive integral solutions satisfying the equation $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 240$.
10. Find the total number of integral solutions of $abc = 24$.
11. There are three piles of identical red, blue and green balls and each pile contains atleast 10 balls. The number of ways of selecting 10 balls, if twice as many red balls as green balls are to be selected.
12. There are 100 different books in a shelf, Number of ways in which 3 books can be selected so that no two of which are neighbours is ${}^{98}C_k$, then find k.
13. Let $n_1 = x_1x_2x_3$ and $n_2 = y_1y_2y_3$ be two 3-digit numbers, then the pairs of n_1 and n_2 can be formed so that n_1 can be subtracted from n_2 without borrowing.
14. Find the total number of integral solutions for the equation $x + y + z + t = 20$, where x, y, z and t are all ≥ -1.
15. Find the number of integer solutions of $x + y + z = 0$ with $x \geq -5$, $y \geq -5$, $z \geq -5$.
16. If $x + y + z = 6$ find number of positive integral solution?

Answer keys

1. 6062 **2.** 27 **3.** 7 **4.** 626 **5.** 756 **6.** 300

7. 27 **8.** 420 **9.** 1750 **10.** 120 **11.** 4 **12.** 3

13. $45 \cdot (55)^2$ **14.** ${}^{27}C_3$ **15.** 136 **16.** 10

Dearrangement

If n letters corresponding to n envelopes, the number of ways in which no letter goes to its correct envelope is given by

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots\dots+(-1)^n\frac{1}{n!}\right)$$

If r things goes to wrong place out of n things then $(n - r)$ things goes to original place $(r < n)$

$$= {}^nC_{n-r}\; r!\left(1-\frac{1}{1!}+\frac{1}{2!}+\dots\dots+(-1)^r\frac{1}{r!}\right)$$

ILLUSTRATION 120

5 students belong to 5 different cities find number of ways in which

(i) no one reach home

SOLUTION

Now we can use derangement formula directly

Required number of ways

$$= 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)$$

$$= 60 - 20 + 5 - 1 = 44$$

(ii) Number of ways by which atleast 3 students do not reach home.

SOLUTION

Lets discuss all possibilities

(a) 3 students do not reach home

Required number of ways

$$= {}^5C_3\, 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right) = 20$$

(b) 4 students do not reach home

Required number of ways

$$= {}^5C_4\, 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 45$$

(c) 5 students do not reach home

Required number of ways

$$=5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)=44$$

Total required number of ways = 20 + 45 + 44 = 109

(iii) Number of ways by which atleast 3 students reach home

SOLUTION

Lets discuss the all possibilities

(a) Exactly 3 students reach home

Here we need to select 3 students and remaining two students should not reach their home, this has only one way that they reach each others home. So no. of ways = ${}^5C_3.1=10$

(b) Exactly 4 students reach home

If 4 students reach their home them 5th student will also reach home so this case is not possible

(c) Exactly 5 students reach home

In this case all students are reaching home so number of ways is 1

Total required no. of ways = 10 + 0 + 1 = 11

EXPONENT OF PRIME *P* IN *N*!

Let p be a given prime and n any positive integer, then the maximum power of a prime 'p' in $n!$ is

$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]....+\left[\frac{n}{p^s}\right]$, where [.] denotes the greatest integer function and S is such that $p^s \le n \le p^{s+1}$.

Remark(s)

The above formula does **not** work for composite numbers.

For example if we have to find the maximum power of 6 present in *32!*, then the answer is not $\left[\frac{32}{6}\right]+\left[\frac{32}{6^2}\right]+\ldots=5$, as 5 is the number of integral multiples of 6 in 1, 2, ..., 32; and 6 can be obtained on multiplying 2 by 3 also. Hence for the required number, we find the maximum powers of 2 and 3 (say *r* and *s*) present in 32!. Using the above formula $r = 31$ and $s = 14$. Hence 2 and 3 will be combined (to form 6) 14 times thus maximum power of 6 presents in 32! is 14.

ILLUSTRATION 121

Find the exponent of all single digit natural numbers except 1 in 100!.

SOLUTION

Let $E_p(n)$ denotes the exponent of p in n.

Then, $E_p(n!)=\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^s}\right]$,

(i) Exponent of 2

$$E_2(100!)=\left[\frac{100}{2}\right]+\left[\frac{100}{2^2}\right]+\left[\frac{100}{2^3}\right]+\left[\frac{100}{2^4}\right]+\left[\frac{100}{2^5}\right]+\left[\frac{100}{2^6}\right]+\left[\frac{100}{2^7}\right]+\left[\frac{100}{2^8}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 + 0 + 0 + \ldots.. = 97$$

(ii) Exponent of 3

$$E_3(100!)=\left[\frac{100}{3}\right]+\left[\frac{100}{3^2}\right]+\left[\frac{100}{3^3}\right]+\left[\frac{100}{3^4}\right]$$

$$= 33 + 11 + 3 + 1 = 48$$

Hence, the exponent of 3 in 100! is 48

(iii) Exponent of 4

Formula can be used as 4 is composite number

$4 = 2^2$

$100! = 2^{97}k = 2.4^{48}k$ so exponent of 4 is 48

(iv) Exponent of 5

$$E_5(100!)=\left[\frac{100}{5}\right]+\left[\frac{100}{5^2}\right]=20+4=24$$

Till now, we have identified that

$100! = 2^{97}.3^{48}.5^{24}.k$

(v) Exponent of 6

Here, 6 is composite number so formula can not be used

$6 = 2.3$

$100! = 2^{97}.3^{48}.5^{24}.k = 2^{49}.(2.3)^{48}k = 6^{48}.q$

So, exponent of 6 is 48

(vi) Exponent of 7

$$E_7(100!)=\left[\frac{100}{7}\right]+\left[\frac{100}{7^2}\right]+\left[\frac{100}{7^3}\right]+\left[\frac{100}{7^4}\right]$$

$$= 7 + 2 + 0 + 0 = 9$$

(vii) Exponent of 8

Here, 8 is composite number so formula cannot be used

$8 = 2^3$

$100! = 2^{97}.k = 2.(2^3)^{32}.k = 2.8^{32}.k$

So, exponent of 8 is 32

(viii) Exponent of 9

Here, 9 is composite number so formula cannot be used

$9 = 3^2$

$100! = 2^{97}.3^{48}.5^{24}.k = (3^2)^{24}.q = 9^{24}.q$

So, exponent of 9 is 24

> **Cypher:** Number of zeroes present at the end of any number
>
> Let number 20030000 then cypher is 4 not 6, we should only count zeroes present at end only.

ILLUSTRATION 122

Find the number of zeros at the end of 100!.

Or

Find cyphers in 100!

SOLUTION

Now, $$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and, $$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

Number of zeros at the end of 100! = exponent of 10 in 100!

Therefore, $100! = 2^{97} \cdot 5^{24}\,(\lambda) = (2 \cdot 5)^{24}\, 2^{75}(\lambda)$

$= 10^{24}\, K$

Thus, the number of zeros at the end of 100! is 24.

ILLUSTRATION 123

If (500)! = k($2^a 3^b 5^c$), (where $a, b, c \in$ N, k does not contain any factor of 2, 3 and 5) then prove that number of zeros at the end of (500)! is equal to 124

SOLUTION

$(500)! = k.(2^a 3^b 5^c)$

$c = \left[\frac{500}{5}\right] + \left[\frac{500}{5^2}\right] + \left[\frac{500}{5^3}\right]$ (Where [.] denotes greatest integer function)

$= 100 + 20 + 4...$

$\therefore \quad c = 124$

Any set of 2 and 5 produces a zero at the end

$\therefore$ Number of zeroes at the end = number of sets of 2 and 5. Clearly $a > c$

$\therefore$ Number of zeros = 124

Test Your Concepts-VI

(Solutions on page H.189)

1. Ajay writes letters to his five friends and addresses the corresponding. Find the number of ways can be letters be placed in the envelops so that atleast two of them are in the wrong envelops.
2. Find number of ways can 10 letters be placed in 10 marked envelops, so that no letters is in the right envelope.
3. Find the prime factorisation of 10!, i.e. if $10! = 2^p \cdot 3^q \cdot 5^r \cdot 7^s$, then find p, q, r and s.
4. Find the exponent of 2 in $x = 20 \times 19 \times 18 \times 17 \times ... \times 11$.
5. Find the number of zero's at the end of (60)!.
6. For some natural number N, the find number of positive integral 'x' satisfying the equation,

   $1! + 2! + 3! + ... + (x!) = (N)^2$
7. Find the number of cyphers at the end of ${}^{2002}C_{1001}$.

Answer keys

1. 119 **2.** $10!\left\{\frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{10!}\right\}$ **3.** $p = 8, q = 4, r = 2, s = 1$

4. 10 **5.** 14 **6.** 2 **7.** 1

SOLVED PROBLEMS

PROBLEM 1

Find the number of times of the digits 4 will be written when listing the integers from 1 to 1000.

SOLUTION

Since 4 does not occur in 1000, we have to count the number of times 4 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz where $0 \le x, y, z, \le 9$.

Let us first count the numbers in which 4 occurs exactly once. i.e. $({}^3C_1)(9)(9) = 243$ and 4 can occurs exactly at two places in $({}^3C_2)(9) = 3 \times 9$ such number.

Lastly, 4 can occur in all three digits in one number only. Hence the number of times 4 occurs is

$$3(9)(9) + 3 \times 9 + 1 = 271$$

PROBLEM 2

Find the number of ways in which 5 boys and 5 girls be seated in a row so that

(i) No two girls may sit together.

(ii) All the girls sit together and all the boys sit together

(iii) All the girls are never together.

SOLUTION

(i) 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in ${}^6P_5$ ways.

$$\mathrm{x}B_1\,\mathrm{x}B_2\,\mathrm{x}B_3\,\mathrm{x}B_4\,\mathrm{x}B_5\,\mathrm{x}$$

Hence, the number of ways in which no two girls sit together $= 5! \times {}^6P_5 = 5! \times 6!$

(ii) The two groups of girls and boys can be arranged in 2! ways 5 girls can be arranged among themselves in 5! Ways. Similarly, 5 boys can be arranged among themselves in 5! ways.

$$\boxed{B_1\,B_2\,B_3\,B_4\,B_5} \qquad \boxed{G_1\,G_2\,G_3\,G_4\,G_5}$$

Hence, by the fundamental principle of counting, the total number of requisite seating arrangements $= 2!\,(5! \times 5!) = 2(5!)^2$.

(iii) The total number of ways in which all the girls are never together

$=$ Total number of arrangements $-$ Total number of arrangements in which all the girls are always together $= 10! - 5! \times 6!$

PROBLEM 3

Ten persons numbered 1, 2, …., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. If $w_1, w_2, \ldots, w_{10}$ are the number of games won by players 1, 2, 3, …, 10 respectively, and $\ell_1, \ell_2, \ldots, \ell_{10}$ are the number of games lost by the players 1, 2, …..10, respectively, then show that

(i) $\Sigma w_i = \Sigma \ell_i = 45$ **(ii)** $w_i + \ell_i = 9$ **(iii)** $\Sigma w_i^2 = \Sigma \ell_i^2$

SOLUTION

Clearly, each player will play 9 games total number of games is ${}^{10}C_2 = 45$. Clearly,

$$w_i + \ell_i = 9 \text{ and } \Sigma w_i = \Sigma \ell_i = 45$$

$$\Rightarrow w_i = 9 - \ell_i \Rightarrow w_i^2 = 81 + \ell_i^2 - 18\ell_i$$

$$\Rightarrow \Sigma w_i^2 = 81 \times 10 + \Sigma \ell_i^2 - 18\Sigma \ell_i$$

$$= 810 + \Sigma \ell_i^2 - 18 \times 45 = \Sigma \ell_i^2$$

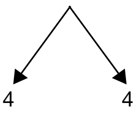
PROBLEM 4

Let there are 5 apples (3 alike and 2 different), 4 mangoes (2 alike and 2 different) and 3 guavas (all different). Find the number of ways in which all of these fruits can be distributed between two children equally and each having exactly two identical fruits.

SOLUTION

$$AAAA_1A_2,\ MMM_1M_2,\ G_1G_2G_3 \Rightarrow 12 \text{ fruits}$$

$$AA_1A_2\ M_1M_2\ G_1G_2G_3 \quad \boxed{AA} \quad \boxed{MM}$$



$$\frac{8!}{4!4!2!} \cdot 2! = \frac{8!}{4! \cdot 4!} = 70$$

If first child receives A in the distribution of 8 fruits then he cannot get $\boxed{AA}$ he has to take $\boxed{MM}$

∴ Distribution of AA and MM can be done by only 1 ways.

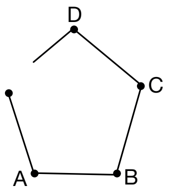
Hence, required number of ways = 70.

PROBLEM 5

Find the number of triangles whose angular points are at the angular points of a given polygon of n sides, but none of whose sides are the sides of the polygon.

SOLUTION

A polygon of n sides has n angular points. Number of triangles formed from these n angular points $= {}^nC_3$.



If one side is common then let assume it is AB then we cannot take C and adjacent to A so we have n-4 vertices left out for 3rd vertex

No. of triangles whose one side common = $n(n-4)$

If two sides are common then we can take ABC, BCD, ... like this it means they are n triangles

Required number of triangles = total – one side common – two side common

$$= {}^nC_3 - n.(n-4) - n$$

$$= \frac{1}{6}n\,(n-4)\,(n-5).$$

PROBLEM 6

Find the number of 5 digit which contains at most

(a) two distinct digits

SOLUTION

Case I:

Without '0' : select a, b from 1 to 9 : ${}^9C_2$ ways

The number are of the form

aaabb, bbbaa, bbbba, aaaab.

Total ways in this case

$$= {}^9C_2\left[\frac{5!}{3!2!}\times 2 + \frac{5!}{4!}\times 2\right] = 36(20+10) = 1080$$

Case II:

All same: 9

Case III:

Using atleast one time '0' (a0000, aa000, aaa00, $aaaa$0 type)

Number of ways = ${}^9C_1({}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3)$

$= 135$

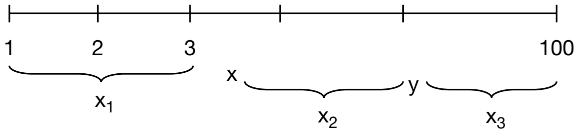
Total ways = 1080 + 9 + 135 = 1224

PROBLEM 7

In how many ways two distinct natural numbers can be selected which are less than or equal to 100 and differ by atleast 10?

SOLUTION

Select two number x, y so that $x - y \geq 10$



Let x_1 be the number of numbers before 1st selection x, x_2 be the numbers between x and y, x_3 be the number of numbers after 2nd selection y

$$x_1 + x_2 + x_3 = 98,\ x_1 \geq 0,\ x_2 \geq 10,\ x_3 \geq 0$$

$$\Rightarrow x_1 + x_2' + x_3 = 88,\ x_1 \geq 0,\ x_2' \geq 0,\ x_3 \geq 0$$

Number of solutions = ${}^{90}C_2$

PROBLEM 8

In an examination, the maximum marks for each of three papers is n and that for the fourth paper is $2n$. Prove that the number of ways in which a candidate can get 3n marks is $\frac{1}{6}(n+1)(5n^2+10n+6)$.

SOLUTION

Let x_1, x_2, x_3, x_4 are marks in given papers.

$$x_1 + x_2 + x_3 + x_4 = 3n$$

$$0 \leq x_1, x_2, x_3 \leq n$$

$$0 \leq x_4 \leq 2n$$

The number of ways of getting $3n$ marks

= coefficient of x^{3n} in $(1 + x + x^2 + \ldots + x^n)^3\,(1 + x + \ldots + x^{2n})$

= coefficient of x^{3n} in

$$\left(\frac{1\cdot 1 - x^{n+1}}{1-x}\right)^3\left(1\cdot\left(\frac{1-x^{2n+1}}{1-x}\right)\right)$$

= coefficient of x^{3n} in $(1 - x^{n+1})^3\,(1 - x^{2n+1})\cdot\ (1-x)^{-4}$

= coefficient of x^{3n} in

$$(1 - 3x^{n+1} + 3x^{2n+2} - x^{3n+3})\,(1 - x^{2n+1})\,(1-x)^{-4}$$

ignoring higher power

= coefficient of x^{3n} in

$$(1 - 3x^{n+1} - x^{2n+1} + 3x^{2n+2})\,(1-x)^{-4}$$

$$= {}^{3n+3}C_{3n} - 3.\ {}^{2n+2}C_{2n-1} + 3.\ {}^{n+1}C_{n-2} - {}^{n+2}C_{n-1}$$

$$= \frac{(3n+3)!}{3!(3n)!} - 3\,.\frac{(2n+2)!}{3!(2n-1)!} + 3\frac{(n+1)!}{3!(n-2)!} - \frac{(n+2)!}{3!(n-1)!}$$

$$= \frac{1}{6}(n+1)\,(27n^2 + 27n + 6 - 24n^2 - 12n + 3n^2 - 3n - n^2 - 2n)$$

$$= \frac{1}{6}(n+1)\,(5n^2 + 10n + 6)$$

PROBLEM 9

Four lettered words are formed using letters of word 'PAPAMUMMY'. If m denotes number of selections of four letters and n denotes number of four lettered words, then find $m + n$.

SOLUTION

PP, AA, MMM, U, Y

Lets first consider selection of 4 letters

(i) 4 different → ${}^5C_4 = 5$

(ii) 2 same, 2 different. → ${}^3C_1 \times {}^4C_2 = 18$

(iii) 2 same, 2 same $\rightarrow {}^3C_2 = 3$
(iv) 3 same, 1 different $\rightarrow {}^1C_1 \cdot {}^4C_1 = 4$

Hence, $m = 30$

Lets consider 4 letter words (we need to consider arrangement also along with selection)

$$n = {}^5C_4 \cdot 4! + {}^3C_1 \cdot {}^4C_2 \cdot \frac{4!}{2!} + {}^3C_2 \cdot \frac{4!}{(2!)^2} + {}^4C_1 \cdot \frac{4!}{3!} = 370$$

now $m + n = 400$

We can also use multinomial approach for it.

PROBLEM 10

Find number of ways in which 15 indistinguishable oranges can be distributed in 3 different boxes so that every box has atmost 8 oranges?

SOLUTION

Let x, y, z are quantities of oranges in 3 Boxes

$x + y + z = 15; x \le 8; y \le 8, z \le 8$

Total solution $= {}^{17}C_2 = 136$

Suppose x or y or z is given 9 oranges

$x + y = 6$ (let $z = 9$)

Number of possible solution of $x, y = 7$

Suppose x or y or z is given 10 oranges

$x + y = 5$ (let $z = 10$)

Number of possible solution of $x, y = 6$

Suppose x or y or z is given 11 oranges

$x + y = 4$ (let $z = 11$)

Number of possible solution of $x, y = 5$

Similarly, rejected cases $= {}^3C_1(7 + 6 + 5 + 4 + 3 + 2 + 1) = 84$

Required number of ways $= 136 - 84 = 52$

PROBLEM 11

We have 19 identical gems available with us which are needed to be distributed among A, B and C such that A always gets an even number of gems. The number of ways this can be done is

SOLUTION

$A + B + C = 19$

Case I:

If $A = 0$ then number of ways $= {}^{20}C_1$

Case II:

If $A = 2$ then number of ways $= {}^{18}C_1$

Case III:

If $A = 4$ then number of ways $= {}^{16}C_1$

:   :   :   :
:   :   :   :

Case X:

If $A = 18$ then number of ways $= {}^2C_1$

$\therefore$ Total no. of ways $= {}^{20}C_1 + {}^{18}C_1 + {}^{16}C_1 + \ldots + {}^2C_1 = 110$

PROBLEM 12

How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \ge 0$?

SOLUTION

We have: $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

and $x_1 + x_2 + x_3 = 5$

These two equations reduce to

$$x_4 + x_5 = 15 \qquad \ldots(1)$$

and, $$x_1 + x_2 + x_3 = 5 \qquad \ldots(2)$$

Since corresponding to each solution of (1) there are solutions of equation (2). So, total number of solutions of the given system of equations

= No. of solutions of (1) × No. of solutions of (2)

$$= {}^{15+2-1}C_{2-1} \times {}^{5+3-1}C_{3-1} = {}^{16}C_1 \times {}^7C_2 = 336.$$

PROBLEM 13

How many integers between 1 and 10^6 have the sum of their digits equal to 18?

SOLUTION

Let $n = x_6\, x_5\, x_4\, x_3\, x_2\, x_1$ be a number between 1 and 10^6, where $x_1, x_2, x_3, \ldots x_6$ are digits at one's, tens, hundreds… places. Clearly $0 \le, x_i \le 9$, $i = 1, 2, \ldots, 6$.

The total number of values of n is the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $0 \le x_i \le 9; i = 1, 2, \ldots\ldots\ldots 9$

The number of solutions of the above equation is

Coefficient of x^{18} in $(x^0 + x^1 + x^2 + \ldots + x^9)^6$

$$= \text{Coefficient of } x^{18} \text{ in } \left(\frac{1 - x^{10}}{1 - x}\right)^6$$

$$= \text{Coefficient of } x^{18} \text{ in } (1 - x^{10})^6 (1 - x)^{-6}$$

$$\text{Coefficient of } x^{18} \text{ in } (1 - {}^6C_1 x^{10} + {}^6C_2 x^{20} \ldots)(1 - x)^{-6}$$

$$= \text{Coefficient of } x^{18} \text{ in } (1 - x)^{-6} - {}^6C_1 \times \text{Coefficient of } x^8 \text{ in } (1 - x)^{-6}$$

$$= {}^{18+6-1}C_{6-1} - {}^6C_1 \times {}^{8+6-1}C_{6-1}$$

$$= {}^{23}C_5 - 6 \times {}^{13}C_5$$

$$= 33649 - 7722$$

$$= 25927.$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A box contains two white, three black and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw? (All balls are distinct)
(A) 45 (B) 64
(C) 32 (D) 48

2. A five-digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is
(A) 216 (B) 240
(C) 600 (D) 3125

3. Sum of all three-digit numbers (no digit being zero) having the property that all digits are perfect squares, is
(A) 3108 (B) 6216
(C) 13986 (D) 2828

4. Number of 4-digit positive integers if the product of their digits is divisible by 3 is.
(A) 2700 (B) 5464
(C) 6628 (D) 7704

5. Among the 8! permutations of the digits 1, 2,…..8, consider those arrangements which have the following property. If we take any five consecutive positions, the product of digits in these positions is divisible by 5. Then no. of such arrangements is.
(A) 7! (B) $2 \times 7!$
(C) ${}^{7}C_{4}$ (D) none

6. The number of ordered pairs of positive integers (m, n) satisfying $m \le 2n \le 60$, $n \le 2m \le 60$ is
(A) 240 (B) 480
(C) 960 (D) none

7. Let the product of all the divisors of 1440 be P. If P is divisible by 24^{x}, then maximum value of x is
(A) 28 (B) 30
(C) 32 (D) 36

8. The number of ordered pairs of positive integers (a, b) such that the least common multiple(l cm) of a and b is $2^{3}5^{7}11^{13}$ is
(A) 2835 (B) 273
(C) 448 (D) 2100

9. I have tiled my square bathroom wall with congruent square tiles. All the tiles are red, except those along the two diagonals, which are all blue. If I used 121 blue tiles, then the number of red tiles, I used are
(A) 900 (B) 1800
(C) 3600 (D) 7200

10. Let $A = \{x_1, x_2, \ldots, x_7\}$, $B = \{y_1, y_2, y_3, y_4\}$. The total number of functions $f : A \to B$ that are onto and there are exactly three elements in A such that $f(x) = y_1$ is equal to
(A) 245 (B) 490
(C) 1050 (D) 1260

11. If three positive real numbers a, b, c are in A.P. such that $abc = 4$, then the minimum possible value of b is
(A) $2^{\frac{3}{2}}$ (B) $2^{\frac{2}{3}}$
(C) $2^{\frac{1}{3}}$ (D) $2^{\frac{5}{2}}$

12. There are 10 different coloured flags. There are 4 flag poles. Number of ways of arranging all these flags on poles such that no pole is empty is
(A) 13 ! (B) $\frac{13!}{3!}$
(C) (10!) (84) (D) (13!)(6)

13. The number of all positive integers of not more than 10 digits that can be formed by using the digits 0, 1, 2, 3, 4 is
(A) ${}^{10}C_{5} - {}^{9}C_{4}$ (B) $5^{10} - 4^{10}$
(C) $5^{10} - 1$ (D) $10^{5} - 1$

14. In a plane there are 37 straight lines of which 13 passes through the point A and 11 passes through the point B. No three lines passes through one point, no lines passes through both points A and B, and no two are parallel, then the number of intersection points the lines have is equal to
(A) 495 (B) 535
(C) 601 (D) 728

15. A, B, C, D develop 18 items. 5 items jointly by A and C, 4 items by A and D, 4 items by B and C and 5 items by B and D. The number of ways of selecting 8 items out of 18 so that the selected ones belong equally to A, B, C, D is
(A) 5226 (B) 5626
(C) 4418 (D) 4936

16. Three ladies have each brought their one child for admission to a school. The principal wants to interview the six persons one by one subject to the condition that no mother is interviewed before her chid. The number of ways in which interviews can be arranged is
(A) 6 (B) 36
(C) 72 (D) 90

17. The number of ways in which six boys and six girls can be seated at a round table so that no two girls sit together and two particular girls do not sit next to a particular boy is

(A) 6! 4! (B) 2.5! 4!
(C) 2.6! 4! (D) 5! 4!

18. If all permutations of the letters of the word AGAIN are arranged as in dictionary the 47th word is
(A) NAAGI (B) INGAA
(C) NAAIG (D) INAGA

19. Out of n students a committee of 12 students is formed in all possible ways. It is observed that the number of such committees containing 2 particular students A, B is 3 times the number of committees containing another 3 particular students D, E, F. Then n =
(A) 32 (B) 38
(C) 45 (D) 72

20. If a set A has 4 elements and a set B has 7 then the number of function from A to B that are not injections is
(A) ${}^7P_4 - {}^7C_4$ (B) 1561
(C) $7^4 - {}^7C_4$ (D) 1651

21. There are 12 persons sitting around a circular table. Then the number of ways of selecting 3 out of them such that no two persons are adjacent to each other
(A) 220 (B) 108
(C) 112 (D) 124

22. The sum of all the divisors of the number 2520 other than 1 and the given number is
(A) 6839 (B) 9360
(C) 5042 (D) 756

23. If '4' points are taken on each of three coplanar parallel lines then maximum no.of triangles at these points is
(A) 64 (B) 27
(C) 208 (D) 144

24. The number of solutions of $x_1 + x_2 + x_3 = 51$ (x_1, x_2, x_3 being odd natural numbers)
(A) 300 (B) 330
(C) 325 (D) 51.52.50

25. Number of tiplets (x, y, z) is chosen from the set {1, 2, 3, ...10} such that $x \le y < z$ is
(A) 120 (B) 145
(C) 165 (D) 160

26. In a shooting competition a man can score 5, 4, 3, 2 or 0 points for each shot. Then the number of different ways in which he can score 30 in seven shots is __
(A) 419 (B) 418
(C) 420 (D) 421

27. Total no. of integers 'n' such that $2 \le n \le 2000$ and H.C.F of n and 36 is one are equal to
(A) 666 (B) 667
(C) 665 (D) None

28. There are 720 permutations of the digits 1,2,3,4,5,6. Suppose these permutations are arranged from smallest to largest numerical values, beginning from 1 2 3 4 5 6 and ending with 6 5 4 3 2 1. What is the position of the number 321546?
(A) 267 (B) 243
(C) 271 (D) 145

29. In a certain test there are n questions. In this test $2n$-i students gave wrong answers to atleast i questions where i = 1,2,3…..n. If the total number of wrong answers given is 2047, then n is
(A) 10 (B) 11
(C) 12 (D) 13

30. The no. of ways of distribution of 10 different games among 10 children so that exactly two children get no game is
(A) ${}^{10}C_8 \dfrac{10!.25}{3}$ (B) ${}^{10}C_8(10)^8$
(C) ${}^{10}C_8 \dfrac{10!.75}{3!}$ (D) ${}^{10}C_8 \dfrac{10!.25}{3!}$

31. The number of ways of arranging eight persons (having A, B, C, D, E, F among them) in a row so that A, B, C, D, E, F are always in order ABCDEF (not necessarily consecutive) is
(A) ${}^8P_6$ (B) 112
(C) 28 (D) 56

32. The no. of ways in which an examiner can assign 30 marks to 8 questions giving not less than 2 marks to any question, is
(A) ${}^{21}C_7$ (B) ${}^{21}C_2$
(C) ${}^{25}C_8$ (D) ${}^{30}C_8$

33. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$ then the number of ordered quadruples $(a, b, c, (D)$ is
(A) 145 (B) 165
(C) 135 (D) 155

34. If the natural numbers are written as a sequence 12345678910111213141516 .... then 2007th digit is
(A) 0 (B) 5
(C) 7 (D) 4

35. The number of ways of assigning 20 marks to four questions, with a maximum of 10 marks for each question is
(A) 891 (B) 981
(C) 881 (D) 791

36. Consider the set S = {1, 2, ..., 999, 1000}. D_n is the subset of S containing the elements that are divisible by n. What is the cardinality of the set $(D_2 \cup D_5) \cap D'_4$?
(A) 350 (B) 451
(C) 450 (D) 351

37. The number of ways of forming an arrangement of 4 letters from the letters of the word "IITJEE" is
(A) 66 (B) 96
(C) 102 (D) 180

38. If $n = 10800$, then the number of divisions of the form $4m + 2$ $(m \in N)$,
(A) 10 (B) 12
(C) 14 (D) 16

39. The total number of ways in which 15 identical blankets can be distributed among four persons so that each of them gets atleast two blankets is equal to
(A) 10_{C_3} (B) 9_{C_3}
(C) 11_{C_3} (D) $15_{C_3}.7_{P_4}$

40. Let a be factor of 120, then the number of positive integral solutions of $x_1x_2x_3 = a$ is

(A) 160 (B) 320
(C) 480 (D) 960

41. The number of ways in which the letters of the word PESSIMISTIC can be arranged so that no two Ss are together, no two Is are together and letters S and I are never together is

(A) 86400 (B) 48000
(C) 2400 (D) 5480

42. The area of regular polygon of n sides with length of side $\sqrt{3}$ where $n > 1$ and satisfies the relation $\sum_{r=0}^{n} \frac{n^2 - 3n + 3}{2^n C_r} = \sum_{r=0}^{n} \frac{r}{^nC_r}$ is

(A) $30\sqrt{3}$ (B) $\frac{3\sqrt{3}}{4}$
(C) $\frac{5\sqrt{3}}{4}$ (D) $\frac{3\sqrt{3}}{2}$

43. Rajadhani express going from Bombay to Delhi stops at five intermediate stations. 10 passengers enter the train during the journey with 10 different ticket of three classes. The number of different sets of tickets they may have is

(A) $^{15}C_{10}$ (B) $^{25}C_{10}$
(C) $^{45}C_{10}$ (D) $^{35}C_{10}$

44. The number of ways to select 6 people out of 6 married couples in which exactly one couple is present, is

(A) 60 (B) 480
(C) 660 (D) 240

45. The number of ways in which 6 different rings can be arranged on 4 fingers so that there are exactly 2 rings on one particular finger is

(A) $^6P_2 \times 3^4$ (B) $^6C_2 \times \frac{6!}{2!}$
(C) $^6C_2 \times 3^4$ (D) $4^6 - 4 \times 3^6 + 60.2^6 - 4$

46. The number of ways of arranging 15 students $A_1, A_2, \ldots, A_{15}$, in a row such that neither A_2 nor A_3 be seated before A_1 is

(A) $\frac{2\underline{|15}}{\underline{|3}}$ (B) $\frac{\underline{|15}}{\underline{|3}}$
(C) $\underline{|2}\,\underline{|15}$ (D) None

47. The number of words that can be formed using all the letters of the word REGULATIONS such that G must come after R, L must come after A and S must come after N

(A) $\frac{\underline{|11}}{8}$ (B) $\underline{|11}$
(C) $^{11}P_6$ (D) None

48. The number of four letter words that can be formed using the letters of the word RAMANA is

(A) 70 (B) 71
(C) 72 (D) 73

49. The number of ways in which three numbers in A.P. can be selected from 1, 2, 3, ..., 21 is

(A) 55 (B) 45
(C) 100 (D) None

50. The number of proper divisors of $\underline{|15}$ is

(A) 4032 (B) 4030
(C) 4031 (D) None

51. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(A) 55 (B) 66
(C) 77 (D) 88

52. The number of integral solutions for the equation $x + y + z + t = 20$, where x, y, z, t are all ≥ -1, is

(A) $^{23}C_4$ (B) $^{23}C_3$
(C) $^{27}C_4$ (D) $^{27}C_3$

53. The number of the positive integer pairs (x, y) such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2007}$ where $x < y$ is

(A) 5 (B) 6
(C) 7 (D) 8

54. The number of three-digit natural number divisible by 3 in which atleast one of the digits is repeated

(A) 64 (B) 66
(C) 68 (D) 72

55. The number of 4-digit natural numbers in which digit at thousand place is 1 and exactly one of the digit comes twice are

(A) 108 (B) 21
(C) 432 (D) 864

56. In the process of numbering the pages of a large book, the printer uses 1890 digits. The numbers of pages in the book is

(A) 600 (B) 666
(C) 659 (D) 660

57. Total number of ways in which 256 identical balls can be placed in '16' numbered boxes (1, 2, 3,......16) such that r^{th} box contains at least 'r' ball is $(1 \leq r \leq 16)$

(A) $^{120}C_{15}$ (B) $^{135}C_{15}$
(C) $^{256}C_{15}$ (D) $^{255}C_{15}$

58. Number of ordered triplets (a, b, c) where $1 \leq a, b, c \leq 10$, such that $2^a + 3^b + 5^c$ is a multiple of 4 is

(A) 1000 (B) 500
(C) 250 (D) 125

59. If a man climbs either one step or two steps at a time, the total number of ways in which he can climb up a staircase of $2n$ steps (starting from the bottom) is

(A) $\sum_{r=0}^{n} {}^{2n-r}C_r$
(B) $\sum_{r=0}^{2n} {}^{2n-r}C_r$
(C) $\sum_{r=0}^{2n} {}^{n-r}C_r$
(D) $\sum_{r=0}^{n} {}^{n-r}C_r$

60. Number of four-digit numbers with all digits different and containing the digit 7 is
(A) 2016
(B) 1828
(C) 1848
(D) 1884

61. We have 30 objects of which 10 are identical and the other 20 are all distinct among themselves and also from the identical ones. No. of ways to select 10 objects out of this group are
(A) $\dfrac{2^{20} - {}^{20}C_{10}}{2}$
(B) $\dfrac{2^{10} + {}^{20}C_{10}}{2}$
(C) $\dfrac{2^{10} - {}^{20}C_{10}}{2}$
(D) $\dfrac{2^{20} + {}^{20}C_{10}}{2}$

62. In a one-day cricket match between India and Pakistan, the Indians require 10 runs in the last 3 balls to win. If any of the scores 0, 1,2, 3, 4, 5, 6 can be made from a ball and no wides or no-balls are bowled, then the number of different ways, such that batsmen make exactly 10 runs is
(A) 15
(B) 21
(C) 17
(D) none

63. The number of words that can be formed using the letters of the word ARTICLE which either begin with A or end with E or the middle letter I is
(A) 344
(B) 1824
(C) 720
(D) 252

64. The number of ways in which nine boys and five girls can be arranged in two vans each having numbered seat, three in the front and five at the back such that at least four girls always sit together is
(A) ${}^{11}P_9$ 5! 46
(B) ${}^{11}P_9$. 5!
(C) ${}^{11}P_9$ 6!. 2
(D) ${}^{10}P_9$ 5!.38

65. A wooden cube with edge length 'n' (>2) units is painted red all over. By cutting parallel to faces, the cube is cut into n^3 smaller cubes each of unit edge length. If the number of smaller cubes with just one face painted Red is equal to the number of smaller cubes completely un painted, then $n =$
(A) 2
(B) 7
(C) 8
(D) 6

66. The number of different words that can be formed using all the letters of the word 'SHASHANK' such that in any word the vowels are separated by atleast two consonants, is
(A) 2700
(B) 1800
(C) 900
(D) 600

67. A (1, 2) and B(5, 5) are two points. Starting from A, line segments of unit length are drawn either rightwards or upwards only, in each step, until B is reached. Then, the number of ways of connecting A and B in this manner is
(A) 35
(B) 40
(C) 45
(D) 50

68. The number of 5-digit numbers which are divisible by 3 that can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9, when repetition of digits is allowed, is
(A) 3^9
(B) 4.3^8
(C) 5.3^8
(D) 7.3^8

69. The number of 4-digit numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 such that the least digit used is 4, when repetition of digits is allowed, is
(A) 617
(B) 671
(C) 716
(D) 761

70. The number of distinct rational numbers of the form p/q, where $p, q \in \{1, 2, 3, 4, 5, 6\}$ is
(A) 23
(B) 32
(C) 36
(D) 28

71. How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent?
(A) $6.8.{}^7C_4$
(B) $7.{}^6C_4.{}^8C_4$
(C) $8.{}^6C_4.{}^8C_4$
(D) $6.7.{}^8C_4$

72. The number of arrangements of $A_1, A_2, \ldots, A_{10}$ in a line so that A_1 is always above than A_2, is
(A) $2 \times \lfloor\!\underline{10}$
(B) $\dfrac{1}{2} \times \lfloor\!\underline{10}$
(C) ${}^{10}P_2$
(D) ${}^{10}C_2$

73. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five question. The number of choices available to him is
(A) 140
(B) 196
(C) 280
(D) 346

74. Let there be 9 fixed points on the circumference of a circle. Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that atmost 2 straight lines meet in any interior point of the circle. The number of such interior intersection points is
(A) 126
(B) 351
(C) 756
(D) 526

75. There are three piles of identical red, blue and green balls and each pile contains at least 10 balls. The number of ways of selecting 10 balls if twice as many red balls as green balls are to be selected is
(A) 3
(B) 4
(C) 6
(D) 5

76. How many times is the digit 5 written when listing all numbers from 1 to 1,00,000?
(A) 5×10^4
(B) $1 + 10 + 100 + 1000 + 10000$
(C) 5×10^3
(D) $1 + 10 + 100 + 1000$

77. There are 10 stations on a circular path. A train has to stop at 3 stations such that no two stations are adjacent. The number of such selections must be

(A) 50 (B) 84
(C) 126 (D) None

78. A flight of stairs has 10 steps. A person can go up the steps one at a time, two at a time or any combination of 1*s* and 2*s*. The total number of ways in which the person can go up the stairs is

(A) 75 (B) 79
(C) 85 (D) 89

79. How many combinations can be made up of 3 hens, 4 ducks and 2 geese so that each combination has hens, ducks and geese? (birds of same kind all different)

(A) 305 (B) 315
(C) 320 (D) 325

80. There are 6 balls of different colours and 6 boxes of colours same as those of the balls. The number of ways in which the balls, one each in a box could be placed such that a ball does not go to a box of its own colour is

(A) 275 (B) 265
(C) 285 (D) 305

81. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is

(A) 38 (B) 21
(C) 5 (D) $^8C_3$

82. The number of point (x, y, z)in space, whose each coordinate is a negative integer such that $x + y + z + 12 = 0$, is:

(A) 55 (B) 110
(C) 75 (D) 100

83. At an election, a voter may vote for any number of candidates not greater than the number to be elected. There are ten candidates and 4 are to be elected. If a voter votes for at least one candidate, then the number of ways in which he can vote, is

(A) 6210 (B) 385
(C) 1110 (D) 5040

84. In the next world cup of cricket there will be 12 teams, divided equally in two groups. Teams of each group will play a match against each other. From each group 3 top teams will qualify for the next round. In this round each team will play against others once. Four top teams of this round will qualify for the semifinal round, where each team will play against the others once. Two top teams of this round will go into the final round, where they play one match. The minimum number of matches in the next world cup will be

(A) 54 (B) 53
(C) 52 (D) 55

85. If $x, y \in (0, 30)$ such that $\left[\frac{x}{3}\right]+\left[\frac{3x}{2}\right]+\left[\frac{y}{2}\right]+\left[\frac{3y}{4}\right] = \frac{11}{6}x + \frac{5}{4}y$ (where $[x]$ denotes greatest integer $\le x$). Then the number of ordered pairs (x,y) is

(A) 0 (B) 2
(C) 4 (D) 28

86. Let $x_1, x_2, \ldots\ldots, x_k$ are the total divisors of positive integer n. If $\sum_{i=1}^{k} x_i = 93$ and $\sum_{i=1}^{k} \frac{1}{x_i} = \frac{93}{50}$ then the value of k is

(A) 4 (B) 3
(C) 6 (D) 7

87. Number of positive integer n, less than 17, for which $n! + (n + 1)! + (n + 2)!$ is an integral multiple of 49 is

(A) 0 (B) 3
(C) 5 (D) 2

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. All the five-digit numbers in which each successive digit exceeds its predecessor are arranged in the increasing order. The 105*th* number does not contain the digit.

(A) 1 (B) 3
(C) 4 (D) 5

2. There are 6 boys and 5 girls. There are two round tables with 7 chairs and 4 chairs.

(A) Number of ways of arranging all of them is $\frac{11!}{28}$
(B) Number of ways of arranging all of them so that all girls are at same table is $\frac{(6!)^2}{8}$
(C) Number of ways of arranging all of them so that all boys are at same table is $\frac{5!6!}{4}$
(D) Number of ways of arranging all of them so that all boys are at one table or all girls are at one table is 6!(120)

3. A contest consists of ranking 10 songs of which 6 are Indian classic and 4 are western songs. Number of ways of ranking so that

(A) There are exactly 3 Indian classic songs in top 5 is $(5!)^3$
(B) Top rank goes to Indian classic song is 6(9!)
(C) The ranks of all western songs are consecutive is 4! 7!
(D) The 6 Indian classic songs are in a specified order is $^{10}P_4$

4. The number of ways in which 6 different objects may be distributed among 4 children so that each child gets at least one object

(A) 7200
(B) 1560

(C) $\frac{6!}{(2!)^2}4! + \frac{6!}{3!11} \times 4!$

(D) ${}^6C_1 \cdot {}^5C_1 \times {}^4C_1 \times \frac{4!}{3!} + {}^6C_1 \times {}^5C_1 \times {}^4C_2 \times \frac{4!}{2!2!}$

5. If S = {1, 3, 5, 7} then which of the following is/are true?
   (A) Sum of all four-digit numbers if repetition of digits is not allowed is equal to 3!.16(1111)
   (B) Sum of all four-digit numbers if repetition of digits is allowed is equal to 432(1111)
   (C) Sum of all three-digit numbers if repetition of digits is not allowed is equal to 5328
   (D) Sum of all the digits in the units place of all four digited numbers if repetition of digits is allowed is equal to 432

6. The 9 horizontal and 9 vertical lines on an 8 × 8 chessboard form '*r*' rectangles and '*s*' squares. Then which of the following is/are true?
   (A) No. of rectangles = 1296
   (B) No. of squares = 204
   (C) No. of non-congruent rectangles = 36
   (D) No. of non-congruent squares = 1

7. If $S = 0! + 1.1! + 2.2! + 3.3! + \ldots n.n!$ then which of the following is / are true
   (A) n divides S
   (B) S is divisible by $n + 1$
   (C) $n!$ divides S
   (D) S is divisible by $(n + 1)!$

8. If $n = 4^5.3^7.5^{11}$ then which of the following is / are true
   (A) The number of divisors of n which are perfect squares is 144
   (B) The number of proper divisors of n is 1054
   (C) The number of odd divisors of n is 96
   (D) The number of even divisors of n is 958

9. The digits 1, 2, 3, 4 and 5 are arranged in all possible manner to yield 5-digit numbers. Consider those 5-digit numbers which are not divisible by 5. Let S be their set.
   (A) If the numbers in S are arranged in ascending order, then the rank of 34152 is 62^{nd};
   (B) The cardinality of S is 120; (Cardinality is the number of elements)
   (C) If $a, b \in S$ then $a + b \in S$;
   (D) The difference in the greatest and least elements of S is 41967.

10. The position vector of a point P is $\vec{r} = xi + yj + zk$, where x, $y, z \in N$ and $\vec{a} = i + j + k$. If $\vec{r}.\vec{a} = 10$, then the no. of possible positions of P is
    (A) 36 (B) 72
    (C) 66 (D) ${}^9C_2$

11. Let 'n' be an integer with $n \geq 3$. Let $P_1P_2P_3 - P_n$ be a regular n-sided polygon inscribed in a circle 'C'. Three points P_i, P_j and P_k are randomly chosen, where i, j and k are distinct integers between 1 and n inclusive. The number of triangles such that the triangle $P_iP_jP_k$ is obtuse is
    (A) $\frac{n(n-2)(n-4)}{8}$ (B) $\frac{(n+2)(n+3)}{2}$
    (C) $\frac{n(n-1)(n-2)}{6}$ (D) $\frac{n(n+1)(n+2)}{6}$

12. Before. I can open my bank locker, I must remember the combination. Two of the numbers of the three-term sequence (–, –, –) are 17 and 24, but I have forgotten the third as well as the order of the numbers. The possibilities for the third number are any number between 1 and 40 (both inclusive). At 10 sec, per try, at most how long will it take for me to exhaust all possibilities? (in min)
    (A) 40 (B) 39
    (C) 20 (D) 21

13. 'n' locks and 'n' corresponding keys are available. But the actual combination is not known. The maximum number of trials that are needed to assign the keys to their corresponding locks are
    (A) ${}^nC_2$ (B) $\sum_{k=2}^{n}(k-1)$
    (C) $\sum_{k=1}^{n}k$ (D) ${}^{n+1}C_2$

14. The number of selections of 4 letters taken from the word 'COLLEGE' must be
    (A) 18
    (B) 22
    (C) Coefficient of x^4 in the expansion of $(1 + x)^3 (1 + x + x^2)^2$
    (D) Coefficient of x^4 in the expansion of $(1 + x)^2 (1 + x + x^2)^3$

15. A forecast is to be made of the result of five cricket matches, each of which can be win or a draw or a loss for Indian team let
    p = number of forecasts with exactly 1 going wrong
    q = number of forecasts with exactly 3 going wrong
    r = number of forecasts with all 5 going wrong
    Then which of the following are/is true
    (A) $2q = 5r$ (B) $8p = q$
    (C) $8p = 5r$ (D) $2(p + r) > q$

16. Let $S = x_1 + x_2 + x_3 + x_4 + x_5$ and $K = x_1 + x_2 + x_3$ then
    (A) Number of positive integral solutions of the equation S = 21 is 4845
    (B) Number of integral solutions of the equation S = 21 such that $x_i \geq 2\ \forall i = 1, 2, 3, 4, 5$ is 1365.
    (C) Number of integral solutions of the equation S = 20 is 42404
    (D) Number of integral solutions $(x_1, x_2, x_3, x_4, x_5)$ such that S = 20 and K = 5 is 336

17. Four balls numbered 1, 2, 3, 4 are to be placed into five boxes numbered 1, 2, 3, 4, 5, such that exactly one box remains empty and no ball goes to its own numbered box. The no. of ways is
    (A) $5!\sum_{r=0}^{5}\frac{(-1)^r}{r!}$ (B) $4!\sum_{r=0}^{4}\frac{(-1)^r}{r!}$
    (C) $4!\sum_{r=0}^{4}\frac{(-1)^r}{r!} + 5!\sum_{r=0}^{5}\frac{(-1)^r}{r!}$ (D) 54

18. If X = 144, then
(A) no. of divisors (including 1 and X) of X = 15
(B) Sum of divisors (including 1 and X) of X = 403
(C) Product of divisors (including 1 and X) of X = 12^{15}
(D) Sum of reciprocals of divisors (including 1 and X) of X $= \frac{403}{144}$

19. Letters of the word SUDESH can be arranged in
(A) 120 ways when two vowels are together
(B) 180 ways when two vowels occupy in alphabetical order
(C) 24 ways when vowels and consonants occupy their respective places
(D) 240 ways when vowels do not occur together

20. An unlimited number of coupons bearing the letters A, B and C are available, then the number of ways of choosing 10 of these coupons so that they can't used to spell BAC
(A) $3(2^{10} - 1)$
(B) $2(3^{10} - 1)$
(C) $2^{10} - 1$
(D) 2^{10}

21. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth numbers is marked (ie. 1, 16, 31 etc. This process is continued until a number is reached which has already been marked, then unmarked numbers are
(A) 200
(B) 400
(C) 600
(D) 800

22. Number of ways of choosing triplets (x, y, z) such that $Z > \max\{x, y\}$ and $x, y, z \in \{1, 2, 3, ...n, n+1\}$ is
(A) ${}^{n+1}C_3 + {}^{n+2}C_3$
(B) $\frac{n(n+1)(2n+1)}{6}$
(C) $\sum_{n=1}^{n} n^2$
(D) ${}^{n+2}C_3 - {}^{n+1}C_2$

23. Consider the following statements.
(i) In a 12 storeyed house, 10 people enter the lift cabin at ground floor. It is known that they will leave lift in groups of particular 2, 3 and 5 people at different storey. The number of ways this can be done if the lift does not stop at first and second floors is 720.
(ii) Each of three ladies have brought their one child for admission to a school. The principal wants to interview the six persons one by one, subject to the condition that no mother is interviewed before her child. The number of ways in which interviews can be arranged is 90.
(iii) The number of ways in which one can put three balls numbered 1, 2, 3 in three boxes labelled a, b, c such that at most one box is empty is equal to 18.
(iv) A box contains 5 different red balls and 6 different white balls. The total number of ways in which 4 ball can be selected, taking atleast 1 ball of each colour is 310.
(A) Statements (i), (ii) are correct.
(B) Statements (ii) and (iv) are correct.
(C) Statements (i) and (iii) are correct.
(D) All statements are correct.

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

A is a set containing n elements. A subset S_1 of A is chosen. The set A is reconstructed by replacing the elements of S_1. Again, a subset S_2 of A is chosen and again the set is reconstructed by replacing the elements of S_2. The number of ways of choosing S_1 or S_2 where

1. S_1 and S_2 have one element common is
(A) 3^{n-1}
(B) $n \cdot 3^{n-1}$
(C) 2^{n-1}
(D) n

2. $S_1 \cup S_2 = A$ is
(A) 3^n
(B) $n \cdot 3^n$
(C) 4^n
(D) 4^{n-1}

Comprehension 2

If $x_1 + x_2 + x_3 ..... + x_r = n$

Then number of solutions of equation ${}^{n+r-1}C_n$ when x_i are $(i = 1, 2, 3 .... r)$ non-negative integers and ${}^{n-1}C_{r-1}$ when x_i are $(i = 1, 2, 3 .... r)$ positive integers

3. If a, b, c be three natural numbers in A.P. then number of solution of $a + b + c = 21$ is
(A) 15
(B) 14
(C) 13
(D) 16

4. Number of ways of distributing 22, identical toys among 4 children when each child must get odd number of toys is equal to
(A) ${}^{8}C_3$
(B) ${}^{12}C_9$
(C) ${}^{21}C_3$
(D) ${}^{25}C_{22}$

Comprehension 3

Given are six 0's, five 1's and four 2's. consider all possible permutations of all these numbers. [A permutation can have its leading digit 0].

5. How many permutations have the first 0 preceding the first 1?
(A) ${}^{15}C_4 \times {}^{10}C_5$
(B) ${}^{15}C_5 \times {}^{10}C_4$
(C) ${}^{15}C_6 \times {}^{10}C_5$
(D) ${}^{15}C_5 \times {}^{10}C_5$

6. In how many permutations does the first 0 preceded the first 1 and the first 1 preceded first 2.
(A) ${}^{14}C_5 \times {}^{8}C_6$
(B) ${}^{14}C_5 \times {}^{8}C_4$
(C) ${}^{14}C_6 \times {}^{8}C_4$
(D) ${}^{14}C_6 \times {}^{8}C_6$

Comprehension 4

When letters of any word are written in all possible ways using all letters of given word and these words are arranged as in a dictionary then the position of the word in that list in called the rank of the word.

7. What is the rank of word 'PARALLEL'
(A) 2353 (B) 2629
(C) 2593 (D) 2623

8. If list of all words is made by using the letters of the word IITJEE in dictionary order then the word at 141th position is
(A) JEE IIT (B) JII TEE
(C) JIE TIE (D) JII EET

9. If all numbers starting from 1 are listed in increasing order by using the digits 0, 1, 2, 3, 4, 5, 6 then the number whose rank is 8000 is
(A) 32605 (B) 32116
(C) 32216 (D) 33316

Comprehension 5

Suppose there are 5 mangoes, 4 apples and 3 oranges in a bag, fruits of same variety being identical. Then

10. The number of ways few fruits are selected
(A) 120 (B) 119
(C) 2^{12-1} (D) 121

11. The number of ways can a selection of fruits be made if at least 2 mangoes be included is
(A) 80 (B) 120
(C) $2^7 (26)$ (D) $2^{12} - 12$

12. The number of ways can a selection of fruits be made if atleast one fruit each kind is to be included is
(A) $(2^5 - 1)(2^4 - 1)(2^3 - 1)$ (B) 120
(C) 80 (D) 60

Comprehension 6

Let 'S' be the set of first 18 natural numbers. The number of ways of selecting from 'S'

13. Three numbers such that they form an AP is
(A) 60 (B) 64
(C) 72 (D) 80

14. Two number such that the sum of their cubes is divisible by 3 is …
(A) 21 (B) 31
(C) 45 (D) 51

Comprehension 7

There are 18 points in a plane out of which 3 of them fall on a line $L_1 = 0$, another 4 of then fall on another line $L_2 = 0$, another five lie on a different line $L_3 = 0$. No three points other than those stated above are collinear.

15. Number of lines formed by joining the points on the plane
(A) 134 (B) 137
(C) 124 (D) 130

16. Number of triangles formed by joining the points on the plane
(A) 801 (B) 800
(C) 821 (D) none of these

17. If the six points that do not lie on $L_1 = 0, L_2 = 0, L_3 = 0$ are now removed. Find the number of n-gon's (polygons with n sides) where $n > 3$ such that not more than two points are selected from each line
(A) 848 (B) 940
(C) 948 (D) none of these

Comprehension 8

Let $x_1 x_2 x_3 x_4 x_5 x_6$ be a six digit number find the number of such numbers such that

18. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ is
(A) ${}^{9}C_{3}$ (B) ${}^{10}C_{3}$
(C) ${}^{10}C_{6}$ (D) ${}^{10}C_{5}$

19. $x_1 < x_2 < x_3 = x_4 < x_5 < x_6$ is
(A) ${}^{9}C_{3}$ (B) ${}^{9}C_{4}$
(C) ${}^{10}C_{3}$ (D) ${}^{10}C_{4}$

20. $x_1 < x_2 < x_3 \leq x_4 < x_5 < x_6$ is
(A) ${}^{9}C_{4}$ (B) ${}^{9}C_{3}$
(C) ${}^{10}C_{4}$ (D) ${}^{10}C_{3}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1.

COLUMN-I	COLUMN-II
(A) Number of arrangements of the letters of the word ENGINEER so that all vowels are together but all *Es* are not together is	(p) 22
(B) Number of ways of arrangement the letters of the word ELEVEN so that no two vowels are together and no two consonants are together is k then $10k$ equals	(q) 44
(C) Number of ways of keeping 5 letters in 5 addressed envelopes such that no letter goes into its envelope is p then p is divisible by	(r) 59
(D) Number of 6 digits numbers greater than 123000 using 1, 2, 2, 3, 3, 3 is	(s) 5!

2.

COLUMN-I	COLUMN-II
(A) The number of positive integral solutions of $xyz = 24$ is λ then λ is divisible by	(p) 2
(B) Let m be a factor of 120. If λ be the number of positive integral solutions of $x_1x_2x_3 = m$ then λ is divisible by	(q) 3
(C) If λ_1, be the number of positive integral solutions of the equation $x + y + z + w = 10$ then λ_1 is divisible by	(r) 5
(D) If λ_2 be the number of ways of distributing 20 fruits among 5 people, so that no one receives less than 3 fruits then λ_2 is divisible by	(s) 7

3. Match the following

COLUMN-I	COLUMN-II
(A) If 6 letter words are formed using the letters of the word NUMBER and the words are arranged in dictionary order, then the rank of the word NUMBER is	(p) 576
(B) Four identical dice are rolled once. The number of ways of getting only prime numbers on them, is	(q) 12
(C) The largest integer n, such that 100! Is divisible by 100^n, is	(r) 15
(D) The number of 4-digit numbers which contain not more than two different digits, is	(s) 469

4.

COLUMN-I	COLUMN-II
(A) If n be the number of ways in which 12 different books can be distributed equally among 3 persons, then $\frac{(4!)^4 n}{12!}$ is divisible by	(p) 4
(B) If n be the number of ways in which 12 different things can be distributed in three groups. Then $\frac{(4!)^3(3!)^2}{12!}n$ is divisible by	(q) 6
(C) If n be the number of ways in which 12 different things can be distributed in 5 sets of 2, 2, 2, 3, 3 things. Then $\frac{(3!)^3(2!)^4 5!}{12!}n$ is divisible by	(r) 12
(D) The number of positive integral solutions of the equation $x_1 x_2 x_3 x_4 x_5 = 1050$ is 'λ' then 'λ' is divisible by	(s) 24
	(t) 5

5. Match the following

COLUMN-I	COLUMN-II
(A) The number of ways, in which 12 red balls, 12 black balls and 12 white balls be given to 2 children each 18, is	(p) 125
(B) The number of ways of forming one team having 5 numbers chosen from 5 boys and 5 girls, so that girls are in majority and atleast one boy is there in the team.	(q) 127
(C) Six bundles of books are to be kept in 6 boxes one in each box. If 2 of the boxes are too small for three of the bundles, the number of ways keeping the bundles in the boxes is	(r) 135
(D) A bag contains 6 black, 6 blue, 6 red, 6 green and 6 white balls. The balls are numbered 1 to 6 in each colour. The number of ways of drawing 2 balls from the bag such that the balls are of the same colour or the numbers on them are same, is	(s) 144

6. There are 2 Indian couples, 2 American couples and one unmarried person

COLUMN-I	COLUMN-II
(A) The total number of ways in which they can sit in a row such that an Indian wife and an American wife are always on either side of the unmarried person, is	(p) 22680
(B) The total number of ways in which they can sit in a row such that the unmarried person always occupy the middle position, is	(q) 5760

(*Continued*)

COLUMN-I	COLUMN-II
(C) The total number of ways in which they can sit around a circular table such that an Indian wife and an American wife are always on either side of the unmarried person, is	(r) 40320
(D) If all the nine persons are to be interviewed one by one then the total number of ways of arranging their interviews such that no wife gives interview before her husband, is	(s) 24320

7. Match the following:

Consider all possible permutations of the letters of the word MASTERBLASTERS

COLUMN-I	COLUMN-II
(A) The number of permutations containing the word RAAT is	(p) $\frac{(7!)^2}{3!(2!)^4}$
(B) The number of permutations in which S occurs in first place and R occurs in the last place is	(q) $\frac{11!\times 4!}{3!(2!)^2}$
(C) The number of permutations in which none of the letters S, T, R occur in first 7 positions is	(r) $\frac{11!}{3!\times 2!}$
(D) The number of permutations in which the letters A, S, R occur in even positions is	(s) $\frac{12!}{(2!)^4}$

8. Match the following

COLUMN-I	COLUMN-II
(A) Number of ways selecting 8 balls out of an unlimited collection of Red , blue, green and yellow balls is	(p) ${}^{21}C_3$
(B) Number of 4 digit numbers having the sum of the digits equal to 9 is	(q) Number of ways of distributing 12 apples to four people with each one getting at least one
(C) Number of ways of arranging 3 identical red balls 20 identical white balls in a row so that no two red balls are together is	(r) 6
(D) Exponent of 3 in 17! is	(s) ${}^{11}C_3$

(*Continued*)

9. Match the following

COLUMN-I	COLUMN-II
(A) If n be the numbers between 500 and 4000 can be formed with the digits 2, 3, 4, 5, 6 when repetition is not allowed, then n is divisible by	(p) 2
(B) If n be the number of even numbers between 200 and 3000 can be formed with the digit 0, 1, 2, 3, 4 when repetition is not allowed, then n is divisible by	(q) 3
(C) If n be the number of words that can be made by arranging the letters of the word ROORKEE that neither begin with R nor end with E, then n is divisible by	(r) 5
(D) In a class tournament when the participants where to play one game with another, two class players fell ill, having played 3 games each. If the total number of games played is 84, the number of participants at the beginning was k then k is divisible by	(s) 11
	(t) 17

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. The number of 4-digited numbers of the form *abcd* where $a, c, d < b$ and $a \neq 0$ is a four-digit number then its tens place is

2. The number of ways in which 4 married couples can be sit four on each side of a long table so that males on one side and females on the other side and no wife is infront of her husband is k then sum of digits of k is

3. There are 3 rows containing 2 seats in each row. The number of ways in which 3 persons can be seated such that no row remains empty is p then $\frac{p}{16}$ equals

4. Consider $S=\{1,2,3,4,.......,10\}$. Then sum of all products of numbers by taking two or more from S is $(11!-k)$ then $\left[\frac{k}{11}\right]$ where [.] is G.I. F is

5. Number of cyphers at the end of ${}^{2002}C_{1001}$ is

6. Nine hundred distinct n-digit numbers are to be formed using only the 3 digits 2, 5, 7. The smallest value of n for which this is possible is

7. There are 5 concurrent lines and another line parallel to one of them the number of different triangles will be formed by 6 lines is ___

8. Six straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then no. of parts in to which these lines divide the plane is $k + 15$ then k =___

9. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, find the value of n.

10. A seven-digit number divisible by '9' is to be formed by using 7 out of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}. The number of ways in which this can be done is (Repetition is not allowed) $= n(7!)$. Then n =

11. Number of four-digit numbers formed by the digits 1, 2, 3, 4, 5, 6, 7 which are divisible by 25 is 8m then m = ___

12. There are four balls of different colours and four boxes of the same colours as those of the balls. The number of ways in which the balls could be placed one each in a box, such that no ball goes into the box of its own colour, is ......

13. If the number of different words that can be formed using all the letters of the word "MAHAVEER" where two of the vowels are together and the other two are also together but separated from the first two is S, then S – 1432 = ...

14. Number of ways in which 5A's and 6B's can be arranged in a row which reads the same backwards and forwards, is $10 + k$ where 'k' is

15. Suppose that each of 5 people knows exactly one piece of information and all 5 pieces of information are different. Every time person A phone calls to person B and tells everything what he knows, while B tells A nothing. The minimum number of phone calls between pairs of people needed for everyone to know everything is

16. No. of positive integers 'n' such that n! contains six zeroes is

17. Let $A_1, A_2, A_3 \cdots A_n$ are the vertices of a polygon of n sides. If the number of pentagons that can be constructed by joining those vertices such that none of the sides of the polygon is also a side of the pentagon is 36, then the value of $(n - 3)$ is equal to.

18. The number of acute triangles whose vertices are chosen from the vertices of a rectangular block are___

19. Let $N = 2011 \times 2012$ and M be the number of positive integer pairs (x, y) such that $x < y$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$. The sum of the digits of M is

20. If total number of 7 digit numbers $x_1x_2x_3x_4x_5x_6x_7$ having the property $x_1 \le x_2 < x_3 < x_4 \le x_5 < x_6 < x_7$ is ${}^{N}C_{K}$ then N – K is

21. 12 small sticks of length 1cm each are distributed into three children A, B and C. These children join the sticks in the form of line segments individually. If n is the number of ways in which the sticks can be distributed to the children so that the line segments joined by them form a triangle then the value of $n - 1$ is_____

22. A cricket player played n $(n > 1)$ matches during his career and made a total of $\frac{(n+1)(2^{n+1} - n - 2)}{4}$ runs. If the player made $k.2^{n-k+1}$ runs in the k^{th} match $(1 \le k \le n)$, the value of 'n' is equal to

23. If n is the number of ways in which 15 identical blankets can be distributed among six beggars such that everyone gets atleast one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets, then the value of $n/2$ is

24. If n is the number of quadruplets (a, b, c, d) such that a, b, c, d are odd natural numbers satisfying $a + b + c + d = 20$ then $\left[\frac{n}{20}\right] =$ (where $[x]$ denotes the greatest integer function)

25. Out of 5 apples, 10 mangoes and 15 oranges, the number of ways of distributing 15 fruits each to two persons, is n then $\frac{n}{22}$ is equal to

26. If n! has exactly (20) zeroes at the end, then number of such possible numbers is ______________

27. Let N is the total number of four-digit natural number beginning with 1, that has exactly two identical digits. Then the sum of digits of N is ____________

28. Consider the set of all triangles OPQ where 'O' is the origin and P and Q are distinct points in the plane with non-negative coordinates (x, y) such that $5x + y = 99$. Number of such distinct triangles whose area is a positive integer, is 10 m. Find m.

29. The number of triangles whose vertices are the vertices of an octagon but none of whose sides happen to come from the octagon is 8K then K =

30. The number of different ordered triplets $(a, b, c, \in I)$ such that these can represent sides of a triangle whose perimeter is 21, is $9k + 10$, then k is _____.

31. Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if balls are identical but boxes are different.

32. Number of ways in which 5 boys and 4 girls can be arranged on a circular table such that no two girls sit together and two particular boys are always together is $72k$, then k =

33. Number of quadrilaterals that can be made using the vertices of a polygon of 10 sides as their vertices had having 3 sides common with the polygon is $(k + 3)$, then k =

34. Number of triangles that can be made using the vertices of a polygon of 10 sides as their vertices and having exactly one side common with the polygon is $10k$, then k =

35. Let X = {1,2,3,......,100} and Y be a subset of X such that the sum of no two element in Y is divisible by 7.If the maximum possible number of element in Y is $40 + \lambda$ then λ is

36. If the number of different four digits number can be formed with the digits {0,1,2,.....8} so that each number contains only one digits as 5 is in the form ABCD then B =.

37. If one quarter of all the three element subsets of A = $\{a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_n\}$, contains the element a_3 then $\frac{n}{3}$ =_________

38. The number natural numbers less than 10^8 with sum of the digits equal to 7 is ${}^{14}C_r$ then r =______

39. The number of three digited numbers of the form xyz such that $x \geq y > z$ is k then $\left[\dfrac{k}{20}\right]$ = _____ ([.] is the GIF).

40. Kiran wrote letters to his five friends and inserted in the corresponding addressed envelopes. If 'n' is the number of ways in which atleast three letters are inserted in wrong addressed envelops then the unit digit of n is ________

41. The digit at units place in the number $(2137)^{754}$ is

42. A building contains 2 rooms each on first and second floor and 3 rooms on the ground floor. If the number of ways in which 5 persons can be allotted one room each so that no floor is empty is k then the value of $\sqrt{\dfrac{K}{10} - 164}$ is ______

43. The number of onto functions which are non decreasing from A = {1, 2, 3, 4, 5} to B = $\{7, 8, 9\}^{\circ}$ is __.

44. If number of numbers greater than 3000, which can be formed by using the digits 0, 1, 2, 3, 4, 5 without repetition, is n then $\dfrac{n}{230}$ is equal to

45. Let S = {1, 2, 3, … n}, If X denotes the set of all subsets of S containing exactly two elements, then the value of $\sum\limits_{A \in X} (\min A)$ is ${}^{n+1}C_{\lambda}$ then λ = ____

46. The number of numbers from 1 to 10^6 (both inclusive) in which two consecutive digits are same is equal to 402128 + K where K is a single digit number then K must be equal to ______.

47. The number of polynomials of the form $x^3 + ax^2 + bx + c$ which are divisible by $x^2 + 1$ where $a, b, c \in \{1, 2, 3, \ldots, 10\}$ is 10K, then K is ______.

48. The number of ways of arranging 11 objects A, B, C, D, E, F, α, α, α, β, β so that every β lie between two α (not necessarily adjacent) is K × 6! × ${}^{11}C_5$, then K is ____.

49. The no. of positive integer solutions of $x + y + z = 10$, where x, y, z are unequal is (20 + K) then K is

50. If one test (on screening paper basis) was conducted on Batch A, maximum number of marks is (90 × 3) = 270. 4 students get the marks lower than 80. Coaching institute decide to inform their guardians, that is why their result card were sent to their home. The number of ways, in which all the letters were put in wrong envelopes, is

51. The exponent of 7 in ${}^{100}C_{50}$ is :

52. Nine hundred distinct n-digit numbers are to be formed using exactly the three digits. 2, 5 and 7. The smallest value of n for which this is possible is.

53. The number of divisors of the form $2n + 1$ ($n \geq 1$) of the integer 120 is

ARCHIVE: JEE MAIN

1. **[JEE-Main 2020]**
An urn contains 5 red marbles, 4 black marbles and 3 white marbles, Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is _______.

2. **[JEE-Main 2020]**
The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word 'EXAMINATION' is __________.

3. **[JEE-Main 2020]**
If the letters of the word 'MOTHER' be permuted and all the words so formed (with or without meaning) be listed as in a dictionary, then the position of the word 'MOTHER' is __________.

4. **[JEE-Main 2020]**
A test consists of 6 multiple choice questions, each having 4 alternative answers of which only one is correct. The number of ways, in which a candidate answers all six questions such that exactly four of the answers are correct, is __________.

5. **[JEE-Main 2020]**
The total number of 3-digit numbers, whose sum of digits is 10, is _______.

6. **[JEE-Main 2020]**
The number of words, with or without meaning, that can be formed by taking 4 letters at a time from the letters of the word 'SYLLABUS' such that two letters are distinct and two letters are alike, is ______.

7. **[JEE-Main 2020]**
The number of words (with or without meaning) that can be formed from all the letters of the word "LETTER" in which vowels never come together is _______.

8. **[JEE-Main 2020]**
Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear, is :

(A) $\dfrac{5}{2}(6!)$
(B) 5^6
(C) $\dfrac{1}{2}(6!)$
(D) 6!

9. **[JEE-Main 2020]**
If the number of five digit numbers with distinct digits and 2 at the 10th place is $336k$, then k is equal to :

(A) 8
(B) 4
(C) 7
(D) 6

10. **[JEE-Main 2020]**
Let $n > 2$ be an integer. Suppose that there are n Metro stations in a city located along a circular path. Each pair of stations is connected by a straight track only. Further, each pair of nearest stations is connected by blue line, whereas all remaining pairs of stations are connected by red line. If the number of red lines is 99 times the number of blue lines, then the value of n is

(A) 199
(B) 101
(C) 201
(D) 200

11. [JEE-Main 2020]

The value of (2. $^1P_0 - 3.^2P_1 + 4.^3P_2 - \ldots$ up to 51th term) + (1! – 2! + 3! – …. Up to 51th term) is equal to

(A) $1 + (51)!$ (B) $1 - 51(51)!$
(C) $1 + (52)!$ (D) 1

12. [JEE-Main 2020]

There are 3 sections in a question paper and each section contains 5 questions. A candidate has to answer a total of 5 questions, choosing at least one question from each section. Then the number of ways, in which the candidate can choose the questions, is :

(A) 1500 (B) 2255
(C) 3000 (D) 2250

13. [JEE-Main 2020]

Two families with three members each and one family with four members are to be seated in a row. In how many ways can they be seated so that the same family members are not separated?

(A) 2!3!4! (B) $(3!)^3.(4!)$
(C) $(3!)^2.(4!)$ (D) $3!(4!)^3$

14. [JEE-Main 2019]

There are m men and two women participating in a chess tournament. Each participant plays two games with every other participant. If the number of games played by the men between themselves exceeds the number of games played between the men and the women by 84, then the value of m is:

(A) 12 (B) 9
(C) 7 (D) 11

15. [JEE-Main 2019]

All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is

(A) 180 (B) 175
(C) 160 (D) 162

16. [JEE-Main 2019]

The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed) is

(A) 306 (B) 360
(C) 310 (D) 288

17. [JEE-Main 2019]

A committee of 11 members is to be formed from 8 males and 5 females. If m is the number of ways the committee is formed with at least 6 males and n is the number of ways the committee is formed with at least 3 females, then:

(A) $m + n = 68$ (B) $m = n = 78$
(C) $m = n = 68$ (D) $n + m - 8$

18. [JEE-Main 2019]

The number of 6 digit numbers that can be formed using the digits 0, 1, 2, 5, 7 and 9 which are divisible by 11 and no digit is repeated is:

(A) 36 (B) 60
(C) 72 (D) 48

19. [JEE-Main 2019]

The number of natural numbers less than 7,000 which can be formed by using the digits 0, 1, 3, 7, 9 (repetition of digits allowed) is equal to

(A) 372 (B) 375
(C) 250 (D) 374

20. [JEE-Main 2019]

Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team, is

(A) 300 (B) 500
(C) 200 (D) 350

21. [JEE-Main 2019]

Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number of beams is:

(A) 180 (B) 170
(C) 190 (D) 210

22. [JEE-Main 2019]

The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is:

(A) $2^{20} - 1$ (B) 2^{20}
(C) 2^{21} (D) $2^{20} + 1$

23. [JEE-Main 2019]

A group of students comprises of 5 boys and n girls. If the number of ways, in which a team of 3 students can randomly be selected from this group such that there is at least one boy and at least one girl in each team, is 1750, then n is equal to:

(A) 24 (B) 25
(C) 27 (D) 28

24. [JEE-Main 2019]

Consider three boxes, each containing, 10 balls labelled 1, 2, ….10 suppose one ball is randomly drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i^{th} box, $(i = 1, 2, 3)$. Then, the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is:

(A) 164 (B) 240
(C) 82 (D) 120

25. [JEE-Main 2019]

The number of functions f from {1, 2, 3, ….20} onto {1, 2, 3, ……., 20} such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is:

(A) $6^5 \times (15)!$ (B) $5^6 \times 15$
(C) $(15)! \times 6!$ (D) $5! \times 6!$

26. [JEE-Main 2018 and 2009]

From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is

(A) less than 500
(B) at least 500 but less than 750
(C) at least 750 but less than 1000
(D) at least 1000

27. [JEE-Main 2017]

A man X has 7 friends, 4 of them are ladies and 3 are men, his wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friend then total number of ways in which X and Y can throw a party together inviting 3 men and 3 Ladies so that 3 friend each of X and Y are in party

(A) 484
(B) 485
(C) 468
(D) 469

28. [JEE-Main 2016]
If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is:
(A) 58th
(B) 46th
(C) 59th
(D) 52nd

29. [JEE-Main 2015]
The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is:
(A) 820
(B) 780
(C) 901
(D) 861

30. [JEE-Main 2015]
The number of integers greater than 6000 that can be formed, using the digits 3,5,6,7 and 8 without repetition, is:
(A) 120
(B) 72
(C) 216
(D) 192

31. [JEE-Main 2013]
Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is
(A) 256
(B) 220
(C) 219
(D) 211

32. [JEE-Main 2013]
Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is.
(A) 7
(B) 5
(C) 10
(D) 8

33. [AIEEE 2012]
Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is
(A) 879
(B) 880
(C) 629
(D) 630

34. [AIEEE 2011]
There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then:
(A) $N \geq 190$
(B) $N \leq 100$
(C) $100 < N \leq 140$
(D) $140 < N \leq 190$

35. [AIEEE 2011]
Statement - 1: The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.
Statement - 2: The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.
(A) Statement-1 is true, Statement-2 is false.
(B) Statement-1 is false, Statement-2 is true
(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(D) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

36. [AIEEE 2010]
There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is
(A) 3
(B) 36
(C) 66
(D) 108

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. [JEE Adv. 2017]
Let S = {1, 2, 3, ….., 9} For $k = 1, 2, \ldots, 5$, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$
(A) 210
(B) 252
(C) 125
(D) 126

2. [JEE Adv. 2016]
A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is:
(A) 380
(B) 320
(C) 260
(D) 95

3. [JEE Adv. 2014]
Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 in always placed in envelope numbered 2. Then the number of ways it can be done is
(A) 264
(B) 265
(C) 53
(D) 67

4. [JEE 2012]
The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is
(A) 75
(B) 150
(C) 210
(D) 243

5. [JEE 2009]
The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is:
(A) 55
(B) 66
(C) 77
(D) 88

6. [JEE 2007]
The letters of the word **COCHIN** are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word **COCHIN** is :
(A) 360
(B) 192
(C) 96
(D) 48

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

1. [JEE 2012]

The value of b_6 is

(A) 7 (B) 8

(C) 9 (D) 11

2. [JEE 2012]

Which of the following is correct?

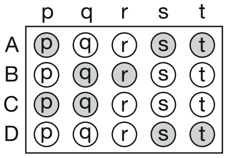
(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$

(C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

Matrix Match/Column Match Type Questions

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. [JEE 2008]

Consider all possible permutations of the letters of the word ENDEANOEL

Match the statements/Expression in **Column-I** with the statements/Expressions in **Column-II**.

COLUMN-I	COLUMN-II
(A) The number of permutations containing the word ENDEA is	(p) 5!
(B) The 'number of permutations in which the letter E occurs in the first and the last position is	(q) 2 × 5!
(C) The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(r) 7 × 5!
(D) The number of permutations in which the letters A, E, O occurs only in odd positions is	(s) 21 × 5!

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE Adv. 2020]

An engineer is required to visit a factory for exactly four days during the first 15 days every month and it is mandatory that no two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1 – 15 June 2021 is ___

2. [JEE Adv. 2020]

In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is ___.

3. [JEE Adv. 2019]

Five persons A, B, C, D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is _____

4. [JEE Adv. 2018]

The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is ___

5. [JEE Adv. 2017]

Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated, and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$

6. [JEE Adv. 2015]

Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is:

7. [JEE Adv. 2014]

Let $n \geq 2$ b an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is

8. [IIT-JEE 2014]

Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. The number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** A	**3.** C	**4.** D	**5.** B	**6.** B	**7.** B	**8.** A	**9.** C	**10.** D
11. B	**12.** C	**13.** C	**14.** B	**15.** B	**16.** D	**17.** C	**18.** D	**19.** A	**20.** B
21. C	**22.** A	**23.** C	**24.** C	**25.** C	**26.** C	**27.** A	**28.** A	**29.** B	**30.** A
31. D	**32.** A	**33.** B	**34.** B	**35.** A	**36.** A	**37.** C	**38.** B	**39.** A	**40.** B
41. C	**42.** B	**43.** C	**44.** B	**45.** A	**46.** A	**47.** A	**48.** C	**49.** C	**50.** B
51. C	**52.** D	**53.** C	**54.** D	**55.** C	**56.** B	**57.** B	**58.** B	**59.** A	**60.** C
61. D	**62.** D	**63.** B	**64.** A	**65.** C	**66.** A	**67.** A	**68.** A	**69.** B	**70.** A
71. B	**72.** B	**73.** B	**74.** A	**75.** B	**76.** A	**77.** A	**78.** D	**79.** B	**80.** B
81. B	**82.** A	**83.** B	**84.** C	**85.** D	**86.** C	**87.** C			

Multiple Correct Choice Type Questions

1. A, B, C, D	**2.** A, B, C, D	**3.** A, B, C, D	**4.** B, D	**5.** A
6. A, B, C	**7.** A, B, C, D	**8.** A, B, C	**9.** D	**10.** A, D
11. A, C	**12.** B	**13.** C, D	**14.** A, C	**15.** A, B, D
16. A, B, D	**17.** C	**18.** A, B, C, D	**19.** A, B, C, D	**20.** A
21. D	**22.** A, B, C	**23.** A, B		

Linked Comprehension Type Questions

1. B	**2.** A	**3.** C	**4.** B	**5.** A	**6.** B	**7.** D	**8.** B	**9.** C	**10.** A
11. A	**12.** D	**13.** C	**14.** D	**15.** B	**16.** A	**17.** C	**18.** A	**19.** B	**20.** C

Matrix Match/Column Match Type Questions

1. A → (s)	B → (s)	C → (p, q)	D → (r)
2. A → (p, q, r)	B → (p, q, r)	C → (p, q, s)	D → (p, q, s)
3. A → (s)	B → (r)	C → (q)	D → (p)
4. A → (p, q, r, s)	B → (q)	C → (p, q, r, s, t)	D → (p, q, r, s)
5. A → (q)	B → (p)	C → (s)	D → (r)
6. A → (r)	B → (r)	C → (q)	D → (p)
7. A → (r)	B → (s)	C → (p)	D → (p)
8. A → (q, s)	B → (q, s)	C → (p)	D → (r)
9. A → (p, q)	B → (q, t)	C → (p, q, r, s)	D → (q, r)

Integer/Numerical Answer Type Questions

1. 4	**2.** 9	**3.** 3	**4.** 5	**5.** 1	**6.** 7	**7.** 6	**8.** 7	**9.** 8	**10.** 4
11. 5	**12.** 9	**13.** 8	**14.** 0	**15.** 8	**16.** 5	**17.** 9	**18.** 8	**19.** 4	**20.** 4
21. 9	**22.** 7	**23.** 6	**24.** 8	**25.** 3	**26.** 5	**27.** 9	**28.** 9	**29.** 2	**30.** 5
31. 6	**32.** 4	**33.** 7	**34.** 6	**35.** 5	**36.** 8	**37.** 4	**38.** 7	**39.** 8	**40.** 9
41. 9	**42.** 8	**43.** 6	**44.** 6	**45.** 3	**46.** 2	**47.** 1	**48.** 3	**49.** 4	**50.** 9
51. 0	**52.** 7	**53.** 3							

ARCHIVE: JEE MAIN

1. 490	**2.** 2454	**3.** 309	**4.** 135	**5.** 54	**6.** 240	**7.** 120	**8.** A	**9.** A	**10.** C
11. C	**12.** D	**13.** B	**14.** A	**15.** A	**16.** C	**17.** B	**18.** B	**19.** D	**20.** A
21. B	**22.** B	**23.** B	**24.** D	**25.** C	**26.** D	**27.** B	**28.** A	**29.** B	**30.** D
31. C	**32.** B	**33.** A	**34.** B	**35.** C	**36.** D				

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. D **2.** A **3.** C **4.** B **5.** C **6.** C

Linked Comprehension Type Questions

1. B **2.** A

Matrix Match/Column Match Type Questions

1. A → (p) B → (s) C → (q) D → (q)

Integer/Numerical Answer Type Questions

1. 495 **2.** 1080 **3.** 30 **4.** 625 **5.** 5 **6.** 5 **7.** 5 **8.** 7

From the Author's Desk

“ Dear Learners,

Probability is concept-based chapter from Algebra. From this chapter, questions are always asked in JEE Mains and Advanced. Conditional Probability, Venn-diagram based problems, Independent events, Geometry based probability, Binomial probability, Bayes theorem are the key concepts to practice more. Many students find it difficult to prepare. So in this book there are ample of illustration followed by solved exercises, which will surely boost your concepts.

As each question requires thinking hence practice is the key to handle it. Students are advised to solve adequate questions covering various concepts in prescribed time limit.

Do not forget to attempt PYQs from JEE Mains and Advanced. ”

Good Luck!!!

CHAPTER 7

Probability

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) The concept of probability
(b) Venn-diagram based probability
(c) Conditional probability
(d) Independent events
(e) Geometrical probability
(f) Bayes theorem
(g) Testimony problems
(h) Binomial probability
(i) Mean, variance, expectation

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

In this Chapter Probability, first we will learn few terms which are very essential to understand this chapter, hence learn these terms, their meaning and definition.

RANDOM EXPERIMENT

It is a process which is being conducted in unbiased manner, all outcomes of it are having their natural chances, without being partial to any specific outcome.
e.g. Tossing a coin, throwing a dice (without making any effort to get any particular outcome)

SAMPLE SPACE

It is the set formed by all possible outcomes of a random experiment.
e.g. {H, T} is the sample space associated with tossing of a coin.
In set notation it can be interpreted as the universal set of outcomes of an experiment.

ILLUSTRATION 1

Write sample space of following experiments

(a) Throwing of a dice

SOLUTION

It is a random experiment as it leads to fall of one of the outcome from {1, 2, 3, 4, 5, 6}. Hence sample space is {1, 2, 3, 4, 5, 6} as it has all possible outcomes

(b) Taking a card from a pack of 52 cards is also a random experiment:

SOLUTION

Sample space = {Diamond ace, diamond king, Diamond Queen, diamond Jack, diamond 10, …….diamond 2, heart ace, heart king, heart Queen, heart Jack, heart 10, ……., heart 2, Spade ace, spade king, spade Queen, spade Jack, spade 10, …….spade 2, club ace, club king, club Queen, club Jack, club 10, …….club 2,}
Here are 52 elements in sample space.

(c) Tossing a pair of coins

SOLUTION

Sample space = {Head-Head, Head-Tail, Tail-Head, Tail-Tail}

(d) A pair of dice are thrown

SOLUTION

Sample space = {11, 12, 13, 14, 15, 16, ……. 61, 62, 63, 64, 65, 66}

There are 36 elements are present

(e) A coin is tossed and then a dice is thrown

SOLUTION

Sample space = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

Where H means Head and T means Tail

This sample space contains 12 elements

(f) A coin is tossed, if it is head, then coin is again tossed upon else a dice is thrown

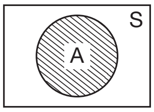
SOLUTION

Sample space = {HH, HT, T1, T2, T3, T4, T5, T6}

This sample space contains 8 elements

Event:

It is subset of sample space. e.g. getting a head on tossing a coin or getting a prime number in throwing a dice.

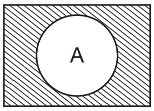


Sample space = $\{a_1, a_2, a_3, \ldots, a_n\}$

No. of total events possible = 2^n

Complement of event:

The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\bar{A}$ or A^c



Simple event:

Individual element present in sample space is simple event e.g. getting a head followed by a tail in throwing of a coin 2 times is a simple event.

Compound event:

When two or more than two events occur simultaneously, the event is said to be a compound event. Symbolically $A \cap B$ or AB represent the occurrence of both A and B simultaneously. "$A \cup B$" or A + B represent the occurrence of either A or B

For example:

(i) Participation of player in the game to win a game, is a trial but winning or losing is a simple event.

(ii) Tossing of a fair coin is a trial and turning up head or tail are simple events.

(iii) Throwing of a dice is a trial and appearance of even number (2 or 4 or 6) is a compound event.

(iv) Drawing a card from a pack of playing cards is a trial and getting "an ace or a queen" is compound event.

TYPES OF EVENTS

(a) Mutually Exclusive or Disjoint events:

Two or more events are said to be **mutually exclusive**, if the occurrence of one event prevents or precludes the occurrence of the other events. In other words they cannot occur together.

$A_i \cap A_j = \phi$ for $i \neq j$

For example:

(i) In tossing of a coin, getting of 'H' or 'T' are two mutually exclusive events because then can not happen together.

(ii) In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six mutually exclusive events.

(iii) In drawing a card from a pack of cards, getting a card of diamond or heart or club or spade are four mutually exclusive events.

(b) Equally-likely(Probable) events:

To or more events are said to be **equally likely events** if they have same number of favourable cases.

For example:

(i) The result of drawing a card from a well shuffled pack of cards, any card may appear in a draw, so 52 different cases are equally likely.

(ii) In tossing of a coin, getting of 'H' or 'T' are two equally likely events.

(iii) In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six equally likely events.

(c) Exhaustive system of events:

If each outcome of an experiment is associated with at least one of the events $E_1, E_2, E_3, \ldots E_n$, then collectively the events are said to be exhaustive mathematically we write $E_1 \cup E_2 \cup E_3 \ldots\ldots E_n = S$. (Sample space)

In simple words, if set of events can form complete sample space collectively then they are known as exhaustive set of events

For example:

(i) Tossing a coin has 2 exhaustive cases i.e. either head or tail may come upward.

(ii) Throwing of a dice has 6 exhaustive cases because any one of six digits 1, 2, 3, 4, 5, 6 may come upward.

(iii) Throwing of a pair of dice has 36 exhaustive cases because any one of six digits 1, 2, 3, 4, 5, 6 may come upward on any dice so total number of exhaustive cases = 6 × 6 = 36.

ILLUSTRATION 2

In following illustration events are explained related to experiments in first column. These events are categorised in Equally likely, Mutually Exclusive and Exhaustive events.

SOLUTION

Experiment	Events	Equi-likely events	Mutually exclusive	Exhaustive
1. Throwing of a dice	A: Throwing an odd face {1, 3, 5} B: Throwing a composite face {4, 6}	No	Yes	No
2. A ball is drawn from an urn containing 2 White, 3 Red and 4 Green balls	E_1 : getting a White ball E_2 : getting a Red ball E_3 : getting a Green ball	No	Yes	Yes
3. Throwing a pair of dice	A : throwing a doublet {11, 22, 33, 44, 55, 66} B : throwing a total of 10 or more {46, 64, 55, 56, 65, 66}	Yes	No	No
4. From a well shuffled pack of cards a card is drawn	E_1 : getting a heart E_2 : getting a spade E_3 : getting a diamond E_4 : getting a club	Yes	Yes	Yes
5. From a well shuffled pack of cards a card is drawn	A = getting a heart B = getting a face card	No	No	No

Working Rule

Playing Cards:

A pack of cards consists of 52 cards in 2 colours and 4 suits, i.e.,

(a) spades	(♠)
(b) clubs	(♣)
(c) Hearts	(♥)
(d) diamonds	(♦)

Each suit consists of 13 cards. Out of these, spades and clubs are black faced cards, while hearts and diamonds are red-faced cards.

There are 12 **court cards** (or **face cards**) i.e. (4 kings, 4 queens, 4 jacks).

There are 16 **Honour Cards** (4 aces, 4 kings, 4 queens, 4 jacks.)

ILLUSTRATION 3

From 52 playing cards one card is drawn randomly from it. Identify following mentioned events in mutually exclusive, equi-likely events and exhaustive set of events pairwise

Event A → Red card is drawn
Event B → Black card is card
Event C → King is drawn
Event D → Diamond is drawn
Event E → Spade is drawn

SOLUTION

Event A → Red card is drawn	(26 cards)
Event B → Black card is card	(26 cards)
Event C → King is drawn	(4 cards)
Event D → Diamond is drawn	(13 cards)
Event E → Spade is drawn	(13 cards)

Mutually exclusive events pairwise : A and B , A and E, B and D, D and E

Equi-likely events pairwise : A and B, D and E

Exhaustive events pairwise : A and B

Classical Definition of Probability

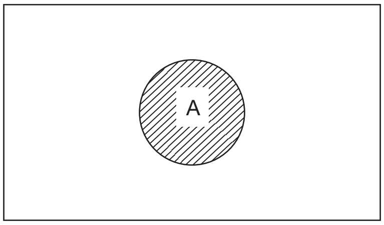
It means 'chance' or 'possibilities measure' of occurrence of an event. Its higher value tells that it has more chances to occur whereas its lower value tells that it has less possibilities of occurrence

Let there are $(m + n)$ exhaustive, and equally likely cases. for an event A and m of those are favourable to it and n of those are not in favour of it, then probability of happening of the event A which is denoted by P(A). Thus

$$P(A) = \frac{m}{m+n}$$

$$= \frac{\text{No. of favourable cases to A}}{\text{Total no. of exhaustive cases to experiment}}$$

$$= \frac{\text{Number of elements in A}}{\text{Number of elements in Sample space}}$$



Note:

(i) Clearly $0 \le P(A) \le 1$

(ii) If $P(A) = 1$ then an event A must happen (100%)

(iii) If $P(A) = 0$ then event A is impossible to happen (0%)

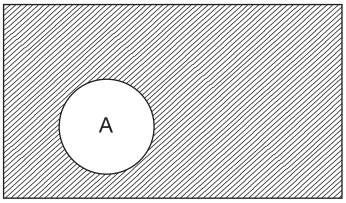
Axiomatic Probability

In every experiment, it is not possible that all events are equi-likely so in those cases we can not simply use no. of elements in favour of event A, so we have to get favourable cases and total cases according to given axiom or rule defined in experiment but actual fundamental about probability remains same, and that is

$$P(A) = \frac{\text{No. of favourable cases to A}}{\text{Total no. of exhaustive cases to experiment}}$$

Complement of Event A (A' or A^C or $\bar{A}$):

If Sample space (U) is the universal set and A is a set such that $A \subset U$. Then the complement of A with respect to U is denoted by A' or A^C or $U - A$



A' or A^C or $U - A \quad = \{x : x \in U \text{ and } x \notin A\}$.

Further, $\bar{A}$ denotes that event A does not happen,

$$P(\bar{A}) = \frac{n}{m+n} = 1 - \frac{m}{m+n} = 1 - P(A)$$

$$\therefore P(A) + P(\bar{A}) = 1$$

Odds in favour or in against are another way of telling about their chances to occur

Odds in favour of event A is m: n then Probability $P(A) = \frac{m}{m+n}$

Odds against event A is n: m then Probability $P(A) = \frac{m}{m+n}$

ILLUSTRATION 4

If U = {1, 2, 3, 4, 5, 6, 7} and A = {1, 3, 5, 7} find P(A) and $P(A^c)$.

SOLUTION

U = {1, 2, 3, 4, 5, 6, 7} and A = {1, 3, 5, 7} and $A' = U - A =$ {2, 4, 6}

$$P(A) = 4/7$$

$$P(A^c) = 1 - P(A) = 3/7$$

ILLUSTRATION 5

A fair dice is rolled.

(a) Find the probability of getting number 5 on the upper face.

(b) Find the probability of getting even number.

SOLUTION

Exhaustive outcomes are 1, 2, 3, 4, 5, 6. Therefore total number of cases = 6 = n.

(a) $\therefore$ probability of getting number 1 on the upper most of dice $= P(1) = \frac{1}{6}$

(b) Let E be the event of getting even number then favourable cases for E are 2, 4, 6

$\therefore m = 3$

Thus, $P(E) = \frac{m}{n} = \frac{3}{6} = \frac{1}{2}$

ILLUSTRATION 6

Find the probability that a leap year selected randomly will have 53 Sundays.

SOLUTION

A leap year contain 366 days i.e. 52 weeks and 2 days, clearly there are 52 Sundays in 52 weeks.
For the remaining two days, we may have any of the two days.
(i) Sunday-Monday, (ii) Monday-Tuesday,
(iii) Tuesday-Wednesday, (iv) Wednesday-Thursday,
(v) Thursday-Friday, (vi) Friday-Saturday and
(vii) Saturday-Sunday.

Now for 53 Sundays, one of the two days must be Sundays, hence required probability = 2/7

ILLUSTRATION 7

A cricket club has 15 members, among whom only 5 can bowl. What is the probability of forming a team of 11 to have at least 3 bowlers?

SOLUTION

Total number of ways of forming the team = ${}^{15}C_{11} = {}^{15}C_4$

Of these, number of ways of formation of the team

(i) with one bowler = ${}^{5}C_1 . {}^{10}C_{10} = 5$

(ii) with two bowlers = ${}^{5}C_2 . {}^{10}C_9 = 100$

Probability that at least 3 bowlers are in the team $= 1 - \dfrac{105}{{}^{15}C_4} = 1 - \dfrac{1}{13} = \dfrac{12}{13}$

ILLUSTRATION 8

The letters of the word "LATENT" are written randomly in all possible manner. Find the probability that the arrangement of these letters will form the word "TALENT".

SOLUTION

Total possible words with L, A, T, E, N, T

$n(s) = \dfrac{6!}{2!} = 360$

'TALENT' is one of them. $n(A) = 1$

Required probability = $P(A) = \dfrac{1}{360}$

ILLUSTRATION 9

Three distinct numbers are selected from 100 natural number. Find the probability that all the three numbers are divisible by 2 and 3.

SOLUTION

The numbers should be divisible by 6. Thus the number of favourable ways is (as there are 16 numbers {6, 12, 18, 24, …, 90, 96} in first 100 natural numbers, divisible by 6).

Required probability $= \dfrac{{}^{16}C_3}{{}^{100}C_3} = \dfrac{16 \times 15 \times 14}{100 \times 99 \times 98} = \dfrac{4}{1155}$.

ILLUSTRATION 10

A number is drawn at random from first 100 natural numbers. What is the probability of getting a prime number.

SOLUTION

Total number of outcomes = 100 = n

Now the list of prime numbers in the first 100 natural numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71,73, 79, 83, 89, 97.

∴ The number of favourable cases = 25 = m

$\therefore$ P(prime number) $= \dfrac{m}{n} = \dfrac{25}{100} = \dfrac{1}{4}$

ILLUSTRATION 11

Ten items out of a set of 100 are defective. What is the probability that 3 out of any four chosen are defective?

SOLUTION

Total ways of selection = ${}^{100}C_4$

Favourable ways = ${}^{90}C_1 \cdot {}^{10}C_3$

Required probability $= \dfrac{{}^{90}C_1 . {}^{10}C_3}{{}^{100}C_4} = \dfrac{144}{52283}$

ILLUSTRATION 12

Seven students are to sit at one side of a straight table. Find the probability of two particular student sitting together.

SOLUTION

7 student can sit in 7! ways.

∴ Total ways = 7!

Now, if two students want to sit together, they can sit in 6! ways

$\boxed{S_1 S_2} \quad S_3\ S_4\ S_5\ S_6\ S_7$

But, they can interchange their seats as $S_1\ S_2$ or $S_2\ S_1$

∴ Favourable cases = 2 × 6!

Hence, probability $= \dfrac{2 \times 6!}{7!} = \dfrac{2}{7}$

ILLUSTRATION 13

A party of 23 persons take their seats at a round table. Find the odds against two particular persons sitting together.

SOLUTION

Total no. of ways = 22!

P1 and P2 want to be together no. of ways = 21!.2! (they can interchange their seats.)

Required probability

$= \dfrac{(21)!\,2!}{(22)!} = \dfrac{1}{11} = \dfrac{1}{1+10}$. $\therefore$ odd against = 10 : 1.

ILLUSTRATION 14

Let M be the set of all 4 digited numbers whose sum is 34. If a number is selected from M then find the probability that the selected number is even.

SOLUTION

Since sum of 4 digited number is 34.

It must contain atleast two '9's since 4 × 8 = 32

So, the 4 digits of the number will be as follows.

(i) 9, 9, 8, 8 **(ii)** 9, 9, 9, 7

The total number of 4-digited numbers

$$\text{whose sum is } 34 = \frac{\lfloor 4}{\lfloor 2 \lfloor 2} + \frac{\lfloor 4}{\lfloor 3} = 6 + 4 = 10$$

Among these 10 numbers,

$$\text{4-digited numbers ended with } 8 = \frac{\lfloor 3}{\lfloor 2} = 3$$

∴ The number of 4 digited even numbers with sum of digits 34 = 3

$$\text{The required probability} = \frac{3}{10}$$

ILLUSTRATION 15

Rajdhani Express stops at six intermediate stations between Kota and Jaipur. Five passengers board at Kota and train departs. Each passengers can get down at any station till Jaipur. Find the probability that all five passengers will get down at different stations.

SOLUTION

Kota x x x x x x Jaipur

$n(S) = 7^5$ (each passenger can get down at any one of the 7 stations in 7 ways etc.)

$n(A) = {}^7C_5\, 5!$ (5 passengers can get down at 7 stations with their possible arrangements)

$$\text{Required probability} = \frac{{}^7C_5 \cdot 5!}{7^5}$$

ILLUSTRATION 16

A subset from the set of 9 smallest natural numbers is chosen randomly. Find probability that the chosen subset contains the number 5.

SOLUTION

$$n(S) = {}^9C_0 + {}^9C_1 + {}^9C_2 + \ldots + {}^9C_9 = 2^9$$

S = {1, 2, 3, 4, 5, 6, 7, 8, 9}

N(A) = (5 is one of the elements. Now we can choose none, one, few or all from remaining 8)

$$= {}^8C_0 + {}^8C_1 + {}^8C_2 + \ldots + {}^8C_7 + {}^8C_8 = 2^8$$

$$\text{Required Probability } = \frac{2^8}{2^9} = 1/2$$

ILLUSTRATION 17

Three electric bulb holders are fixed in a room. 3 bulbs are chosen at random from a set of 20 bulbs of which 16 are good and fitted to be holders. What is the probability that the room is lighted.

SOLUTION

P(room gets light)

= P(choosing atleast one good bulb)

= 1 – P(choosing no good bulb)

$$= 1 - \frac{{}^4C_3}{{}^{20}C_3} = 1 - \frac{1}{285} = \frac{284}{285}$$

ILLUSTRATION 18

A game consists of tossing a coin 3 times and noting its outcome. A boy wins if all tosses give the same outcome and losses otherwise. Find the probability that the boy losses the game.

SOLUTION

Since a coin is tossed 3 times, $n(S) = 2^3 = 8$

Let A be the event that the boy losses the game (all the tosses do not give the same out come)

Then $\bar{A} = \{HHH, TTT\}$

$n(\bar{A}) = 2$; $P(\bar{A}) = 2/8 = 1/4$

$$\therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4}$$

ILLUSTRATION 19

The letters of the word 'ASSASSIN' are written down at random in a row. Find the probability that no two S occur together.

SOLUTION (B)

$$\text{Total ways of arrangements} = \frac{8!}{2!\,4!}.\ (A \to 2, S \to 4)$$

x *A*x*A*x*I*x*N*x

We have to put 2A's, one I and one N as shown above. Now 'S' can be placed between the Gaps. Gap selection is ${}^5C_4$.

$$\text{Therefore, favourable ways } = {}^5C_4\left(\frac{4!}{2!}\right).$$

$$\text{Hence, required probability} = \frac{5.4!\,2!\,4!}{2!\,8!} = \frac{1}{14}$$

ILLUSTRATION 20

In rolling three fair dice, find the probability of getting

(i) a sum of points on the dice as 9

(ii) the number on dice are same

(iii) sum of two numbers on two dice equal to a number on remaining dice

SOLUTION

The total number of outcomes is 216.

(i) Favourable cases are:

D_1	D_2	D_3	Cases
2	1	6	6
2	2	5	3
3	1	5	6
2	3	4	6
1	4	4	3
3	3	3	1

The number of favourable cases = 25

$$P(\text{sum is } 9) = \frac{25}{6^3} = \frac{25}{216}$$

(ii) There are 6 favourable cases for our event which are (1, 1, 1), (2, 2, 2), … , (6, 6, 6)

$$P(\text{the numbers on dice are same}) = \frac{6}{6^3} = \frac{1}{36}$$

(iii) Favourable cases :

D_1	D_2	D_3	Cases
1	1	2	3
1	2	3	6
2	2	4	3
1	3	4	6
2	3	5	6
1	4	5	6
1	5	6	6
2	4	6	6
3	3	6	3

The number of favourable cases for required event is 45

P(sum of two numbers on two dice is equal to number on remaining dice) $= \dfrac{45}{6^3} = \dfrac{5}{24}$

ILLUSTRATION 21

Two numbers are selected at random from 1, 2, 3, …., 100 and multiplied. Find the probability that the product thus obtained is divisible by 3.

SOLUTION

Here, A^C is the event of selecting 2 numbers whose product is not divisible by 3.

Favourable ways $= n(A^C) = {}^{67}C_2$

($\therefore$ product of 2 numbers is not divisible by 3 if both of them are not divisible by 3)

$$\therefore P(A^C) = \frac{{}^{67}C_2}{{}^{100}C_2} = \frac{66 \times 67}{100 \times 99} = \frac{67}{150}$$

$$\therefore P(A) = 1 - P(A^C) = 1 - \frac{67}{150} = \frac{83}{150}$$

ILLUSTRATION 22

Three cards are drawn from a well shuffled pack of 52 cards randomly. What is the probability that the cards

(i) all are aces

(ii) with at least one ace

(iii) in which two are same colour and other is different.

(iv) have same faces

(v) have different denomination

(vi) have different denomination with same colour.

SOLUTION

The total number of cases $= {}^{52}C_3$

(i) $P(\text{all are aces}) = \dfrac{{}^4C_3}{{}^{52}C_3} = \dfrac{1}{5525}$

(ii) $P(\text{at least one ace}) = 1 - P(\text{no ace}) = 1 - \dfrac{{}^{48}C_3}{{}^{52}C_3} = \dfrac{1201}{5525}$

(iii) P(two are of same colour and other is different)

= P(2 Red, 1 Blue) + P(1 Red, 2 Blue)

$$= \frac{{}^{26}C_2 \cdot {}^{26}C_1 + {}^{26}C_1 \cdot {}^{26}C_2}{{}^{52}C_3} = \frac{13}{17}$$

(iv) P(have same face)

P(all 3 Jacks) + P(all 3 Kings) + P(all 3 Queens)

$$= \frac{{}^4C_3 + {}^4C_3 + {}^4C_3}{{}^{52}C_3} = \frac{3}{5525}$$

(v) $P(\text{different denomination}) = \dfrac{{}^{13}C_3 . {}^4C_1 . {}^4C_1 . {}^4C_1}{{}^{52}C_3}$

$= \dfrac{352}{425}$ we have to select suits as well

(vi) P(different denomination with same colour) =

$$\frac{{}^2C_1(\text{Red/Black}) \,.\, {}^{13}C_3(\text{diff denomination}) \,.\, {}^2C_1(\text{Red/Black}) \,.\, {}^2C_1(\text{Red/Black}) \,.\, {}^2C_1(\text{Red/Black})}{{}^{52}C_3}$$

$$= \frac{88}{425}$$

ILLUSTRATION 23

Consider the word "GOOGLE". Among all the 6 letter word that are formed using all the letter of word "GOOGLE". A word is randomly selected. Find the probability that both O's remain between G's in the randomly selected word.

SOLUTION

GOOGLE

Total possible words $n(S) = \dfrac{6!}{2! \cdot 2!} = 180$

First we should select 4 places to satisfy this relative order, then there is only one way to arrange these four letter, remaining two alphabets L and E can be arranged as 2!

Favourable words $n(\text{A}) = {}^6\text{C}_4.2! = 30$

$$\therefore \text{P(E)} = \frac{30}{180} = \frac{1}{6}$$

ILLUSTRATION 24

Out of 30 consecutive integers, three integers are drawn at random, Find the probability that their product is

(i) an odd number **(ii)** an even number

SOLUTION

(i) Now $n(\text{S}) = {}^{30}\text{C}_3$. There are 15 odd integers and 15 even integers from 1 to 30.

The product of the three integers is odd if all of them are odd

$\therefore n(\text{A}) = {}^{15}\text{C}_3$

$$\therefore \text{P(A)} = \frac{n(\text{A})}{n(\text{S})} = \frac{{}^{15}\text{C}_3}{{}^{30}\text{C}_3} = \frac{15\times14\times13}{30\times29\times28} = \frac{13}{116}$$

ii) The probability that the product of the three integers is even

$$= \text{P}(\bar{\text{A}}) = 1 - \text{P(A)} = 1 - \frac{13}{116} = \frac{103}{116}$$

ILLUSTRATION 25

Two numbers are selected at random from 1, 2, 3, ….. 100 without replacement. Find the probability that the minimum of the two numbers is less than 70.

SOLUTION

Since two numbers are selected at random from 1, 2, …., 100 we have, $n(\text{S}) = {}^{100}\text{C}_2$

Let A be the event that the minimum of the two numbers is less than 70;

Then the complement of A i.e., A^C is the event of selecting both the numbers from {70, 71, …, 100}

$n(\text{A}^\text{C}) = {}^{31}\text{C}_2$

$$\text{P}(\text{A}^\text{C}) = \frac{{}^{31}\text{C}_2}{{}^{100}\text{C}_2} \qquad \Rightarrow \text{P(A)} = 1 - \frac{{}^{31}\text{C}_2}{{}^{100}\text{C}_2}$$

ILLUSTRATION 26

An ordered pair is selected at random from $\{(x, y); x + y = 90, x, y \in \text{N}\}$. Find probability that the numbers of ordered pair have Greatest Common Divisor as 6.

SOLUTION

The number of ordered pairs satisfying $x + y = 90$, $x, y \in \text{N}$ is 89.

$\therefore n(\text{S}) = {}^{89}\text{C}_1 = 89.$

Let $x = 6a$, $y = 6b$, where $(a, b) = 1$

$\therefore 6a + 6b = 90 \Rightarrow a + b = 15,\ a, b \geq 1$

The total number of ordered pairs satisfying $a + b = 15$, is 14 among which

(3, 12), (12, 3), (6, 9), (9, 6), (5, 10), (10, 5) are not satisfying that a and b should be co-prime $(a, b) = 1$

$$\therefore \quad \text{P(E)} = \frac{8}{89}$$

ILLUSTRATION 27

Let $n = 51! + 1$. A number is selected at random from $\{n + 1, n + 2 \ldots.. 50 + n\}$, then find the probability that it is a prime.

SOLUTION

None of the numbers $n + 1$, $n + 2$ …….. $n + 50$ is prime

$\therefore \quad n + k = 51! + (k + 1),\ 1 \leq k \leq 50,$

$1.2.3 \ldots k(k + 1) \ldots 50.51 + (k + 1)$

$= (k + 1)\ (1.2.3 \ldots k \ldots 50.51 + 1)$

$= (k + 1)\ (\lambda + 1)$

$n + k$ is divisible by $k + 1$

So required probability = 0

ILLUSTRATION 28

Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the chance that the numbers on them are in A.P.

SOLUTION

Total number of ways $= {}^{21}\text{C}_3 = 1330$

Favourable cases:

If common difference of the A.P. is to be 1, then the possible groups are 1, 2, 3; 2, 3, 4; .... 19, 20, 21. (19 APs)

If the common difference is 2, then possible groups are 1, 3, 5; 2, 4, 6; ..... 17, 19, 21. (17 APs)

Proceeding in the same way, if the common difference is 10, then the possible group is 1, 11, 21. (only 1 AP)

Thus if the common difference of the A.P. is to be 11, obviously there is no favourable case

Hence, total number of favourable cases = 19 + 17 + 15 + ... + 3 + 1 = 100

$$\text{Hence, required probability} = \frac{100}{1330} = \frac{10}{133}$$

Alternate:

From 1 to 21

odd numbers = {1, 3, 5, 7, 9, ….., 21} 11 numbers

Even number = {2, 4, 6, 8, ….., 20} 10 numbers

For three numbers to be in AP, any two numbers should be both from even or from odd numbers

$$\text{Required probability} = = \frac{{}^{11}C_2 + {}^{10}C_2}{{}^{21}C_3} = \frac{10}{133}$$

ILLUSTRATION 29

A set of 3 numbers are chosen from the set of numbers 1, 2, 3, ..., $(2n+1)$. What is the probability that the numbers chosen are in A.P.?

SOLUTION

Any three numbers can be chosen in ${}^{(2n+1)}C_3$ ways. Regarding the favourable cases; if three numbers a, b, c chosen are to be in A.P. then $a + c = 2b$. Thus the sum of the extremes being $2b$ should be always even. The two extreme numbers given are both odd or even number and hence there are $(n+1)$ odd numbers and there are n even numbers.

$$\Rightarrow \quad \text{Required probability is} = \frac{{}^{n+1}C_2 + {}^{n}C_2}{{}^{2n+1}C_3}$$

$$= \frac{\dfrac{n(n+1)}{2} + \dfrac{n(n-1)}{2}}{\dfrac{(2n+1)2n(2n-1)}{6}} = \frac{3n}{(4n^2-1)}.$$

ILLUSTRATION 30

Of all the non-empty subsets of {1, 2, 3, 4, 5, 6, 7}, a non-empty subset is randomly selected. Find probability that it does not contain two consecutive numbers.

SOLUTION

$$n(S) = 2^7 - 1 = 127$$

Now, we are looking for no two consecutive numbers to be selected, for this we should use gap method

No. of singleton subset = ${}^7C_1$

No. of two element subset = ${}^6C_2$

No. of three element subset = ${}^5C_3$

No. of four element subset = ${}^4C_4$

Now, more than 4 numbers can not be chosen as they will have consecutive numbers among them

$$n(A) = {}^7C_1 + {}^6C_2 + {}^5C_3 + {}^4C_4 = 33$$

$$\text{Hence, } P(E) = \frac{33}{127}$$

ILLUSTRATION 31

If four whole numbers taken at random are multiplied together, show that chance that the last digit in the product is 1, 3, 7 or 9 is $\frac{16}{625}$.

SOLUTION

Total number of digits used in any number at the last (i.e. at unit's) place is 10 since, 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 can occur at that place. It is clear that if the last digit in any of the four numbers cannot be even (0, 2, 4, 5, 6 or 8)

All four numbers should be odd numbers, 1, 3, 5, 7, 9. But if any of one number is 5 then multiplication will have 5 as last digit on multiplication with remaining 3 odd digits but its not desirable case, so favourable case are 1, 3, 7 or 9 only

$$\text{The required probability} = \frac{4.4.4.4}{10.10.10.10} = \left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

ILLUSTRATION 32

Two fair dice rolled.

(A) Find the probability of getting a sum of points on the two dice as n where $n = 2, 3, 4, \ldots, 12$.

(B) Find the probability of getting sum of points is

(i) atleast 7 (ii) atmost 5 (iii) a prime

SOLUTION

The total number of outcomes are 36 which are

((1, 1), ….. (1, 6), (2, 1), ….., (2, 6), ….., (6, 1), ….(6, 6))

(A) Let E be the event that the sum of points on the two dice is $n \in [2, 12]$, $n \in$ N. Then

To get sum 2, favourable cases are (1, 1)

To get sum 3, favourable cases are (1, 2), (2, 1)

To get sum 4, favourable cases are (1, 3), (3, 1), (2, 2)

To get sum 5, favourable cases are (1, 4), (4, 1), (2, 3), (3, 2) and so on.

$p(n) = \dfrac{n-1}{36}$ for $2 \le n \le 7$ and $p(n) = \dfrac{13-n}{36}$ for $8 \le n \le 12$

(B) **(i)** P (sum is atleast 7)

= P (sum is 7) + P (sum is 8) + …….. + P (sum is 12)

$$= \frac{6}{36} + \frac{5}{36} + \ldots + \frac{1}{36} = \frac{21}{36} = \frac{7}{12}$$

(ii) P (sum is at most 5)

= P (sum is 2) + P (sum is 3) + P (sum is 4) + P (sum is 5)

$$= \frac{1+2+3+4}{36} = \frac{10}{36} = \frac{5}{18}$$

(iii) P (sum is prime) = P (sum is 2) + P (sum is 3) + P (sum is 5) + P (sum is 7) + P (sum is 11)

$$= \frac{1+2+4+6+2}{36} = \frac{15}{36} = \frac{5}{12}$$

ILLUSTRATION 33

What is the probability that in a group of n people, at least two of them will have the same birthday in non-leap year?

(i) If $n > 365$ **(ii)** $n < 365$

SOLUTION

(i) if $n > 365$ then its 100% surety that two person must have birthday on same day so $P(A) = 1$

(ii) Let us find the probability that no two persons have the same birthday and then subtract the result from 1. Excluding leap years, there are 365 different birthdays possible. Any person might have any one of the 365 days of the year as a birthday. A second person may likewise have anyone of 365 birthday and so on. Hence, the total number of ways is given by $= 365^n$.

And the number of possible ways for none of the n birthdays to coincide is

$$m = 365 \,.\, 364 \,\ldots\ldots\ldots\, (365 - (n-1))$$

$$= \frac{365!}{(365-n)!}$$

$\therefore$ The probability that no two birthdays coincide is $=$

$$\frac{365!}{365^n(365-n)!}$$

Hence, the required probability

$$= \left(1 - \frac{365!}{365^n(365-n)!}\right)$$

ILLUSTRATION 34

Suppose m boys and m girls take their seats randomly round a circle. Find probability of their sitting when

(i) no two boys sit together

(ii) no two girls sit together

(iii) boys and girls sit alternately

SOLUTION

Total ways = (2m – 1)!

(i) All girls can be arranged in $(m-1)!$ Then 'm' gaps are present.

In these gaps 'm' boys can be arranged in $m!$ ways

$$P(E) = \frac{m!(m-1)!}{(2m-1)!} = \frac{1}{{}^{2m-1}C_m}$$

(ii) As counting of girls and boys are same,

P(No two girls sits together) = P(Boys and girls are alternate)

$$= \frac{1}{{}^{2m-1}C_m}$$

Test Your Concepts-I

(Solutions on page H.213)

1. Two dice are thrown. The events A, B, C are as follow:

(A) Getting an even number on the first dice.

(B) Getting an odd number on the first dice.

(C) Getting a sum of 2.

State whether the following statements are true or false.

(i) A is a simple event.

(ii) C is a simple event.

(iii) A and B are mutually exclusive.

(iv) A′ and B′ are mutually exclusive.

(v) A = B′

(vi) A and B are mutually exhaustive.

2. 3 letters are placed in 3 envelopes randomly. Find the probability that only one letter goes in right envelope.

3. If a party of n persons sit at a round table, then find the odds against two specified individuals sitting next to each other.

4. Three cards are drawn successively, without replacement from a pack of 52 cards. Then find the probability that the drawn cards are face cards of same suits.

5. An unbiased dice marked with 1, 2, 2, 3, 3, 3 is rolled 3 times find the probability of getting a total score of 4 or 6.

6. Find the probability that the two-digit number formed by digits 1, 2, 3, 4, 5 is divisible by 4 (while repetition of digit is allowed).

7. Three persons A, B and C are to speak at a function along with the 5 others. If they all speak in random order, then find probability that A speaks before B and B speaks before C.

8. Out of 40 consecutive natural numbers, two are chosen at random. Find probability that sum of the numbers is odd.

9. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others then find probability that all three apply for the same house.

10. There are 4 defective items in a lot consisting of 10 items. From this lot, we select 5 items at random. The probability that there will be 2 defective items among them is $\frac{m}{n}$, then find $(3m - n)$.

11. Two red counters, 3 green counters and 4 blue counters are placed in a row in random order. The probability that no 2 blue counters are adjacent is $\frac{m}{n}$, then find $|8m - n|$.

12. Three integers are chosen at random from the first 20 integers. Find the probability that their product is even.

13. 6 married couples are standing in a room. If 4 people are chosen at random, then find the chance that exactly one married couple is among the 4.

14. Three distinct numbers are selected from first 100 natural numbers, then find the probability that all the three numbers are divisible by 2 and 3.

Answer keys

1. (i) False (ii) True (iii) True (iv) True (v) True (vi) True

2. $\frac{1}{2}$ **3.** $(n-3):2$ **4.** $\frac{1}{5525}$ **5.** $\frac{50}{216}$ **6.** $\frac{1}{5}$ **7.** $\frac{1}{6}$ **8.** $\frac{20}{39}$

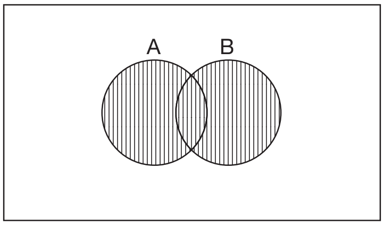
9. $\frac{1}{9}$ **10.** 9 **11.** 2 **12.** $\frac{17}{19}$ **13.** $\frac{16}{33}$ **14.** $\frac{4}{1155}$

Addition Theorem on Probability

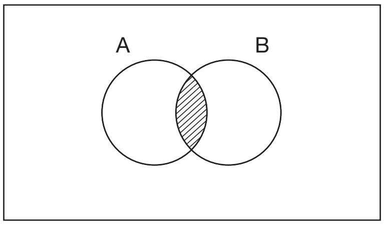
(i) $P(A \cup B)$ = Probability of either event A or event B to occur

= Probability of happening of the events A or B or both

= Probability of occurrence of at least one event A or B



(ii) $P(A \cap B)$ = Probability of events A and B together to occur = $P(AB)$.



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

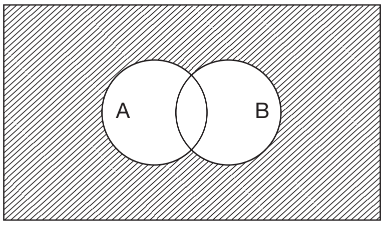
If A and B are mutually exclusive events, then $n(A \cap B) = 0 \Rightarrow P(A \cap B) = 0$

$\therefore \quad P(A \cup B) = P(A) + P(B)$.

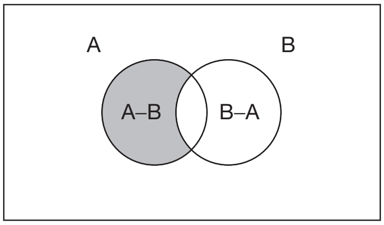
(iii) Probability of occurrence of neither A nor B

= 1 – P(either A or B)

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B).$$



(iv) Let A and B two events associated with a random experiment, then



(a) Probability of A occurs but B does not

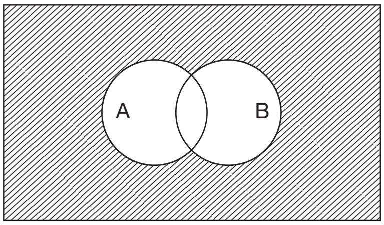
$$P(A - B) = P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

(b) Probability of B occurs but A does not

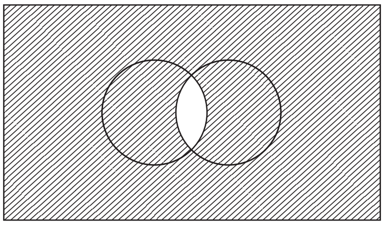
$$P(B - A) = P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

(v) **De Morgan's Law:**

(a) $P(A \cup B)' = P(A' \cap B')$



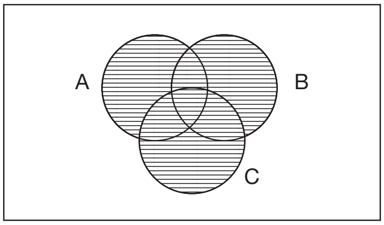
(b) $P(A\cap B)'=P(A'\cup B')$



(vi) For any three arbitrary events A, B, C

Probability of occurrence of atleast one event A or B or C to occur

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$



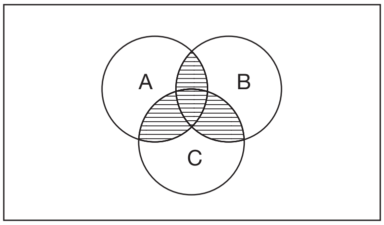
If three events A, B, C are mutually exclusive

$$P(A\cap B)=P(B\cap C)=P(C\cap A)=P(A\cap B\cap C)=0$$

$$\therefore\ P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

(vii) Probability of occurrence of atleast two events occur

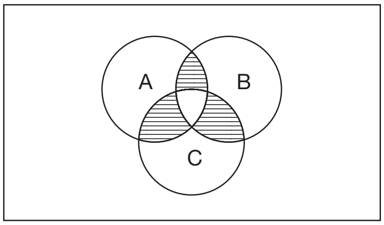
$$=P(A\cap B)+P(B\cap C)+P(C\cap A)-2P(A\cap B\cap C)$$



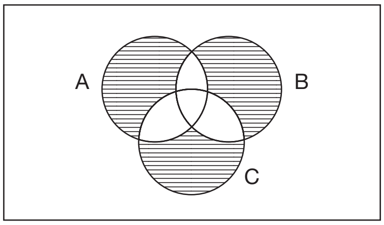
(viii) Probability of occurrence of exactly two events occur

$$=\begin{pmatrix}\text{Probability of}\\ \text{occurrence of atleast}\\ \text{two events occur}\end{pmatrix}-\begin{pmatrix}\text{Probability of}\\ \text{occurrence of three}\\ \text{events occur}\end{pmatrix}$$

$$=P(A\cap B)+P(B\cap C)+P(C\cap A)-3P(A\cap B\cap C)$$



(ix) Probability of occurrence of exactly one event occurs



$$=\begin{pmatrix}\text{Probability of}\\ \text{occurrence of atleast}\\ \text{one event occur}\end{pmatrix}-\begin{pmatrix}\text{Probability of}\\ \text{occurrence of atleast}\\ \text{two events occur}\end{pmatrix}$$

$$=P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(C\cap A)+3P(A\cap B\cap C)$$

Verbal description of event	Equivalent Notation
(i) Only A occurs	(i) $A\cap\bar{B}\cap\bar{C}$
(ii) Both A and B, but not C occur	(ii) $A\cap B\cap\bar{C}$
(iii) All the three events occur	(iii) $A\cap B\cap C$
(iv) At least one occur	(iv) $A\cup B\cup C$
(v) At least two occurs	(v) $(A\cap B)\cup(B\cap C)\cup(A\cap C)$
(vi) Exactly two of A, B and C occur	(vi) $(A\cap B\cap\bar{C})\cup(\bar{A}\cap B\cap C)\cup(A\cap\bar{B}\cap C)$
(vii) None occurs	(vii) $\bar{A}\cap\bar{B}\cap\bar{C}$

Generalization of the Addition Theorem

If $A_1, A_2, \ldots\ldots A_n$ are n events associated with a random experiment, then

$$P\left(\bigcup_{i=1}^{n}A_i\right)=\sum_{i=1}^{n}P(A_i)-\sum_{\substack{i,j=1\\ i\neq j}}^{n}P(A_i\cap A_j)+\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}P(A_i\cap A_j\cap A_k)+\ldots+(-1)^{n-1}P(A_1\cap A_2\cap\ldots\cap A_n)$$

If all the events $A_i(i=1,2\ldots., n)$ are mutually exclusive, then $P\left(\bigcup_{i=1}^{n}A_i\right)=\sum_{i=1}^{n}P(A_i)$

i. e. $P(A_1\cup A_2\cup\ldots\ldots\cup A_n)=P(A_1)+P(A_2)+\ldots\ldots+P(A_n)$.

While solving a question, formation of events is a best approach.

ILLUSTRATION 35

If $P(E) = 0.15$, $P(F) = 0.09$ and $P(E \cup F) = 0.19$, then find the value of $P(E^c \cap F)$.

SOLUTION

$$P(E \cap F) = P(E) + P(F) - P(E \cup F)$$

$$P(E \cap F) = 0.15 + 0.09 - 0.19 = 0.05$$

$$P(E^c \cap F) = P(F) - P(E \cap F)$$

$$P(E^C \cap F) = 0.09 - 0.05 = 0.04$$

ILLUSTRATION 36

A card is drawn from a well shuffled pack of playing cards. What is the probability that it is either king or red card?

SOLUTION

Let E_1, E_2 be respectively events of drawing king and red card.

Then $n(E_1) = 4$, $n(E_2) = 13$. $n(E_1 \cap E_2) = 1$

(∵ there is only one red king)

$$P(E_1) = \frac{4}{52}, P(E_2) = \frac{13}{52}, P(E_1 \cap E_2) = \frac{1}{52}$$

$$\therefore \quad P(E_1 \cup E_2) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

ILLUSTRATION 37

Two events A and B have the probabilities 0.25 and 0.5 respectively. The probability that both A and B occur simultaneously is 0.14. Find the probability that neither A nor B occurs.

SOLUTION

Given,

$P(A) = 0.25$; $P(B) = 0.5$; $P(A \cap B) = 0.14$

P(neither A nor B occurs)

$= P(\bar{A} \cap \bar{B})$

$= 1 - P(A \cup B)$ (By De morgan law)

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= 1 - \{0.25 + 0.5 - 0.14\}$

$= 1 - \{0.75 - 0.14\} = 1 - 0.61 = 0.39$

ILLUSTRATION 38

A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a nail?

SOLUTION

Let A be the event that the item chosen is rusted and B be the event that the them chosen is a nail.

$$\therefore \; P(A) = \frac{8}{16}, P(B) = \frac{6}{16} \text{ and } P(A \cap B) = 3/16$$

Required probability

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{8}{16} + \frac{6}{16} - \frac{3}{16} = \frac{11}{16}$$

ILLUSTRATION 39

If three dice are rolled once then find the probability of getting a sum of 16 or 6 on the first dice.

SOLUTION

Here $n(S) = 6^3 = 216$

Let A be the event of getting sum 16;

$A = \{(4, 6, 6), (5, 6, 5), (5, 5, 6), (6, 5, 5), (6, 6, 4), (6, 4, 6)\}$

Here $n(A) = 6$;

Let B be the event of getting 6 on first dice.

$B = \{(6, x_2, x_3) \,/\, 1 \le x_2 \le 6; 1 \le x_3 \le 6\}$

$n(B) = 36$;

$A \cap B = \{(6, 5, 5), (6, 6, 4), (6, 4, 6)\}$; $n(A \cap B) = 3$

Therefore $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{6}{216} + \frac{36}{216} - \frac{3}{216} = \frac{39}{216} = \frac{13}{72}$$

ILLUSTRATION 40

The probability of a horse A winning a race is $\frac{1}{3}$ and that of a horse B winning the same race is $\frac{1}{5}$, find the probability that

(i) either of them will win,

(ii) none of them will win.

SOLUTION

Let E_1, E_2 be respectively the events that horse A and B win the race

Then, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{5}$

Since, both the horses A and B take part in the same race, therefore, both of them cannot win at a time. Hence E_1 and E_2 are mutually exclusive events

(i) $P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$

(ii) $P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2) = 1 - \frac{8}{15} = \frac{7}{15}$

ILLUSTRATION 41

Given two events A and B, if the odds against A are 2 to 1, and those in favour of $A \cup B$ are 3 to 1, then find range of P(B).

SOLUTION

Odds against event A are 2 : 1

$$\therefore \quad P(A) = \frac{1}{3}$$

Odds in favour of $A \cup B$ are 3 : 1

$$\therefore \quad P(A \cup B) = \frac{3}{4}$$

Given that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) - P(A) \le P(B) \le P(A \cup B)$$

$$\Rightarrow \quad \frac{5}{12} \le P(B) \le \frac{3}{4}$$

ILLUSTRATION 42

If A and B are events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$, $P(\bar{A}) = 2/3$, then find $P(\bar{A} \cap B)$.

SOLUTION

$$P(A \cup B) = \frac{3}{4}, P(A \cap B) = \frac{1}{4}, P(\bar{A}) = \frac{2}{3} \Rightarrow P(A) = \frac{1}{3}.$$

$$\therefore \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\Rightarrow \quad \frac{1}{4} = \frac{1}{3} + P(B) - \frac{3}{4} \Rightarrow P(B) = \frac{2}{3}.$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

ILLUSTRATION 43

A and B are two candidates seeking admission in IIT Bombay. The probability that A is selected is 0.5 and the probability that both A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.9.

SOLUTION

Let E_1, E_2 be respectively the events that A and B are selected

Given that, $P(E_1) = 0.5$ and $P(E_1 \cap E_2) \le 0.3$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$\Rightarrow \quad P(E_1 \cup E_2) \ge 0.5 + P(E_2) - 0.3$$

$$[\because \text{Max. value of } P(E_1 \cap E_2) = 0.3]$$

$$\therefore \quad P(E_2) \le P(E_1 \cup E_2) - 0.2 \le 1 - 0.2 = 0.8$$

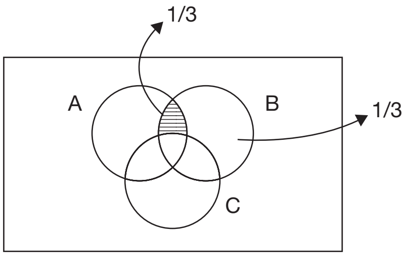
Hence $P(E_2) \ne 0.9$

ILLUSTRATION 44

If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and $P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, then find the value of $P(B \cap C)$.

SOLUTION

From Venn diagram, we can see that



$$P(B \cap C) = P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}.$$

ILLUSTRATION 45

The probabilities of a student getting I, II and III division in an examination are respectively $\frac{1}{10}, \frac{3}{5}$ and $\frac{1}{4}$. Then find probability that the student fails in the examination.

SOLUTION

Let A denotes the event getting I division; B denotes the event getting II division.

C denotes the event getting III division; and D denotes the event that a student fails.

Obviously, these four events are mutually exclusive and exhaustive, therefore

$$P(A) + P(B) + P(C) + P(D) = 1 \Rightarrow P(D) = 1 - 0.95 = 0.05.$$

ILLUSTRATION 46

If A, B, C are events such that $P(A) = 0.3$; $P(B) = 0.4$; $P(C) = 0.8$; $P(A \cap B) = 0.08$; $P(A \cap C) = 0.28$; $P(A \cap B \cap C) = 0.09$ and if $P(A \cup B \cup C) \ge 0.75$, then show that $P(B \cap C)$ lies in the interval $0.23 \le x \le 0.48$.

SOLUTION

Let $P(B \cap C) = x$

$$P(A \cup B \cup C) \ge 0.75$$

i.e., $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) \ge 0.75$

i.e., $0.3 + 0.4 + 0.8 - 0.08 - 0.28 - x + 0.09 \ge 0.75$

i.e., $x \le 0.48$ ...(1)

Also $P(A \cup B \cup C) \le 1$

$\therefore \quad 0.3 + 0.4 + 0.8 - 0.08 - 0.28 - x + 0.09 \le 1$

$$x \ge 1.5 - 1 + 0.09 - 0.36 \quad \Rightarrow \quad x \ge 0.23 \qquad \ldots(2)$$

(1) $\cap$ (2)

$\therefore \quad 0.23 \le x \le 0.48$

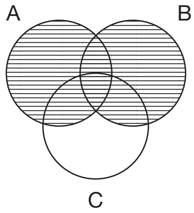
Hence proved.

ILLUSTRATION 47

If A, B and C are exhaustive events satisfying

$P(A \cup B \cap \bar{C}) = \frac{1}{5}$, $P(B \cap C) - P(A \cap B \cap C) = \frac{1}{15}$ and $P(A \cap C) = \frac{1}{10}$, then find $P(C \cap (A \cup B)')$.

SOLUTION



From figure,

$$P(C \cap (A \cup B)') = 1 - P(C' \cup (A \cup B))$$

$$= 1 - (P(A \cup B \cap \bar{C}) + P(B \cap C) - P(A \cap B \cap C) + P(A \cap C)$$

$$P(C \cap (A \cup B)') = 1 - \left(\frac{1}{10} + \frac{1}{15} + \frac{1}{5}\right) = \frac{30 - 3 - 2 - 6}{30} = \frac{19}{30}$$

ILLUSTRATION 48

In a swimming competition, only 3 students A, B and C are taking part. The probability of A's winning or the probability of B's winning is three times the probability of C's winning. Find the probability of the event such that either B or C to win.

SOLUTION

Suppose that A, B and C denote the events that the 3 students A, B and C win the competition respectively. By data, P(A) = P(B) = 3P(C)

Here A, B and C are mutually exclusive and exhaustive ($\because$ one and only one of A, B and C can win the competition)

$\therefore \quad P(A) + P(B) + P(C) = 1$

$\Rightarrow \quad 3P(C) + 3P(C) + P(C) = 1$

$\Rightarrow \quad P(C) = 1/7$ and $P(A) = P(B) = 3/7$

$$P(B \cup C) = P(B) + P(C) = \frac{3}{7} + \frac{1}{7} = \frac{4}{7}$$

$[\because B \cap C = \phi]$

ILLUSTRATION 49

For the three events A, B and C, P(exactly one of A and B occurs) = P(exactly one of B and C occurs) = P(exactly one of A and C occurs) = p and P (all the three events occur simultaneously) = p^2 where $0 < p < \frac{1}{2}$. Find the probability that atleast one of the three events occurs. Also find the value of p if $P(A \cup B \cup C) = \frac{11}{18}$.

SOLUTION

P(exactly one of A and B occurs)

$= P(A) + P(B) - 2P(A \cap B) = p$

P(exactly one of B and C occurs)

$= P(B) + P(C) - 2P(B \cap C) = p$

P(exactly one of A and B occurs)

$= P(A) + P(C) - 2P(A \cap C) = p$

By adding all, yields,

$= 2\Sigma P(A) - 2\Sigma P(A \cap B) = 3p$

$= \Sigma P(A) - \Sigma P(A \cap B) = \frac{3p}{2}$

Also given that $P(A \cap B \cap C) = p^2$

Hence, $P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$

$$= \frac{3p}{2} + p^2 = \frac{3p + 2p^2}{2} = \frac{11}{18}$$

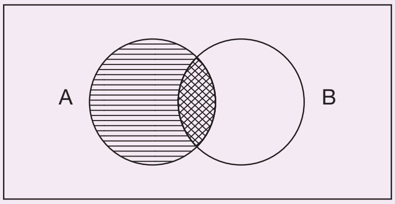
$\Rightarrow \quad 18p^2 + 27p - 11 = 0 = (3p - 1)(6p + 11) = 0$

$\Rightarrow \quad p = \frac{1}{3} \quad (\because \quad 0 < p < \frac{1}{2})$

Conditional Probability

Probability of B when event A has already happened is called conditional probability of B with respect to A and it is denoted by P(B/A).

It is denoted as $P(B/A) = \frac{P(A \cap B)}{P(A)}$



In such situation, as event A has already occurred so new sample space is shrinking to A only, now we will look for event B to occur, so favourable case are equivalent to $A \cap B$

Note: For two mutually exclusive events $P(B/A) = 0$

ILLUSTRATION 50

A fair dice is rolled. What is the probability that the outcome is 6 if it is given that outcome is even?

SOLUTION

Let $S = \{1, 2, 3, 4, 5, 6\}$

A : Outcome is even. Then $A = \{2, 4, 6\}$

B: Outcome is 6. Then $B = \{6\}$, $\therefore\ A \cap B = \{6\}$

$$P(A) = \frac{3}{6} = \frac{1}{2},\ P(B) = \frac{1}{6},\ P(A \cap B) = \frac{1}{6}$$

$$\text{By definition } P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

ILLUSTRATION 51

A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, then find the probability that the coin is tossed thrice.

SOLUTION

S = {H, TH, TTH, TTT}, Let 'A' the event that head does not occur on the first toss,

A = {TH, TTH, TTT}. 'B' be the event that the coin being tossed three times B = {TTH, TTT},

$$P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)} = \frac{2}{3}$$

ILLUSTRATION 52

If A and B are two events such that $P(A \cup B) = P(A \cap B)$ then show that $P(A) + P(B) = 2P(A).P\left(\frac{B}{A}\right)$.

SOLUTION

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow\ P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$(\because\ P(A \cap B) = P(A \cup B))$$

$$\Rightarrow 2P(A \cap B) = P(A) + P(B) \Rightarrow 2P(A)P\left(\frac{B}{A}\right) = P(A) + P(B).$$

ILLUSTRATION 53

A fair dice is tossed three times. If the sum of the first two tosses equals the third, then find the probability that 2 is tossed once.

SOLUTION

Different possibilities are as follows:

1^{st} toss	2^{nd} toss	3^{rd} toss
1	1	2
1	2	3
2	1	3
2	2	4
2	3	5
3	2	5
2	4	6
4	2	6

$\therefore$ Total favourable cases = 7

$$\begin{aligned} n(S) &= x + y \le 6;\ 2 \le x, y \le 6 \\ &= x' + y' \le 4 \text{ (Assign 1 to each } x \text{ and } y) \\ &= x' + y' + z' = 4,\ 0 \le x', y', z' \le 4 \\ &= {}^6C_2 = 15 \end{aligned}$$

$$\therefore \text{ Probability} = \frac{7}{{}^6C_2} = \frac{7}{15}$$

ILLUSTRATION 54

The probability of event A is $\frac{5}{6}$ and probability of event B, given that A occurs is $\frac{2}{3}$. The probability of event A, given that B occurs is $\frac{7}{9}$. Find the probability that neither event occurs.

SOLUTION

Given, $P(A) = \frac{5}{6};\ P(B/A) = \frac{2}{3};\ P(A/B) = \frac{7}{9}$

To find $P(A^C \cap B^C) = P(\overline{A \cup B})$

Now, $P(B/A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = \frac{5}{6}\left(\frac{2}{3}\right)$

$$\Rightarrow\quad P(A \cap B) = \frac{5}{9}$$

Again $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{9}$

$$\Rightarrow\quad P(B) = \frac{5}{9}\left(\frac{9}{7}\right) = \frac{5}{7}$$

Now, $P(A^C \cap B^C) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \frac{5}{6} - \frac{5}{7} + \frac{5}{9} = \frac{1}{6} + \frac{5}{9} - \frac{5}{7} = \frac{1}{126}$$

ILLUSTRATION 55

If two events A and B are such that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^c) = 0.5$, then find $P(B/A \cup B^c)$.

SOLUTION

$P(A) = 1 - P(A^c) = 0.7$; $P(B^c) = 1 - P(B) = 0.6$

$P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$

$= 0.7 + 0.6 - 0.5 = 0.8,$

Since ∴ $P(A \cap B) + P(A \cap B^c) = P(A)$

∴ $P(A \cap B) = 0.7 - 0.5 = 0.2$

Also $B \cap (A \cup B^c) = (B \cap A) \cup (B \cap B^c)$

$= (B \cap A) \cup \phi = B \cap A$

Now required

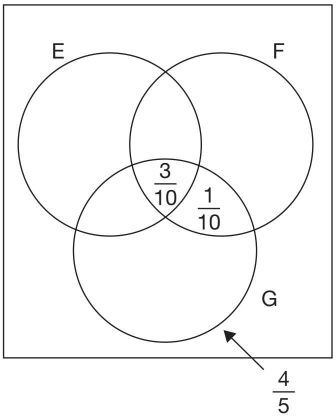
$$= P(B / A \cup B^c) = \frac{P(B \cap (A \cup B^c))}{P(A \cup B^c)} = \frac{P(B \cap A)}{P(A \cup B^c)}$$

$$= \frac{0.2}{0.8} = \frac{1}{4} = 0.25$$

ILLUSTRATION 56

If $P(F) = \frac{1}{2}$, $P(G) = \frac{4}{5}$, $P(E \cap F \cap G) = \frac{3}{10}$ and $P(E^c \cap F \cap G) = \frac{1}{10}$. Find the value of P(F/G).

SOLUTION



$$P(F/G) = \frac{P(F \cap G)}{P(G)} = \frac{P(E \cap F \cap G) + P(E^c \cap F \cap G)}{P(G)}$$

$$= \frac{\frac{3}{10} + \frac{1}{10}}{\frac{4}{5}} = \frac{4}{10} \cdot \frac{5}{4} = \frac{1}{2}$$

ILLUSTRATION 57

If A and B are two events defined on a sample space such that $P(A) = 0.3$; $P(B) = 0.5$; $P(A/B^C) = 0.2$ and $P(B/A^C) = \frac{3}{7}$, then find the value of $P((A \cap B^C) \cup (A \cap B^C) \cup (A^C \cap B))$.

SOLUTION

Given, $P(A) = 0.3$; $P(B) = 0.5$ and $P(A/B^C) = 0.2$

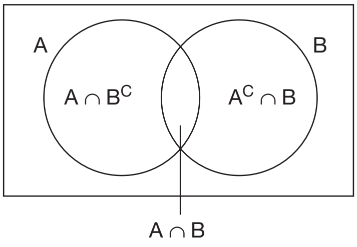
or $\quad \dfrac{P(A \cap B^C)}{P(B^C)} = 0.2$

$\Rightarrow P(A \cap B^C) = (0.2)\,(0.5) = 0.1$

$\Rightarrow P(A) - P(A \cap B) = 0.1$

$\Rightarrow P(A \cap B) = 0.3 - 0.1 = 0.2$ …(1)

Let $P((A \cap B^C) \cup (A^C \cap B)) = P(A) + P(B) - 2P(A \cap B)$ …(2)



∴ $P(X) = 0.3 + 0.5 - 0.4 = 0.4$

Test Your Concepts-II

(Solutions on page H.214)

1. If A and B are two mutually exclusive, Mark following relations as True/False

(i) $P(A) \le P(\overline{B})$ (ii) $P(\overline{A} \cap \overline{B}) = P(\overline{A}) + P(B)$

(iii) $P(\overline{A} \cup \overline{B}) = 0$ (iv) $P(\overline{A} \cap B) = P(A)$

2. A dice is thrown. Let A be the event that the number obtained is greater than 3 and B be the event that the number obtained is less than 5, then find $P(A \cup B)$.

3. There are 4 red, 3 pink, 5 yellow and 8 white roses in a lawn. What is the probability that a blind man will touch a red or a white rose?

4. A committee of five is to be chosen from a group of a 9 people. Find probability that a certain married couple will either serve together or not at all.

5. 3 students A, B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Then, find the probability that B or C wins. (Assume no two reach the winning point simultaneously)

6. In an examination, one hundred candidates took paper in Physics and Chemistry. Twenty five candidates failed in Physics only. Twenty candidates failed in Chemistry only. Fifteen failed in both Physics and Chemistry. A candidate is selected at random. Find the probability that he failed either in Physics or in Chemistry but not in both.

7. If A, B and C are three events such that $P(B) = 3/4$, $P(A \cap B \cap C') = 1/3$ and $P(A' \cap B \cap C') = 1/3$, then find $P(B \cap C)$.

8. If $4P(A) = 6P(B) = 10P(A \cap B) = 1$, then find $P(B/A)$.

9. Two dice are thrown. Then, find the probability that the numbers appeared have a sum of 8, if it known that the second dice always exhibits 4.

10. A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one-by-one, then find the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag.

11. Three numbers are chosen at random without replacement from 1, 2, 3, ... 10. Find the probability that the minimum of the chosen number is 4 or their maximum is 8.

12. If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is

(A) $P(C/D) < P(C)$ (B) $P(C/D) = \dfrac{P(D)}{P(C)}$

(C) $P(C/D) = P(C)$ (D) $P(C/D) \geq P(C)$

13. For a biased dice, the probability for the different faces to turn up are given below:

Faces	1	2	3	4	5	6
Probabilities	0.12	0.32	0.21	0.15	0.05	0.17

The dice is tossed and you are told that either face one or face two has turned up then find probability that it faces one.

14. The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen is 0.0015 then find probability that a stolen automobile will be found in one week.

15. The probability that a person will get an electric contract is 2/5 and the probability that he will not get plumbing contract is 4/7. If the probability of getting atleast one contract is 2/3 then find the probability that he will get both.

Answer keys

1. (i) True (ii) False (iii) False (iv) False **2.** 1 **3.** $\frac{3}{5}$ **4.** $\frac{4}{9}$ **5.** $\frac{3}{5}$

6. $\frac{9}{20}$ **7.** $\frac{1}{12}$ **8.** $\frac{2}{5}$ **9.** $\frac{1}{6}$ **10.** $\frac{7}{40}$ **11.** 11/40 **12.** (D)

13. 5/21 **14.** 0.4 **15.** $\frac{17}{105}$

INDEPENDENT EVENTS

Two events A and B are said to be independent if occurrence or non-occurrence of one event does not affect the occurrence or non-occurrence of other event

Lets consider conditional probability case when event A is already occurred and we want probability of event B to occur but if we consider them independent event then occurrence of event A should not make any impact to occurrence of event B hence its probability should be same as earlier as P(B)

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B)$$

$$\Rightarrow \quad P(A \cap B) = P(A).\ P(B)$$

It is taken as definition of independent events

If the occurrence of one event affects the probability of the occurrence of the other event then the events are said to be Dependent or Contingent.

$P(A \cap B) \neq P(A).\ P(B)$ for dependent events

For example:

(i) If we toss two coins, then the occurrence of head on one coin does not influence the occurrence of head or tail on the other coin in any way. Hence these events are independent.

(ii) Suppose a bag contains 5 white and 4 black balls. Two balls are drawn one by one then two events that first ball is white and second ball is black are independent if the first ball is replaced before drawing the second ball. If the first ball is not replaced then these two events will be dependent because second draw will have only 8 balls in bag.

Mutually Exclusive Events are Not Independent Events

If event A and B are independent then

(a) A′ and B are also independent
(b) A and B′ are also independent
(c) A′ and B′ are also independent

Three events A, B and C are independent if and only if all the following **four conditions** hold:

(i) $P(A \cap B) = P(A) . P(B)$;
(ii) $P(B \cap C) = P(B) . P(C)$
(iii) $P(C \cap A) = P(C) . P(A)$
(iv) $P(A \cap B \cap C) = P(A) . P(B) . P(C)$

If (i), (ii), (iii) are true then they are pairwise independent. If all four are true then they must be **pairwise** as well as **mutually independent**.
Similarly for n events $A_1, A_2, A_3, \ldots\ldots A_n$ to be independent, the number of these conditions $= {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - n - 1$.

Probability of at least one of the *n* independent events:

If $p_1, p_2, p_3, \ldots\, p_n$ are the probabilities of n independent events $A_1, A_2, A_3 \ldots A_n$ then the probability of happening of at least one of these event is

$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = 1-$ Probability of none of the events occur

$$= 1 - P(\bar{A}_1)\, P(\bar{A}_2)\, P(\bar{A}_3) \ldots P(\bar{A}_n)$$
$$= 1 - [(1 - P_1)(1 - P_2) \ldots (1 - P_n)]$$

ILLUSTRATION 58

The probability that a man will be alive in 20 years is 3/5 and the probability that his wife will be alive in 20 years is 2/3. Then find the probability that at least one will be alive in 20 years.

SOLUTION

Let A be the event that the husband will be alive in 20 years. B be the event that the wife will be alive in 20 years. Clearly A and B are independent events.

$\therefore P(A \cap B) = P(A)P(B)$.

Given $P(A) = \frac{3}{5}. P(B) = \frac{2}{3}$.

The probability that at least one of them will be alive in 20 years is

$$P(A \cup B) = P(A) + P(B) - P(A).P(B)$$
$$= \frac{3}{5} + \frac{2}{3} - \frac{3}{5}.\frac{2}{3} = \frac{9+10-6}{15} = \frac{13}{15}$$

ILLUSTRATION 59

A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II, III are p, q and $\frac{1}{2}$ respectively. If the probability that the student is successful is 1/2 then find relation in p and q.

SOLUTION

Let A, B and C be the events that the student is successful in test I, II and III respectively, then P (the student is successful)

$$= P[(A \cap B \cap C')] + P[(A \cap B' \cap C)] + P[(A \cap B \cap C)]$$
$$= P(A).P(B).P(C') + P(A).P(B').P(C) + P(A).P(B).P(C)$$

[$\because$ A, B, C are independent]

$$= pq\left(1 - \frac{1}{2}\right) + p(1-q)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)pq = \frac{1}{2}p(1+q) = \frac{1}{2}$$
$$\Rightarrow p(1+q) = 1.$$

This equation has infinitely many values of p and q.

ILLUSTRATION 60

A bag contains 5 black and 7 red balls. One ball is drawn at random and is replaced, then a second ball is drawn. What is the probability of having red balls in both drawings.

SOLUTION

A : drawing a red ball in 1st draw.

B : drawing a red ball in 2nd draw.

$$P(A) = \frac{7}{12}, P(B) = \frac{7}{12} \text{ and } P\left(\frac{B}{A}\right) = \frac{7}{12}$$

$$\therefore P(A \cap B) = P(A)\, P(B) = \frac{7}{12}.\frac{7}{12} = \frac{49}{144}$$

ILLUSTRATION 61

Let A_i, where $i = 1, 2, 3, \ldots\ldots n$, be an independent event such that $P(A_i) = \frac{1}{i+1}$, then find probability that none of the event $A_1, A_2 \ldots A_n$ occur.

SOLUTION

$$P(\bar{A}_i) = 1 - \frac{1}{i+1} = \frac{i}{i+1}$$

$$\text{P(none of event occur)} = P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots\ldots \cap \bar{A}_n)$$

$$= P(\bar{A}_1)\, P(\bar{A}_2) \ldots. P(\bar{A}_n)$$

$$= \frac{1}{2}\cdot\frac{2}{3}\ldots\frac{n}{n+1} = \frac{1}{n+1}$$

ILLUSTRATION 62

Given two independent events, if the probability that exactly one of them occurs is 26/49 and the probability that none of them occurs is 15/49, then find the probability of more probable of the two events.

SOLUTION

Let A = first event occurs

B = second event occurs

P(Exactly A or B occur)

$$= P(A) + P(B) - 2P(A)\,P(B) = \frac{26}{49} \qquad \ldots(1)$$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B}) = (1 - P(A))(1 - P(B)) = \frac{15}{49}$$

$$\Rightarrow \quad P(A) + P(B) - P(A)\,P(B) = \frac{34}{49} \qquad \ldots(2)$$

From (1) and (2)

$$P(A) + P(B) = \frac{6}{7}$$

$$P(A)P(B) = \frac{8}{49}$$

$$\therefore \quad P(A) = \frac{4}{7},\ P(B) = \frac{2}{7} \text{ or } P(A) = \frac{2}{7},\ P(B) = \frac{4}{7}$$

More probable event with probability $\frac{4}{7}$

ILLUSTRATION 63

Three persons roll a six sided fair dice once. Let A_{ij} denotes the event that the persons i and j roll the same face, then show that the events A_{12}, A_{23} and A_{31} are pairwise independent but not mutually independent.

SOLUTION

$P(A_{12}) = P(\text{both } P_1 \text{ and } P_2 \text{ have the same throw})$

$$= \frac{6}{36} = \frac{1}{6}$$

Similarly $P(A_{23}) = P(A_{31}) = \frac{1}{6}$

$P(A_{12} \cap A_{23}) = P(A_1, A_2 \text{ and } A_3 \text{ have the same score})$

$$= \frac{6}{216} = \frac{1}{36} = P(A_{12}) \cdot P(A_{23}) \qquad \ldots(1)$$

$$\text{Similarly } P(A_{23} \cap A_{31}) = \frac{1}{36} = P(A_{23}) \cdot P(A_{31}) \qquad \ldots(2)$$

$$\text{And } P(A_{31} \cap A_{12}) = \frac{1}{36} = P(A_{31}) \cdot P(A_{12}) \qquad \ldots(3)$$

Now $P(A_{12} \cap A_{23} \cap A_{31}) = P(P_1, P_2 \text{ and } P_3 \text{ have the same throw})$

$$= \frac{6}{216} = \frac{1}{36} \neq P(A_{12}) \cdot P(A_{23}) \cdot P(A_{31}) = \frac{1}{216}$$

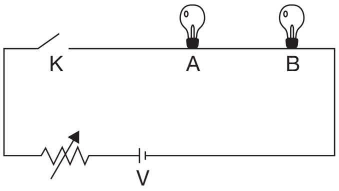
$\therefore$ A_{12}, A_{23}, A_{31} are pairwise independent but not mutually independent.

ILLUSTRATION 64

Two lights bulbs are connected in series in a circuit. Find probability that an increase in voltage above its rated value will break the circuit if, under these assumptions, the chance that a bulb burns out is 0.4 for each of the two bulbs.

SOLUTION

P(bulb burns) = 0.4



Hence P(circuit breaks) = 1 – P(Both bulbs work fine)

$$= 1 - [(0.6)(0.6)]$$

$$= 1 - \frac{1}{36}$$

$$= \frac{64}{100} = 0.64$$

ILLUSTRATION 65

Suppose two fair dice are rolled.

Let A = {1st roll is 1, 2 or 3}, B = {1st roll is 3, 4 or 5}

C = {the sum of two rolls is 9} be the event then check whether these events are independent or not.

SOLUTION

A = {(1, 1)(1, 2), …, (1, 6), (2, 1), (2, 2), …, (2, 6), (3, 1), …, (3, 6)}

B = {(3, 1)(3, 2), …, (3, 6), (4, 1), (4, 2), …, (4, 6), (5, 1), (5, 2), …, (5, 6)}

C = {(3, 6), (4, 5), (6, 3), (5, 4)}

$$P(A) = \frac{18}{36} = \frac{1}{2};\ P(B) = \frac{18}{36};\ P(C) = \frac{4}{36} = \frac{1}{9}$$

We have, $P(A \cap B) = \frac{1}{6} \neq \frac{1}{2}\cdot\frac{1}{2} = P(A)\,P(B)$

$$P(A \cap C) = \frac{1}{36} \neq \frac{1}{2}\cdot\frac{4}{36} = P(A)\,P(C)$$

$$P(B \cap C) = \frac{1}{12} \neq \frac{1}{2}\cdot\frac{4}{36} = P(B)\,P(C)$$

$$P(A \cap B \cap C) = \frac{1}{36} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{36} = P(A)P(B)P(C)$$

They are not pairwise independent.
Hence dependent events

ILLUSTRATION 66

Two fair dice are thrown independently. Three events A, B and C are defined as follows:

A: Odd faces with first dice

B: Odd faces with 2nd dice

C: Sum of points on two dice is odd

Are the events A, B and C mutually independent?

SOLUTION

$$P(A) = \frac{3\times 6}{36} = \frac{1}{2} \qquad \because \underset{\{1,3,5\}}{D_1} \qquad \underset{\{1,2,3,4,5,6\}}{D_2}$$

$$P(B) = \frac{3\times 6}{36} = \frac{1}{2} \qquad \because \underset{\{1,2,3,4,5,6\}}{D_1} \qquad \underset{\{1,3,5\}}{D_2}$$

$$P(C) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4} = P(A)\,P(B)$$

$$P(A \cap C) = \frac{9}{36} = \frac{1}{4} = P(A)\,P(C)$$

$$P(B \cap C) = \frac{9}{36} = \frac{1}{4} = P(B)\,P(C)$$

$$P(A \cap B \cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C)$$

Hence, the events A,B and C are pairwise independent but not mutually independent events.

ILLUSTRATION 67

Let E^C denotes the complement of an event E. Let E, F, G be pairwise independent events with P(G) > 0 and $P(E \cap F \cap G) = 0$, then find $P\left(\frac{E \cup F}{G}\right)$.

SOLUTION

As E, F, G are pairwise independent then

$$P(E \cap F) = P(E)P(F)$$
$$P(F \cap E) = P(F)P(E)$$
$$P(E \cap G) = P(E)P(G)$$

$$P(E \cap F/G) = \frac{P((E \cup F) \cap G)}{P(G)} = \frac{P((E \cap G) \cup (F \cap G))}{P(G)}$$

$$= \frac{P(E \cap G) + P(F \cap G) - 0}{P(G)} = P(E) + P(F)$$

ILLUSTRATION 68

A problem in mathematics is given to the three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be

(i) solved

(ii) solved by exactly two of A, B, C

(iii) solved by exactly one of A, B, C

SOLUTION

Let A, B, C denote the events that the problem is solved by the students A, B and C respectively.

Then $P(A) = \frac{1}{2}, P(B) = \frac{3}{4}, P(C) = \frac{1}{4}$

(i) The problem will be solved if at least one of them solved the problem. Thus we have to calculate the probability of occurrence of at least one of the three events A, B, C i.e.

$$P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C) = \Sigma P(A) - \Sigma P(A)\,P(B) + P(A)\,P(B)\,P(C)$$

$$= \frac{29}{32}$$

Alternate

$$P(A \cup B \cup C) = 1 - P(A^C \cap B^C \cap C^C)$$
$$= 1 - \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{29}{32}$$

(ii) Required Probability =

$$P(A \cap B \cap C^C) + P(A \cap B^C \cap C) + P(A^C \cap B \cap C)$$
$$= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4} = \frac{13}{32}$$

(iii) Required Probability =

$$P(A \cap B^C \cap C^C) + P(A^C \cap B \cap C^C) + P(A^C \cap B^C \cap C)$$
$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{13}{32}$$

ILLUSTRATION 69

A fair dice is rolled. Find the probability that the first time 1 occurs at the even throw.

SOLUTION

In rolling A fair dice, 1 has to occur at 2nd, 4th, 6th, 8th, …… roll.

Let E be the event of getting 1 on a dice.

$$P(E) = \frac{1}{6} \text{ and } P(\bar{E}) = \frac{5}{6}$$

$$\bar{E}\,E \qquad \longrightarrow \qquad 5/6 \cdot 1/6$$

$$\bar{E}\,\bar{E}\,\bar{E}\,E \qquad \longrightarrow \qquad (5/6)^3 \cdot 1/6$$

$$\bar{E}\,\bar{E}\,\bar{E}\,\bar{E}\,\bar{E}\,E \qquad \longrightarrow \qquad (5/6)^5 \cdot 1/6$$

$$\text{Required probability} = \left(\frac{5}{6}\right)\frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6} + \ldots$$

$$= \frac{\frac{5}{36}}{1-\frac{25}{36}} = \frac{5}{11}$$

ILLUSTRATION 70

A biased coin with probability p $(0 < p < 1)$ of falling tails is tossed until appears for the first time. If the probability that tail comes in odd number of trials is $\frac{1}{2}$, then find p.

SOLUTION

We have $P(T) = P\ p(H) = P(\bar{T}) = 1 - p$

$$\text{T} \qquad \longrightarrow \qquad p$$

$$\text{HHT} \qquad \longrightarrow \qquad (1-p)^2 p$$

$$\text{HHHHT} \qquad \longrightarrow \qquad (1-p)^4 p$$

like this more cases can be formed.

$\therefore$ Required probability $= p + (1-p)^2 p + (1-p)^4 p + \ldots$

$$\Rightarrow \frac{p}{1-(1-p)^2} = \frac{p}{2p-p^2} = \frac{2}{3} \quad \Rightarrow \quad 2p^2 = p \Rightarrow p = \frac{1}{2}$$

ILLUSTRATION 71

A pair of fair dice are rolled together a sum of either 3 or 5 or 7 is obtained. Find the probability that 3 comes before 5 and 7.

SOLUTION

E_1: getting a sum 3 $\quad \Rightarrow P(E_1) = \frac{2}{36} = \frac{1}{18}$

(1, 2)(2, 1)

E_2: getting a sum 5 $\quad \Rightarrow P(E_2) = \frac{4}{36} = \frac{1}{9}$

(2, 3)(3, 2)(1, 4)(4, 1)

E_3 : getting a sum 7 $\quad \Rightarrow P(E_3) = \frac{6}{36} = \frac{1}{6}$

(1, 6)(6, 1)(3, 4)(4, 3)(2, 5)(5, 2)

$$P(E_1^C \cap E_2^C \cap E_3^C) = 1 - \left(\frac{1}{18} + \frac{1}{9} + \frac{1}{6}\right)$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

now P (E_1 comes before E_2 and E_3)

$$\frac{1}{18} + \frac{2}{3}\cdot\frac{1}{18} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{18} + \ldots = \frac{\frac{1}{18}}{1-\frac{2}{3}} = \frac{1}{6}$$

ILLUSTRATION 72

A sportsman's chance of shooting an animal at a distance $r(> a)$ is a^2/r^2 He fires when $r = 2a$ and if he misses, he reloads and fires when $r = 3a$, $4a$, $5a$..... If he misses at a distance na, the animal escapes. What are the odds against the sportsman?

SOLUTION

$$P(r) = \frac{a^2}{r^2}$$

$$P(2a) = \frac{1}{4},\ P(3a) = \frac{1}{9},\ P(4a) = \frac{1}{16} \text{ etc.}$$

The sportsman succeeds if

(a) he hits the first time or

(b) misses the first time but succeeds at the second or

(c) misses the first and second time but succeeds in the third and so on.

$\therefore$ probability of success, $p = \frac{1}{4} + \frac{3}{4} \times \frac{1}{9} + \frac{3}{4} \times \frac{8}{9} \times \frac{1}{16}$

$+ \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{1}{25} + \ldots\ldots (n-1)$ terms

$\Rightarrow \quad p = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \ldots.$ upto $(n-1)$ terms

$2p = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots.$ upto $(n-1)$ terms

$$p = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots\ldots \right.$$

$$\left. + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right] = \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{n-1}{2n}$$

$\therefore$ Odds against the sportsman $= \frac{n+1}{n-1}$

ILLUSTRATION 73

There are ninety cards in a box numbered 10, 11, 12, …… 98, 99. Three cards are drawn from the box one by one with replacement. Find the probability that product of the digits on the cards will be 12 at least once.

SOLUTION

The cards whose product of digits is 12 are 26, 34, 43, 62, therefore the number of favourable cases of the event that product of the digits is 12 is 4.

The probability that the product of the digits is not 12 in one trail is $\frac{86}{90}$.

The probability that the product of the digits is not 12 in none of 3 trails $\left(\frac{86}{90}\right)^3$.

The probability that the product of the digits is 12, at least one in 3 trails is $1-\left(\frac{86}{90}\right)^3$.

ILLUSTRATION 74

A random selector can only select one of the nine integers 1, 2, 3, …, 7, 8, 9 and it makes these selections with equal probability. Find the probability that after 10 selections, the product of 10 numbers selected will be divisible by 10.

SOLUTION

$$P(1,3,5,7,9)=\left(\frac{5}{9}\right)^{10};\ P(1,3,7,9);=\left(\frac{4}{9}\right)^{10};$$

P(5 is always taken and product of numbers is not divisible by 10)

$$=P(5)=\left(\frac{5}{9}\right)^{10}-\left(\frac{4}{9}\right)^{10};$$

$$P(1,2,3,4,6,7,8,9)=\left(\frac{8}{9}\right)^{10}$$

P(Product of number is not divisible by 10)

$$=\left(\frac{8}{9}\right)^{10}+\left(\frac{5}{9}\right)^{10}-\left(\frac{4}{9}\right)^{10}$$

$$P(\text{div. by }10)=1-\left[\left(\frac{8}{9}\right)^{10}+\left(\frac{5}{9}\right)^{10}-\left(\frac{4}{9}\right)^{10}\right]$$

ILLUSTRATION 75

If n distinct biscuits are distributed among N different beggars, what is the probability that a particular beggar gets exactly r biscuits ?

SOLUTION

Total no. of ways = N^n.

[The first biscuit can be given to any of the N beggars. Similarly, second biscuit can also be given to any of the N beggars and so on.]

Out of n biscuits, r biscuits can be given to a particular beggar in ${}^nC_r$ ways and the remaining $n-r$ biscuits can be distributed among the remaining $N-1$ beggars in $(N-1)^{n-r}$ ways. Favourable no. of ways is ${}^nC_r.\ (N-1)^{n-r}$. Hence, the required probability $=\dfrac{{}^nC_r.(N-1)^{n-r}}{N^n}$

ILLUSTRATION 76

A determinant of the second order is made with the elements 0 and 1, then find

(i) P(det is negative)

(ii) P(det is zero)

SOLUTION

The determinant is of the form, $\begin{vmatrix} - & - \\ - & - \end{vmatrix}$ each box can be filled in 2 ways (0 (or) 1)

$\therefore n(S)=16$ let E : det < 0

$$E=\left\{\begin{vmatrix}0&1\\1&0\end{vmatrix},\begin{vmatrix}0&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&0\end{vmatrix}\right\}\qquad \therefore P(E)=\frac{3}{16},$$

$$\text{det is zero}\ \begin{vmatrix}0&1\\0&0\end{vmatrix}\begin{vmatrix}0&0\\1&0\end{vmatrix}\begin{vmatrix}1&0\\0&0\end{vmatrix}\begin{vmatrix}0&0\\0&1\end{vmatrix}\begin{vmatrix}0&0\\0&0\end{vmatrix}\begin{vmatrix}1&1\\1&1\end{vmatrix}$$

$$\begin{vmatrix}1&1\\0&0\end{vmatrix}\begin{vmatrix}1&0\\1&0\end{vmatrix}\begin{vmatrix}0&0\\1&1\end{vmatrix}\begin{vmatrix}0&1\\0&1\end{vmatrix}$$

$$P(\text{det is zero})=\frac{10}{16}=\frac{5}{8}$$

ILLUSTRATION 77

Consider the system of equations $px+qy=0$, $rx+sy=0$ where $p, q, r, s \in \{-1, 1\}$

(i) Find probability that the system of equation has a unique solution

(ii) Find probability that the system of equation has no solution

SOLUTION

We have

$$S=\begin{vmatrix}p&q\\r&s\end{vmatrix}\ \text{as } p,q,r,s\in\{-1,1\}\ \therefore n(S)=16$$

(i) For unique solution, determinant should not be zero

$$E=\left\{\begin{vmatrix}1&-1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\-1&1\end{vmatrix},\begin{vmatrix}-1&1\\-1&-1\end{vmatrix},\begin{vmatrix}-1&-1\\1&-1\end{vmatrix},\begin{vmatrix}-1&1\\1&1\end{vmatrix},\begin{vmatrix}1&1\\1&-1\end{vmatrix},\right.$$

$$\left.\begin{vmatrix}1&-1\\-1&-1\end{vmatrix},\begin{vmatrix}-1&-1\\-1&1\end{vmatrix}\right\}$$

Hence probability that the system of equations has a unique solution $=\dfrac{n(E)}{n(S)}=\dfrac{8}{16}=\dfrac{1}{2}$

(ii) As we can see (0, 0) is always a solution of these equation so it will always have a solution irrespective of p, q, r, s

Hence probability that the system of equation has no solution = 0

ILLUSTRATION 78

Two integers R and S (not necessarily distinct) are chosen at random between 1 and 100 inclusively. Find the chance that the ratio $\frac{R}{S}$ is greater than unity.

SOLUTION

R and S each can be selected in 100 ways, given $\{1, 2, 3, \ldots, 99, 100\}$

Total ways $= 100 \cdot 100 = 10000$

For $\frac{R}{S} > 1$, then they both must be distinct

$^{100}C_2 =$ selecting two numbers

$$P\left(\frac{R}{S} > 1\right) = \frac{^{100}C_2}{100^2} = \frac{100 \cdot 99}{2(100^2)} = \frac{99}{200}$$

ILLUSTRATION 79

Three dice are thrown, the numbers appearing on them are respectively a, b, c. Find chance that the roots of the quadratic equation $ax^2 + bx + c = 0$ are

(i) real and equal

(ii) non real

SOLUTION

For real roots

$$b^2 - 4ac \geq 0 \Rightarrow \frac{b^2}{ac} \geq 4$$

b	(a, c)	fav. cases
2	(1, 1)	1
3	(1, 1), (1, 2), (2, 1)	3
4	(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2)	8
5	(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2)	14
6	(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (3, 2), (4, 2), (2, 3), (2, 4), (2, 2), (3, 3)	17
	Total	43

$P(\text{real roots}) = 43/216$

$P(\text{real and equal roots}) = 5/216$

5 cases

$\{a, b, c\} \equiv \{1, 2, 1\}, \{2, 4, 2\}, \{3, 6, 3\}, \{1, 4, 4\}, \{4, 4, 1\}$

P(non-real roots)

$$= 1 - \text{P(real roots)} = 1 - \frac{43}{216} = \frac{173}{216}$$

ILLUSTRATION 80

In rolling three fair dice, find the probability of getting

(i) the number on the dice that form an isosceles triangle, but not equilateral triangle

(ii) the numbers on the dice will not form a triangle

SOLUTION

(i) (x, y, z) form an isosceles triangle if two of them are equal and sum of two is greater than third.

(2, 2, 1) – 3 ways
(2, 2, 3) – 3 ways
(3, 3, 1 to 5 (but not 3)) – 3×4 ways
(4, 4, 1 to 6 (but not 4)) – 3×5 ways
(5, 5, 1 to 6 (but not 5)) – 3×5 ways
(6, 6, 1 to 5) – 3×5 ways

Total number of favourable cases $= 21 \times 3$

P(the number on the dice form an isosceles triangle)

$$= \frac{21 \times 3}{6^2} = \frac{7}{24}$$

(ii) (2, 3, 4) (2, 4, 5), (2, 5, 6), (3, 4, 5), (3, 4, 6), (4, 5, 6), (3, 5, 6) are all form a scalene triangle.

Hence total number of triangle formed by the points on the dices = number of scalene triangles + number of isosceles triangles + number of equilateral triangles $= 7 \times 6 + 21 \times 3 + 6 = 111$

$$\text{P(triangle will formed)} = \frac{111}{216}$$

P(the number on the dice will not form a triangle)

$$= 1 - \frac{111}{216} = \frac{35}{72}$$

ILLUSTRATION 81

Out of $3n$ consecutive integers, three are selected at random. Find the chance that their sum is divisible by 3.

SOLUTION

Let the sequence of numbers start with the integer m so that $3n$ consecutive integers are

$m, m + 1, m + 2, m + 3, \ldots\ldots., m + 3n - 1.$

Now, they can be classified ∞ →.

$m, m + 3, m + 6, \ldots\ldots\ldots., m + 3n - 3$

$m + 1, m + 4, m + 7, \ldots\ldots\ldots., m + 3n - 2.$

$m + 2, m + 5, m + 8, \ldots\ldots\ldots., m + 3n - 1.$

The sum of the three numbers shall be divisible by 3 if either all the three numbers are from the same row or all the three numbers are from different rows. The number of ways that the three numbers are from the same row is 3. $^nC_3$ and the number of ways that the number are from different rows, is $n \times n \times n = n^3$ since a number can be selected from each row in n ways.

Hence the favourable no. of ways $= 3.\ ^nC_3 + n^3$.

And the total number of ways $= {}^{3n}C_3$

$\therefore$ The required probability

$$= \frac{3.{}^{n}C_3 + n^3}{{}^{3n}C_3} = \frac{3n^2 - 3n + 2}{(3n-1)(3n-2)}$$

ILLUSTRATION 82

If 'p' is the probability that a man aged x years will die in a year, find the probability that out of n men A_1, A_2, ..., A_n each aged x years, A_1 will die in a year and will be the first to die.

SOLUTION

Since the probability that a man aged x years will die in a year is 'p', therefore the probability of that man not to die in a year $= 1 - p$. Hence, the probability that none out of n men aged x year will die in a year $= (1 - p)^n$. It follows that the probability that out of n men, at least one die in a year $= 1 - (1 - p)^n$.
Also, the probability that out of n men, the man to die first is $A_1 = 1/n$.

$\therefore$ The required probability $= \dfrac{1}{n}[1 - (1 - p)^n]$.

GEOMETRICAL PROBABILITY

As we understand the fundamental of probability, we can associate its application to geometrical problems also .

(i) Probability $= \dfrac{\text{Favourable length}}{\text{Total length}}$ (One-Dimension)

(ii) Probability $= \dfrac{\text{Favourable area}}{\text{Total area}}$ (Two-Dimension)

(iii) Probability $= \dfrac{\text{Favourable area}}{\text{Total area}}$ (Three-Dimension)

ILLUSTRATION 83

If a be chosen at random in the interval [0, 5], show that the probability that the equation $4x^2 + 4ax + (a + 2)$ 0, to have real roots is 3/5.

SOLUTION

Equation $4x^2 + 4ax + (a + 2) = 0$ will have real roots, if its discriminant $D \geq 0$

$\Rightarrow 16a^2 - 16(a + 2) \geq 0$

i.e., $a^2 - a - 2 \geq 0$ or $(a - 2)(a + 1) \geq 0$

$\therefore a \geq 2$ or $a \leq -1$

But in the interval [0, 5], $a > 2 \Rightarrow a \in [2, 5]$

$\therefore$ Required probability $= \dfrac{\text{length of interval } [2, 5]}{\text{length of interval } [0, 5]} = \dfrac{3}{5}$.

ILLUSTRATION 84

A real number a is picked at random from the interval [–6, 6]. What is the probability that it satisfies the inequality $x^2 + 2x + a^2 - 4a - 4 \geq 0\ \forall\ x \in R$?

SOLUTION

Here, $s = [-6, 6]$. Total length = 12

$x^2 + 2x + a^2 - 4a - 4 \geq 0\ \forall\ x \in R$.

if $b^2 - 4ac \leq 0$

i.e., $4 - 4(a^2 - 4a - 4) \leq 0 \quad \Rightarrow \quad (a - 5)(a + 1) \geq 0$

$\Rightarrow a \leq -1$ or $a \geq 5$

i.e., $a \in [-6, -1] \cup [5, 6]$

favourable length = 5 + 1 = 6

Hence, required probability $= \dfrac{6}{12} = \dfrac{1}{2}$

ILLUSTRATION 85

On a chess board if two squares of smallest size are chosen, what is the probability that they have a side in common?

SOLUTION

Let S be the sample space and E be the event that the two squares selected at random on a chess-board have a side in common.

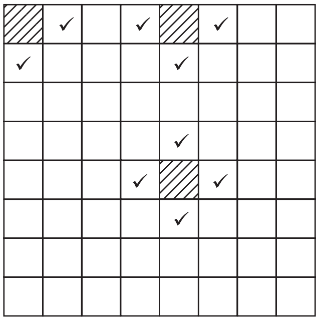
Now, $n(S)$ = total number of selections of 2 squares out of 64 squares on a chess board.

$$= {}^{64}C_2 = \frac{64 \times 63}{2} = 32 \times 63$$

$n(E)$ = number of ways of selecting two consecutive squares from rows or from columns
= (7 + 7 + ... to 8 terms) + (7 + 7 + ... to 8 terms)
= 56 × 2 = 112

Required probability $= \dfrac{112}{32 \times 63} = \dfrac{1}{18}$

Alternative:



n(E) = number of ways of selecting 2 squares on the chess board which have a side common

$= \frac{1}{2}[4 \times {}^2C_1$ (where one of the four corner squares are selected) $+ 24 \times {}^3C_1$ (When one of the edge square excluding corner square is selected) $+ 36 \times {}^4C_1$ (when one of the interior squares is selected)

$$= \frac{1}{2}.\,(4 \times 2 + 24 \times 3 + 36 \times 4) = 112$$

∴ Required probability

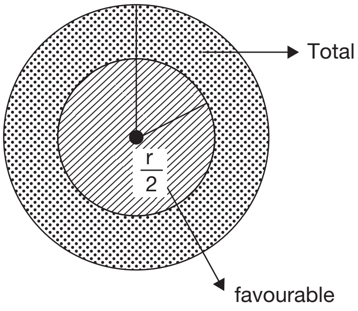
$$P(E) = \frac{n(E)}{n(S)} = \frac{112}{32 \times 63} = \frac{1}{18}$$

ILLUSTRATION 86

A point is selected at random insider a circle. Find the probability that the point is closer to the centre of the circle than its circumference.

SOLUTION

Let r be the radius of the circle. Consider a concentric circle of radius $\frac{r}{2}$. Any point inside the annular is closer to the centre than the circumference.



$$\text{Hence, } = \frac{\text{Area of inner circle}}{\text{Area of outer circle}} = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

ILLUSTRATION 87

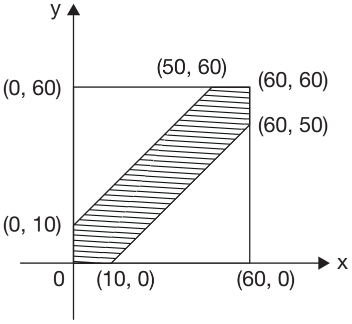
Ashu and Ravi make an appointment to meet on coming Sunday in London, between 7 PM to 8 PM and they have understanding that each is to wait not longer than ten minutes for the other. Assuming that each is independently equally likely to arrive at any time during the hour, find the probability that they meet.

SOLUTION

Denote the times of arrival of Ashu by x and of Ravi by Y. For the meeting to take place it is necessary and sufficient that $|x - y| < 10$

We depict x and y as Cartesian coordinates in the plane: for the scale unit we take one minute $0 < x < 60$; $0 < y < 60$ total conditions

Favourable condition $|x - y| < 10$



$$\text{Required Probability} = \frac{\text{shaded area}}{\text{Total square area}}$$

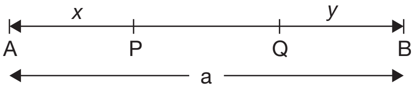
$$= \frac{60 \times 60 - 2\left(\frac{1}{2}50(50)\right)}{60 \times 60} = \frac{60^2 - 50^2}{60^2} = \frac{11}{36}$$

ILLUSTRATION 88

A line is divided at random into three parts. What is the probability that they form the sides of a triangle?

SOLUTION

Let AP $= x$, BQ $= y$ and AB $= a$, PQ $= a - x - y$



Since the sum of two sides of a triangle is greater than the third side,

$$AP + PQ > QB \Rightarrow x + a - x - y > y \Rightarrow y < \frac{a}{2}$$

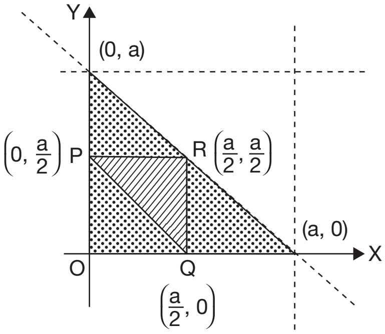
$$QB + PQ > AP \Rightarrow y + a - x - y > x \Rightarrow x < \frac{a}{2}$$

$$AP + QB > PQ \Rightarrow x + y > a - x - y \Rightarrow x + y > \frac{a}{2}$$

For all possible cases of dividing the line,

$$0 < x < a,\ 0 < y < a \text{ and } 0 < x + y < a \qquad \text{...(2)}$$

Condition (2) corresponds to the triangular region OXY and condition (1) corresponds to the triangular region PQR.



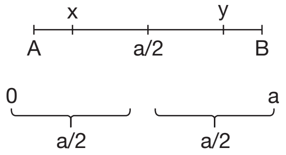
$$p = \frac{\text{Area of } \Delta\text{ABC}}{\text{Area of } \Delta\text{OXY}} = \frac{\frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}{\frac{1}{2}(a)(a)} = \frac{1}{4}.$$

ILLUSTRATION 89

Two points are selected at random on a line AB of length a so as to lie on the opposite sides of its mid point. Find the probability that the distance between them is less than $\frac{a}{3}$.

SOLUTION

Let $x \in [0, a/2], y \in [a/2, a]$ or vice-versa

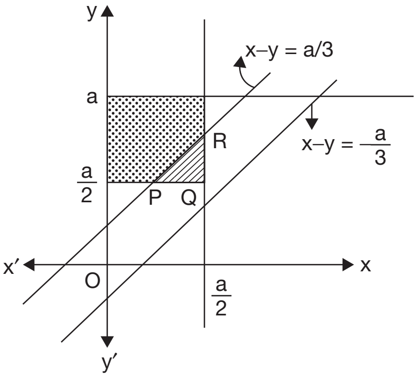


Favourable condition $|x-y| < a/3$

$$\Rightarrow -\frac{a}{3} < x-y < \frac{a}{3}$$

$$\Rightarrow x-y-\frac{a}{3} < 0 \text{ and } x-y+\frac{a}{3} > 0$$

$$Q\left(\frac{a}{2}, \frac{a}{2}\right) \; P\left(\frac{5a}{6}, \frac{a}{2}\right) R\left(\frac{a}{2}, \frac{5a}{6}\right)$$



$$\text{Required probability} = \frac{\text{Area of } \Delta PQR}{\text{Area of}}$$

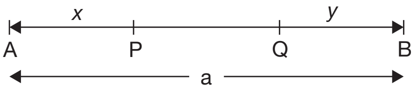
$$= \frac{\frac{1}{2} PQ\, QR}{\left(\frac{a}{2}\right)^2} = \frac{\frac{1}{2} \cdot \frac{a}{3} \cdot \frac{a}{3}}{\frac{a^2}{4}} = \frac{2}{9}$$

ILLUSTRATION 90

On a straight line of length a two points are taken at random. Find the chance that no part is greater than *b*.

SOLUTION

Let $AB = a$, $AP = x$ and $BQ = y$.



For favourable cases,

$$0 < x < b, 0 < y < b \text{ and } 0 < a-x-y < b$$

$$\Rightarrow \quad a-b < x+y < b \qquad ...(1)$$

For all possible cases,

$$0 < x < a, 0 < y < a, 0 < a-x-y < a$$

$$\Rightarrow \quad 0 < x+y < a \qquad ...(2)$$

The ΔOXY where X = $(a, 0)$, Y = $(0, a)$ represents all cases.

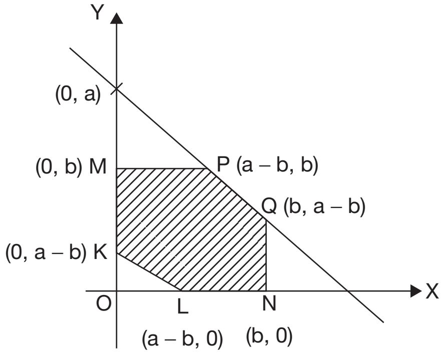
Let M be $(0, b)$, N be $(b, 0)$ and equation of KL be $x+y=a-b$.

Then the region KLNQPM represents the favourable cases.

Two cases arise.

Case I: $b > \frac{a}{2}$

The conditions of (1) are represented by the region KLNQPM in figure and hence



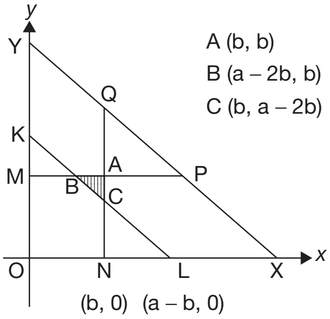
since OL = $a-b$, NX = $a-b$ and MY = $a-b$

$$p = \frac{\frac{1}{2}\{a^2 - 3(a-b)^2\}}{\frac{1}{2}a^2}$$

$$= 1 - 3\left(\frac{a-b}{a}\right)^2$$

Case II: $b < \frac{a}{2}$

$$ON = b < \frac{a}{2}; OM = b < \frac{a}{2}$$



In this case, the favourable conditions correspond to the shaded region ABC

CA = NA – NC

$= b - (a - 2b)$, since (ON, NC) satisfies the equation $x + y = a - b$

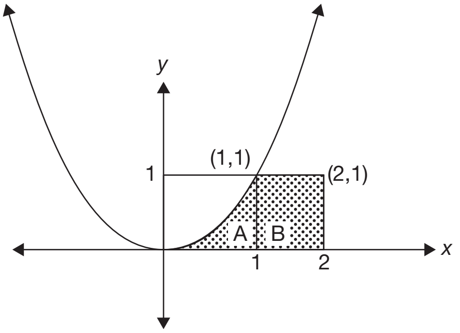
$= 3b - a$

$$\therefore \text{ probability} = \frac{\frac{1}{2}\text{CA} \times \text{BA}}{\frac{1}{2}\text{OX} \times \text{OY}} = \frac{\frac{1}{2}(3b-a)^2}{\frac{1}{2}a^2} = \frac{(3b-a)^2}{a^2}$$

ILLUSTRATION 91

If x and y are chosen randomly from the intervals [0, 2] and [0, 1] respectively, then the probability that $y \le x^2$ is 2/3.

SOLUTION



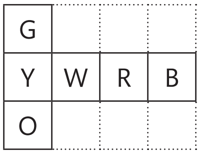
$$\text{Where A} = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}; \text{B} = |x| = 1$$

$$\text{Required Probability} = \frac{\frac{1}{3}+1}{2\times 1} = \frac{2}{3}$$

Test Your Concepts-III

(Solutions on page H.215)

1. Events A and C are independent. If the probabilities relating A, B and C are P(A) = 1/5, P(B) = 1/6, P(A ∩ C) = 1/20 and P(B ∪ C) = 3/8, then prove that events B and C are independent.
2. A bag contains 4 red and 3 blue balls. Two balls are drawn twice. If the drawn balls are replaced before the second draw, then find the probability that first two balls are red and second two balls are blue.
3. The probabilities of happening of three independent events are p_1, p_2 and p_3, then find the probability of happening atleast one of them.
4. Four persons independently solve a certain problem correctly with probabilities 1/2, 3/4, 1/4, 1/8 then find the probability that the problem is solved correctly by atleast one of them.
5. A man can kill a bird once in three shots on this assumption he fires three shots then find the probability that the bird is killed.
6. The odds against solving a problem by A are 4 : 3 and in favour of b are 7 : 5, then find the probability that only one will solve the problem.
7. A butterfly randomly lands on one of the six squares of T-shaped figure shown and then randomly moves to an adjacent square then find the probability that the butterfly ends up on the R square.

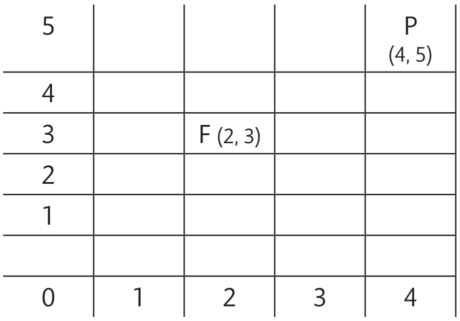


8. If all the mappings that can be defined from the set A : {1, 2, 3, 4} → B : {5, 6, 7, 8, 9} a mapping is randomly selected then find the chance that the selected mapping is strictly monotonic.
9. The natural number X is randomly selected from the set of first 100 natural numbers. Then, the probability that it satisfies the inequality $x + \frac{100}{x} > 50$ is $\frac{m}{n}$, then find $|m - n|$.
10. 5 different marbles are placed in 5 different boxes randomly. Then, the probability that exactly two boxes remain empty is $\frac{m}{n}$, then find $(2m - n)$.
11. Two points are selected at random on a line of length 'a'. What is the probability that none of three sections in which the line is thus divided is less than $a/4$?
12. A bag contains 4 red and 4 blue balls. Four balls are drawn one-by-one, from the bag, then find the probability that the drawn balls are in alternate colour.
13. The number a is randomly selected from the set {0, 1, 2, 3, ..., 98, 99}. The number b is selected from the same set

find the probability that the number $3^a + 7^b$ has a digit equal to 8 at the unit place.

14. Two dice are thrown thrice. Find the probability that first throw shows 10, second 11 and third 12.

15. Mr. A lives at origin on the Cartesian plane and has office at (4, 5). His friend lives at (2, 3) on the same plane (as shown in figure). Mr. A can go his office travelling one block at a time either in the + y or + x direction. If all possible paths are equally likely, then find the probability that Mr. A passed by his friends house.



Answer keys

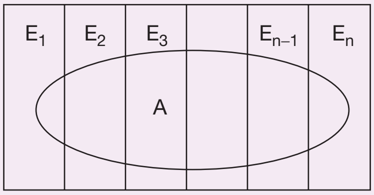
2. $\frac{2}{49}$ **3.** $1-(1-p_1)(1-p_2)(1-p_3)$ **4.** $\frac{235}{256}$ **5.** $\frac{19}{27}$ **6.** $\frac{43}{84}$

7. $\frac{1}{4}$ **8.** $\frac{2}{125}$ **9.** 9 **10.** 1 **11.** $\frac{1}{16}$ **12.** $\frac{6}{35}$

13. $\frac{3}{16}$ **14.** $\frac{1}{7776}$ **15.** $\frac{10}{21}$

TOTAL PROBABILITY & BAYES' THEOREM (INVERSE PROBABILITY)

If $E_1, E_2, \ldots\ldots\ldots, E_n$ be n mutually exclusive and exhaustive events associated to a random experiment. A be any event related to these n events.



Total probability for event A to occur

$$= P(A) = \sum_{i=1}^{n} P(E_i).P(A/E_i) \qquad \ldots(1)$$

$$= P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + \ldots P(E_k).P(A/E_k) + \ldots + P(E_n).P(A/E_n)$$

Lets consider that event A has occurred and it belongs to event E_k

Then required probability

$$P(E_k/A) = \frac{P(E_k).P(A/E_k)}{\sum_{i=1}^{n} P(E_i).P(A/E_i)} \qquad \ldots(2)$$

This result (2) is **Bayes' Theorem (Inverse Probability)**

Baye's Theorem or Total probability can be easily understood by visualising a tree structure, that we will learn from illustrations

ILLUSTRATION 92

A is one of the 6 horses entered for a race, and is to be ridden by one of the two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win; if C rides A, his chance is tripled; what are the odds against his winning?

SOLUTION

Let E_1 be the event that B rides A, E_2 be the event that C rides A and E the event that A wins. Then, according to the question,

$$P(E_1) = \frac{2}{3}, \qquad P(E_2) = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P(E/E_1) = \frac{1}{6}$$

Since, all the 6 horses are equally likely to win when B rides A

$$P(E/E_2) = 3 \times \frac{1}{6} = \frac{1}{2}$$

Since, A's chance of winning is trebled when C rides A.

$$\therefore\ P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2).$$

$$= \frac{2}{3}.\frac{1}{6} + \frac{1}{3}.\frac{1}{2} = \frac{5}{18}$$

Hence, probability of A's win is 5/18 so that odds against A's win are as 18–5 : 5, that is 13 : 5.

ILLUSTRATION 93

A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at a random one by one without replacement and tested till all the defective articles are found. What is the probability that the testing procedure ends at the twelfth testing?

SOLUTION

Let A_1 be the events that the lot contains 2 defective articles and A_2 be the event the lot contains 3 defective articles. Also let A be the event that the testing procedure ends at the twelfth testing. Then according to the question;

$$P(A_1) = 0.4 \quad \text{and} \quad P(A_2) = 0.6$$

Since, $0 < P(A_1) < 1, 0 < P(A_2) < 1,$

and $P(A_1) + P(A_2) = 1$

the events A_1, A_2 form a partition of the sample space. Hence by the theorem of total probability for compound events, we have

$$P(A) = P(A_1)\, P(A/A_1) + P(A_2)\, P(A/A_2) \qquad \ldots(1)$$

$$\text{But} \quad P(A/A_1) = \left(\frac{{}^{18}C_{10} \times {}^{2}C_1}{{}^{20}C_{11}}\right) \times \left(\frac{1}{9}\right) \qquad \ldots(2)$$

$$= \frac{2}{9} \times \frac{18!}{10!\,8!} \times \frac{11! \times 9!}{20!} = \frac{11}{190}$$

Note that out of 20 articles, first 11 draws must contain 10 non-defective and 1 defective article and the 12th draw must give a defective article.

$$\text{Similarly, } P(A/A_2) = \frac{{}^{17}C_9 \times {}^{3}C_2}{{}^{20}C_{11}} \times \frac{1}{9}$$

$$= \frac{1}{3} \times \frac{17!}{9!\,8!} \times \frac{11!\,9!}{20!} = \frac{11}{228}$$

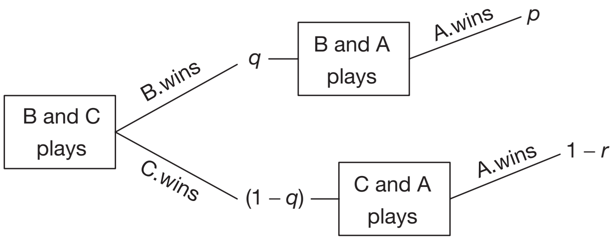
$$P(A) = 0.4 \times \frac{11}{190} + 0.6 \times \frac{11}{228}$$

$$= \frac{11}{475} + \frac{11}{380} = \frac{99}{1900}$$

ILLUSTRATION 94

In a table tennis singles, two players play, and one of them must win. Probability that A beats B is p, B beats C is q and C beats A is r. If B plays C and then the winner plays with A, then find the probability that A will be the final winner.

SOLUTION



P(A is final winner) = $qp + (1 - q)(1 - r)$

ILLUSTRATION 95

A bag contains $(n + 1)$ coins. It is known that one of these coins shows head on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is 7/12, then find the value of n.

SOLUTION

Let E_1, denotes the event that a coin with the heads is selected and E_2 denotes the event that a fair coin is selected. Let A be the event that the toss results in heads. Then,

$$P(E_1) = \frac{1}{n+1},\ P(E_2) = \frac{n}{n+1},\ P\left(\frac{A}{E_1}\right) = 1;\ P\left(\frac{A}{E_2}\right) = \frac{1}{2}$$

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)$$

$$\Rightarrow \quad \frac{7}{12} = \frac{1}{n+1} \times 1 + \frac{n}{n+1} \times \frac{1}{2}$$

$$\Rightarrow \quad 12 + 6n = 7n + 7 \Rightarrow n = 5.$$

ILLUSTRATION 96

A box contains 100 balls. All number of white or non-white balls in the box are equally probable. A white ball is dropped into the box and the box is shaken. Now a ball is drawn from the box. Find the probability that the drawn ball is white.

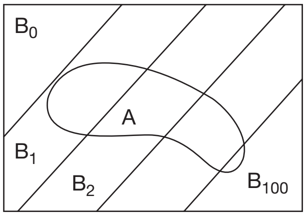
SOLUTION

Originally the number of white balls in the bag vary from 0 to 100.

Initially

B_0 :	Bag has no W ball
B_1 :	Bag has 1 W ball
B_2 :	Bag has 2 W ball
⋮	
B_{100} :	Bag has 100 W ball

$$P(B_0) = P(B_1) = P(B_{100}) = \frac{1}{101}$$



After the white has been dropped in the bag

$A = A \cap B_0 + A \cap B_1 + \ldots + B_{100}$

$P(A) = P(B_0) \cdot P(A/B_0) + P(B_1) + P(A/B_2) + \ldots + P(B_{100}) \cdot P(A/B_{100})$

$$= \frac{1}{101}\left[\frac{1}{101} + \frac{2}{101} + \ldots + \frac{101}{101}\right]$$

$$= \frac{1}{101}\,\frac{1}{101}\left[\frac{(101)(102)}{2}\right] = \frac{51}{101}$$

ILLUSTRATION 97

In a test Akshay either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he make a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$.The probability that his answer is correct given that he copied it, is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered it.

SOLUTION

Let A_1, A_2, A_3 be the events that Akshay guesses copies or know the answer respectively

A : Akshay answers correctly.

according to question, $P(A_1) = \frac{1}{3}; P(A_2) = \frac{1}{6}$

As any one happens out of A_1, A_2, A_3 these are mutually exclusive and exhaustive events.

$\therefore \quad P(A_1) + P(A_2) + P(A_3) = 1$

$\Rightarrow \quad P(A_3) = \frac{1}{2}$

Also we have $P(A/A_1) = \frac{1}{4}$

[∵ out of 4 choices only one is correct].

$P(A/A_2) = \frac{1}{8}$

[If Akshay knows the answer, it is correct]

$P\left(\frac{A}{A_3}\right) = 1$

By Bayes' theorem,

$P(A_3/A)$

$$= \frac{P\left(\frac{A}{A_3}\right)P(A_3)}{P\left(\frac{A}{A_1}\right)P(A_1) + P\left(\frac{A}{A_2}\right)P(A_2)\cdot P\left(\frac{A}{A_3}\right)P(A_2)}$$

$$= \frac{1\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{3} + \frac{1}{8}\cdot\frac{1}{6} + 1\cdot\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{29}{48}}$$

$\Rightarrow \quad P(A_3/A) = \frac{24}{29}$

ILLUSTRATION 98

Gauri's gardener is not dependable, the probability that he will forget to water the rose bush is 2/3. The rose bush is in questionable condition, anyhow, if watered, the probability of its withering is 1/2, if not watered, the probability of its withering is 3/4. Gauri went out of station and upon returning, she finds that the rose bush has withered. What is the probability that the gardener did not water the bush?

SOLUTION

(Here result is known that the rose bush has withered. Therefore bayes' theorem should be used)

Let

A = the event that the rose bush has withered.

A_1 = the event that the gardener did not water.

A_2 = the event that the gardener watered.

By Bayes' theorem, required probability,

$$P(A_1/A) = \frac{P(A_1)P(A/A_1)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2)} \qquad \ldots(1)$$

Given, $P(A_1) = \frac{2}{3} \quad \therefore\ P(A_2) = \frac{1}{3}$

$P\ (A/A_1) = \frac{3}{4}, P(A/A_2) = \frac{1}{2}$

From (1), $P(A_1/A) = \dfrac{\frac{2}{3}\cdot\frac{3}{4}}{\frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{3}{4}$

ILLUSTRATION 99

An unbiased coin is tossed. If the result is head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3 ..... 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?

SOLUTION

Let E_1 be the event "toss results in a head",

E_2 be the event "toss results in a tail"

A be the event "the noted number is 7 or 8"

We have $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A/E_1) = P(7) + P(8) = \frac{6}{36} + \frac{5}{36} = \frac{11}{36}$

Since $7 = \{1+6, 2+5, 3+4, 4+3, 5+2, 6+1\}$ and

$8 = \{2+6, 3+5, 4+4, 5+3, 6+2\}$

$P(A/E_2) = \frac{2}{11}$

We know that $P(A)=P(A/E_1)\times P(E_1)+P(A/E_2)$

$$=\frac{11}{36}\times\frac{1}{2}+\frac{2}{11}\times\frac{1}{2}=\frac{193}{792}$$

ILLUSTRATION 100

Bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B.

SOLUTION

Let the event of drawing a red ball be denoted by L and the event of choosing bag A be denoted by M and the event of choosing bag B be denoted by N.

We have $P(M)=P(N)=\frac{1}{2}$

Now $P(L/M)$ = Probability of drawing a red ball from bag A $=\frac{3}{5}$ and $P(L/N)=\frac{5}{9}$.

Required probability $=P(N/L)$

$$=\frac{P(N).P(L/N)}{P(N).P(L/N)+P(M).P(L/M)} \text{ (Bayes' formula).}$$

$$=\frac{\frac{1}{2}.\frac{5}{9}}{\frac{1}{2}.\frac{5}{9}+\frac{1}{2}.\frac{3}{5}}=\frac{25}{52}$$

ILLUSTRATION 101

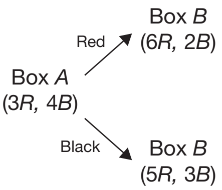
Box A contains 3 red chips and 4 black chips. Box B contains 5 red chips and 2 black chips. A chip is drawn randomly from box A and placed in B, and then a chip is randomly drawn from box B. Find the probability that the chip drawn from box B is black.

SOLUTION

E_1 : chips from A to B is red

E_2 : chips from A to B is black

E : chips drawn from B to A is black



$P(E) = P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2)$

$$=\frac{3}{7}\cdot\frac{2}{8}+\frac{4}{7}\cdot\frac{3}{8}=\frac{18}{56}=\frac{9}{28}$$

ILLUSTRATION 102

There are 18 hunters which are divided into four groups as,

G_1: Consisting of 5 hunters each of whom can hit a target with probability 0.8.

G_2: Consisting of 7 hunters each hitting a target with probability 0.7.

G_3: Consisting of 4 hunters each hitting a target with probability 0.6.

G_4: Consisting of 2 hunters each hitting a target with probability 0.5. A randomly selected hunter fires a shot and failed to hit the target. Find most probable group of the hunter.

SOLUTION

$$P(G_1)=\frac{5}{18};P(G_2)=\frac{7}{18};P(G_3)=\frac{4}{18};P(G_4)=\frac{2}{18}$$

F : hunter failed to hit the target

$P(F) = P(G_1)\cdot P(F/G_1) + P(G_2) \cap F)\cdot P(F/G_2) + P(G_3)\cdot P(F/G_3) + P(G_4)\cdot P(F/G_4)$

$$=\frac{5}{18}\cdot\left(\frac{2}{10}\right)+\frac{7}{18}\left(\frac{3}{10}\right)+\left(\frac{4}{18}\right)\left(\frac{4}{10}\right)+\frac{2}{18}\left(\frac{5}{10}\right)$$

$$=\frac{10+21+16+10}{180}=\frac{57}{180}$$

$$P(G_1/F)=\frac{10}{57};P(G_2/F)=\frac{21}{57};P(G_3/F)=\frac{16}{57};P(G_4/F)=\frac{10}{57}$$

Hence, most probably group of hunter is G_2

ILLUSTRATION 103

A is one among the 8 horses in a race. A is to be ridden by one of the 3 jockeys P, Q, R. If P rides A all the horses are equally likely to win, if Q rides A his chances are doubled and if R rides A his chances are tripled. A die is thrown if 1 or 2 or 3 appears then P rides A, if 4 or 5 appears then Q rides A other-wise R rides A. Then find the probability that A wins.

SOLUTION

Let E_1, E_2, E_3 be respectively Events that P, Q, R ride the horse A.

A = Event that horse A win the race

$$P(E_1)=\frac{1}{2};P(E_2)=\frac{1}{3};P(E_3)=\frac{1}{6}$$

$$P\left({}^{A}\!/_{E_1}\right)=\frac{1}{8};P\left({}^{A}\!/_{E_2}\right)=\frac{2}{8};P\left({}^{A}\!/_{E_3}\right)=\frac{3}{8}$$

$$P(A)=P(E_1)P\left({}^{A}\!/_{E_1}\right)+P(E_2)P\left({}^{A}\!/_{E_2}\right)+P(E_3)P\left({}^{A}\!/_{E_3}\right)$$

$$=\frac{1}{2}\times\frac{1}{8}+\frac{1}{3}\times\frac{2}{8}+\frac{1}{6}\times\frac{3}{8}=\frac{5}{24}$$

ILLUSTRATION 104

In a factory manufacturing bolts; machines A, B and C manufacture respectively 20%, 30% and 50% of the total production. And from their outputs, 2%, 3% and 5% are defective bolts respectively. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine C?

SOLUTION

Let A, B, C be the events that a bolt selected at random is manufactured respectively by machines A, B and C respectively so that $P(A) = 2/10$; $P(B) = 3/10$ and $P(C) = 5/10$.

Let D be the event that the bolt selected at random is defective, so that

$P(D/A) = 2/100$; $P(D/B) = 3/100$; $P(D/C) = 5/100$;

$$P(C/D) = \frac{P(D/C).P(C)}{P(D/A).P(A) + P(D/B).P(B) + P(D/C).P(C)}$$

$$= \frac{\frac{5}{10}\cdot\frac{5}{100}}{\frac{2}{10}\times\frac{2}{100}+\frac{3}{10}\cdot\frac{3}{100}+\frac{5}{10}\cdot\frac{5}{100}} = \frac{5\times 5}{4+9+25} = \frac{25}{38}$$

Testimony Probability

If p_1 and p_2 are the probabilities of speaking the truth of two independent witnesses A and B and they both assert of giving same statement then

P(their combined statement is true)

$$= \frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)}.$$

In this case it has been assumed that we have no knowledge of the event except the statement made by A and B.

If p is the probability of the happening of a certain event before their statement and they both assert of giving same statement then

P(their combined statement is true)

$$= \frac{pp_1p_2}{pp_1p_2 + (1-p)(1-p_1)(1-p_2)}$$

Here it has been assumed that the statement given by all the independent witnesses can be given in two ways only, so that if all the witnesses tell falsehoods they agree in telling the same falsehood.

If this is not the case and c is the chance of their coincidence testimony then the

Probability that the statement is true = pp_1p_2

Probability that the statement is false = $(1 - p).\ c(1 - p_1)(1 - p_2)$

Here c stands for **chance factor** for telling same lie if there more than one way of falsehood

However chance of coincidence testimony is taken only if the joint statement is not contradicted by any witness.

ILLUSTRATION 105

The probability that A speaks truth is 4/5 while this probability for B is 3/4. Find the probability that they contradict each other when asked to speak on a fact.

SOLUTION

Let E be the event that B speaks truth and F be the event that A speaks truth.

Now $P(E) = \frac{3}{4}$ and $P(F) = \frac{4}{5}$

∴ P(A and B contradict each other)

= P[(B tells truth and A tells lie) or (B tells lie and A tells truth)]

$= P[(E \cap \bar{F}) \cup (\bar{E} \cap F)] = P(E).P(\bar{F}) + P(\bar{E}).P(F)$

$= \frac{3}{4}\times\frac{1}{5}+\frac{1}{4}\times\frac{4}{5} = \frac{7}{20}$

ILLUSTRATION 106

A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six.

SOLUTION

Let E denote the event that a 6 occurs and A be the event that the man reports that it is a 6 we have

$P(E) = \frac{1}{6}$, $P(E') = \frac{1}{4}$; $P(T) = \frac{3}{4}$; $P(F) = \frac{1}{4}$

Where P(A/E) = probability that the man reports that there is a six on the die given that 6 has occurred on the die.

$= \frac{1}{6}\cdot\frac{3}{4}$

$P\left(A/\bar{E}\right) = \frac{5}{6}\cdot\frac{1}{4}\cdot\frac{1}{5}$

where $\frac{1}{5}$ is chance factor of telling 1 lie out of 5 lies

now, $P(E/A) = \frac{\frac{1}{6}\cdot\frac{3}{4}}{\frac{1}{6}\cdot\frac{3}{4}+\frac{5}{6}\cdot\frac{1}{4}\cdot\frac{1}{5}} = \frac{3}{4}$

ILLUSTRATION 107

A speaks the truth is 3 times out of 4 and B speaks the truth 2 times out of 3. A dice is thrown. Both assert that the number turned up is 1. Find the probability of the truth of their assertion?

SOLUTION

Let A and B be the events that A speaks the truth and B speaks the truth respectively. Let C be the event the number turned up is not 1 but both agree to the some conclusion that the die has turned up.

Then $P(A) = \frac{3}{4}, P(B) = \frac{2}{3}$ and $P(C) = \frac{1}{5} \cdot \frac{1}{5}$

There are two hypothesis:

(i) The dice turns up 1

(ii) The dice does not turn up 1

Let these events E_1 and E_2 respectively, then

$$P(E_1) = \frac{1}{6}, P(E_2) = \frac{5}{6}$$

Now, let E be the event that statements made by A and B come to the same conclusion.

Then $P(E/E_1) = P(A)\,P(B) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$ and

When 1 has not turned up then there are 5 ways to tell lie so A and B can die in 25 ways hence they said that 1 has appeared so chance factor $= \frac{1}{25}$

$$P(E/E_2) = P(\bar{A})\,P(\bar{B})\,P(C) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{25} = \frac{1}{300}$$

Thus, $P(E) = P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)$

$$= \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{25} = \frac{31}{360}$$

$$\therefore \quad P(E_1/E) = \frac{\frac{1}{6} \cdot \frac{3}{4} \cdot \frac{2}{3}}{\frac{31}{360}} = \frac{30}{31}$$

Test Your Concepts-IV

(Solutions on page H.217)

1. A purse contains 4 copper 3 silver coins another purse contains 6 copper and 2 silver coins. One coin is drawn from any one of these two purses find the probability that it is a copper coin.
2. Two cubes have their faces painted either red or blue. The first cube has five red faces and one blue face. When the two cubes are rolled simultaneously, then probability that the two top faces show the same colour is (1/2). Find number of red faces on the second cube.
3. Suppose families always have one, two or three children, with probabilities 1/4, 1/2 and 1/4, respectively. Assume everyone eventually gets married and has children, then find probability of a couple having exactly four grand children.
4. Sixteen players $S_1, S_2, ..., S_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair, a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength then find probability that 'exactly one of the two players S_1 and S_2 is among the eight winners'.
5. The probability of a number n showing in a throw of a die marked 1 to 6 is proportional to n then find the probability of the number 3 showing in a throw.
6. A bag contains 3 red and 3 green balls and a person draws out 3 at random. Then, he drops 3 blue balls into the bag and again draws out 3 at random find the chance that the 3 later balls being all of different colours.
7. Let A consists of 3 good and 2 defective articles. Let B consists of 4 good and 1 defective article. A new to C is formed by taking 3 articles from A and 2 from B, find the probability that an article chosen at random form C is defective.
8. A real estate man has eight master keys to open several new houses. Only one master key will open a given house. If 40% of these houses are usually unlocked then find probability that the real estate man can get into a specific houses, if he selects three master keys at random.
9. A bowl has 6 red marbles and 3 green marbles then find the probability that a blind folded person will draw a red marble on the second draw from the bowl without replacing the marble from the first draw.
10. A bag A contains 2 white balls, 3 red balls and bag B contains 4 white balls, 5 red balls. One ball is drawn at random from a randomly chosen bag and is found to be red then find probability that it was drawn from bag B.
11. Box I contains 5 red and 2 blue balls, while box II contains 2 red and 6 blue balls. A fair coin is tossed. If it turns up head, a ball is drawn from box I, else a ball is drawn from box II then find the probability ball drawn is from box I, if it is blue.
12. A letter is known to have come either from 'KRISHNAGIRI' or 'DHARMAPURI'. On the post mark only the two consecutive letters 'RI' are visible then find the chance that it came from 'KRISHNAGIRI'.
13. A purse contains 100 coins of unknown value, a coin drawn at random is found to be a rupee then find the chance that it is the only rupee in the purse. (Assume all numbers of rupee coins in the purse to be equally likely)
14. A and B are two inaccurate arithmeticians whose chance of solving a given question correctly are 1/8 and 1/12, respectively then find probability that they solve a problem and obtained the same result. If it is 1000 to 1 against their making the same mistake.

Answer keys

1. $\frac{37}{56}$
2. 3
3. $\frac{27}{128}$
4. $\frac{8}{15}$
5. $\frac{1}{7}$
6. $\frac{27}{100}$
7. $\frac{8}{25}$
8. $\frac{5}{8}$
9. $\frac{2}{3}$
10. $\frac{27}{52}$
11. $\frac{8}{29}$
12. $\frac{9}{14}$
13. $\frac{1}{5050}$
14. $\frac{13}{14}$

BINOMIAL PROBABILITY

Suppose that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. Each toss or selection is called a trial. In any single trial there will be a probability associated with a particular event such as head on the coin, 4 on the die, or selection of a red marble. In some cases this probability will not change from one trial to the next (as in tossing a coin or die). Such trials are then said to be independent and are often called Bernoulli trials.

The probability of getting exactly r success in n independent trials is given by

$$P(r) = {}^nC_r\, p^r\, q^{n-r}$$

where p = probability of success in a single trial

q = probability of failure in a single trial

Note that $p + q = 1$

Remark(s)

When outcome of a trial are not changing with respect to number of times experiment is being conducted then it belongs to binomial probability

ILLUSTRATION 108

8 coins are tossed simultaneously. Find the probability of getting at least 6 heads.

SOLUTION

$$\text{P(head)} = \frac{1}{2};\quad q = \frac{1}{2}$$

The required probability = P(Exactly 6 heads) + P(Exactly 7 heads) + P(Exactly 8 heads)

$$= {}^8C_6\left(\frac{1}{2}\right)^6\cdot\left(\frac{1}{2}\right)^2 + {}^8C_7\left(\frac{1}{2}\right)^7\cdot\left(\frac{1}{2}\right) + {}^8C_8\left(\frac{1}{2}\right)^8$$

$$= \frac{37}{256}.$$

ILLUSTRATION 109

An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained. What is the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5?

SOLUTION

P(minimum face value is not less than 2 and maximum face value is not greater than 5)

$$= \text{P(2 or 3 or 4 or 5)} = \frac{4}{6} = \frac{2}{3}.$$

Hence required probability $= {}^4C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{16}{81}.$

ILLUSTRATION 110

One hundred identical coins each with probability p of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then find the value of p.

SOLUTION

We have ${}^{100}C_{50}p^{50}(1-p)^{50} = {}^{100}C_{51}p^{51}(1-p)^{49}$

or $\frac{1-p}{p} = \frac{100!}{51!\,49!}\times\frac{50!\,50!}{100!} = \frac{50}{51}$ and $51 - 51p = 50p$

$$\Rightarrow p = \frac{51}{101}.$$

ILLUSTRATION 111

A coin has probability 'P' of showing head when tossed. It is tossed n times. Let P_n denotes the probability that no two (or more) consecutive heads occur. Find the value of P_4.

SOLUTION

Case I: All 4 tails T T T T $P(H) = p$; $P(T) = 1 - p$

Probability $= (1 - p)^4$

Case II: 3 tails + 1 head (4 cases)

(T T T H, T T H T, T H T T, H T T T)

Probability $= 4p\,(1 - p)^3$

Case III: 2 tails + 2 heads

T H T H, H T H T, H T T H

Probability $= 3p^2 (1-p)^2$

Hence, total probability $= (1-p)^4 + 4p(1-p)^3 + 3p^2(1-p)^2$

$= (1-p)^2 [(1-p)^2 + 4p(1-p) + 3p^2]$

$= (1-p)^2 [1 - 2p + p^2 + 4p - 4p^2 + 3p^2]$

$= (1-p)^2 [1 + 2p]$

$= (1 - 2p + p^2)(1 + 2p)$

$= 1 - 4p^2 + p^2 + 2p^3 = 1 - 3p^2 + 2p^3$

ILLUSTRATION 112

Ten fair coins are tossed. Find the probability of getting:

(i) 5 heads, 5 tails comes alternately.

(ii) more number of heads on the first five coins than on the last five coins.

SOLUTION

(HTHTHTHTHT or THTHTHTHTH)

(i) P(5 heads, 5 tails comes alternately)

$$= 2\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{2^9}$$

(ii)

number of H's on 1st five coins	number of H's on last five coins
5	0 or 1 or 2 or 3 or 4
4	0 or 1 or 2 or 3
3	0 or 1 or 2
2	0 or 1
1	0

Here we have to count case for heads in first 5 tosses and last 5 tosses as mentioned in table.

Required probability =

$$\frac{{}^5C_5\sum_{r=0}^{4}{}^5C_r + {}^5C_4\sum_{r=0}^{3}{}^5C_r + {}^5C_3\sum_{r=0}^{2}{}^5C_r + {}^5C_2\sum_{r=0}^{1}{}^5C_r + {}^5C_1}{2^{10}}$$

ILLUSTRATION 113

A coin that comes up head with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip, is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Find p.

SOLUTION

We have ${}^8C_3\, p^3 (1-p)^5 = \frac{1}{25}\, {}^8C_5\, p^5 (1-p)^3$

or $1 - p = \frac{(1)(p)}{5}$

So, we find $p = \frac{5}{6}$

ILLUSTRATION 114

Suppose A and B are two equally strong table tennis players. Which of the following two events is more probable?

(A) A beats B in exactly 3 games out of 4 or

(B) A beats B in exactly 5 games out of 8

SOLUTION

Since A and B are equally strong players, the probability that A beats B is 1/2 and the probability that A loses to B is also 1/2.

(A) A beats B in exactly three games out of 4.
In this case, A has to lose one game and win three for which the probability is ${}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 = \frac{1}{4}$

(B) A beats B in exactly 5 games out of 8.
In this case, A has to lose 3 games out of 8.

$\Rightarrow$ Probability $= {}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 = \frac{7}{32}$

Now $\frac{1}{4} = \frac{8}{32} > \frac{7}{32}$

Hence, the first event is more probable than the second.

ILLUSTRATION 115

An irregular six faced die is thrown and the expectation that in 10 throws it will give five even number is twice that expectation that it will give four even numbers. Write probability of getting 3 even number in 10 throws.

SOLUTION

Let p be the probability of getting an even number in a throw of die. Then the probability of getting x even numbers in ten throws of a die is

$$p(X = x) = \binom{10}{x} p^x q^{10-x},\ x = 0, 1, 2, 3, \ldots 10$$

where $p + q = 1$

we are given that $p(X = 5) = 2p(X = 4)$

$$\binom{10}{5} p^5 q^5 = 2\binom{10}{4} p^4 q^6 \Rightarrow \frac{P}{5} = \frac{2q}{6} = \frac{q}{3}$$

$\Rightarrow 3p = 5q \qquad \Rightarrow p = \frac{5}{8},\ q = \frac{3}{8}$

Hence, $p(X = 3) = {}^{10}C_3\left(\frac{5}{8}\right)^3\left(\frac{3}{8}\right)^7$

ILLUSTRATION 116

A test is made up of 5 questions, for each question there are 4 possible answers and only one is correct. For every right choice you gain 1 mark while for each wrong choice there is a penalty of 1 mark. Find the probability of getting atleast 3 marks answering to every question in a random way.

SOLUTION

To get atleast 3 marks, atleast 4 must be right answers.

Now, $n = 5; p = \frac{1}{4}; q = \frac{3}{4}$

$r = 4$ or 5

$$P\ (r \geq 4) = {}^5C_4\left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right) + {}^5C_5\left(\frac{1}{4}\right)^5$$

$$= 5 \cdot \frac{1}{256} \cdot \frac{3}{4} + \frac{1}{4^5} = \frac{16}{4 \cdot 256} = \frac{1}{64}$$

ILLUSTRATION 117

If X is following a binomial distribution with parameters $n = 8$ and $p = 1/2$, then find value of $P(|X-4| \leq 2)$.

SOLUTION

$$p = q = \frac{1}{2}; n = 8$$

We have, $P(|X-4| \leq 2) = P(-2 \leq X-4 \leq 2) = P(2 \leq X \leq 6$

$$= P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= {}^8C_2\left(\frac{1}{2}\right)^8 + {}^8C_3\left(\frac{1}{2}\right)^8 + {}^8C_4\left(\frac{1}{2}\right)^8 + {}^8C_5\left(\frac{1}{2}\right)^8$$

$$+ {}^8C_6\left(\frac{1}{2}\right)^8$$

$$= \frac{1}{2^8}[28+56+70+56+28] = \frac{238}{2^8} = \frac{119}{128}$$

ILLUSTRATION 118

A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

SOLUTION

The man will be one step away from the starting point if (i) either he is one step ahead or (ii) one step behind the starting point.

∴ The required probability = P(i) + P(ii)

The man will be one step ahead at the end of eleven steps if he moves six step forward and five steps backward.

The probability of this event is ${}^{11}C_6(0.4)^6(0.6)^5$.

The man will be one step behind at the end of eleven steps if he moves six steps backward and five steps forward.

The probability of this event is ${}^{11}C_6(0.6)^6(0.4)^5$.

Hence, the required probability

$$= {}^{11}C_6(0.4)^6(0.6)^5 + {}^{11}C_6(0.6)^6(0.4)^5 = {}^{11}C_6(0.4)^5(0.6)^5$$

$$(0.4+0.6) = {}^{11}C_6(0.24)^5.$$

ILLUSTRATION 119

A man can take a step forward, backward, left or right with equal probability. Find the probability that after nine steps, he will be just one step away from his initial position.

SOLUTION

Let n_F, n_B, n_L, n_R denote the number of steps taken out of 9 steps in the forward, backward, left and right directions respectively.

Let E = the event that after 9 steps the man is one step away from his initial position.

Let E_1 = the event of taking 9 steps such that

$n_F = n_B$ and $n_L = n_R + 1$.

E_2 = the event of taking 9 steps such that $n_F = n_B$ and $n_R = n_L + 1$.

E_3 = the event of taking 9 steps such that $n_L = n_R$ and $n_F = n_B + 1$.

E_4 = the event of taking 9 steps such that $n_L = n_R$ and $n_B = n_F + 1$.

Clearly, $E = E_1 \cup E_2 \cup E_3 \cup E_4$ and E_1, E_2, E_3, E_4 are mutually exclusive events.

$$\therefore P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$$

$$= 4P(E_1) \quad \ldots(1)$$

(Since man can take a step forward, backward, left or right with equal probability.)

But, for E_1 to occur only following cases are possible.

No. of steps taken:

F	B	L	R
0	0	5	4
1	1	4	3
2	2	3	2
3	3	2	1
4	4	1	0

$$\therefore n(E_1) = {}^9C_5\,{}^4C_4 + {}^9C_1\,{}^8C_1\,{}^7C_4\,{}^3C_3$$

$$+ {}^9C_2\,{}^7C_2\,{}^5C_3\,{}^2C_2 + {}^9C_3\,{}^6C_3\,{}^3C_2 \,.\, {}^1C_1 + {}^9C_4\,{}^5C_4\,{}^1C_1$$

$$= \frac{9!}{5!4!} + \frac{9!}{1!\,1!\,3!\,4!} + \frac{9!}{2!\,2!\,2!\,3!} + \frac{9!}{3!\,3!\,1!\,2!} + \frac{9!}{4!\,4!\,1!}$$

$$= 126 + 2520 + 7560 + 5040 + 630 = 15876$$

Since, each of the 9 steps can be taken in 4 ways (in forward, backward, left or right).

$\therefore\ n(S) = 4^9$

$$P(E) = 4P\,(E_1) = 4.\frac{15876}{4^9} = \frac{3969}{4^7}$$

PROBABILITY DISTRIBUTION

A Probability Distribution of an experiment tells us about how a total probability of 1 is distributed over several values of a random variable.

(i) Mean of any probability distribution of a random variable is given by

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i \quad (\text{Since } \Sigma p_i = 1)$$

(ii) Variance of a random variable is given by,

$$\sigma^2 = \Sigma(x_i - \mu)^2\, p_i$$

$$= \sum (x_i^2 + \mu^2 - 2x_i\mu)P_i$$

$$= \sum x_i^2 P_i + \mu^2 \sum P_i - 2\mu \sum x_i P_i$$

$$= \sum x_i^2 P_i + \mu^2\,(I) - 2\mu(\mu)$$

$$\sigma^2 = \Sigma p_i\, x_i^2 - \mu^2 \quad (\text{Note that Standard Deviation is } \sigma)$$

(iii) Binomial Probability Distribution

Mean(μ) = np;

variance(σ^2) = npq where $p + q = 1$

Expectation

Expectation term is related to a person who is participating in an experiment, let suppose P_i probability is associated with amount M_i then expectation of a person from experiment is

$$= \sum P_i M_i$$

ILLUSTRATION 120

A random variable X has the probability distribution:

X :	1	2	3	4	5	6	7	8
P(X) :	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events E = {X is a prime number} and F = {X < 4}, then what is the probability P(E∪F)?

SOLUTION

E = {X is a prime number}

$P(E) = P(2) + P(3) + P(5) + P(7) = 0.62$, $F = \{x < 4\}$

$P(F) = P(1) + P(2) + P(3) = 0.50$ and

$P(E \cap F) = P(2) + P(3) = 0.35$

$$\therefore\ P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.62 + 0.50 - 0.35$$
$$= 0.77$$

ILLUSTRATION 121

A pair of dies is thrown 4 times, getting a sum of 9 is considered as success in a throw. Find mean and variance.

SOLUTION

Sum of numbers on dies as 9, we get (3, 6) (6, 3) (4, 5) (5, 4)

$$P = 4/36 = 1/9 : q = 1 - P = 8/9$$

$$\text{mean} = np = 4/9$$

$$\text{variance} = npq = 32/81$$

ILLUSTRATION 122

In a binomial distribution, difference of mean and variance is 1 and difference of their squares is 11, find probability to get success exactly 3 times.

SOLUTION

mean = np and variance = npq where $p + q = 1$

given $np - npq = 1$ …(1)

$$(np)^2 - (npq)^2 = 11 \Rightarrow (np - npq)\,(np + npq) = 11$$

$\Rightarrow\quad 1(np + npq) = 11 \Rightarrow np + npq = 11$ …(2)

On solving

$np = 6$ and $npq = 5$

$$q = 5/6\,;\ p = 1/6\,;\ n = 36$$

$$p(\text{success for exactly 3 times}) = {}^nC_3\, p^3 q^3 = {}^{36}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3$$

ILLUSTRATION 123

An unbiased dice is thrown, if 2, 3, 5 appear, then equal amount is given to the player, but 1, 4, 6 appear, then equal amount is taken from the player. Find expectation of a player.

SOLUTION

Lets understand above experiment by considering X_i as number shown on die, and P_i is associated probability with amount M_i. (relative to player)

X_i	M_i	$P(X_i)$	$P(X_i)M_i$
1	–1000	$\frac{1}{6}$	$-\frac{1000}{6}$ ₹
2	+2000	$\frac{1}{6}$	$\frac{2000}{6}$ ₹
3	+3000	$\frac{1}{6}$	$\frac{3000}{6}$ ₹
4	–4000	$\frac{1}{6}$	$-\frac{4000}{6}$ ₹
5	+5000	$\frac{1}{6}$	$\frac{5000}{6}$ ₹
6	–6000	$\frac{1}{6}$	$-\frac{6000}{6}$ ₹

$$\text{Expectation} = \sum P(X_i)M_i = -\frac{1000}{6}\text{ ₹}$$

ILLUSTRATION 124

A bag contains two 5-₹ coins and one 10 ₹ coin. Anuj is taking out without replacement till 10 ₹ coin appears. Whatever Anuj takes out during experiment that money is given to him. Find his expectation.

SOLUTION

Lets understand above experiment by considering X_i as number of time experiment is conducted and P_i is associated probability with amount M_i. (relative to player)

X_i	M_i	$P(X_i)$	$P(X_i)M_i$
1 (10 ₹ coin)	10 ₹	$\frac{1}{3}$	$\frac{10}{3}$ ₹
2 (5 ₹ + 10 ₹ coin)	15 ₹	$\frac{2}{3} \times \frac{1}{2}$	5 ₹
3 (5 ₹, 5 ₹, 10 ₹)	20 ₹	$\frac{2}{3} \times \frac{1}{2} \times 1$	$\frac{20}{3}$ ₹

$$\text{Expectation} = \sum P(X_i)M_i = \frac{10}{3} + 5 + \frac{20}{3} = 15\text{ ₹}$$

ILLUSTRATION 125

A special die is so constructed that the probabilities of throwing 1, 2, 3, 4, 5 and 6 are $\frac{(1-k)}{6}, \frac{(1+2k)}{6}, \frac{(1-k)}{6}, \frac{(1+k)}{6}, \frac{(1-2k)}{6}$ and $\frac{(1+k)}{6}$ respectively. If two such dice are thrown and the probability of getting a sum equal to nine lies between 1/9 and 2/9, find the set of integral values of k.

SOLUTION

Let E_i = the event of occurrences of i on a die, where i = 1, 2, 3 ...., 6.

According to question,

$$P(E_1) = \frac{1-k}{6},\ P(E_2) = \frac{1+2k}{6},\ P(E_3) = \frac{1-k}{6}$$

$$P(E_4) = \frac{1+k}{6},\ P(E_5) = \frac{1-2k}{6},\ P(E_6) = \frac{1+k}{6}$$

Let E = the event of getting a sum of numbers equal to 9.

$$\text{Given, } \frac{1}{9} \le P(E) \le \frac{2}{9} \qquad \ldots(1)$$

Sum of numbers will be 9 if

(i) 3 occurs on one die and 6 occurs on the other die.

(ii) 4 occurs on one die and 5 on the other die.

$$P(E) = [P(E_3).\ P(E_6) + P(E_6).\ P(E_3)] + [P(E_4).\ P(E_5) + P(E_5).\ P(E_4)]$$

$$\therefore\ P(E) = 2P(E_3)\,P(E_6) + 2P(E_4).\ P(E_5)$$

$$= 2\left(\frac{1-k^2}{36}\right) + \frac{2(1+k)(1-2k)}{36}$$

$$= \frac{1}{18}(1 - k^2 + 1 - k - 2k^2)$$

$$= \frac{1}{18}(-3k^2 - k + 2)$$

$$\text{since, } \frac{1}{9} \le P(E) \le \frac{2}{9}$$

$$\therefore\quad P(E) \ge \frac{1}{9} \Rightarrow \frac{-3k^2 - k + 2}{18} \ge \frac{1}{9}$$

$$\Rightarrow\ -3k^2 - k \ge 0 \Rightarrow -\frac{1}{3} \le k \le 0$$

Therefore, only possible integral value of k is zero.

Also, when $k = 0$, $P(E) = \frac{1}{9} < \frac{2}{9}$

$\therefore$ Set of integral values of $k = \{0\}$.

Test Your Concepts-V

(Solutions on page H.220)

1. If a fair coin is tossed 10 times, then find the probability of getting

  i. exactly six heads

  ii. atleast six heads

  iii. atmost six heads

2. India and Pakistan play a 5 match test series of hockey, then find the probability that India wins atleast three matches.

3. A coin is tossed 7 times, Each time a man call head. Find the probability that he wins the toss on more than three occasions.

4. If a die is thrown 7 times, then find the probability of obtaining 5 exactly 4 times.

5. Two coins are tossed five times, find the probability that an odd number of heads are obtain.

6. A pair of fair dice is thrown independently three times. Find the probability of getting a score of exactly 9 twice.

Answer keys

1. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$ **2.** $\frac{1}{2}$ **3.** $\frac{1}{2}$ **4.** ${}^{7}C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^3$

5. $\frac{1}{2}$ **6.** $\frac{8}{243}$

SOLVED PROBLEMS

PROBLEM 1

A closet contains n pairs of shoes. $2r$ shoes are choosen at random ($r < n$) then show that

(i) P(all are pairs) $= \dfrac{{}^nC_r}{{}^{2n}C_{2r}}$

(ii) P(out of $2r$ shoes, exactly s pairs come out) $= \dfrac{{}^nC_s\,{}^{n-s}C_{2r-2s}\,2^{2r-2s}}{{}^{2n}C_{2r}}$

(iii) P(no complete pair) $= \dfrac{{}^nC_{2r}\,2^{2r}}{{}^{2n}C_{2r}}$

(iv) P(atleast one pair) $= 1 - \dfrac{{}^nC_{2r}\cdot 2^{2r}}{{}^{2n}C_{2r}}$

SOLUTION

Total number of ways of drawing $2r$ shoes from $2n$ shoes $= {}^{2n}C_{2r}$

(i) P(all are pairs)

$$= \frac{\text{selecting } r \text{ pairs from } n \text{ pairs}}{{}^{2n}C_{2r}} = \frac{{}^nC_r}{{}^{2n}C_{2r}}$$

(ii) To select exactly $s(\le r)$ pairs we have to select s pairs from n pairs. This can be done in ${}^nC_s$ ways

Now there are n-s pairs and remaining shoes to be selected are $2r - 2s$ of which, no two shoes are from same pair. For this we select $2r - 2s$ pairs from n-s pairs. This can be done in ${}^{n-s}C_{2r-2s}$ ways,

Now from each pair, we may choose either left one or right one i.e. there are two ways for each pair, for $2r - 2s$ shoes, there are 2^{2r-2s} ways so that there is no complete pair. Thus

P(out of $2r$ shoes, exactly s pairs come out)

$$= \frac{{}^nC_s\,{}^{n-s}C_{2r-2s}\,2^{2r-2s}}{{}^{2n}C_{2r}}$$

(iii) P(no complete pair) $= \dfrac{{}^nC_{2r}\,2^{2r}}{{}^{2n}C_{2r}}$ (put $s = 0$ in (B))

(iv) P(atleast one pair) $= 1 - \dfrac{{}^nC_{2r}\cdot 2^{2r}}{{}^{2n}C_{2r}}$

PROBLEM 2

A four digit number (numbered from 0000 to 9999) is said to be lucky if sum of its first two digits is equal to the sum of its last two digits. If a four digit number is picked up at random, then find the probability that it is lucky number.

SOLUTION

The total number of ways of choosing the ticket is 10000. Let the four digits number on the ticket be x_1, x_2, x_3, x_4. Note that $0 \le x_1 + x_2 \le 18$ and $0 \le x_3 + x_4 \le 18$. Also, the number of non-negative integral solution of $x + y = m$ (with $0 \le x, y \le 9$) is $m + 1$; if $0 \le m \le 9$ and is $19–m$ if $10 \le m \le 18$.

The number of favourable ways

$$= 1\times1 + 2\times2 + \ldots. + 10\times10 + 9\times9 + 8\times8 + \ldots.. + 1\times1$$

$$= 2\,(1^2 + 2^2 + \ldots. + 9^2) + 10^2$$

$$= 2\cdot\left(\frac{9\cdot10\cdot19}{6}\right) + 100 = 670$$

$$= 2\left\{\frac{9\times10\times19}{6}\right\} + 100 = 670$$

$\therefore$ Probability of required event $= \dfrac{670}{10000} = 0.067$

Alternate:

The total number of ways of choosing a four digit number is $10^4 = 10000$. Let, a_k denote the number of distinct non-negative integral solutions of the equation $x + y = k$ ($0 \le k \le 18$)

$\therefore$ The number of favourable cases

$$= a_0^2 + a_1^2 + \ldots\ldots + a_{18}^2.$$

Suppose $(1 + x + x^2 + \ldots\ldots.. + x^9)^2$

$$= a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots\ldots. + a_{18} x^{18}.$$

Thus, $a_0^2 + a_1^2 + \ldots\ldots. + a_{18}^2$

= coeff. of constant term in

$$(a_0 + a_1 x + \ldots. + a_{18} x^{18}) \times \left(a_0 + \frac{a_1}{x} + \ldots\ldots. \frac{a_{18}}{x^{18}}\right)$$

= coeff. of constant term in $(1 + x + x^2 + \ldots. + x^9)^2$

$$\times\left(1 + \frac{1}{x} + \frac{1}{x^2} + \ldots\ldots\ldots. + \frac{1}{x^9}\right)^2$$

= coeff. of x^{18} in $(1 + x + x^2 + \ldots\ldots\ldots. + x^9)^4$

= coeff. of x^{18} in $(1 - x^{10})^4\,(1 - x)^{-4}$

= coeff. of x^{18} in $(1 - 4x^{10})(1 - x)^{-4}$

$$= (1 - x)^{-4} - 4x^{10}\,(1 - x)^{-4}$$

$$= {}^{21}C_{18} - 4\,.\,{}^{11}C_8$$

[$\because$ Coefficient of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$]

$= 670$

Hence, required probability $= \dfrac{670}{10000} = \dfrac{67}{1000}$

PROBLEM 3

Five ordinary dice are rolled at random and the sum of the numbers shown on them is 16. What is the probability that the numbers shown on each is anyone from, 2, 3, 4, or 5?

SOLUTION

If the integers x_1, x_2, x_3, x_4 and x_5 are shown on the dice, then

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

$$1 \le x_i \le 6$$

where $(i = 1, 2, 3, 4, 5)$

The number of total solutions of this equation

$= \text{coeff. of } x^{16} \text{ in } (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^5$

$= \text{coeff. of } x^{16} \text{ in } x^5\,(1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= \text{coeff. of } x^{11} \text{ in } (1 + x + x^2 + x^3 + x^4 + x^5)^5$

$= \text{coeff. of } x^{11} \text{ in } \left\{\left(\dfrac{1-x^6}{1-x}\right)^5\right\}$

$= \text{coeff. of } x^{11} \text{ in } (1 - x^6)^5\,(1 - x)^{-5}$

$= \text{coeff. of } x^{11} \text{ in } (1 - 5x^6 + ...)\,(1 - x)^{-5}$

$= \text{coeff. of } x^{11} \text{ in } (1 - x)^{-5} - 5x^6\,(1 - x)^{-5}$

$= {}^{15}C_{11} - 5\,{}^{9}C_5 = 1365 - 630 = 735$

If S be the sample space

$$n(S) = 735$$

Let E be the occurence event, then

$n(E)$ = The number of integral solutions of

$x_1 + x_2 + x_3 + x_4 + x_5 = 16$

where $2 \le x_i \le 5$ $(i = 1, 2, 3, 4, 5)$

$= \text{coeff. of } x^{16} \text{ in } (x^2 + x^3 + x^4 + x^5)^5$

$= \text{coeff. of } x^{16} \text{ in } x^{10}\,(1 + x + x^2 + x^3)^5$

$= \text{coeff. of } x^{6} \text{ in } (1 + x + x^2 + x^3)^5$

$= \text{coeff. of } x^{6} \text{ in } \left\{\dfrac{(1-x^4)^5}{(1-x)^5}\right\}$

$= \text{coeff. of } x^{6} \text{ in } (1 - x^4)^5\,(1 - x)^{-5}$

$= \text{coeff. of } x^{6} \text{ in } (1 - 5x^4 + ...)\,(1 - x)^{-5}$

$= \text{coeff. of } x^{6} \text{ in } (1 - x)^{-5} - 5x^4\,(1 - x)^{-5}$

$= {}^{10}C_6 - 5.\,{}^{6}C_2 = 135$

$\therefore$ Required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{135}{735} = \frac{9}{49}$$

PROBLEM 4

If $6n$ tickets numbered 0, 1, 2, ... $(6n - 1)$ are placed in a bag, and three are drawn out, show that the chance that the sum of the numbers on them is equal to $6n$ is $\dfrac{3n}{(6n-1)(6n-2)}$.

SOLUTION

Total no. of ways

$$N = {}^{6n}C_3 = \frac{6n(6n-1)(6n-2)}{1\,.\,2\,.\,3}$$

$$= n\,(6n - 1)\,(6n - 2).$$

We now obtain the favourable number of ways. Let the lowest number drawn be 0. Then the remaining two tickets are to be drawn from the tickets numbered 1, 2, 3, ... $3n - 2$, $3n - 1$, $3n$, $3n + 1$, ..., $6n - 2$, $6n - 1$ such that their sum is $6n$. In this case, there are the following $(3n - 1)$ triples of tickets having the sum $6n$. $(0, 1, 6n - 1)$, $(0, 2, 6n - 2)$, $(0, 3, 6n - 3)$ ...... $(0, 3n - 1, 3n + 1)$.

If the lowest number drawn is 1, then the other two tickets will be chosen from the tickets numbered.

2, 3, 4 .... $(3n - 1)$, $3n$, $3n + 1$, ..., $6n - 1$ such that the sum of all the three tickets is $6n$. There are the following $(3n - 2)$ triplets with their sum $6n$. $(1, 2, 6n - 3)$, $(1, 3, 6n - 4)$, $(1, 4, 6n - 5)$, ... $(1, 3n - 1, 3n)$. Similarly, if the lowest number drawn is 2, then the other two tickets can be chosen from the tickets numbered 3, 4, 5....., $3n - 1$, $3n$, $3n + 1$, ... $6n - 1$ in $(3n - 4)$ ways and so on. There is only one way with the lowest number $2n - 1$.

It is the triplet $(2n - 1, 2n, 2n + 1)$.

$\therefore$ Favourable number of ways

$$\begin{aligned} M &= [(3n - 1) + (3n - 2)] + [(3n - 4) \\ &\quad + (3n - 5)] + \ldots + (5 + 4) + (2 + 1) \\ &= 3 + 9 + 15 + ... + (6n - 9) + (6n - 3) \\ &= 3\,[1 + 3 + 5 + ... + (2n - 1)] \\ &= 3\,.\,\frac{n}{2}\,[2.1 + (n - 1)\,2] = 3n^2 \end{aligned}$$

Hence, the required probability

$$= \frac{M}{N} = \frac{3n^2}{n(6n-1)(6n-2)} = \frac{3n}{(6n-1)(6n-2)}.$$

PROBLEM 5

A player tosses a coin and scores one point for every head turned up and two for every tail. He plays on until his score reaches or passes n. If P_n is the probability of attaining a score of exactly n, then show that

(i) $P_n = \dfrac{1}{2}(P_{n-1} + P_{n-2})$ **(ii)** $P_n = \dfrac{2}{3} + \dfrac{1}{3}\left(-\dfrac{1}{2}\right)^n$

SOLUTION

Event can be materialized in the two mutually exclusive ways

(i) a score of $(n-1)$ points is obtained and a score of 1 point is obtained at the n^{th} trial by getting a head in tossing coin.

(ii) a score of $(n-2)^{th}$ point then getting a tail.

$$\text{Hence } P_n = \frac{1}{2}P_{n-1} + \frac{1}{2}P_{n-2} \qquad \ldots(1)$$

$$P_n + \frac{1}{2}P_{n-1} = P_{n-1} + \frac{1}{2}P_{n-2} = P_{n-2} + \frac{1}{2}P_{n-3}\cdots$$

$$= P_2 + \frac{1}{2}P_1 \qquad \left(\because P_1 = \frac{1}{2}, P_2 = \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}\right)$$

$$\Rightarrow P_n + \frac{1}{2}P_{n-1} = \frac{3}{4} + \frac{1}{4}$$

$$P_n - \frac{2}{3} = -\frac{1}{2}\left(P_{n-1} - \frac{2}{3}\right)$$

$$P_{n-1} - \frac{2}{3} = -\frac{1}{2}\left(P_{n-2} - \frac{2}{3}\right)$$

$$P_2 - \frac{2}{3} = \left(-\frac{1}{2}\right)\left(P_1 - \frac{2}{3}\right)$$

Multiplying all the above equations, we get

$$P_n - \frac{2}{3} = \left(-\frac{1}{2}\right)^{n-1}\left(P_1 - \frac{2}{3}\right) = \left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{2} - \frac{2}{3}\right)$$

$$P_n = \frac{1}{3}\left[2 + (-1)^n\frac{1}{2^n}\right]$$

PROBLEM 6

Cards are drawn one by one at random from a well-shuffled full pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that the probability of obtaining 2 aces when N is equal to n is $\frac{(n-1)(52-n)(51-n)}{50\times 49\times 17\times 13}$, where $2 \le n \le 50$.

SOLUTION

Suppose $N = 2$, (i.e.) the number of cards to be drawn is 2.
In this case, both the cards have to be aces.

$$\therefore \text{Probability} = \frac{4}{52}\cdot\frac{3}{51} = \frac{1}{13}\cdot\frac{1}{17}$$

Suppose $N = 3$
In this case, 2 aces and 1 non-ace have to be drawn.

It could be (NA).A A or A(NA) A and the probability

$$= \frac{48}{52}\cdot\frac{4}{51}\cdot\frac{3}{50} + \frac{4}{52}\cdot\frac{48}{51}\cdot\frac{3}{50} = \frac{48}{13\,.\,17\,.\,25}$$

Suppose $N = 4$

$$\text{Probability} = 3.\frac{48}{52}\times\frac{47}{51}\times\frac{4}{50}\times\frac{3}{49}$$

(In this case, the first ace can be in the first or second or third draw)
Similarly when $N = n$, (nth draw is a definite ace and the first ace can be in any one of the first $(n-1)$ draws.

$$\text{Probability} = (n-1).\frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50}\,\ldots\ldots\,\frac{48-(n-3)}{52-(n-3)}\times\frac{4}{54-n}\times\frac{3}{53-n}$$

$$= \frac{(n-1)(52-n)(51-n)}{52.51.50.49}\times 4\times 3$$

$$= \frac{(n-1)(52-n)(51-n)}{50.49.13.17}, \text{ where } 2 \le n \le 50$$

Alternate:

Probability of making n draws until two aces appear

$$= \left\{\frac{{}^{48}C_{n-2}\times{}^{4}C_{1}}{{}^{52}C_{n-1}}\right\}\times\frac{3}{53-n} = \frac{(n-1)(52-n)(51-n)}{50\cdot 49\cdot 17\cdot 13}$$

The first factor is the probability of drawing $(n-2)$ non-aces and one ace in the first $(n-1)$ draws while the second factor is the probability of drawing an ace in the n^{th} draw.

PROBLEM 7

A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that the statement is true is $\frac{xy}{1-x-y+2xy}$.

SOLUTION

Let E denote that event that A and B agree in a certain statement.

Let E_1 = the event that both A and B speak the truth.

E_2 = the event that both A and B tell a lie.

By Bayes' Theorem, required probability

$$P(E_1/E) = \frac{P(E_1).P(E/E_1)}{P(E_1).P(E/E_1) + P(E_2)P(E/E_2)} \qquad \ldots(1)$$

Let C = the event that A speaks the truth

D = the event that B speaks the truth

The $E_1 = C \cap D$ and $E_2 = C' \cap D'$

Now, $P(E_1) = P(C \cap D) = P(C) \,.\, P(D) = xy$

[$\because$ C and D are independent events]

$$\text{and } P(E_2) = P(C' \cap D') = P(C') \,.\, P(D')$$
$$= \{1 - P(C)\}\,\{1 - P(D)\}$$
$$= (1 - x)\,(1 - y) = 1 - x - y + xy$$

$P(E/E_1)$ = probability that A and B will agree when both of them speak the truth = 1 [Since the event is a sure event]

and $P(E/E_2)$ = probability that A and B agree when both of them tell a lie = 1

From (1), $P(E_1/E) = \dfrac{xy}{xy.1 + (1 - x - y + xy)1}$

$$= \frac{xy}{1 - x - y + 2xy}$$

PROBLEM 8

Each of the two shooters fires 10 shots at their own target. Show that the chance that they will score the same number of hits if the probability of hitting in each shot is 0.5, is given by $\displaystyle\prod_{r=1}^{10}\left(\frac{2r-1}{2r}\right)$

SOLUTION

P(they score the same number of hits)

$$= \left({}^{10}C_0 \frac{1}{2^{10}}\right)\left({}^{10}C_0 \frac{1}{2^{10}}\right) + \left({}^{10}C_1 \frac{1}{2^{10}}\right)$$
$$\left({}^{10}C_1 \frac{1}{2^{10}}\right) + \ldots + \left({}^{10}C_{10} \frac{1}{2^{10}}\right)\left({}^{10}C_{10} \frac{1}{2^{10}}\right)$$

$$= \frac{C_0^2 + C_1^2 + C_2^2 + C_0^2 + \ldots + C_{10}^2}{2^{20}} = \frac{{}^{20}C_{10}}{4^{10}} = \frac{(20)!}{10!\cdot 10!\cdot 4^{10}}$$

$$\frac{1\cdot 2\cdot 3\cdot 4\ldots 19\cdot 20}{10!\,10!\,2^{20}} = \frac{1\cdot 3\cdot 5\ldots 19(2\cdot 4\cdot 6\ldots 20)}{10!\,2^{10}\,(10!\,2^{10})}$$

$$= \frac{1\cdot 3\cdot 5\ldots 19}{2\cdot 4\cdot 6\ldots 18\cdot 20} = \prod_{r=1}^{10}\left(\frac{2r-1}{2r}\right)$$

PROBLEM 9

A mathematician carries one matchbox each in his right and left pockets. Initially, each matchbox contains n matchsticks. Each time he wants a matchbox, he selects at random from the right or left pocket to get it. Consider the moment when the mathematician discovers that a box is empty. Show that the probability that there are exactly r matchsticks in one box when the other box is found empty is ${}^{2n-r}C_r \left(\dfrac{1}{2}\right)^{2n-r}$

SOLUTION

Let p denotes the probability that the box in the left-pocket is selected. Then $p = \dfrac{1}{2}$. The left-pocket box will be empty at the moment when the right-pocket box contains exactly r matchsticks if and only if exactly $(n - r)$ selections of the right-pocket box precede the $(n + 1)$ st selection of the left-pocket box. The probability of finding the left pocket box empty when the right-pocket box contains r matchsticks is

$${}^{2n-r}C_r \left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{2}\right)^{n+1} = {}^{2n-r}C_r \left(\frac{1}{2}\right)^{2n-r+1}$$

Similarly, the probability that the right-pocket box is empty when the left pocket box contains exactly r matchsticks is ${}^{2n-r}C_r \left(\dfrac{1}{2}\right)^{2n-r+1}$.

Hence, the required probability

$$= 2\left[{}^{2n-r}C_r \left(\frac{1}{2}\right)^{2n-r+1}\right] = {}^{2n-r}C_r \left(\frac{1}{2}\right)^{2n-r}$$

PROBLEM 10

A coin is tossed $(m + n)$ times $(m > n)$; show that the probability of at least m consecutive heads is $\dfrac{n+2}{2^{m+1}}$.

SOLUTION

We denote by H the appearance of head and T the appearance of tail and let X denote the appearance of head or tail.

Then, $P(H) = P(T) = \dfrac{1}{2}$ and $P(X) = 1$.

If the sequence of m consecutive heads starts from the first throw, we have

(HHH .... m times) (XXX .... n times)

The chance of this event

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \,\ldots\ldots\, m \text{ times} = \frac{1}{2^m}.$$

[Note that $(m + 1)$ and subsequent throws may be head or tail since we are considering at least m consecutive heads.]

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have T(HHH .... m times) (XXX .... $(n - 1)$) times

The chance of this event $= \dfrac{1}{2}\cdot\dfrac{1}{2^m} = \dfrac{1}{2^{m+1}}$.

If the sequence of heads starts with the $(r + 1)^{\text{th}}$ throw, then the first $(r - 1)$ throws may be head or tail, but r^{th} throw must be tail and we have

(XX .... $\overline{r - 1}$ times) (THHH .... m times) (XX .... $n - m - r$ times)

∴ The probability of this event also

$$=\frac{1}{2}\cdot\frac{1}{2^m}=\frac{1}{2^{m+1}}.$$

Since, all the above cases are mutually exclusive, the required probability

$$=\frac{1}{2^m}+\left(\frac{1}{2^{m+1}}+\frac{1}{2^{m+1}}+\text{.......... to } n \text{ terms}\right)$$

$$=\frac{1}{2^m}+\frac{n}{2^{m+1}}=\frac{n+2}{2^{m+1}}$$

PROBLEM 11

Two players A and B want respectively m and n points for winning a set of games; their chances of winning a single game are 'p' and 'q' respectively where $p+q=1$. The stake is to belong to the player who first makes up his set. Find the probabilities in favour of each player.

SOLUTION

Suppose A wins in exactly $m+r$ games. To do so he must win in the last game and $m-1$ out of the preceding $m+r-1$ games. The chance of this is

$${}^{m+r-1}C_{m-1}\,p^{m-1}\,q^r\,p={}^{m+r-1}C_{m-1}\,p^m\,q^r.$$

Now the set will definitely be decided in $m+n-1$ games, and A may win these m games in exactly m games, or $m+1$ games, ..., or $m+n-1$ games. Hence we shall obtain the chance of A's winning the set by giving to r the values $0, 1, 2, \ldots\ldots, n-1$ in succession in the expression ${}^{m+r-1}C_{m-1}\,p^m$. Hence,

A's probability to win the set is:

$$p^m\left[1+mq+\frac{m(m+1)}{1.2}q^2+\text{..........}+\frac{(m+n-2)!}{(m-1)!(n-1)!}q^{n-1}\right]$$

& B's probability to win the set is:

$$q^n\left[1+np+\frac{n(n+1)}{1.2}p^2+\text{..........}+\frac{(m+n-2)!}{(m-1)!(n-1)!}p^{m-1}\right]$$

PROBLEM 12

Of three independent events, the chance that only the first occurs is a, that only the second occurs is b, and that only the third occurs is c and none of the events occur with probability x. Show that the probabilities of occurrence of these three events are respectively $\frac{a}{a+x}, \frac{b}{b+x}, \frac{c}{c+x}$ where x is a root of the equation $(a+x)(b+x)(c+x)=x^2$.

SOLUTION

Let E_1, E_2, E_3 be three independent events and E_1', E_2', E_3' be their complements. Then

$$P(E_1\cap E_2'\cap E_3')=P(E_1).P(E_2').P(E_3')=a \qquad \ldots(1)$$

Since E_1, E_2, E_3 are independent.

$$P(E_1'\cap E_2\cap E_3')=P(E_1')P(E_2)P(E_3')=b \qquad \ldots(2)$$

$$\text{and } P(E_1'\cap E_2'\cap E_3)=P(E_1')\,P(E_2')P(E_3)=c \qquad \ldots(3)$$

Denote $P(E_1')\,P(E_2')P(E_3')$ by x

Then, $\frac{P(E_1)}{P(E_1')}=\frac{a}{x}$ this implies $\frac{P(E_1)}{1-P(E_1)}=\frac{a}{x}$

or $P(E_1)=\frac{a}{a+x}$

Similarly, we get $P(E_2)=\frac{b}{b+x}$ and $P(E_3)=\frac{c}{c+x}$ …(4)

Multiplying (1), (2) and (3), we get

$$\frac{abc}{(a+x)(b+x)(c+x)}x^2=abc \text{ or } (a+x)(b+x)(c+x)=x^2$$

PROBLEM 13

In a factory, which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts of their outputs 5%, 4% and 2% are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. The probability that it is manufactured by the machine B is $\frac{m}{n}$ then find $\frac{|2m-n|}{13}$.

SOLUTION

Let E_1, E_2, E_3 and A be the events defined as follows

E_1: Bolt is manufactured by machine A

E_2: bolt is manufactured by machine B

E_3: Bolt is manufactured by machine C.

A: Bolt is defection.

Then, $P(E_1)=\frac{25}{100}$, $P(E_2)=\frac{35}{100}$, $P(E_3)=\frac{40}{100}$

$$P\left(\frac{A}{E_1}\right)=\frac{5}{100},\ P\left(\frac{A}{E_2}\right)=\frac{4}{100},\ P\left(\frac{A}{E_3}\right)=\frac{2}{100}$$

$$\therefore \quad \text{Required probability}=P(E_2/A)=\frac{P(E_2)P\left(\frac{A}{E_2}\right)}{\sum_{i=1}^{n}P(E_1)P\left(\frac{A}{E_1}\right)}$$

$$=\frac{\frac{35}{100}\times\frac{4}{100}}{\frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}}$$

$$=\frac{140}{125+140+80}=\frac{140}{345}=\frac{28}{69}$$

But given that, $\frac{m}{n}=\frac{28}{69}$

$$\Rightarrow \quad \left|\frac{2m-n}{13}\right|=\left|\frac{2\times 28-69}{13}\right|=1$$

PROBLEM 14

On a Saturday night, 20% of all drivers in the USA are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a Saturday night smashed into a tree, then find the probability that the driver was under the influence of alcohol.

SOLUTION

Let E_1, E_2 and A be the event defined as follow :

E_1: The driver are not under the influence

E_2: The driver are under the influence of alcohol

A: Car met with an accident

$$\text{Now, } P(E_1) = \frac{80}{100} = \frac{4}{5}$$

$$P(E_2) = \frac{20}{100} = \frac{1}{5}$$

$$P\left(\frac{A}{E_1}\right) = 0.0001$$

$$P\left(\frac{A}{E_2}\right) = 0.001$$

To find, $P\left(\frac{E_2}{A}\right)$

By Bayes' theorem, we have

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{5}\times 0.001}{\frac{4}{5}\times 0.0001+\frac{1}{5}\times 0.001} = \frac{0.001}{0.0004+0.001}$$

$$= \frac{0.001}{0.0014} = \frac{10}{14} = \frac{5}{7}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Three fair and unbiased dice and rolled at a time. The probability that the numbers shown are totally different.

(A) $\frac{5}{9}$ (B) $\frac{27}{16}$

(C) $\frac{4}{9}$ (D) $\frac{2}{3}$

2. A bag contains 7 black and 4 white balls two balls are drawn at a time from the bag. The probability at least one white ball is selected is

(A) $\frac{7}{11}$ (B) $\frac{5}{11}$

(C) $\frac{28}{55}$ (D) $\frac{34}{55}$

3. When 5-boys and 5-girls sit at a round table the probability that no two girls come together

(A) $\frac{1}{120}$ (B) $\frac{1}{126}$

(C) $\frac{3}{47}$ (D) $\frac{4}{7}$

4. There are m-stations on a railway line. A train has to stop at 3 intermediate stations then probability that no two stopping stations are adjacent

(A) $\frac{1}{^{m}C_3}$ (B) $\frac{3}{^{m}C_3}$

(C) $\frac{^{m-2}C_3}{^{m}C_3}$ (D) $\frac{^{m}C_2}{^{m}C_3}$

5. The probability that randomly selected positive integer is relatively prime to 6

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{6}$ (D) $\frac{5}{6}$

6. A and B are events such that P(A) = 0.3, P(A ∪ B) = 0.8. If A and B are independent then P(B) =

(A) $\frac{1}{7}$ (B) $\frac{3}{7}$

(C) $\frac{5}{7}$ (D) $\frac{6}{7}$

7. There are ten pairs of shoes in a cup board out of which 4 are picked up at random one after the other. The probability that there is at least one pair is

(A) $\frac{4}{11}$ (B) $\frac{3}{11}$

(C) $\frac{33}{107}$ (D) $\frac{99}{323}$

8. Out of 10 persons sitting at a round table. Three persons are selected at random one the other. The chance that no two of the selected are together.

(A) $\frac{1}{2}$ (B) $\frac{2}{5}$

(C) $\frac{5}{9}$ (D) $\frac{5}{12}$

9. Seven coupons are selected at random one at a time with replacement from 15 coupons numbered 1 to 15. The probability that the largest number appearing on a selected coupon is 9, is

(A) $\left(\frac{9}{16}\right)^6$ (B) $\left(\frac{8}{15}\right)^7$

(C) $\left(\frac{3}{5}\right)^7$ (D) None of these

10. If the letters of the word MATHEMATICS are arranged arbitrarily, the probability that C comes before E, E before H, H before I and I before S is

(A) $\frac{1}{75}$ (B) $\frac{1}{24}$

(C) $\frac{1}{120}$ (D) $\frac{1}{720}$

11. If three smallest squares are selected at random from chessboard, then the probability that they form the letter 'L' is

(A) $\frac{196}{^{64}C_3}$ (B) $\frac{49}{^{64}C_3}$

(C) $\frac{296}{^{64}C_3}$ (D) $\frac{96}{^{64}C_3}$

12. The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probability of 2 successes is

(A) 37/256 (B) 28/256

(C) 128/256 (D) 219/256

13. For a binomial distribution $\bar{x} = 4$, $\sigma = \sqrt{3}$. Then $P(X = r) =$

(A) $^{16}C_r(1/4)^r\,(3/4)^{16-r}$ (B) $^{12}C_r(1/4)^r\,(3/4)^{12-r}$

(C) $^{16}C_r(3/4)^r\,(1/4)^{16-r}$ (D) $^{12}C_r(3/4)^r\,(1/4)^{12-r}$

14. Two natural numbers a and b are selected at random. The probability that $a^2 + b^2$ is divisible by 7 is

(A) 3/8 (B) 1/7

(C) 3/49 (D) 1/49

15. Thirty-two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players, the better-ranked player wins, the probability that

ranked 1 and ranked 2 players are winner and runner up respectively, is

(A) 16/31 (B) 1/2

(C) 17/31 (D) None

16. A coin is tossed 7 times. Then the probability that at least 4 consecutive heads appear is

(A) 3/16 (B) 5/32

(C) 5/16 (D) 1/8

17. A fair coin is tossed 5 times then probability that two heads do not occur consecutively (No two heads come together)

(A) $\frac{1}{16}$ (B) $\frac{15}{32}$

(C) $\frac{13}{32}$ (D) $\frac{7}{16}$

18. A man throws a die until he gets a number bigger than 3. The probability that he gets 5 in the last throw

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{6}$ (D) $\frac{1}{36}$

19. If the integers m and n are chosen at random from 1 to 100, then the probability that a number of the form $7^n + 7^m$ is divisible by 5 equals

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$

(C) $\frac{1}{8}$ (D) $\frac{1}{6}$

20. A natural number 'x' is chosen at random from the first 1000 natural numbers. If [.] denotes the greatest integer function and the probability that $\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]+\left[\frac{x}{5}\right]=\frac{31x}{30}$

(A) $\frac{31}{1000}$ (B) $\frac{33}{999}$

(C) $\frac{33}{1000}$ (D) $\frac{67}{1000}$

21. If mn distinct coins have been distributed in 'm' purses of different colours, n coins into each then the probability that two specified coins will be found in same purse

(A) $\frac{n+1}{mn+1}$ (B) $\frac{n-1}{mn-1}$

(C) $\frac{m+n}{mn+1}$ (D) $\frac{m-1}{mn+1}$

22. The sum of two natural numbers, n_1 and n_2 is known to be equal to 100. The probability of their product being greater than 1600, is equal to

(A) $\frac{20}{33}$ (B) $\frac{58}{99}$

(C) $\frac{13}{33}$ (D) $\frac{59}{99}$

23. A bag contains 4-balls two balls are drawn from the bag and are found to be white then probability that all balls in the bag are white

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$

(C) $\frac{3}{5}$ (D) $\frac{4}{5}$

24. If the sum of the ordinate and the abscissa of a point P(x, y) is $2n$ (where x, y are natural numbers) then the probability that 'P' does not lie on the line $y = x$ is

(A) $\frac{n-1}{n+3}$ (B) $\frac{{}^{2n}C_n}{2^{2n}}$

(C) $\frac{2n-2}{2n-1}$ (D) $\frac{2n+1}{2n+3}$

25. An urn contains $(2n + 1)$ coins of which n coins have a head on both sides and the remaining $(n + 1)$ coins are fair. A coin is taken out of the urn and tossed. If the probability of obtaining a head be $\frac{37}{50}$, then n is equal to

(A) 10 (B) 11

(C) 12 (D) 13

26. There are 10 tickets numbered 0,1, 2, 3, ... 9. Two tickets are drawn at random. If the number obtained by writing the digits together is a perfect square then the probability that sum of digits is 9 is.

(A) $\frac{1}{3}$ (B) $\frac{5}{7}$

(C) $\frac{4}{9}$ (D) None of these

27. A die is rolled three times. The probability of getting a larger no. than the previous no is

(A) $\frac{2}{7}$ (B) $\frac{5}{31}$

(C) $\frac{5}{54}$ (D) $\frac{5}{19}$

28. A bag contains n white and n red balls. Pairs of balls are drawn with out replacement until the bag is empty. The probability that each pair consists of one white and one red ball is

(A) $\frac{\lfloor\underline{n}\cdot\lfloor\underline{n}}{\lfloor\underline{2n}}\,\frac{\lfloor\underline{n}\cdot\lfloor\underline{n}}{\lfloor\underline{2n}}.2^{n-1}$ (B) $\frac{\left(\lfloor\underline{n}\right).2^n}{\lfloor\underline{2n}}$

(C) $\frac{\left(\lfloor\underline{n}\right)^2.2^n}{\lfloor\underline{2n}}$ (D) $\frac{\left(\lfloor\underline{\boldsymbol{n}}\right).2^{\boldsymbol{n}-1}}{\lfloor\underline{2\boldsymbol{n}}}$

29. If $a \in [-20, 0]$, then the probability that the equation $16x^2 + 8(a + 5)\,x - 7a - 5 = 0$ has imaginary roots is

(A) $\frac{13}{20}$ (B) $\frac{20}{23}$

(C) $\frac{13}{24}$ (D) $\frac{15}{20}$

30. Two distinct numbers *a* and *b* are chosen randomly from the set $\left\{2, 2^2, 2^3, \ldots 2^{25}\right\}$. Then the probability that $\log_a b$ is an integer is

(A) $\frac{131}{300}$ (B) $\frac{31}{300}$

(C) $\frac{21}{300}$ (D) $\frac{62}{300}$

31. An elevator starts with m passengers and stops at 'n' floors ($m \le n$). The probability that no two passengers alight at the same floor is

(A) $\frac{{}^nC_m}{n^m}$ (B) $\frac{m \times n}{n^m}$

(C) $\frac{1}{n^m}$ (D) None

32. A and B are two weak students in Mathematics and their chances of solving a problem correctly are $\frac{1}{8}$ and $\frac{1}{12}$ respectively. They are given a problem and they obtain the same answer. If the probability of a common mistake is $\frac{1}{1001}$, then the probability that the answer was correct is

(A) $\frac{12}{13}$ (B) $\frac{13}{14}$

(C) $\frac{14}{15}$ (D) $\frac{15}{16}$

33. Suppose $f(x) = x^3 + ax^2 + bx + c$, where a, b, c are chosen respectively by throwing a die three times. The probability that $f(x)$ is an increasing function is

(A) $\frac{4}{9}$ (B) $\frac{3}{8}$

(C) $\frac{2}{5}$ (D) $\frac{8}{17}$

34. A and B throw alternatively with a pair of dice. A wins if he throws a sum 6 before B throws 7 and B wins if he throws a 7 before A throws sum 6. If A starts the game, his chance of winning is

(A) $\frac{30}{61}$ (B) $\frac{31}{61}$

(C) $\frac{15}{61}$ (D) $\frac{60}{61}$

35. There are four players A_1, A_2, A_3, and A_4 in pool A and four players B_1, B_2, B_3, and B_4 in pool B all players are equally capable. They are arbitrarily paired in their respective pools to play against each other and one winner is decided in each pair to play in semi-final. It is known that whenever A_1 plays with A_2, A_1 wins the game and whenever B_1 plays with B_2, B_1 wins the game then the probabilities that

i) A_1 and A_2 reach in the semifinal

ii) A_1, A_2 and B_2, B_3 reach semifinal are respectively

(A) $\frac{1}{6}, \frac{1}{72}$ (B) $\frac{1}{6}, \frac{1}{6}$

(C) $\frac{1}{6}, \frac{1}{16}$ (D) none

36. The probability that $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ is an integer $x, y \in \{1, 2, 3, 4\}$ is

(A) $\frac{1}{16}$ (B) $\frac{3}{16}$

(C) $\frac{15}{16}$ (D) $\frac{14}{16}$

37. A signal which can be green or red with probability $\left(\frac{4}{5}\right)$ and $\left(\frac{1}{5}\right)$ respectively is received by the station A and transmitted to B. The probability each station receive signal correctly $= \left(\frac{3}{4}\right)$. If signal in received in B is green. The probability original signal was green.

(A) $\frac{3}{5}$ (B) $\frac{6}{7}$

(C) $\frac{20}{23}$ (D) $\frac{9}{20}$

38. Let w be complex cube root of unity with $w \neq 1$. A fair die is thrown 3 times. If r_1, r_2, r_3 be the numbers obtained on the die the probability that $w^{r_1} + w^{r_2} + w^{r_3} = 0$ is

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$

(C) $\frac{2}{9}$ (D) $\frac{1}{36}$

39. If the product of three positive real numbers is 1 and their sum is greater than sum of their reciprocals then

(A) Exactly one of them exceeds 1

(B) Each of them equals

(C) Exactly one of them less than 1

(D) none of these

40. If two events A and B such that P(A′) = 0.3, P(B) = 0.5 and P(A ∩ B) = 0.3, then P(B | AUB′) is

(A) 3/8 (B) 2/3

(C) 5/6 (D) 1/4

41. A bag contains some white and some black balls, all combinations of balls being equally likely. The total number of balls in the bag is 10. If three balls are drawn at random without replacement and all of them are found to be black, the probability that the bag contains 1 white and 9 black balls is

(A) 14/55 (B) 12/55

(C) 2/11 (D) 22/55

42. There is a five-volume dictionary among 50 books arranged on a shelf in random order. The probability that these volumes stand in increasing order from left to right (the volumes are not necessarily kept side-by-side) is

(A) 1/5 (B) $1/5^{50}$

(C) $1/50^5$ (D) 1/120.

43. If a and b are chosen randomly from the set consisting of numbers 1, 2, 3, 4, 5, 6 with replacement. Then the probability that $\lim\limits_{x\to 0}\left[\left(a^x + b^x\right)/2\right]^{2/x} = 6$ is

(A) 1/3 (B) 1/4
(C) 1/9 (D) 2/9

44. Two numbers x and y are choosen at random without replacement from the set $\{1,2,3,4,\ldots 100\}$. The probability that $x^4 - y^4$ is divisible by 5 is

(A) 67 / 99 (B) 32 / 99
(C) 332 / 495 (D) 134 / 225

45. From a group of n persons arranged in a circle, 3 persons are selected at random. If probability that no two adjacent persons are selected is $\frac{2}{7}$ then $n =$

(A) 6 (B) 7
(C) 8 (D) 9

46. A box has n coins, m of which are fair and the rest are biased, the probability of getting head on biased coin is 2/3. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. Then the probability that the coin drawn is fair is

(A) $\frac{m}{m+8n}$ (B) $\frac{3m}{m+8n}$
(C) $\frac{9m}{m+8n}$ (D) $\frac{n}{8m+n}$

47. A number x is chosen from the set $A=\{33^n : n \in N\}$, the probability that x has 3 in the units place is

(A) 1/2 (B) 1/3
(C) 1/4 (D) 3/5

48. There are 12 railway stations in a railway line A train stops at 4 stations randomly. Then probability that exactly three of them are consecutive is

(A) $\frac{28}{165}$ (B) $\frac{8}{55}$
(C) $\frac{8}{65}$ (D) $\frac{4}{55}$

49. Three numbers are chosen at random from the set $\{1,2,3,\ldots 20\}$. The probability that the minimum of these chosen numbers is 7 is

(A) $\frac{13}{190}$ (B) $\frac{26}{57}$
(C) $\frac{11}{190}$ (D) none of these

50. A and B are events of an experiment such that $0 < P(A), P(B) < 1$. If $P(B^C) > P(A^C)$, then

(A) $P(A \cap B^C) < P(A^C \cap B)$
(B) $P(A \cap B^C) = P(A^C \cap B)$
(C) $P(B \mid A) < P(A \mid B)$
(D) $P(B \mid A) > P(A \mid B)$

51. Four identical oranges and six distinct apples (each a different variety) are distributed randomly into five distinct boxes. The probability that each box gets a total of two objects is

(A) $\frac{813}{109375}$ (B) $\frac{162}{21875}$
(C) $\frac{323}{43750}$ (D) $\frac{151}{21875}$

52. A natural number x is chosen from the first 100 natural numbers. The probability that $\frac{x^2-60x+800}{x-30} < 0$ is

(A) $\frac{3}{25}$ (B) $\frac{1}{50}$
(C) $\frac{7}{25}$ (D) none of these

53. Let A, B, C be three events. If the probability of occurring exactly one event out of A and B is $1 - a$, out of B and C is $1 - 2a$, out of C and A is $1 - a$ and that of occurring three events simultaneously is a^2, then the probability that at least one out of A, B, C will occur is

(A) 1/2 (B) greater than 1/2
(C) less than 1/2 (D) none

54. Two persons each make a single throw with a pair of dice. The probability that their throws are unequal is:

(A) $\frac{1}{6^3}$ (B) $\frac{70}{6^3}$
(C) $\frac{115}{216}$ (D) $\frac{575}{648}$

55. The probability that when 12 distinct balls are distributed among three boxes, the first will contain three ball is

(A) $\frac{2^9}{3^{12}}$ (B) $\frac{{}^{12}C_9 \,.\, 2^9}{9^6}$
(C) $\frac{{}^{12}C_3 \,.\, 2^{12}}{3^{12}}$ (D) None

56. Consider the system of equations $ax + by = 0$ and $cx + dy = 0$ where $a, b, c, d \in \{1, 2\}$. The probability that the system of equations has a unique solution is

(A) 3/8 (B) 5/16
(C) 9/16 (D) 5/8

57. Let A, B, C be any three events in a sample space of a random experiment. Let the events E_1 = exactly one of A, B occurs E_2 = exactly one of B, C occurs, E_3 = exactly one of C, A occurs, E_4 = all of A, B, C occurs, E_5 = atleast one of A, B, C occurs. $P(E_1) = P(E_2) = P(E_3) = 1/3$, $P(E_4) = \frac{1}{9}$ then $P(E_5) =$

(A) $\frac{1}{9}$ (B) $\frac{7}{9}$
(C) $\frac{5}{18}$ (D) $\frac{11}{18}$

58. Box I contains 3 white, 1 black; Box II contains 2 white, 2 black and Box III contains 1 white, 3 black balls. If from each of these boxes one ball is drawn at random, find the probability that 2 white, 1 black balls will be drawn

(A) 13/32 (B) 1/4
(C) 3/16 (D) 1/32

59. In a game of chance a player throws a pair of dice and scores points equal to the difference between the numbers on the two dice. Winner is the person who scores exactly 5 points more than his opponent. If two players are playing

this game only one time, then the probability that neither of them wins, is

(A) $\frac{1}{54}$ (B) $\frac{1}{108}$

(C) $\frac{53}{54}$ (D) $\frac{107}{108}$

60. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was

(A) $\frac{7}{2^{13}}$ (B) $\frac{1}{2^{13}}$

(C) $\frac{13}{14}$ (D) $\frac{13}{2^{13}}$

61. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2 respectively. If they race three times the probability that "the same horse wins all the three races" and the probablity that a, b, c each wins one race, are respectively

(A) $\frac{8}{50}; \frac{9}{50}$ (B) $\frac{16}{100}, \frac{3}{100}$

(C) $\frac{12}{50}; \frac{15}{50}$ (D) $\frac{10}{50}; \frac{8}{50}$

62. Assume that the birth of a boy or girl to a couple to be equally likely, mutually exclusive, exhaustive and independent of the other children in the family. For a couple having 6 children, the probability that their "three oldest are boys" is

(A) $\frac{20}{64}$ (B) $\frac{1}{64}$

(C) $\frac{8}{64}$ (D) none

63. Box A contains 3 red and 2 blue marbles while box B contains 2 red and 8 blue marbles. A fair coin is tossed. If the coin turns up heads, a marble is drawn from A, if it turns up tails, a marble is drawn from bag B. The probability that a red marble is chosen, is

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$

(C) $\frac{3}{5}$ (D) $\frac{1}{2}$

64. If E_1, E_2 are two events such that $P(E_1)=\frac{3}{10}$, $P(E_2)=\frac{1}{4}$ and $P(E_1 \cap E_2)=\frac{1}{5}$ then $P(\overline{\overline{E_1}/\overline{E_2}})=$

(A) $\frac{2}{15}$ (B) $\frac{11}{15}$

(C) $\frac{13}{15}$ (D) $\frac{14}{15}$

65. An even number of cards are drawn from a well shuffled pack of playing cards. The probability that half of them are red and the other half are black is

(A) $\frac{{}^{52}C_{26}-1}{2^{51}-1}$ (B) $\frac{{}^{52}C_{26}-1}{2^{51}}$

(C) $\frac{{}^{52}C_{26}}{2^{51}}$ (D) $\frac{{}^{52}C_{26}}{2^{51}-1}$

66. The altitude through A of ΔABC meets BC at D and the circumscribed circle at E. If $D = (2, 3)$, E = (5, 5) and the ordinate of the orthocentre being a natural number. The probability that the orthocentre lies on the lines $y = 1$, $y = 2$, $y = 3$,… $y = 10$ is

(A) $\frac{2}{5}$ (B) $\frac{1}{5}$

(C) $\frac{3}{5}$ (D) None of these

67. Two real numbers x, y are chosen from the interval [0, 8] then the probability that $y^2 > 2x$ is

(A) $\frac{1}{3}$ (B) $\frac{11}{32}$

(C) $\frac{11}{64}$ (D) $\frac{2}{3}$

68. In a 3×3 matrix the entries a_{ij} are randomly selected from the digits $\{0,1,2,\ldots\ldots.9\}$ with replacement. The probability that the 3-digits number in each row will be divisible by 11 is

(A) $\frac{7^3}{10^6}$ (B) $\frac{13^3}{10^6}$

(C) $\frac{78^3}{10^9}$ (D) $\frac{91^3}{10^9}$

69. A bag contains two red balls and two green balls. A person randomly pulls out a ball, replacing it with a red ball regardless of the colour. What is the probability that all the balls are red after three such replacements?

(A) $\frac{3}{8}$ (B) $\frac{7}{16}$

(C) $\frac{5}{32}$ (D) $\frac{9}{32}$

70. $2n$ girls are randomly divided into two subgroups containing n girls each. The probability that the tallest girls are in different groups is

(A) $\frac{n+1}{2(2n-1)}$ (B) $\frac{n-1}{2n-1}$

(C) $\frac{2n-1}{4n^2}$ (D) None of these

71. 'A' tosses a fair coin. If it shows a tail 'A' is asked to roll a fair die and A's score is the number that die shows. If the coin shows a head, 'A' is asked to toss five more coins and A's score is total number of heads shown (including the first coin). If 'A' tells you that his score is only 3 then the probability that 'A' rolls a die is

(A) $\frac{23}{96}$ (B) $\frac{1}{6}$

(C) $\frac{8}{23}$ (D) $\frac{7}{44}$

72. Fifteen coupons are numbered 1,2,3, ... ,15 respectively. Seven coupons are selected at random one at a time with replacement the probability that the largest number appearing on a selected coupon is 9 is

(A) $\left(\frac{3}{5}\right)^7 - \left(\frac{8}{15}\right)^6$ (B) $\left(\frac{8}{15}\right)^7$

(C) $\left(\frac{3}{5}\right)^7$ (D) $\left(\frac{3}{5}\right)^7 - \left(\frac{8}{15}\right)^7$

73. From 10 pairs of shoes, 6 shoes are selected at random. Then probability that there is atleast one correct pair in the selected shoes is

(A) $\frac{{}^{10}C_6 . 2^6}{{}^{20}C_6}$ (B) $\frac{211}{323}$

(C) $\frac{{}^{10}C_2 . {}^{8}C_2 . 2^2}{{}^{20}C_{10}}$ (D) None

74. A number is chosen at random from the set of all 4-digit numbers each of which contains not more than 2 different digits, probability that it does not contain the digit zero is

(A) $\frac{7}{64}$ (B) $\frac{37}{64}$

(C) $\frac{47}{64}$ (C) $\frac{57}{64}$

75. There are 3 bags. Bag 1 contain 2 red and $a^2 - 4a + 8$ black balls, bag 2 contains 1 red and $a^2 - 4a + 9$ black balls and bag 3 contains 3 red and $a^2 - 4a + 7$ black balls. A ball is draw at random from at random chosen bag. Then maximum value of probability that it is a red ball is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) $\frac{2}{9}$ (D) $\frac{4}{9}$

76. Let p, q be chosen one by one from the set $\{1, \sqrt{2}, \sqrt{3}, 2, e, \pi\}$ with replacement. Now a circle is drawn taking (p, q) as its centre then the probability that atmost two rational points exist on circle is (Rational points are those points whose both co-ordinates are rational)

(A) $\frac{3}{4}$ (B) $\frac{5}{6}$

(C) $\frac{7}{8}$ (D) $\frac{8}{9}$

77. A, B are two independent events such that P(A) > $\frac{1}{2}$ P(B) > $\frac{1}{2}$. If P(A $\cap$ $\bar{B}$) = $\frac{3}{25}$ and P($\bar{A}$ $\cap$ B) = $\frac{8}{25}$ then P(A $\cap$ B) =

(A) 3/4 (B) 2/3
(C) 12/25 (D) 18/25

78. Two persons A and B are throwing 3 dice taking turns. If A throws 8 then the probability that B throws a higher number is

(A) 5/27 (B) 9/17
(C) 8/27 (D) 20/27

79. Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. The probability that the three selected consists of 1 girl and 2 boys is

(A) 5/9 (B) 13/32
(C) 12/19 (D) 25/64

80. From a bag containing 10 distinct balls, 6 balls are drawn simultaneously and replaced. Then 4 balls are drawn. The probability that exactly 3 balls are common to the drawings is

(A) 8/21 (B) 6/19
(C) 5/24 (D) 9/22

81. A bag contains 4 red and 3 blue balls. Two drawings of two balls are made. The probability that the first drawing gives 2 red balls and the second drawing gives two blue balls if the balls are not returned to the bag after the first draw is

(A) 2/49 (B) 3/35
(C) 3/10 (D) 1/4

82. A team has probability 2/3 of winning a game whenever it plays. If the team plays 4 games then the probability that it wins more than half of the games is

(A) 17/25 (B) 15/19
(C) 16/27 (D) 13/20

83. A bag contains 5 balls. Two balls are drawn and found them to be red. Then the prabability that all the balls are red is

(A) $\frac{1}{4}$ (B) 1

(C) $\frac{1}{2}$ (D) $\frac{1}{10}$

84. Let $p(x, y)$ be an interior point of $x^2 + y^2 - 2x - 3 = 0$ or $x^2 + y^2 + 2x - 3 = 0$. Then the probability that p lies in the interior of both the circles.

(A) $\frac{4\pi - 3\sqrt{3}}{8\pi}$ (B) $\frac{8\pi - 3\sqrt{3}}{16\pi + 3\sqrt{3}}$

(C) $\frac{4\pi - 3\sqrt{3}}{20\pi + 3\sqrt{3}}$ (D) $\frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$

85. Four whole numbers, taken at random, are multiplied together. The probability that last digit in the product is 1, 3, 7 or 9 is

(A) $\frac{1}{125}$ (B) $\frac{16}{625}$

(C) $\frac{1}{25}$ (D) $\frac{3}{25}$

86. A die is thrown $2n + 1$ times, $n \in N$. The probability that faces with even numbers show up odd numbers of times is

(A) $\frac{2n+1}{4n+3}$ (B) Less than $\frac{1}{2}$

(C) Greater than $\frac{1}{2}$ (D) $\frac{1}{2}$

87. The probabilities of A, B, C solving a problem independently are respectively $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. If 21 such problems are given to A, B, C then the probability that atleast 11 problems can be solved by them is

(A) ${}^{21}C_{11}\left(\frac{1}{2}\right)^{11}$ (B) $\frac{1}{2}$

(C) $\left(\frac{1}{2}\right)^{11}$ (D) ${}^{21}C_{11}\frac{2^{11}}{3^{21}}$

88. A coin is tossed 20 times. The probability of atleast 11 consecutive heads is

(A) $\frac{11}{2^{12}}$ (B) $\frac{12}{2^{11}}$

(C) $\frac{9}{2^{12}}$ (D) $\frac{12}{2^{9}}$

89. A bag contains a large number of white and black marbles in equal proportions. Two samples of 5 marbles are selected (with replacement) at random. The probability that the first sample contains exactly 1 black marble, and the second sample contains exactly 3 black marbles, is

(A) $\frac{25}{512}$ (B) $\frac{15}{32}$

(C) $\frac{15}{1024}$ (D) $\frac{35}{256}$

90. $x_1, x_2, x_3, \ldots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \ldots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle number is

(A) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (B) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$

(C) $\frac{{}^{19}C_2 \times {}^{31}C_3}{{}^{50}C_5}$ (D) $\frac{{}^{29}C_2 \times {}^{21}C_3}{{}^{50}C_5}$

91. Let P_n is the probability that n throws of a die contain an odd number of sixes, then

(A) $6P_n - 4P_{n-1} = 1$ (B) $6P_n - 5P_{n-1} = 1$

(C) $4P_n - 3P_{n-1} = 1$ (D) $P_n = 2P_n - 1$

92. The probability that an electric bulb will last for 1000 hours or more is 0.7 and that it will last atmost 1500 hours is 0.8. The probability that the bulb will last 1000 to 1500 hours is

(A) 0.7 (B) 0.5

(C) 0.8 (D) 0.75

93. If A and B each toss three coins then the probability that both gets the same number of heads is

(A) $\frac{1}{9}$ (B) $\frac{3}{16}$

(C) $\frac{5}{16}$ (D) $\frac{3}{8}$

94. In a convex hexagon two diagonals are drawn at random. The probability that the diagonals intersect at an interior point of the hexagon, is

(A) $\frac{5}{12}$ (B) $\frac{7}{12}$

(C) $\frac{2}{5}$ (D) $\frac{3}{5}$

95. Suppose there are 8 white and 2 red balls in a packet, each time one ball is drawn and replaced by a white one. Then probability of drawing out all the red balls just in the 4th drawn is:

(A) 0.0534 (B) 0.0434

(C) 0.12 (D) none of these

96. For independent events $A_1, \ldots, A_n$, $P(A_i) = \frac{1}{i+1}$, $i = 1, 2, \ldots, n$. Then the probability that none of the events will occur is

(A) $n/(n+1)$ (B) $n - 1/(n+1)$

(C) $1/(n+1)$ (D) $n+1/(n+2)$

97. Each of the 'n' urns contain 4 white and 6 black balls. The $(n + 1)^{\text{th}}$ urn contains 5 white and 5 black balls. One of the urn is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n + 1)^{\text{th}}$ urn was chosen to draw the balls is $\frac{1}{16}$, then the value of n is

(A) 10 (B) 11

(C) 12 (D) 13

98. An integer from the set of first 100 natural numbers is selected at random. The probability that the number satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$ is

(A) $\frac{3}{25}$ (B) $\frac{19}{25}$

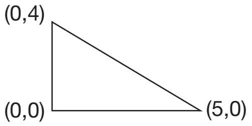
(C) $\frac{2}{25}$ (D) $\frac{7}{25}$

99. Let $A = \{u, v, w, x\}$. A bijective function $f : A \to A$ is chosen at random, then the probability that $f(u) \neq u$, $f(v) \neq v, f(w) \neq w$ and $f(x) \neq x$ is

(A) $\frac{3}{8}$ (B) $\frac{11}{24}$

(C) $\frac{1}{64}$ (D) $\frac{7}{24}$

100. On a wall, a triangle is drawn as shown in figure suppose you have thrown an arrow on the target triangle and it sticks at the interior of the triangle. If it sticks at a point P (x, y) with $x < y$, then you will win the game. The probability that you win the game?



(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{4}{9}$ (D) $\frac{2}{5}$

101. A two headed coin, a two tailed coin and a fair coin are placed in a bag. A coins is drawn at random and flipped; it turns up 'head '. What is the probability that there is a head on the other side of the coin?

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{2}{3}$ (D) $\frac{3}{4}$

102. If a triangle is formed using the vertices of a cube then the probability that the triangle is equilateral is ____

(A) $\frac{2}{7}$ (B) $\frac{3}{8}$

(C) $\frac{1}{7}$ (D) $\frac{5}{7}$

103. A man firing a distant target has 20% chance of hitting the target in one shoot. If P be the probability of hitting the target in 'n' attempts where $20P^2 - 13P + 2 \le 0$, then maximum and minimum value of n is

(A) 3, 2 (B) 5, 4

(C) 4, 2 (D) 2, 2

104. A positive integer k not exceeding 100 is chosen in such a way that it $k \le 50$, then the probability of choosing k is P, and if $k > 50$, then the probability of choosing k is 3P. The probability that a perfect square is chosen is

(A) 0.05 (B) 0.065

(C) 0.08 (D) 0.09

105. In a n sided regular polygon the probability that the two diagonal chosen at random will intersect inside the polygon is

(A) $\frac{2\,{}^{n}C_2}{{}^{{}^nC_2-n}C_2}$ (B) $\frac{{}^{n(n-1)}C_2}{{}^{{}^nC_2-n}C_2}$

(C) $\frac{{}^{n}C_4}{{}^{{}^nC_2-n}C_2}$ (D) none

106. If each face of a cubical die is numbered with a distinct number from among the first six odd numbers, such that the sum of two numbers on any pair of opposite face is 12. If ten such dice are thrown simultaneously, then the probability that the sum of the numbers that turn up is exactly 53, is

(A) $\frac{53}{6^{10}}$ (B) $\frac{153}{6^{10}}$

(C) $\frac{3}{6^{10}}$ (D) none of these

107. A fair die is tossed three times. Given that the sum of the first two tosses equals the third. The probability that at least one '2' has turned up, is equal to

(A) $\frac{1}{6}$ (B) $\frac{91}{216}$

(C) $\frac{1}{2}$ (D) $\frac{8}{15}$

108. The sum of two non-negative real numbers is $2a$. The probability that the product of these two numbers is not less then $\frac{3}{4}$ times the greatest possible product is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) $\frac{9}{16}$

109. Let $S = \{a \in N, a \le 1000\}$. If equation $\left[\tan^2 2x\right] - \tan 2x - a = 0$ has real roots then probability of getting element a from set S which satisfy given equation

(A) $p = \frac{25}{999}$ (B) $p = \frac{25}{999}$

(C) $p = \frac{31}{1000}$ (D) $p = \frac{27}{1000}$

110. Two boxes containing 20 balls each and each ball is either black or white. The total number of black balls is different from the total numbers of white balls. One ball is drawn at random from each box. The probability that both balls are white is 0.21, then the probability that both the balls drawn are black is

(A) 0.23 (B) 0.24

(C) 0.25 (D) 0.26

111. Matrices of order 3 × 3 are formed by using the elements of the set $A = \{-3, -2, -1, 0, 1, 2, 3\}$, then probability that matrix is either symmetric or skew symmetric is

(A) $\frac{1}{7^6} + \frac{1}{7^3}$ (B) $\frac{1}{7^9} + \frac{1}{7^3} - \frac{1}{7^6}$

(C) $\frac{1}{7^3} + \frac{1}{7^9}$ (D) $\frac{1}{7^3} + \frac{1}{7^6} - \frac{1}{7^9}$

112. An integer is selected at random from between 1 and 200, then probability that integer is divisible by 2, 3, 6, or 8 is

(A) $\frac{133}{200}$ (B) $\frac{133}{198}$

(C) $\frac{33}{50}$ (D) $\frac{2}{3}$

113. If the probability of choosing an integer k out of '$2m$' integers 1, 2, 3, ..., $2m$ is inversely proportional to $k^4\,(1 \le k \le 2m)$, then the probability that chosen number is odd, is

(A) > 1 (B) $> \frac{1}{2}$

(C) $= \frac{1}{2}$ (D) $< \frac{1}{3}$

114. Let a function f: $x \to y$ is defined where $x = \{0, 1, 2, 3, \dots, 9\}$, $y = \{0, 1, 2, \dots, 100\}$ and $f(5) = 5$, then the probability that the function of type $f: x \to$ B where $B \subseteq Y$ is of bijection in nature is

(A) $\dfrac{10!}{\sum_{r=1}^{101} r^9 \cdot {}^{100}C_{r-1}}$ (B) $\dfrac{{}^{101}C_9 . 9!}{\sum_{r=1}^{101} r^{10} \cdot {}^{100}C_r}$

(C) $\dfrac{{}^{100}C_9 . 9!}{\sum_{r=1}^{101} r^{10} \cdot {}^{101}C_r}$ (D) $\dfrac{{}^{100}C_9 . 9!}{\sum_{r=1}^{101} r^9 \cdot {}^{100}C_{r-1}}$

115. Two subsets A and B of a set containing n elements are chosen at random. The probability that A $\subseteq$ B is

(A) $\dfrac{1}{2}$ (B) $\dfrac{2^n}{n!}$

(C) $\left(\dfrac{2}{3}\right)^n$ (D) $\left(\dfrac{3}{4}\right)^n$

116. On a given circle 6 points A, B, C, D, E, F are chosen at random, independently and uniformly with respect to arc length. The probability that 2 triangles ABC and DEF are disjoint, i.e. have no common points, is

(A) $\dfrac{7}{10}$ (B) $\dfrac{3}{10}$

(C) $\dfrac{2}{3}$ (D) $\dfrac{1}{3}$

117. A determinant is chosen at random from the set of all determinants of order 2 with elements 1 and 0 only. The probability that the value of the determinant is positive is

(A) $\dfrac{1}{8}$ (B) $\dfrac{3}{16}$

(C) $\dfrac{5}{16}$ (D) $\dfrac{1}{16}$

118. Two numbers x and y are chosen at random from the set {1, 2, 3, …….. $3n$}. The probability that $x^2 - y^2$ is divisible by 3 is

(A) $\dfrac{5n-3}{3(3n-1)}$ (B) $\dfrac{2n}{3(3n-1)}$

(C) $\dfrac{5n}{3(3n-1)}$ (D) $\dfrac{n}{3n-1}$

119. In shuffling a pack of cards 3 are accidentally dropped. The chance that the missing cards are of different suits is.

(A) $\dfrac{169}{425}$ (B) $\dfrac{169}{1700}$

(C) $\dfrac{1}{4}$ (D) $\dfrac{169}{2550}$

120. There are 7 applicants (3 men and 4 women) and two interview boards. Each applicant is interviewed by exactly one board. Each board interviews at least one applicant. The chance that all women are interviewed by same board is

(A) $\dfrac{2}{3}$ (B) $\dfrac{1}{9}$

(C) $\dfrac{1}{18}$ (D) $\dfrac{1}{6}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $O < P(A) < 1, O < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)P(B)$ then which of the following is correct?

(A) $P\left(\dfrac{B}{A}\right) = P(B) - P(A)$

(B) $P(\overline{A} \cup \overline{B}) = P(\overline{A}) + P(\overline{B})$

(C) $P(\overline{A \cup B}) = P(\overline{A})P(\overline{B})$

(D) $P\left(\dfrac{A}{B}\right) = P(A)$

2. Words are formed using all the letters of the word “CANADA”. A word is selected from these words randomly

(A) Probability that the word neither contains the pattern CAN nor ADA is $\dfrac{49}{60}$

(B) Probability that the word neither contains the pattern CAN nor ADA is $\dfrac{43}{60}$

(C) Probability that the vowels and constants occur alternately is $\dfrac{1}{10}$

(D) Probability that all vowels and all consonants occur together is $\dfrac{1}{10}$

3. Cards are drawn one by one in a pack of well shuffled cards without replacement until all the cards are drawn then

(A) Chance of getting a spade at the 13th trail is 1/13

(B) Chance of getting a spade at the 13th trail is 1/4

(C) Chance of getting last card as a spade is 1/4

(D) Chance of getting king at the 5th trail and queen at the 10th trail is 4/663

4. A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one and examined. The one examined are not put back. Then

(A) Probability of getting exactly 3 defectives in the examination of 8 record players is $\frac{{}^{4}C_3 \times {}^{11}C_5}{{}^{15}C_8}$

(B) Probability that 9^{th} one examined is the last defective is 8/195

(C) Probability that 9^{th} examined record player is defective given that there were 3 defectives in the first 8 players examined is 1/7

(D) Probability 9^{th} one examined is last defective is 8/197

5. A rational number is selected at random from the set of all rational numbers from the interval (2010, 2011) all whose digits after the decimal point are non -zero and are in the decreasing order then

(A) The probability that it has exactly seven digits after the decimal point is $\frac{36}{511}$

(B) The probability that it contains the digit 3 after the decimal point is $\frac{256}{511}$

(C) The probability that the last digit after the decimal point is atleast 4 is $\frac{63}{511}$

(D) The probability that the last digit after the decimal point is 4 is $\frac{32}{511}$

6. A bag contains four tickets with numbers 112, 121, 211, 222. One ticket is drawn at random from the bag. Let denote the event that 'i'th digit on the drawn ticket is 2 then

(A) are pair – wise independent
(B) are independent
(C) are not independent
(D) are mutually independent

7. A boy has a collection of blue and green marbles. The number of blue marbles belong to the set $\{2,3,4,\ldots\ldots\ldots\ldots,13\}$. If two marbles are chosen simultaneously and at random from his collection, then the probability that they have different colours is $\frac{1}{2}$. Possible number of blue marbles is

(A) 3 (B) 6
(C) 10 (D) 12

8. A and B are two independent events. The probability that A and B happen simultaneously is $\frac{1}{12}$ and neither A nor B happens is $\frac{1}{2}$, then

(A) $P(A)=\frac{1}{3},\ P(B)=\frac{1}{4}$ (B) $P(A)=\frac{1}{2},\ P(B)=\frac{1}{6}$

(C) $P(A)=\frac{1}{4},\ P(B)=\frac{1}{3}$ (D) $P(A)=\frac{1}{6},\ P(B)=\frac{1}{2}$

9. For two events A and B let P(A) = 3/5, P(B) = 2/3 then which of the following statements is correct?

(A) $P\left(A\cap\bar{B}\right)\le\frac{1}{3}$ (B) $P(A\cup B)\ge\frac{2}{3}$

(C) $\frac{4}{15}\le P(A\cap B)\le\frac{3}{5}$ (D) $\frac{2}{5}\le P(A/B)\le\frac{9}{10}$

10. A number x is chosen at random from the first 100 natural numbers. The probability that it satisfies

(A) $x^2-25x\le 150$ is 0.3
(B) $x^2-17x+30\ge 0$ is 0.88
(C) $30x-x^2$ is a perfect square of a natural number is 0.07
(D) $30x-x^2<0$ is 0.7

11. When a fair die is thrown twice, let (a, b) denote the outcome in which the first throw shows a and the second shows b. Further, let A,B and C be the following events: $A=\{(a,b)\mid a \text{ is odd}\}$, $B=\{(a,b)\mid b \text{ is odd}\}$ and $C=\{(a,b)\mid a+b \text{ is odd}\}$. Then

(A) $P(A\cap B)=1/4$ (B) $P(B\cap C)=1/4$
(C) $P(A\cap C)=1/4$ (D) $P(A\cap B\cap C)=0$

12. The probability that a student passes in mathematics, physics and chemistry are m, p and c respectively. Of these subjects the student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two and a 40% chance of passing exactly two. Which of the following statements is true.

(A) $p+m+c=\frac{19}{20}$ (B) $p+m+c=\frac{27}{20}$

(C) $pmc=\frac{1}{10}$ (D) $pmc=\frac{1}{4}$

13. The probability that a 50-year-old man will be alive at 60 is 0.83 and the probability that a 45 year-old woman will be alive at 55 is 0.87. Then

(A) The probability that both will be alive for the next 10 years is 0.7221
(B) At least one of them will alive for the next 10 years is 0.9779
(C) At least one of them will alive for the next 10 years is 0.8230
(D) The probability that both will be alive for the next 10 years is 0.6320

14. A cubical die is thrown 9 times and the numbers obtained are written as a 9-digit number. The probability that the number.

(A) begins with 246 is $1/6^3$
(B) ends with 135 is $1/6^3$
(C) begins with 246 and ends with 135 is $1/6^6$
(D) begins with 246 or ends with 135 is $431/6^6$.

15. P_1, P_2, P_3 and P_4 are four players playing in a knockout tournament and chance that P_1 will pair up with P_j is proportional to j and P_i wins if $i<j$, then probability that P_2 will reach the next round is:

(A) $\frac{1}{4}$ (B) $\frac{7}{9}$

(C) $\frac{2}{3}$ (D) none of these

16. A and B are independent events such that $P\left(\bar{A}\cap B\right)=2/15$ and $P\left(A\cap\bar{B}\right)=1/6$ then $P(B)$ can be

(A) 1/5 (B) 4/5
(C) 1/6 (D) 5/6

17. A certain coin is tossed with probability of showing head being *p*. Let *Q* denotes the probability that when the coin is tossed four times the number of heads obtained is even. Then

(A) there is no value of p, if $Q=\frac{1}{4}$

(B) there is exactly one value of *p* if $Q=\frac{3}{4}$

(C) there are exactly two values of *p* if $Q=\frac{3}{5}$

(D) there are exactly four values of *p* if $Q=\frac{4}{5}$

18. Two candidates appear for a competitive exam Independently. Let E and F denote the events of their failing in exam. The probability that both fails is $\frac{1}{12}$. The probability that neither fails, is $\frac{1}{2}$ then

(A) $P(E)=\frac{1}{3},\ P(F)=\frac{1}{4}$

(B) $\frac{P(A\cap W)}{P(W)}\frac{\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)}{\frac{1}{3}}\left(\frac{6}{48}\right)\frac{3}{32}$

(C) $P(E)=\frac{1}{6},\ P(F)=\frac{1}{2}$

(D) $P(E)=\frac{1}{4},\ P(F)=\frac{1}{3}$

19. From a pack of cards A man is drawing cards one by one without replacement. Consider the events
E1: The event that the 10th card drawn is the second hearts card.
E2: The event that the 8th card drawn is third Ace
E3 : The event that the 9th card drawn is first king
E4 : The event that the 12th card drawn is clubs '6'

(A) $P(E_1)=\frac{^{39}C_8}{^{51}C_9}$ X 3 (B) $P(E_2)=\frac{594}{7735}$

(C) $P(E_3)=\frac{4\left(^{48}C_8\right)}{9\left(^{52}C_9\right)}$ (D) $P(E_4)=\frac{1}{52}$

20. If *n* different objects are distributed among $n+2$ persons, then

(A) Probability that exactly 2 persons will get nothing is $\frac{(n+1)!}{2.(n+2)^{n-1}}$

(B) Probability that exactly 3 persons will not get anything is $\frac{^{n+2}C_3(n-1)\left(^{n}C_2\right)(n-2)!}{(n+2)^n}$

(C) Probability that exactly 3 persons will not get anything is $\frac{n.(n-1)^2.(n+1)}{12(n+2)^{n-1}}$

(D) Probability that exactly 2 persons will get nothing is $\frac{n^2.(n-1)^2}{(n+2)^{n-1}}$

21. Given that $x\in[0,1]$ and $y\in[0,1]$. Let A be the event of selecting a point (*x*, *y*) satisfying $y^2\le x$ and B be the event selecting a point (*x*, *y*) satisfying $x^2\le y$, then

(A) $P(A\cap B)=\frac{1}{3}$ (B) $A\subset B$

(C) $2P(A)=3P(B)$ (D) $P(B)<P(A)$

22. Two numbers are chosen from {1, 2, 3, 4, 5, 6, 7, 8} one after another without replacement. Then the probability that

(A) The smaller value of two is less than 3 is 13/28

(B) The bigger value of two is more than 5 is 9/14

(C) Product of two number is even is 11/14

(D) None of these

23. K subsets of a set $A\equiv\{1,2,3,\ldots..,n\}$ are chosen at random with replacement. The probability that their intersection is

(A) empty is $\left(\frac{1}{2}\right)^n$ for K = 2

(B) empty is $\left(\frac{3}{4}\right)^n$ for K = 2

(C) empty is $\left(\frac{7}{8}\right)^n$ for K = 3

(D) singleton is $n\left(\frac{7}{8}\right)^{n-1}$ for K = 3

24. In a gambling between Mr. A and Mr. B a machine continues tossing a fair coin until the two consecutive throws either HT or TT are obtained for the first time. If it is HT, Mr, A wins and if it is TT, Mr, B wins. Which of the following is (are) true?

(A) probability of winning Mr. A is $\frac{3}{4}$

(B) Probability of Mr. B winning is $\frac{1}{4}$

(C) Given first toss is head probability of Mr. A winnings is 1

(D) Given first toss is tail, probability of Mr. A winning is $\frac{1}{2}$

25. Contents of the two urns is as given in this table. A fair die is tossed. If the face 1, 2, 4 or 5 comes, a marble is drawn from the urn A otherwise a marble is chosen from the urn B.

Urn	Red Marbles	White marbles	Blue marbles
A	5	3	8
B	3	5	0

Let E_1: Denote the event that a red marble is chosen
E_2: Denote the event that a white marble is chosen
E_3: Denote the event that a blue is chosen
Then

(A) The event E_1, E_2 and E_3 are equiprobable

(B) $P(E_1)$, $P(E_2)$, $P(E_3)$ are in A.P

(C) If the marble drawn is red, the probability that it came from the urn A is $\frac{1}{2}$

(D) If the marble drawn is white, the probability that the face 5 appeared on the die is $\frac{3}{32}$

26. Let a, b, c be integers and $a^2+b^2+c^2=2$.

Let $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. Consider the system of simultaneous equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(A) The probability that the system of equations has unique solution is $\frac{1}{2}$

(B) The number of triplets (a, b, c) for which the system of equations has infinitely many solutions is 6

(C) If $a=0$, the number of ordered pairs (b, c) for which the system of equations has no solution is 2

(D) The number of elements in the range of $ab+bc+ca$ is 2

27. If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement, then the probability that the roots of the equation $x^2+px+q=0$.

(A) are real is $\frac{33}{50}$

(B) are imaginary is $\frac{19}{50}$

(C) are real and equal is $\frac{3}{100}$

(D) are real and distinct is $\frac{59}{100}$

28. If A, B are exhaustive events in a sample space such that probabilities of the events $A\cap B$, A, B and $A\cup B$ are in A.P. If P (A) = K, where 0 < K < 1, then

(A) $P(B)=\frac{K+1}{2}$ (B) $P(A\bigcap B)=\frac{3K-1}{2}$

(C) $P(A\cup B)=1$ (D) $P(A'\cup B')=\frac{3(1-K)}{2}$

29. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

(A) $p_1=1/9$ (B) $p_1=1/16$

(C) $p_2=1/3$ (D) $p_2=1/4$

30. A forecast is to be made of the result of five cricket matches, each of which can be win or a draw or a loss for Indian team let

p = number of forecasts with exactly 1 going wrong
q = number of forecasts with exactly 3 going wrong
r = number of forecasts with all 5 going wrong

Then which of the following are / is true

(A) $2q=5r$ (B) $8p=q$

(C) $8p=5r$ (D) $2(p+r)>q$

31. A bag initially contains 1 red & 2 blue balls. An experiment is performed consisting of selecting a ball at random, noting its colour & replacing it together with an additional ball of some colour. If three such trials are made, then

(A) Probability that atleast One blue ball is drawn is 0.9

(B) Probability that exactly One blue ball is drawn is 0.2

(C) Probability that all the drawn balls are red given that all the drawn balls are of same colour is 0.2

(D) Probability that atleast One red ball is drawn is 0.6

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

A slip of paper is given to a person A who marks it either with a plus sign or a minus sign. The probability of his writing a plus sign is 1/3. A passes the slip to B, who may either leave it alone or change the sign before passing it to C. Next C passes the slip to D after perhaps changing the sign. Finally D passes it to a referee after perhaps changing the sign. B, C, D each change the sign with probability 2/3.

1. The probability that the referee observes a plus sign on the slip if it is known that A wrote a plus sign is

(A) 14/27 (B) 16/27

(C) 13/27 (D) 17/27

2. The probability that the referee observes a plus sign on the slip if it is known that A wrote a minus sign is

(A) 16/27 (B) 14/27

(C) 13/27 (D) 11/27

3. If the referee observes a plus sign on the slip then the probability that A originally wrote a plus sign is

(A) 13/41 (B) 19/27

(C) 17/25 (D) 21/37

Comprehension 2

A chess match between two grandmasters X and Y is won by whoever first wins a total of two games. X's chances of winning, drawing of losing any particular game are a, b, c respectively. The games are independent and $a + b + c = 1$.

4. The probability that X wins the match after $(n + 1)$ games $(n \neq 1)$ is

(A) na^2n^{n-1}

(B) $a^2(nb^{n-1} + n(n-1)b^{n-2}c)$

(C) na^2bc^{n-1}

(D) $(n-1)a^2b^{n-2}(b + nc)$

5. The probability that Y wins the match in the 4th game is

(A) $3bc^2(b + 2a)$ (B) $bc^2\,(3b + a)$

(C) $2ac^2\,(b + c)$ (D) $abc\,(2a + 3b)$

6. The probability that X wins the match is

(A) $\dfrac{a^3 + a^2c}{(a+c)^3}$ (B) $\dfrac{a^3 + 3a^2c}{(a+c)^3}$

(C) $\dfrac{a^3}{(a+c)^3}$ (D) $\dfrac{a(a^2 + 3c)}{(1-b)^3}$

Comprehension 3

A box contains nine slipe bearing the numbers –4, –3, –2, –1, 0, 1, 2, 3, 4 (one number for one slip)

7. A slip is chosen at random, its number is noted and replaced in the box. If this experiment is repeated 5 times then the probability that exactly 3 numbers appear in the noted list

(A) $\dfrac{{}^9C_3 \times 90}{9^5}$ (B) $\dfrac{{}^9C_3 \times 243}{9^5}$

(C) $\dfrac{{}^9C_3 \times 150}{9^5}$ (D) $\dfrac{{}^9C_3 \times 240}{9^5}$

8. If a slip is chosen at random from the box and the experiment is repeated nine times (with replacement) to form a 3×3 matrix. (The chosen numbers are arranged row wise). The probability that the matrix so formed is skew symmetric

(A) $\left(\dfrac{1}{9}\right)^9$ (B) $\left(\dfrac{1}{9}\right)^6$

(C) $\left(\dfrac{1}{9}\right)^3$ (D) $\left(\dfrac{1}{9}\right)^4$

9. A slip is chosen at random from the box, its number is noted and replaced in the box. If the experiment is repeated 5 times then the probability that sum of five noted number is zero

(A) $\dfrac{3941}{9^5}$ (B) $\dfrac{3923}{9^5}$

(C) $\dfrac{3978}{9^5}$ (D) $\dfrac{3951}{9^5}$

Comprehension 4

A JEE aspirant estimates that she will be successful with an 80 percent chance if she studies 10 hours per day, with a 60 percent chance if she studies 7 hours per day and with a 40 percent chance if she studies 4 hours per day. She further believes that she will study 10 hours, 7 hours and 4 hours per day with probabilities 0.1, 0.2 and 0.7, respectively

10. The chance she will be successful, is

(A) 0.28 (B) 0.38

(C) 0.48 (D) 0.58

11. Given that she is successful, the chance she studied for 4 hours, is

(A) $\dfrac{6}{12}$ (B) $\dfrac{7}{12}$

(C) $\dfrac{8}{12}$ (D) $\dfrac{9}{12}$

12. Given that she does not achieve success, the chance she studied for 4 hour, is

(A) $\dfrac{18}{26}$ (B) $\dfrac{19}{26}$

(C) $\dfrac{20}{26}$ (D) $\dfrac{21}{26}$

Comprehension 5

Urn - I contains 5 red balls and 1 blue ball, Urn - II contains 2 red balls and 4 blue balls. A fair die is tossed. If it results in an even number, balls are repeatedly drawn one at a time with replacement from urn-I. If it is an odd number, balls are repeatedly drawn one at a time with replacement from urn-II. Given that the first two draws both are resulted in blue balls.

13. If the probability that the Urn - I is being used is p and q is the corresponding probability from Urn - II then.

(A) $q = 3p$ (B) $q = 4p$

(C) $q = 16p$ (D) $q = 2p$

14. The probability of getting a red ball in the third draw, is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$

(C) $\dfrac{41}{102}$ (D) $\dfrac{37}{102}$

Comprehension 6

Let S be the set of first 18 natural Numbers. Then attempt the following.

15. The probability of choosing $\{x, y\} \subseteq$ S such that $x^3 + y^3$ is divisible by 3.

(A) 1/3 (B) 1/6

(C) 1/5 (D) 1/4

16. The probability of choosing $\{x, y, z\} \subseteq$ S such that x, y, z are in A.P is

(A) 1/17 (B) 2/17

(C) 5/34 (D) 3/34

17. The probability of choosing $\{x, y, z\} \subseteq$ S such that no two of the numbers x, y, z are consecutive is

(A) 35/51 (B) 2/17

(C) 4/17 (D) 6/17

Comprehension 7

If H_1 is one of 6 horses entered for a race; and is to be ridden by one of two persons A and B. It is 2 to 1 that A rides H_1, in which case all the horses are equally likely to win. If B rides H_1 his chances of wining is trebled, where $E \rightarrow$ the event that horse H_1 wins

$E_1 \rightarrow$ the even that person A rides H_1.

$E_2 \rightarrow$ the even that person B rides H_1.

If $A_1 = E_1 \cap E$ and $A_2 = E_2 \cap E$, then

18. The value of $P\left(\frac{E}{E_2}\right)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$

(C) $\frac{1}{2}$ (D) None

19. The value of $P(A_2)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$

(C) $\frac{1}{2}$ (D) None

Comprehension 8

An unbiased dice with faces numbers 1, 2, 3, 4, 5, 6 is thrown five times and list of five numbers showing up is noted then.

20. The probability that among the numbers 1, 2, 3, 4, 5 and 6 only four numbers appear in the list is.

(A) $\frac{883}{7776}$ (B) $\frac{3600}{7776}$

(C) $\frac{483}{1296}$ (D) $\frac{883}{1296}$

21. The probability that the five numbers appearing on the list are in decreasing order, is

(A) $\frac{1}{1296}$ (B) $\frac{1}{7776}$

(C) $\frac{1}{216}$ (D) $\frac{1}{6}$

22. The probability that the sum of five numbers on the list is 29 is

(A) $\frac{256}{7776}$ (B) $\frac{358}{7776}$

(C) $\frac{5}{7776}$ (D) $\frac{728}{7776}$

Comprehension 9

There are n black balls and 2 red balls in a bag. One by one balls are drawn from the bag X wins as soon as 2 black balls are drawn and Y wins as soon as 2 red balls are drawn. The game continuous until one of 2 wins. Let X(n) and Y(n) denotes the probability that X and Y wins respectively

23. Then the value of Y(n) is

(A) $\frac{1}{(n+1)(n+2)}$ (B) $\frac{6}{(n+1)(n+2)}$

(C) $\frac{6}{n(n+1)(n+2)}$ (D) $\frac{1}{n(n+1)(n+2)}$

24. The value of $\lim\limits_{n\to\infty}\left(Y(1)+Y(2)+Y(3)+\cdots+Y(n)\right)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) 2 (D) 3

25. The value of $\lim\limits_{n\to\infty}\left(X(2).X(3)X(4)...X(n)\right)$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{10}$

(C) $\frac{1}{5}$ (D) $\frac{1}{20}$

Comprehension 10

Cubics are formed using $b,\ c,\ d\ \in\{0,1,2,3,4,5\}$ of the type $x^3+bx^2+cx+d=0$ then

26. The probability that the cubic is monotonic is

(A) 1/9 (B) 2/3

(C) 3/7 (D) 2/27

27. The odds in favour of the cubic possessing one local maximum and one local minimum is

(A) 2 : 5 (B) 3 : 7

(C) 1 : 7 (D) 1 : 11

28. Given that cubic possesses a local maximum and a local minimum, the chance that they both are on the negative side of x-axis is

(A) 3/4 (B) 2/3

(C) 1/4 (D) 13/18

Comprehension 11

Let E_1, E_2, E_3 are three mutually exclusive events such that $p(E_i)=p_i$ for $i=1, 2, 3$

29. If $p_1=\frac{1}{2}(1-p)$, $p_2=\frac{1}{3}(1+2p)$, $p_3=\frac{1}{5}(2+3p)$, then p belongs to

(A) $(-1,2)$ (B) $\left(-\frac{1}{3},\frac{2}{3}\right)$

(C) $\left(-1,\frac{1}{47}\right)$ (D) $\left(-\frac{1}{2},\frac{-7}{23}\right)$

30. If p_1, p_2, p_3 are roots of $27x^3-27x^2+ax-1=0$ then the value of 'a' is

(A) 9 (B) 6

(C) 3 (D) $\left[0,\frac{1}{5}\right]$

Comprehension 12

A packet of blades contains a coupon which has any one of the three letters A, B, C printed upon it. m-packets are purchased $(m \geq 3)$

31. The probability that the word CAB may be spelt by using the letters printed upon the coupons is (assuming that coupons are all distinct)

(A) $\dfrac{3.2^m - 6}{3^m}$ (B) $\dfrac{3^m - 3.2^m + 3}{3^m}$

(C) $\dfrac{m^2 - 3m + 2}{m^2 + 3m + 2}$ (D) $\dfrac{2}{m+2}$

32. The probability that the word CAB may be spelt by using the letters printed upon the coupons is (assuming that coupons of the same letter are identical)

(A) $\dfrac{3.2^m - 6}{3^m}$ (B) $\dfrac{3^m + 3.2^m + 3}{3^m}$

(C) $\dfrac{m^2 - 3m + 2}{m^2 + 3m + 2}$ (D) $\dfrac{2}{m+2}$

33. The probability that only two of the lettres A, B, C are obtained (assuming that coupons of same letter are identical) is

(A) $\dfrac{2}{m+2}$ (B) $\dfrac{4}{m+2}$

(C) $\dfrac{6}{(m+1)(m+2)}$ (D) $\dfrac{6(m-1)}{(m+2)(m+1)}$

Comprehension 13

Twenty persons among whom are A and B, sit at random at a round table.

34. The probability that there are exactly 4 persons between A and B is

(A) $\dfrac{2}{19}$ (B) $\dfrac{1}{19}$

(C) $\dfrac{4}{19}$ (D) $\dfrac{3}{19}$

35. The probability that there are at least 4 persons between A and B is

(A) $\dfrac{11}{38}$ (B) $\dfrac{10}{19}$

(C) $\dfrac{11}{19}$ (D) $\dfrac{5}{19}$

Comprehension 14

Only three horses A, B and C are in a horse race. The event that "A beats B" will be denoted by AB. The event that "A beats B and B beats C" will be denoted by ABC etc. Suppose it is known that P(AB) = 2/3, P(BC) = 1/2, P(AC) = 2/3 and that P(ABC) = P(ACB), P(BAC) = P(BCA), P(CAB) = P(CBA). [Here the symbol P(E) means the probability of the event E]

36. The probability that A wins is

(A) 5/9 (B) 2/9

(C) 1/3 (D) 1/9

37. The probability that B wins is

(A) 1/3 (B) 2/9

(C) 4/9 (D) 5/9

38. Only one of the following statements is false. Which one is it?

(A) P(BA) = 1/3

(B) P(CA) = 1/3

(C) P(CB) = 1/2

(D) the events AB, AC and CB are independent

Comprehension 15

There are four boxes A_1, A_2, A_3 and A_4. Box A_i has i balls and on each ball a number is printed, the numbers are from 1 to i. A box is selected randomly, the probability of selection of box A_i is $\dfrac{i}{10}$ and then a ball is drawn. Let E_i represents the even that a ball with number 'i' is drawn

39. The value of $\sum_{i=1}^{4} P(E_i \cap A_i)$ is

(A) $\dfrac{1}{5}$ (B) $\dfrac{1}{10}$

(C) $\dfrac{2}{5}$ (D) $\dfrac{1}{4}$

40. $P\left(\dfrac{A_3}{E_2}\right)$ is equal to

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{3}$

(C) $\dfrac{2}{3}$ (D) $\dfrac{1}{2}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match the following:

COLUMN-I	COLUMN-II
(A) 7 identical white balls and 3 identical black balls are placed in a row at random.The probability that no two black balls are adjacent is	(p) $\frac{5}{11}$
(B) 4 gentlemen and 4 ladies take seats at random round a table. The probability that they are sitting alternately is	(q) $\frac{1}{16}$
(C) 10 different books and 2 different pens are given to 3 boys, so that each gets equal number of things. The probability that the same boy does not receive both the pens, is	(r) $\frac{7}{15}$
(D) A fair coin is tossed repeatedly. The probability of getting a result in the fifth toss different from those obtained in the first four tosses is	(s) $\frac{1}{35}$
	(t) $\frac{3}{16}$

2. Match the data in Column -I with the corresponding expressions/ properties/ conditions in column - II

COLUMN-I	COLUMN-II
(A) A and B are two independent events such that P(A) = 1/2 and P(B) = 1/5	(p) $P(A \cup B) \geq \frac{3}{5}$
(B) A and B are two independent events such that P(B) = 0.2 and $P(A \cup B) = 0.6$	(q) $P(A \mid B) = \frac{1}{2}$
(C) A and B are two events such that P(A) = 1/2 and P(B) = 2/3	(r) $P(A \cap B) \leq \frac{1}{2}$
(D) A and B are two independent events such that P(A) = 3/7 and P(B) = 1/2	(s) P(Exactly one of A, B) = 0.5

3.

COLUMN-I	COLUMN-II
(A) six different balls are put in 3 different boxes, no box being empty. The probability of putting balls in boxes in equal nos.in	(p) 20/27
(B) six letters are posted in 3 letter boxes. The probability that no letter box remains empty is	(q) 1/6
(C) Two persons A and B throw two dice each. If A throws a sum of 9 then the probability of B throwing. sum greater than A is	(r) 2/7
(D) If A and B are independent and P(A) = 0.3 and P$\left(A \bigcup \bar{B}\right) = 0.8$ then P(B) =	(s) 1/4

4. A bag contains some white and some black balls, all combinations being equally likely. The total number of balls in the bag is 12. Four balls are drawn at random from the bag at random without replacement. Now match the entries from the following two columns:

COLUMN-I	COLUMN-II
(A) probability that all the four balls are black is equal to	(p) $\frac{14}{33}$
(B) if the bag contains 10 black and 2 white ball then the probability that all four balls are black is equal to	(q) $\frac{1}{5}$
(C) if all the four balls are black then the probability that the bag contains 10 black balls is equal to	(r) $\frac{70}{429}$
(D) probability that two balls are black and two are white is equal to	(s) $\frac{13}{165}$

5. Let A and B are two independent events such that P(A) = $\frac{1}{3}$ and P(B) = $\frac{1}{4}$. Now match the entries from the following two columns :

COLUMN-I	COLUMN-II
(A) P $(A \cup B)$ is equal to	(p) $\frac{1}{12}$
(B) P $(A/A \cup B)$ is equal to	(q) $\frac{1}{2}$
(C) P $(B/A' \cap B')$ is equal to	(r) $\frac{2}{3}$
(D) P (A'/B) is equal to	(s) 0

6.

COLUMN-I	COLUMN-II
(A) A number is chosen randomly from the set {1, 2, 3, … 15}. If chosen number is a two digit number, then the probability that the number is a prime numbers, is	(p) 3/8
(B) A positive integer is chosen at random. Probability that the chosen number is a multiple of 4, is	(q) 1/3
(C) If the integers m and n are chosen at random from 1 and 100, then the probability that a number of from $7^m + 7^n$ is divisible by 5, equals	(r) 1/4
(D) A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. The probability that it is actually a six is	(s) 1/8
	(t) 3/4

7. In a tournament there are twelve players S_1, S_2, …… S_{12} and divided into six pairs at random. From each game a winner is decided on the basis of a game played between the two players of the pair. Assuming all the pairs are of equal strength, then match following:

COLUMN-I	COLUMN-II
(A) Probability that S_2 is among the losers is	(*p*) $\frac{5}{22}$
(B) Probability that exactly one S_3 and S_4 is among the losers, is	(*q*) $\frac{10}{11}$
(C) Probability that both S_2 & S_4 are among the winners is	(*r*) $\frac{1}{2}$
(D) Probability of S_4 & S_5 not playing against each other is	(*s*) $\frac{6}{11}$

8.

COLUMN-I	COLUMN-II
(A) A problem is given to three students whose chances of solving it are $\frac{1}{4}, \frac{1}{5}$ and $\frac{1}{6}$ respectively. The probability that the problem is solved is	(*p*) $\frac{1}{7}$
(B) If a coin is tossed three times, then the probability of getting at least two heads is	(*q*) $\frac{1}{9}$
(C) The probability that an ordinary year has 53 sundays is	(*r*) $\frac{1}{2}$
(D) Three coins are tossed. Then the probability that all coins show head, if at least one of the coins shows head is	(*s*) $\frac{1}{3}$

9. Match the following:

COLUMN-I	COLUMN-II
(A) An urn contains five balls, two balls are drawn and are found to be white. If probability of all the balls in urn are white is 'k', then	(*p*) $\frac{2}{5} < k < \frac{3}{4}$
(B) Out of 15 consecutive integers three are selected at random, then the probability of the sum is divisible by 3 is 'k', then	(*q*) $\frac{1}{6} < k < \frac{1}{3}$
(C) If 3 cards are placed at random and independently in 4 boxes lying in a straight line. Then the probability of the cards going into 3 adjacent boxes so that each box contain one card, is 'k', then	(*r*) $\frac{2}{3} < k < 1$
(D) A box contains 4 balls which are either red or black, 2 balls are drawn and found to be red if these are replaced, then the probability that next draw will result in a red ball is 'k', then	(*s*) $\frac{1}{7} < k < \frac{2}{3}$
	(*t*) $\frac{1}{6} < k < \frac{1}{2}$

(*Continued*)

10. Match the functions given in Column I with their domain in Column II A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of the subset P. A subset Q of A is again chosen at random then the probability that

COLUMN-I	COLUMN-II
(A) $P \cap Q = \phi$	(*p*) $\frac{n\left(3^{n-1}\right)}{4^n}$
(B) $P \cap Q$ is a singleton	(*q*) $(3/4)^n$
(C) $P \cap Q$ contain 2 elements	(*r*) $\frac{{}^{2n}C_n}{4^n}$
(D) $\lvert P \rvert = \lvert Q \rvert$ where $\lvert X \rvert$ = number of elements in X	(*s*) $\frac{3^{n-2}(n-1)n}{2(4^n)}$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. In a multiple-choice question, there are five alternative answers, of which one or more than one is correct. A candidate will get marks on the question if he ticks all the correct answers. So, he decides to tick answers at random, if the least number of chances, he should be allowed so that the probability of his getting marks on the question exceeds 1/8 is K, then K (the student always attempts the question)

2. If $n(X) = (k + 1)$, then the probability of selecting 2 subsets A and B of the set 'X' such that $B = A^c$ is equal to $\frac{1}{2^{m-1}}$ where $m - k$ is equal to

3. A bag contains 10 different balls. Five balls are drawn simultaneously and then replaced and then seven balls are drawn. The probability that exactly three balls are common to the two drawn is p, then the value of $12p$ is

4. Triangles are formed with vertices of a regular polygon of 20 sides. The probability that no side of the polygon is a side of the triangle is $\frac{\lambda}{57}$ Then $\frac{\lambda}{40}$ is ____

5. In a group of people, if 4 are selected at a random, the probability that the any two of the four do not have same month of birth is p then $\frac{96p}{11}$ is equal to

6. Functions are formed form $A = \{1, 2, 3,\}$ to set $B = \{1, 2, 3, 4, 5\}$ and one function is elected at random. If P the probability that function satisfying $f(i) \le f(j)$ whenever $i < j$ then value of 25 p is equal to

7. If the sides of triangle are decided by throwing a die thrice, the probability that the triangle is isosceles or equilateral is $\frac{1}{k}$ then $k =$

8. Four identical dice are rolled once the probability that all the members on them are primes is $\frac{L}{8L+2}$ then $L =$

9. If $\{x, y\}$ is a subset of the first 30 natural numbers, then the probability, that $x^3 + y^3$ is divisible by 3, is $\frac{S}{9}$ then $S =$

10. If p, q are chosen randomly with replacement from the set $\{1, 2, 3, \ldots, 10\}$. the probability, that the roots of the equation $x^2 + px + q = 0$ are real, is $\frac{k^2+6}{50}$ then $k =$

11. An unbiased coin is tossed 12 times. The probability that at least 7 consecutive heads show up is $\frac{K}{256}$ then $K =$

12. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 3, 4, ... 12 is picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is $\frac{P}{792}$ then the digit in tens place of P is

13. Two persons X and Y go to a hotel. There are two hotels having three rooms each and one hotel having four rooms each, The probability that X and Y are in the same hotel having four rooms is $k/15$ then k is

14. If A and B throw a die each. The probability that A's throw is not greater than B's throw is K/12, then K =

15. The probability that a teacher will give a surprise test in a class is 1/5. If a student is absent twice, the probability that he will miss atleast one test will be $k/25$, then k must be

16. A box contains 24 balls of which 12 are black and 12 are white. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is K/32 , then K =

17. Two non-negative integers are chosen at random. The probability that the sum of their squares is divisible by 5 is $k/25$ then k is

18. If the integers m and n are chosen at random from $\{1, 2, 3, \ldots, 100\}$ then the probability that a number of the form $7^m + 7^n$ is divisible by 5 is equal to $\frac{1}{k}$. The numerical value of k is

19. A man parks his car among n cars standing in a row, his car not being parked at an end, on his return he finds that exactly m of the n cars are still there, probability that both the cars parked on two sides of his car, have left is $\frac{(n-m)(n-m-1)}{(n-A)(n-B)}$ then A + B is

20. Five horses are in a race. Mr. A selects two of the horses at random and bets on them. If the probability that Mr. A selected the winning horse is $\frac{P}{5}$ then the value of P is

21. You are given a box with 20 cards in it. 10 cards of this have the letter I printed on them. The other ten have the letter T printed on them. If you pick up 3 cards at random and keep them in the same order, the probability of making the word IIT is $k/38$. The numerical value of k is

22. One mapping reflected at random from all mappings of set $S = \{1, 2, 3, \ldots, 2n\}$ in to itself. If the probability that the mapping is many one is $\frac{29}{32}$, then value of n is

23. A die is rolled three times, the probability of getting a large number than the previous number is $k/54$ then the value of k is

24. Let a be the number which is selected from the set $\{a : a \in I, |a| \le 10$. If the probability that the expression $x^2 + 2ax + 10 - 3a > 0\ \forall x \in R$ is K then the value of 14K is______

25. A coin is tossed $m + n$ times $(m > n)$, the probability of getting m consecutive heads is $\frac{n+k}{2^{m+2}}$ then $k =$ ___

26. X wants to put 5 different rings into three different pockets is that no pocket is empty. Then the probability of such event occurring is $\frac{1}{P}.[P] =$ ________________ ([.] is the greatest integer function).

27. Two distinct numbers are chosen at random from set $\{1, 2, \ldots\ldots 3n\}$. The probability that $x^2 - y^2$ is divisible by 3 is $\frac{pn+q}{r(3n-1)}$ then $p + q + r =$

28. 3 numbers are chosen at random without replacement from $\{1, 2, 3 \ldots\ldots 14\}$

Let A = { min of chosen number is 5}

B = { max of chosen number is 11 }

$P(A \cup B) = \frac{K+11}{91}$ then K = _____

29. In a bag there are 15 balls of either red or green colour. Let G_k be the event that it contains exactly k green balls and its probability is proportional to k^2. Now a ball is drawn at random. Let P(A) be the probability that the ball drawn is green. If $P(A) = p/q$ in lowest form then $q - p$ is _____.

30. The probability that a random chosen 3 digit number has exactly 3 factors is $\frac{p}{900}$ (where $p \in$ N) then the value of p is _____.

31. Die A has four red and two white faces whereas die B has two red and four white faces. A coin is flipped once. If it falls a head, the game continues by throwing die A, if it falls tail then die B is to be used. If the probability that die A used is $\frac{32}{33}$ when it is given that red turns up every time in first n throws, then the value of n is _____.

32. A positive integer that reads the same backwards and forwards is called a palindrome. A number greater than 10 and less than 10^6 is chosen at random. If it is known that it is a five-digit number and the probability that it is a palindrome is P, then the value of 300 P is equal to

33. A drawer contains several pairs of socks. Not wanting to count Mr. A asked miss X "how many pairs of socks are there in the drawer?". Miss X, not wanting to give answer replied, "well, each sock has exactly one matching sock and the probability that two socks drawn from the drawer form a matching pair is $\frac{1}{15}$". Then the answer to Mr. A's question is

34. If A and B are two events of an experiment with $P(A) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{2}$, then $3P\left(\frac{B}{A}\right) =$

35. The sum of two positive quantities is equal to $2n$. The probability that their product is not less than $\frac{3}{4}$ times their greatest product is p then $2p =$ _____

36. If the papers of 4 students can be checked by any one of the 7 teachers. If the probability that all the 4 papers are checked by exactly 2 teachers is A, then the value of 49 A must be

37. A coin that comes up head with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip, is flipped eight times. Suppose the probability of three head and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p = \frac{m}{n}$, where m and n relative prime positive integers. The value of $(m + n - 5)$ equals.

38. If a and b are selected at random from the range of y (a, $b \in$ N, $a \neq b$), and if the probability of selecting distinct ordered pairs (a, b) of prime integers from the range of y, where $y = \frac{147}{x + \frac{1}{x} + 5} \forall x \neq 0$ is $\frac{k}{15}$ then $k =$ _______

39. One function is selected at random from all function of set S = {1, 2, 3, ... n} to itself. If, probability that it is one-one is $\frac{3}{32}$, then $n =$

40. A man has three coins A, B, C out of which A is fair. The probability of head appearing when tossed are $\frac{2}{3}$ and $\frac{1}{3}$ for B and C respectively. If one of the coins chosen at random, is tossed three times giving 2 heads and one tail. If the probability that the chosen coin was A is $\frac{k}{25}$, then $k =$ ____

41. From a pack of 52 playing cards, n cards are missing. From the remaining cards, 2 cards are drawn at random and are found to be spades. If the probability that all the n missing cards are spades is $\frac{11}{245}$ then $n =$ ____

42. The probability of a bomb hitting a bridge is 1/2 and two direct hits are needed to destroy it. The least number of bombs required so that the probability of the bridge being destroyed is greater than 0.9 is

43. 21 tickets are numbered from1 to 21 serially. Three are drawn at random. The chance that the numbers on the selected tickets are in A.P is 'P'. Then $\left[\frac{1}{P} - 7\right] =$ ____where [.] is G.I.F

44. The probability of man hitting a target in one fire is 1/5. Then the minimum number of fire he must follow in order to make his chance of hitting the target more than $\frac{3}{4}$ is

45. Three natural numbers are taken at random from the set $A = \{x / 1 \le x \le 100, x \in N$ and the probability that the A.M of the numbers taken is 25 is $\frac{K}{{}^{100}C_3}$ then the digit in units place of k is

46. A box contains 9 balls. One ball is selected at random and is known to be white. If the probability that the box contains exactly K white ball is $\frac{1}{9}$ then $k =$

47. A bag contains n identical red balls, $2n$ identical black balls and $3n$ identical white balls. If probability of drawing n balls of same colour is greater than or equal to $\frac{1}{6}$, then minimum number of red balls in the bag is equal to ____________

48. A letter is selected from each of the words "ARTICLE and TRIANGLE" The probability of selecting same letter is $\frac{k}{28}$ then $k =$ __________

49. A man throws a die until he gets a number bigger than 3. The probability that he gets 5 in the last throw is $\frac{a}{b}$ then $a + b =$ __________

50. If four integers are selected at random from the first 23 natural numbers then the probability that their product is even, if it is given that their sum is odd is __________

51. If a is an integer lying in [–5, 30], then the probability that the graph of $y = x^2 + 2(a + 4)x - 5a + 64$ is strictly above the x-axis is $\frac{2}{k}$, then k is

52. A letter is taken at random from the letters of the word "STATISTICS" and another letter is taken at random from the letters of the word "ASSISTANT". The probability that they are the same letter is $\frac{k}{l}$, then $k + l - 100$ is_____(k, l are relative primes)

53. A bag contains $n+1$ coins. If it is known that one of the coins shows head on both sides whereas the other coins are fair. One coin is selected at random and tossed. If the probability of getting head is $\frac{7}{12}$ then $n =$

54. The probability that the sum of the squares of the two numbers which show up when two fair dice are thrown is even, is $\frac{k}{2}$, then $k =$ ____________

55. A box contains 100 tickets numbered 1, 2, ..., 100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum number on them is 5 with probability $1/k$ then k is

56. There are two balls in an urn. Each ball is equally likely to be black or white. A white ball is now put in the urn. If the chance of drawing a white ball now is P then 6P=

57. There are 10 cards labelled from 1 to 10. Three cards denoted by a, b, c $(a > b > c)$ are drawn from the cards at the same time. If $R = \{a / \int_0^a (x^2 - 2bx + 3c)dx = 0\}$ then $n(R) =$

58. A pair of unbiased dice is rolled together till a sum of either 5 (or) 7 is obtained. If the probability that 5 comes before 7 is $\frac{m}{n}$ where m, n are primes then $m+n$ is ___

59. A fair coin is tossed n times. Let x denote the number of times head occurs. If $P(x=4)$, $P(x=5)$ and $P(x=6)$ are in A.P then sum of values of $n =$____

60. A bag contains "b" black balls and r red balls. Two balls are drawn at random. The probability of drawing two red balls is 5 times the probability of drawing two black balls. Further the probability of drawing one ball from each colour is 6 times the probability of drawing two black balls then $r - b =$

61. The sum of three positive integers is 20. The probability that they form the sides of a triangle is P then 19P = ____________

62. A positive integer n is randomly guessed. Let the probability that $x^{n+1} - x^n + 1$ is divisible by $x^2 - x + 1$ be P. The value of 12P is ____________

63. A drawer contains red and black socks, when two socks are drawn at random, the probability that both are red is $\frac{1}{2}$. Minimum number of socks in the drawer is ____________

ARCHIVE: JEE MAIN

1. **[JEE Main 2020]**
In a bombing attack, there is 50% chance that a bomb will hit the target. At least two independent hits are required to destroy the target completely. Then the minimum number of bombs, that must be dropped to ensure that there is at least 99% chance of completely destroying the target, is _____ .

2. **[JEE Main 2020]**
Four fair dice are thrown independently 27 items. Then the expected number of times, at least two dice show up a three or a five, is ____.

3. **[JEE Main 2020]**
The probability of a man hitting a target is $\frac{1}{10}$. The least number of shots required, so that the probability of his hitting the target at least once is greater than $\frac{1}{4}$, is ______.

4. **[JEE Main 2020]**
In a workshop, there are five machines and the probability of any one of them to be out of service one day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to:

(A) $\frac{17}{2}$　　(B) 4

(C) $\frac{17}{8}$　　(D) $\frac{17}{4}$

5. **[JEE Main 2020]**
Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Then, which of the following is TRUE?

(A) $P(A'/B') = \frac{1}{3}$　　(B) $P(A/B') = \frac{1}{3}$

(C) $P(A/B) = \frac{2}{3}$　　(D) $P(A/(A \cup B)) = \frac{1}{4}$

6. **[JEE Main 2020]**
Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is:

(A) 0.10　　(B) 0.02

(C) 0.20　　(D) 0.01

7. **[JEE Main 2020]**
In a box, there are 20 cards, out of which 10 are labelled as A and the remaining 10 are labelled as B. Cards are drawn at random, one after the other and with replacement, till a second A-card is obtained. The probability that the second A-card appears before the third B-card is:

(A) $\frac{15}{16}$　　(B) $\frac{9}{16}$

(C) $\frac{13}{16}$　　(D) $\frac{11}{16}$

8. **[JEE Main 2020]**
A random variable X has the following probability distribution

X:	1	2	3	4	5
P(K):	K^2	2K	K	2K	$5K^2$

then $P(X > 2)$ is equal to:

(A) $\frac{23}{36}$　　(B) $\frac{1}{6}$

(C) $\frac{1}{36}$　　(D) $\frac{7}{12}$

9. **[JEE Main 2020]**

Let E^C denotes the complement of an event E. Let E_1, E_2 and E_3 be any pairwise independent events with $P(E_1) > 0$ and $P(E_1 \cap E_2 \cap E_3) = 0$. The $P\left(E_2^C \cap E_3^C / E_1\right)$ is equal to:

(A) $P(E_3^C) - P(E_2)$ (B) $P(E_2^C) + P(E_3)$

(C) $P(E_3^C) - P(E_2^C)$ (D) $P(E_3) - P(E_2^C)$

10. **[JEE Main 2020]**

Box I contains 30 cards numbered 1 to 30 and Box II contains 20 cards numbered 31 to 50. A box is selected at random and a card is drawn from it. The number on the card is found to be a non-prime number. The probability that the card was drawn from Box 1 is:

(A) $\frac{8}{17}$ (B) $\frac{2}{3}$

(C) $\frac{4}{17}$ (D) $\frac{2}{5}$

11. **[JEE Main 2020]**

A die is thrown two times and the sum of the scores appearing on the die is observed to be a multiple of 4. Then the conditional probability that the score 4 has appeared atleast once is:

(A) $\frac{1}{8}$ (B) $\frac{1}{9}$

(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

12. **[JEE Main 2020]**

In a game two players A and B take turns in throwing a pair of fair dice starting with player A and total of scores on the two dice, in each throw is noted. A wins the game if the throws a total of 6 before B throws a total of 7 and B wins the game if the throws a total of 7 before A throws a total of six. The game stops as soon as either of the players wins. The probability of A winning the game is:

(A) $\frac{31}{61}$ (B) $\frac{5}{6}$

(C) $\frac{5}{31}$ (D) $\frac{30}{61}$

13. **[JEE Main 2020]**

The probability that a randomly chosen 5-digit number is made from exactly two digits is:

(A) $\frac{121}{10^4}$ (B) $\frac{150}{10^4}$

(C) $\frac{135}{10^4}$ (D) $\frac{134}{10^4}$

14. **[JEE MAIN 2020]**

An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for $k = 3, 4, 5$, otherwise X takes the value -1. Then the expected value of X, is:

(A) $-\frac{3}{16}$ (B) $\frac{3}{16}$

(C) $\frac{1}{8}$ (D) $-\frac{1}{8}$

15. **[JEE MAIN 2020]**

The probabilities of three events A, B and C are given by $P(A) = 0.6$, $P(B) = 0.4$ and $P(C) = 0.5$,

If $P(A \cup B) = 0.8$, $P(A \cap C) = 0.3$, $P(A \cap B \cap C) = 0.2$, $P(B \cap C) = \beta$ and $P(A \cup B \cup C) = \alpha$. Where $0.85 \le \alpha \le 0.95$, then β lies in the interval

(A) [0.36, 0.40] (B) [0.35, 0.36]

(C) [0.25, 0.35] (D) [0.20, 0.25]

16. **[JEE MAIN 2020]**

Out of 11 consecutive natural numbers if three numbers are selected at random (without repetition), then the probability that they are in A.P. with positive common difference, is:

(A) $\frac{15}{101}$ (B) $\frac{5}{101}$

(C) $\frac{5}{33}$ (D) $\frac{10}{99}$

17. **[JEE Main 2020]**

If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is:

(A) $\frac{945}{2^{11}}$ (B) $\frac{965}{2^{11}}$

(C) $\frac{965}{2^{10}}$ (D) $\frac{945}{2^{10}}$

18. **[JEE Main 2019]**

In a game, a man wins ₹ 100 if he gets 5 or 6 on a throw of a fair die and loses ₹ 50 for getting any other number on the die. If he decides to throw the die either till he gets a five or a six or to a maximum of three throws, then his expected gain/loss (in rupees) is:

(A) $\frac{400}{3}$ loss (B) 0

(C) $\frac{400}{9}$ loss (D) $\frac{400}{3}$ gain

19. **[JEE Main 2019]**

Let a random variable X have a binomial distribution with mean 8 and variance 4. If $P(X \le 2) = \frac{k}{2^{16}}$, then k is equal to:

(A) 17 (B) 1

(C) 137 (D) 121

20. **[JEE Main 2019]**

A person throws two fair dice. He wins ₹ 15 for throwing a doublet (same numbers on the two dice), wins ₹ 12 when the throw results in the sum of 9, and loses ₹ 6 for any other outcome on the throw. Then the expected gain/loss (in ₹) of the person is:

(A) $\frac{1}{4}$ loss (B) $\frac{1}{2}$ gain

(C) $\frac{1}{2}$ loss (D) 2 gain

21. **[JEE-Main 2019]**

Let $S = \{1, 2, \ldots. 20\}$. *A* subset *B* of *S* is said to be ''nice'', if the sum of the elements of *B* is 7. Then the probability that a randomly chosen subset of *S* is ''nice'' is

(A) $\frac{5}{2^{20}}$ (B) $\frac{7}{2^{20}}$

(C) $\frac{4}{2^{20}}$ (D) $\frac{6}{2^{20}}$

22. **[JEE-Main 2019]**
Two integers are selected at random from the set {1, 2, …, 11}. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is:

(A) $\frac{2}{5}$ (B) $\frac{1}{2}$

(C) $\frac{7}{10}$ (D) $\frac{3}{5}$

23. **[JEE-Main 2019]**
In a class of 60 students, 40 opted for NCC, 30 opted for NSS and 20 opted for both NCC and NSS. If one of these students is selected at random, then the probability that the students selected has opted neither for NCC nor for NSS is:

(A) $\frac{1}{3}$ (B) $\frac{1}{6}$

(C) $\frac{2}{3}$ (D) $\frac{5}{6}$

24. **[JEE-Main 2019]**
In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to:

(A) $\frac{200}{6^5}$ (B) $\frac{225}{6^5}$

(C) $\frac{150}{6^5}$ (D) $\frac{175}{6^5}$

25. **[JEE-Main 2019]**
Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Let X denote the random variable of number of aces obtained in the two drawn cards. Then $P(X = 1) + P(X = 2)$ equals:

(A) 25/169 (B) 24/169

(C) 49/169 (D) 52/169

26. **[JEE-Main 2019]**
An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a second ball is drawn at random from it. The probability that the second ball is red, is

(A) $\frac{27}{49}$ (B) $\frac{26}{49}$

(C) $\frac{32}{49}$ (D) $\frac{21}{49}$

27. **[JEE-Main 2019]**
An unbiased coin is tossed. If the outcome is a head then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail then a card from a well-shuffled pack of nine cards numbered 1, 2, 3, …….. 9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is

(A) $\frac{19}{36}$ (B) $\frac{15}{72}$

(C) $\frac{13}{36}$ (D) $\frac{19}{72}$

28. **[JEE-Main 2019]**
If the probability of hitting a target by a shooter, in any shot, is $\frac{1}{3}$, then the minimum number of independent shots at the target required by him so that the probability of hitting the target atleast once is greater than $\frac{5}{6}$, is

(A) 4 (B) 6

(C) 5 (D) 3

29. **[JEE-Main 2019]**
Let A and B be two non-null events such that $A \subset B$. Then, which of the following statements is always correct?

(A) $P(A \mid B) = P(B) - P(A)$ (B) $P(A \mid B) \geq P(A)$

(C) $P(A \mid B) = 1$ (D) $P(A \mid B) \leq P(A)$

30. **[JEE-Main 2019]**
The minimum number of times one has to toss a fair coin so that the probability of observing at least one head is at least 90% is

(A) 4 (B) 5

(C) 3 (D) 2

31. **[JEE-Main 2019]**
Four persons can hit a target correctly with probabilities $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{8}$ respectively. If all hit at the target independently, then the probability that the target would be hit, is:

(A) $\frac{25}{192}$ (B) $\frac{25}{32}$

(C) $\frac{7}{32}$ (D) $\frac{1}{192}$

32. **[JEE-Main 2019]**
Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls is:

(A) $\frac{1}{10}$ (B) $\frac{1}{17}$

(C) $\frac{1}{11}$ (D) $\frac{1}{12}$

33. **[JEE-Main 2019]**
Minimum number of times a fair coin must be tossed so that the probability of getting at least one head is more than 99% is:

(A) 6 (B) 7

(C) 8 (D) 5

34. **[JEE-Main 2019]**
If three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is:

(A) $\frac{1}{10}$ (B) $\frac{3}{10}$

(C) $\frac{3}{20}$ (D) $\frac{1}{5}$

35. [JEE-Main 2019]

For an initial screening of an admission test, a candidate is given fifty problems to solve. If the probability that the candidate solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problems is

(A) $\frac{164}{25}\left(\frac{1}{5}\right)^{48}$ (B) $\frac{316}{25}\left(\frac{4}{5}\right)^{48}$

(C) $\frac{201}{5}\left(\frac{1}{5}\right)^{49}$ (D) $\frac{54}{5}\left(\frac{4}{5}\right)^{49}$

36. [JEE-Main 2018]

A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

(A) $\frac{2}{5}$ (B) $\frac{1}{5}$

(C) $\frac{3}{4}$ (D) $\frac{3}{10}$

37. [JEE-Main 2017]

If two different numbers are taken from the set (0, 1, 2, 3…, 10), then the probability that their sum as well as absolute difference are both multiple of 4, is

(A) $\frac{7}{55}$ (B) $\frac{6}{55}$

(C) $\frac{12}{55}$ (D) $\frac{14}{55}$

38. [JEE-Main 2017]

For three events A, B and C, P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs) = P(Exactly one of C or A occurs) = $\frac{1}{4}$ and P(All the three events occurs simultaneously) = $\frac{1}{16}$. Then the probability that at least one of the events occurs, is

(A) $\frac{3}{16}$ (B) $\frac{7}{32}$

(C) $\frac{7}{16}$ (D) $\frac{7}{64}$

39. [JEE-Main 2016]

Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four,E_2 is event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is **NOT true**?

(A) E_1, E_2 and E_3 are independent.
(B) E_1 and E_2 are independent.
(C) E_2 and E_3 are independent.
(D) E_1 and E_3 are independent.

40. [JEE-Mains 2014]

Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A. Then the events A and B are:

(A) mutually exclusive and independent.
(B) equally likely but not independent.
(C) independent but not equally likely.
(D) independent and equally likely.

41. [JEE-Main 2013]

A multiple-choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is:

(A) $\frac{17}{3^5}$ (B) $\frac{13}{3^5}$

(C) $\frac{11}{3^5}$ (D) $\frac{10}{3^5}$

42. [AIEEE 2012]

Three numbers are chosen at random without replacement from {1, 2, 3 .... 8}. The probability that their minimum is 3, given that their maximum is 6, is

(A) $\frac{2}{5}$ (B) $\frac{3}{8}$

(C) $\frac{1}{5}$ (D) $\frac{1}{4}$

43. [AIEEE -2011]

If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is

(A) $P(C \mid D) = P(C)$ (B) $P(C \mid D) = \frac{P(D)}{P(C)}$

(C) $P(C \mid D) < P(C)$ (D) $P(C \mid D) \geq P(C)$

44. [AIEEE -2011]

Consider 5 independent Bernoulli trials each with probability of success p. If the probability of at least one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval:

(A) $\left[0, \frac{1}{2}\right]$ (B) $\left(\frac{11}{12}, 1\right]$

(C) $\left(\frac{1}{2}, \frac{3}{4}\right]$ (D) $\left(\frac{3}{4}, \frac{11}{12}\right]$

45. [AIEEE 2010]

An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours is

(A) $\frac{1}{3}$ (B) $\frac{2}{7}$

(C) $\frac{1}{21}$ (D) $\frac{2}{23}$

46. **[AIEEE 2010]**

Four numbers are chosen at random (without replacement) from the set {1, 2, 3, ..., 20}.

Statement -1: The probability that the chosen numbers when arranged in some order will form an AP is $\frac{1}{85}$.

Statement -2: If the four chosen numbers form an AP, then the set of all possible values of common difference is { ± 1, ± 2, ± 3, ± 4, ± 5}

(A) Statement-1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.
(B) Statement -1 is true, Statement-2 is true Statement -2 is not a correct explanation for Statement -1.
(C) Statement-1 is true, Statement-2 is false.
(D) Statement -1 is false, Statement -2 is true.

47. **[AIEEE 2009]**

One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equal:

(A) $\frac{5}{14}$ (B) $\frac{1}{50}$
(C) $\frac{1}{14}$ (D) $\frac{1}{7}$

48. **[AIEEE 2009]**

In a binomial distribution $B\left(n, p=\frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than:

(A) $\frac{9}{\log_{10}4-\log_{10}3}$ (B) $\frac{4}{\log_{10}4-\log_{10}3}$
(C) $\frac{1}{\log_{10}4-\log_{10}3}$ (D) $\frac{1}{\log_{10}4+\log_{10}3}$

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Single Correct Choice Type Problems

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. **[IIT-JEE Advanced 2020]**

Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$
(C) $\frac{1}{2}$ (D) $\frac{1}{4}$

2. **[JEE-Advanced 2017]**

Three randomly chosen non-negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$
(C) $\frac{1}{2}$ (D) $\frac{5}{11}$

3. **[JEE-Advanced 2016]**

A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that

P(computer turns out to be defective given that is produced in plant T_1)

= 10P(computer turns out to be defective given that it is produced in plant T_2)

where P(E) denotes the probability of an event E. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is-

(A) $\frac{36}{73}$ (B) $\frac{47}{79}$
(C) $\frac{78}{93}$ (D) $\frac{75}{83}$

4. **[JEE-Advanced 2014]**

Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$
(C) $\frac{2}{3}$ (D) $\frac{3}{4}$

5. **[JEE-Advanced 2013]**

Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one or them is

(A) $\frac{235}{256}$ (B) $\frac{21}{256}$
(C) $\frac{3}{256}$ (D) $\frac{253}{256}$

6. **[JEE-2012]**

Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1, 2, 3, 4, 5, and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(A) $\frac{91}{216}$ (B) $\frac{108}{216}$

(C) $\frac{125}{216}$ (D) $\frac{127}{216}$

7. **[IIT-JEE 2010]**

Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

(A) $\frac{1}{18}$ (B) $\frac{1}{9}$

(C) $\frac{2}{9}$ (D) $\frac{1}{36}$

8. **[IIT-JEE 2010]**

A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is

(A) $\frac{3}{5}$ (B) $\frac{6}{7}$

(C) $\frac{20}{23}$ (D) $\frac{9}{20}$

9. **[IIT-JEE 2008]**

An experiment has 10 equally likely outcomes. Let A and B be non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is

(A) 2, 4 or 8 (B) 3, 6 or 9

(C) 4 or 8 (D) 5 or 10

10. **[IIT-JEE 2007]**

One Indian and four American men and their wives are to be seated randomly around a circular table. Then the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{2}{5}$ (D) $\frac{1}{5}$

11. **[IIT-JEE 2007]**

Let E^c denote the complement of an event E. Let E,F,G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P((E^C \cap F^C)/G)$ equals

(A) $P(E^C) + P(F^C)$ (B) $P(E^C) - P(F^C)$

(C) $P(E^C) - P(F)$ (D) $P(E) - P(F^C)$

Multiple Correct Choice Type Problems

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. **[JEE-Advanced 2019]**

There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}, \frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following option is/are correct?

(A) Probability that the chosen balls is green equals $\frac{39}{80}$

(B) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(C) Probability that the selected bag is B_3, given that the chosen ball is green, equals $\frac{5}{13}$

(D) Probability that the selected bag is B_3 and the chosen balls is green equals $\frac{3}{10}$

2. **[JEE-Advanced 2017]**

Let X and Y be two events such that $P(X) = \frac{1}{3}, P(X \mid Y) = \frac{1}{2}$ and $P(Y \mid X) = \frac{2}{5}$. Then

(A) $P(Y) = \frac{4}{15}$ (B) $P(X' \mid Y) = \frac{1}{2}$

(C) $P(X \cap Y) = \frac{1}{5}$ (D) $P(X \cup Y) = \frac{2}{5}$

3. **[IIT-JEE 2012]**

A ship is fitted with three engines E_1, E_2 and E_3, The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

(A) $P\left[X_1^C \mid X\right] = \frac{3}{16}$

(B) P[Exactly two engines of the ship are functioning $\mid X$] $= \frac{7}{8}$

(C) $P[X \mid X_2] = \frac{5}{16}$

(D) $P[X \mid X_1] = \frac{7}{16}$

4. **[IIT-JEE 2012]**

Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}$, $P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$, Which of the following is (are) correct?

(A) $P(X \cup Y) = \frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P(X^C \cap Y) = \frac{1}{3}$

5. [IIT-JEE 2011]

Let *E* and *F* be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If *P*(*T*) denotes the probability of occurrence of the event T, then

(A) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ (B) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(A) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (D) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

[JEE-Advanced 2018]

There are five students S_1, S_2, S_3, S_4 and S_5 in a music class and for them there are five seats R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to student S_i, $i = 1, 2, 3, 4, 5$ but on the examination day, the five students are randomly allotted five seats.

1. The probability that, on examination day, the student S_1 gets the previously allotted seat R_1, and **NONE** of the remaining students gets the seat previously allotted to him/her is

(A) $\frac{3}{40}$ (B) $\frac{1}{8}$

(C) $\frac{7}{40}$ (D) $\frac{1}{5}$

2. For $i = 1, 2, 3, 4$, let T_i denote the event that the student S_i and S_{i+1} do **NOT** sit adjacent to each other on the day of the examination. Then, the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is

(A) $\frac{1}{15}$ (B) $\frac{1}{10}$

(C) $\frac{7}{60}$ (D) $\frac{1}{5}$

Comprehension 2

[JEE-Advanced 2016]

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$ respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2, respectively, after two games

3. $P(X > Y)$ is

(A) $\frac{1}{4}$ (B) $\frac{5}{12}$

(C) $\frac{1}{2}$ (D) $\frac{7}{12}$

4. $P(X = Y)$ is

(A) $\frac{11}{36}$ (B) $\frac{1}{3}$

(C) $\frac{13}{36}$ (D) $\frac{1}{2}$

Comprehension 3

[JEE-Advanced 2015]

Let n_1 and n_2 be the number of red and black balls respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

5. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

(B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$

(D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

6. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$ then the correct option(s) with the possible values of n_1 and n_2 is (are)

(A) $n_1 = 4$ and $n_2 = 6$ (B) $n_1 = 2$ and $n_2 = 3$

(C) $n_1 = 10$ and $n_2 = 20$ (D) $n_1 = 3$ and $n_2 = 6$

Comprehension 4

[JEE-Advanced 2014]

Box 1 contains three cards bearing numbers, 1, 2, 3; box 2 contains five cards bearing numbers 1, 2, 3, 4, 5; and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7, A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the i^{th} box, $i = 1, 2, 3$

7. The probability that $x_1 + x_2 + x_3$ is odd, is

(A) $\frac{29}{105}$ (B) $\frac{53}{105}$

(C) $\frac{57}{105}$ (D) $\frac{1}{2}$

8. The probability that x_1, x_2, x_3 are in an arithmetic progression, is

(A) $\frac{9}{105}$ (B) $\frac{10}{105}$

(C) $\frac{11}{105}$ (D) $\frac{7}{105}$

Comprehension 5

[JEE-Advanced 2013]

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

9. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

(A) $\frac{116}{181}$ (B) $\frac{126}{181}$

(C) $\frac{65}{181}$ (D) $\frac{55}{181}$

10. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3 the probability that all 3 drawn balls are of the same colour is

(A) $\frac{82}{648}$ (B) $\frac{90}{648}$

(C) $\frac{558}{648}$ (D) $\frac{566}{648}$

Comprehension 6

[IIT-JEE 2011]

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

11. The probability of the drawn ball from U_2 being white is

(A) $\frac{13}{30}$ (B) $\frac{23}{30}$

(C) $\frac{19}{30}$ (D) $\frac{11}{30}$

12. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is

(A) $\frac{17}{23}$ (B) $\frac{11}{23}$

(C) $\frac{15}{23}$ (D) $\frac{12}{23}$

Comprehension 7

[IIT-JEE 2009]

A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

13. The probability that $X = 3$ equals

(A) $\frac{25}{216}$ (B) $\frac{25}{36}$

(C) $\frac{5}{36}$ (D) $\frac{125}{216}$

14. The probability that $X \geq 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{36}$

(C) $\frac{5}{36}$ (D) $\frac{25}{216}$

15. The conditional probability that $X \geq 6$ given $X > 3$ equals

(A) $\frac{125}{216}$ (B) $\frac{25}{216}$

(C) $\frac{5}{36}$ (D) $\frac{25}{36}$

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [IIT-JEE Advanced 2020]
The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is **NOT** less than 0.95, is ____________

2. [IIT-JEE Advanced 2020]
Two fair dice, each with faces numbered 1, 2, 3, 4, 5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of $14p$ is ____________

3. [JEE-Advanced 2019]
Let S be the sample space of all 3×3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by $E_1 = \{A \in S\text{: det } A = 0\}$ and $E_2 = \{A \in S\text{: sum of entries of } A \text{ is } 7\}$. If a matrix is chosen at random from S, then the conditional probability $P(E_1 | E_2)$ equals _____

4. [JEE-Advanced 2019]
Let $|X|$ denote the number of elements in a set X. Let $S = \{1, 2, 3, 4, 5, 6\}$ be a sample space, where each element is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A, B) such that $1 \leq |B| < |A|$, equals

5. [JEE-Advanced 2015]
The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is

6. [JEE-Advanced 2013]
Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γLet the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)\, p = \alpha\beta$ and $(\beta - 3\gamma)\, p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1).

Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of ocurrence of } E_3} =$

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** D	**3.** B	**4.** C	**5.** B	**6.** C	**7.** D	**8.** D	**9.** D	**10.** C
11. A	**12.** B	**13.** A	**14.** D	**15.** A	**16.** B	**17.** C	**18.** A	**19.** A	**20.** C
21. B	**22.** D	**23.** C	**24.** C	**25.** C	**26.** A	**27.** C	**28.** C	**29.** A	**30.** B
31. D	**32.** B	**33.** A	**34.** A	**35.** A	**36.** B	**37.** C	**38.** C	**39.** A	**40.** A
41. A	**42.** D	**43.** C	**44.** A	**45.** C	**46.** C	**47.** C	**48.** B	**49.** A	**50.** C
51. B	**52.** C	**53.** B	**54.** D	**55.** B	**56.** D	**57.** D	**58.** A	**59.** C	**60.** A
61. A	**62.** C	**63.** B	**64.** A	**65.** A	**66.** C	**67.** D	**68.** D	**69.** D	**70.** D
71. C	**72.** D	**73.** B	**74.** D	**75.** A	**76.** D	**77.** C	**78.** D	**79.** B	**80.** A
81. B	**82.** C	**83.** C	**84.** D	**85.** B	**86.** D	**87.** B	**88.** A	**89.** A	**90.** B
91. A	**92.** B	**93.** C	**94.** A	**95.** B	**96.** C	**97.** A	**98.** C	**99.** A	**100.** C
101. C	**102.** C	**103.** D	**104.** C	**105.** C	**106.** D	**107.** D	**108.** A	**109.** C	**110.** D
111. D	**112.** D	**113.** B	**114.** D	**115.** D	**116.** B	**117.** B	**118.** A	**119.** A	**120.** B

Multiple Correct Choice Type Questions

1. C, D	**2.** B, C, D	**3.** B, C, D	**4.** A, B, C	**5.** A, B, C, D
6. A, B	**7.** A, B, C	**8.** A, C	**9.** A, B, C, D	**10.** A, B, D
11. A, B, C, D	**12.** B, C	**13.** A, B	**14.** A, B, C, D	**15.** B
16. B, C	**17.** A, C	**18.** A, D	**19.** A, C, D	**20.** A, B
21. A	**22.** A, B, C	**23.** B, C	**24.** A, B, C, D	**25.** A, B, D
26. A, C, D	**27.** B, C, D	**28.** A, B, C, D	**29.** A, C	**30.** A, B, D
31. A, B, C, D				

Linked Comprehension Type Questions

1. C	**2.** B	**3.** A	**4.** B	**5.** A	**6.** B	**7.** C	**8.** B	**9.** D	**10.** C
11. B	**12.** D	**13.** C	**14.** D	**15.** A	**16.** D	**17.** A	**18.** C	**19.** B	**20.** B
21. A	**22.** C	**23.** B	**24.** D	**25.** B	**26.** D	**27.** D	**28.** D	**29.** D	**30.** A
31. B	**32.** C	**33.** D	**34.** A	**35.** C	**36.** A	**37.** B	**38.** D	**39.** C	**40.** B

Matrix Match/Column Match Type Questions

1. A → (r)	B → (s)	C → (p)	D → (q)
2. A → (p, q, r, s)	B → (p, q, r, s)	C → (p, r)	D → (r, s)
3. A → (q)	B → (p)	C → (q)	D → (r)
4. A → (q)	B → (p)	C → (r)	D → (q)
5. A → (q)	B → (r)	C → (s)	D → (r)
6. A → (q)	B → (r)	C → (r)	D → (t)
7. A → (r)	B → (s)	C → (p)	D → (q)
8. A → (r)	B → (r)	C → (p)	D → (p)
9. A → (p, s)	B → (s, t)	C → (q, s, t)	D → (r)
10. A → (q)	B → (p)	C → (s)	D → (r)

Integer/Numerical Answer Type Questions

1. 4	**2.** 2	**3.** 5	**4.** 1	**5.** 5	**6.** 7	**7.** 8	**8.** 5	**9.** 3	**10.** 5
11. 7	**12.** 9	**13.** 2	**14.** 7	**15.** 9	**16.** 5	**17.** 9	**18.** 4	**19.** 3	**20.** 2
21. 5	**22.** 2	**23.** 5	**24.** 4	**25.** 3	**26.** 1	**27.** 5	**28.** 8	**29.** 7	**30.** 7
31. 5	**32.** 3	**33.** 8	**34.** 1	**35.** 1	**36.** 6	**37.** 6	**38.** 2	**39.** 4	**40.** 9
41. 2	**42.** 7	**43.** 6	**44.** 7	**45.** 1	**46.** 5	**47.** 5	**48.** 3	**49.** 4	**50.** 1
51. 9	**52.** 9	**53.** 5	**54.** 1	**55.** 9	**56.** 4	**57.** 3	**58.** 7	**59.** 21	**60.** 3
61. 4	**62.** 2	**63.** 4							

ARCHIVE: JEE MAIN

1. 11	**2.** 11	**3.** 3	**4.** C	**5.** B	**6.** A	**7.** D	**8.** A	**9.** A	**10.** A
11. B	**12.** D	**13.** C	**14.** C	**15.** C	**16.** C	**17.** D	**18.** B	**19.** C	**20.** C
21. A	**22.** A	**23.** B	**24.** D	**25.** A	**26.** C	**27.** D	**28.** C	**29.** B	**30.** A
31. B	**32.** C	**33.** B	**34.** A	**35.** D	**36.** A	**37.** B	**38.** C	**39.** A	**40.** C
41. C	**42.** C	**43.** D	**44.** A	**45.** B	**46.** C	**47.** C	**48.** C		

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. B	**2.** B	**3.** C	**4.** A	**5.** A	**6.** A	**7.** C	**8.** C	**9.** D	**10.** C
11. C									

Multiple Correct Choice Type Questions

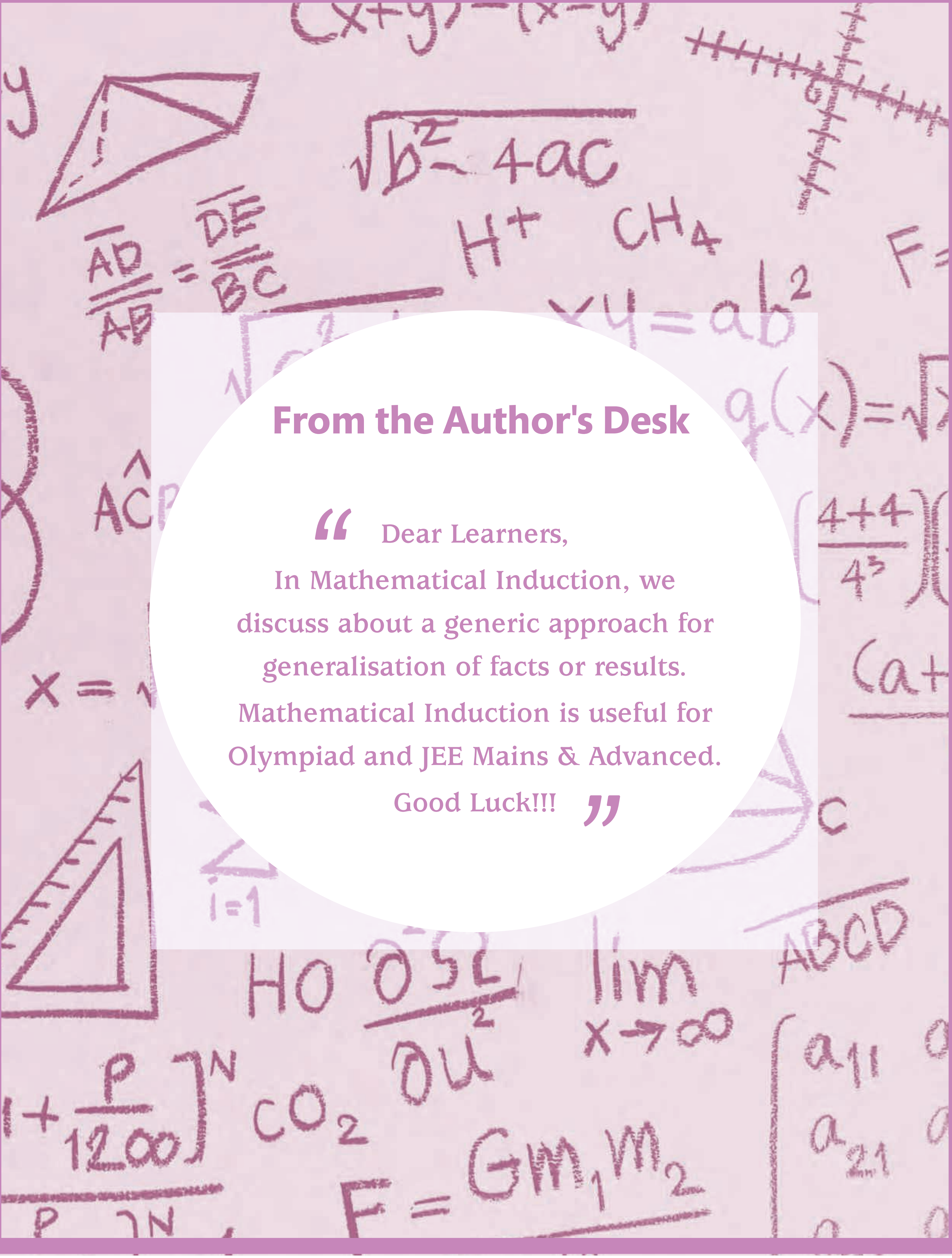
1. A, B	**2.** A, B	**3.** B, D	**4.** A, B	**5.** A, D

Linked Comprehension Type Questions

1. A	**2.** C	**3.** B	**4.** C	**5.** A,B	**6.** C, D	**7.** B	**8.** C	**9.** D	**10.** A
11. B	**12.** D	**13.** A	**14.** B	**15.** D					

Integer/Numerical Answer Type Questions

1. 6	**2. 8**	**3.** 0.50	**4.** 422.00	**5.** 8	**6.** 6



$(x+y)^2-(x-y)^2$
y
$\sqrt{b^2-4ac}$
$\frac{AD}{AB}=\frac{DE}{BC}$
H+
CH_4
F=
$xy=ab^2$
$g(x)=\sqrt{x}$
$\hat{ACB}$
$\left(\frac{4+4}{4^3}\right)$
x =
(a+
c
i=1
ABCD
$\frac{\partial^2}{\partial u^2}$
$\lim_{x\to\infty}$
a_{11}
a_{21}
$\left[1+\frac{P}{1200}\right]^N$
CO_2
$F=Gm_1m_2$

From the Author's Desk

"Dear Learners,

In Mathematical Induction, we discuss about a generic approach for generalisation of facts or results. Mathematical Induction is useful for Olympiad and JEE Mains & Advanced.

Good Luck!!!"

CHAPTER

8

Principle of Mathematical Induction

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Mathematical induction concept
(b) Problems based on mathematical induction

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

In algebra there are certain results that are formulated with n number of terms in them, where n is a natural number (i.e. a positive integer). Those results can be proved by a specific technique, known as the principle of mathematical induction. We use the symbol $P(n)$ (read "P of n") to denote some proposition which depends on the positive integer n.

Type I

If $P(n)$ is a statement depending upon n, then to prove $P(n)$ by induction, we procced as follows :

(i) Verify the validity of $P(n)$ for $n = 1$
(ii) Assume that $P(n)$ is true for any positive integer m and then using it establish the validity of $P(n)$ for $n = m + 1$.
Then $P(n)$ is true for each $n \in N$

Type II

If $P(n)$ is a statement depending upon n but beginning with any positive integer k, then to prove $P(n)$ by Induction,

we proceed as follows :
(i) Verify the validity of $P(n)$ for $n = k$.
(ii) Assume that the $P(n)$ is true for $n = m \geq k$.
then using it establish the validity of $P(n)$ for $n = m + 1$.
Then $P(n)$ is true for each $n \geq k$

SOME USEFUL RESULT BASED ON PRINCIPLE OF MATHEMATICAL INDUCTION

For any natural number n

(i) $1 + 2 + 3 + \ldots + n = \sum_{n=1}^{n} n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum_{n=1}^{n} n^3 = (\sum_{n=1}^{n} n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2.$

(iv) $2 + 4 + 6 + \ldots + 2n = \sum_{n=1}^{n} 2n = n(n+1)$

(v) $1 + 3 + 5 + \ldots + (2n-1) = \sum_{n=1}^{n} (2n-1) = n^2$

(vi) Product of r successive integers is divisible by $r!$

(vii) For $x \neq y$, $x^n - y^n$ is divisible by
  (a) $x + y$, if n is even
  (b) $x - y$, if n is even or odd

(viii) $x^n + y^n$ is divisible by $x + y$, if n is odd

(ix) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$

(x) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \ldots - xy^{n-2} + y^{n-1})$
when n is odd positive integer

SOLVED PROBLEMS

PROBLEM 1

Prove that $P(n) = n^2 + n + 1$ is an odd natural number $\forall$ $n \in N$.

SOLUTION

$P(n)$: $n^2 + n + 1$

$P(1) : 1 + 1 + 1 = 3$, which is an odd number

Let $P(k) : k^2 + k + 1 = 2m + 1$ (an odd number)
We need to prove

$\therefore \quad P(k+1) : (k+1)^2 + (k+1) + 1 = 2\lambda + 1 \quad \therefore \lambda \in I$

L.H.S. of $P(k+1) = (k+1)^2 + (k+1) + 1$

$= k^2 + 1 + 2k + k + 1 + 1$

$= (k^2 + k + 1) + 2k + 2$

$= (2m + 1) + 2k + 2$ [Assuming $P(k)$ to be true]

$= 2(m + k + 1) + 1$

$= 2\lambda + 1$ (an odd number)

By mathematical induction we can say P$(k + 1)$ is true.
Hence $P(n) = n^2 + n + 1$ is an odd natural number for $n \in N$.

PROBLEM 2

By mathematical induction, prove that

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}.$$

SOLUTION

$$\text{Let } P(n) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{n(n+1)(n+2)}$$
$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

for $n = 1$

$$\text{L.H.S.} = \frac{1}{1.2.3} = \frac{1}{6} \text{ and R.H.S.} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$$

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true, then

$$P(k) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{k(k+1)(k+2)}$$
$$= \frac{k(k+3)}{4(k+1)(k+2)} \qquad \ldots(1)$$

For $n = k + 1$, we need to prove $P(k + 1)$

$$P(k+1) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\text{L.H.S.} = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

[From Eq.(1)]

$$= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)} = \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \text{RHS}$$

Hence, $P(k + 1)$ is true.
Hence, by the principle of mathematical induction for all $n \in N$, $P(n)$ is true.

PROBLEM 3

Use the principle of mathematical induction to show that $5^{2n+1} + 3^{n+2} \cdot 2^{n-1}$ divisible by 19 for all natural numbers n.

SOLUTION

Let $P(n) = 5^{2n+1} + 3^{n+2} \cdot 2^{n-1}$

STEP-1: For $n = 1$

$P(1) = 5^{2+1} + 3^{1+2} \cdot 2^{1-1}$

$= 125 + 27$

$= 152$, which is divisible by 19

STEP-2: Assume that the result is true for $n = k$,

i.e., $P(k) = 5^{2k+1} + 3^{k+2}. 2^{k-1}$ is divisible by 19.

$\Rightarrow$ $P(k) = 19r$, where r is an integer,

STEP-3: For $n = k + 1$

$P(k+1) = 5^{2(k+1)+1} + 3^{k+1+2} \cdot 2^{k+1-1}$

$= 5^{2k+3} + 3^{k+3} \cdot 2^{k}$

$= 25.5^{2k+1} + 3.3^{k+2} \cdot 2.2^{k-1}$

$= 25.5^{2k+1} + 6.3^{k+2} \cdot 2^{k-1}$

Now, $25[5^{2k+1}] + 6.3^{k+2}\, 2^{k-1}$

$= 25 \cdot [19r - 3^{k+2} \cdot 2^{k-1}] + 6.3^{k+2} \cdot 2^{k-1}$

$\Rightarrow$ $19[25r - 3^{k+2}\, 2^{k-1}]$

i.e., $P(k+1) = 25P(k) - 19.3^{k+2} \cdot 2^{k-1}$

But we know that $P(k)$ is divisible by 19.
Also $19.3^{k+2}\, 2^{k-1}$ is clearly divisible by 19.

Hence $P(k+1)$ is divisible by 19. This shows that the result is true for $n = k+1$. Hence by the principle of mathematical induction, the result is true for all $n \in N$.

PROBLEM 4

Use the principle of mathematical induction to show that $1.3 + 2.4 + \ldots + n.(n+2) = \frac{1}{6}\,n(n+1)\,(2n+7)$.

SOLUTION

Let $P(n) : 1.3 + 2.4 + \ldots\ n.(n+2) = \frac{1}{6}n(n+1)\,(2n+7)$

STEP-1: For $n = 1$

$$\text{L.H.S. of } P(1) = 1.3 = 3 = \frac{1}{6}.1.2.9$$

$$= \frac{1}{6}.1(1+1)\,(2.1+7) = \text{RHS of } P(1)$$

So, $P(1)$ is true.

STEP-2: Now assume $P(k)$ is true, for some natural number k, i.e.,

$$1.3 + 2.4 + \ldots + k.(k+2) = \frac{1}{6}k(k+1)\,(2k+7)$$

Now deduce $P(k+1)$.

LHS of $P(k+1) = 1.3 + 2.4 + \ldots + k.(k+2) + (k+1).\,(k+1+2)$

$= (\text{LHS of } P(k)) + (k+1)\,(k+3)$

$= (\text{RHS of } P(k)) + (k+1)\,(k+3)$, by inductive assumption)

$$= \frac{1}{6}k(k+1)\,(2k+7) + (k+1)\,(k+3)$$

$$= \frac{1}{6}(k+1)\,(k(2k+7) + 6(k+3))$$

$$= \frac{1}{6}(k+1)\,(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)\,(k+2)\,(2k+9)$$

$$= \frac{1}{6}(k+1)\,(k+1+1)\,(2(k+1)+7)$$

$= \text{RHS of } P(k+1)$.

So $P(k+1)$ is true, if $P(k)$ is true.
Hence by induction $P(n)$ is true for all natural numbers n.

PROBLEM 5

Use the principle of mathematical induction to show that for any positive integer number n, $n^3 + 2n$, is divisible by 3.

SOLUTION

Statement $P(n)$ is defined by $n^3 + 2n$ is divisible 3.

STEP-1: We first show that $P(1)$ is true. Let $n = 1$ and calculate $n^3 + 2n$.

$$1^3 + 2(1) = 3$$

Hence $P(1)$ is true.

STEP-2: We now assume that $P(k)$ is true $k^3 + 2k$ is divisible by 3. is equivalent to $k^3 + 2k = 3M$, where M is a positive integer.
We now consider the algebraic expression $(k+1)^3 + 2(k+1)$; expand it and group like terms.

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 5k + 3$$

$$= [k^3 + 2k] + [3k^2 + 3k + 3]$$

$$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$$

Hence $(k+1)^3 + 2(k+1)$ is also divisible by 3 and therefore statement $P(k+1)$ is true.
By mathematical induction, we can say that $n^3 + 2n$ is divisible by 3.

PROBLEM 6

Using the principle of mathematical induction, prove that $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$ for all $n \in N$.

SOLUTION

$$P(n){:}\ 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n}$$

$$= \frac{2n}{n+1}$$

For $n = 1$,

L.H.S. $= 1$ and R.H.S. $= \frac{2\times 1}{1+1} = \frac{2}{2} = 1$

Thus, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

i.e., $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k}$

$$= \frac{2k}{k+1} \quad \ldots(1)$$

Now, we have to prove that $P(n)$ is true for $n = k+1$.

i.e., $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+(k+1)}$

$$= \frac{2(k+1)}{k+2}$$

Adding $\frac{1}{1+2+3+\ldots+(k+1)}$ on both sides of (1), we get

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+(k+1)}$$

$$= \frac{2k}{k+1} + \frac{1}{1+2+3+\dots+(k+1)}$$

$$= \frac{2k}{k+1} + \frac{1}{\dfrac{(k+1)(k+2)}{2}}$$

$$\left[\text{Using } 1+2+3+\dots+n = \frac{n(n+1)}{2}\right]$$

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)} = \frac{2}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)$$

$$= \frac{2(k+1)^2}{(k+1)(k+2)} = \frac{2(k+1)}{(k+2)} = \text{RHS}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers.

PROBLEM 7

Using principle of mathematical induction, prove that for all $n \in N, \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{3n+1} > 1$.

SOLUTION

$$P(n): \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{3n+1} > 1, \text{ for all } n \in N$$

for $n = 1$,

L.H.S. $= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$ thus, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

i.e., $$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{3k+1} > 1 \qquad \dots(1)$$

Adding $\frac{1}{3k+4}$ on both sides, we get

$$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{3k+1}$$
$$+ \frac{1}{3k+4} > 1 + \frac{1}{3k+4}$$

Clearly, $1 + \frac{1}{3k+4} > 1$.

So, $$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{3k+1} + \frac{1}{3k+4} > 1$$

or $$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{3(k+1)+1} > 1$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers.

PROBLEM 8

Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$. Then use mathematical induction to show that

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \text{ for all } n > 1.$$

SOLUTION

Given that $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$, $n \geq 1$

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$\therefore \quad u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= \frac{1}{\sqrt{5}}(1 \times \sqrt{5}) = 1$$

Thus, u_n is true for $n = 2$ (as given $u_2 = 1$).

Let it be true for $n = k > 2$. Then

$$u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

Consider that $u_{k+1} = u_k + u_{k-1}$ [Using $u_{n+2} = u_{n+1} + u_n$]

$$\therefore \quad u_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] +$$
$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\right]$$
$$- \left[\left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2} + 1\right\}\right]$$
$$- \left[\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2} + 1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6+2\sqrt{5}}{4}\right\}\right.$$

$$-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{6-2\sqrt{5}}{4}\right\}\Bigg]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{\sqrt{5}+1}{2}\right)^{2}\right.$$

$$\left.-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{\sqrt{5}-1}{2}\right)^{2}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

So, u_{k+1} is also true.

Hence, by principle of mathematical induction u_n is true $\forall\, n > 1$.

PROBLEM 9

A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k.
Show that $b_n = 5 + 4n$, for all natural numbers n using mathematical induction.

SOLUTION

We have a sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k.

Let $P(n) : b_n = 5 + 4n$, for all natural numbers n

For $n = 1$,

$$b_1 = 5 + 4 \times 1 = 9$$

Also $b_0 = 5$

$$\therefore \quad b_1 = 4 + b_0 = 4 + 5 = 9$$

Thus, $P(1)$ is true.
Now, assume that $P(n)$ is true for $n = k$.

$$\therefore \quad P(k) : b_k = 5 + 4k \qquad \ldots(1)$$

Now, to prove $P(k + 1)$ is true, we have to show that

$$P(k+1) : b_{k+1} = 5 + 4(K + 1)$$

$$b_{k+1} = 4 + b_{k+1-1} \qquad (\text{as } b_k = 4 + b_{k-1})$$

$$= 4 + b_k$$

$$= 4 + 5 + 4k = 5 + 4(k + 1)$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.
So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. For all $n \in N$, $(3)(5^{2n+1}) + 2^{3n+1}$ is divisible by
   (A) 17 (B) 19
   (C) 21 (D) 23
2. If $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$, then n is
   (A) a positive integer (B) an even positive integer
   (C) an odd positive integer (D) none of these
3. When $P(n) = 9^n - 8^n$ is divided by 8, then the remainder is
   (A) 2 (B) 3
   (C) 1 (D) 7
4. Let $P(n) = 2^{3n} - 7n - 1$, then $P(n)$ is divisible by
   (A) 63 (B) 36
   (C) 49 (D) 25
5. For all $n \in N$, $a^{2n-1} + b^{2n-1}$ is divisible by
   (A) $a + b$ (B) $(a + b)^2$
   (C) $a^3 + b^3$ (D) none of these
6. For all $n \in N$, $10^n + 3{\cdot}4^{n+2} + 5$ is divisible by
   (A) 23 (B) 3
   (C) 7 (D) 1
7. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is
   (A) 1 (B) 2
   (C) 3 (D) 4
8. The sum of n terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots$
   (A) $\frac{n(n+1)^3(n+2)}{24}$ (B) $\frac{n(2n^2+9n+13)}{24}$
   (C) $\frac{4n^2+1}{5}$ (D) None of these
9. If $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then for $n \in N$, A^{10} is equal to
   (A) $\begin{pmatrix} 1 & 10x \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & x^{10} \\ 0 & 1 \end{pmatrix}$
   (C) $\begin{pmatrix} 10 & x \\ 0 & 10 \end{pmatrix}$ (D) $\begin{pmatrix} 10x & x \\ 0 & 10x \end{pmatrix}$
10. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ to n terms =
    (A) $\frac{(n+1)(n+2)(n+3)}{4}$
    (B) $\frac{(n+2)(n-2)(n-3)(n+3)}{4}$
    (C) $\frac{n(n+1)(n+2)(n+3)}{4}$
    (D) None of these
11. The sum of n terms of the series $1 \cdot 3^2 + 2 \cdot 5^2 + 3 \cdot 7^2 + \ldots$ is
    (A) $\frac{n}{6}(n+1)\,(6n^2+14n+7)$ (B) $\frac{n}{6}(n+1)\,(2n+1)(3n+1)$
    (C) $4n^3 + 4n^2 + n$ (D) None of these
12. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ upto $(n + 1)$ terms is equal to
    (A) $\frac{n}{n+1}$ (B) $\frac{2n}{n+1}$
    (C) $\frac{2n}{n(n+1)}$ (D) $\frac{2n}{n+2}$
13. For every positive integer n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is
    (A) an integer (B) a rational number
    (C) a negative real number (D) an odd integer
14. If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then for $n \in N$, A^n is equal to
    (A) $\begin{pmatrix} \cos^n\theta & \sin\theta \\ -\sin^n\theta & \cos\theta \end{pmatrix}$ (B) $\begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$
    (C) $\begin{pmatrix} n\cos\theta & n\sin\theta \\ -n\sin\theta & n\cos\theta \end{pmatrix}$ (D) None of these
15. $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \ldots$ to n terms =
    (A) $\frac{1}{5n-1}$ (B) $\frac{1}{n+4}$
    (C) $\frac{n}{3n+1}$ (D) $\frac{n}{5n-1}$
16. If $1 + 5 + 12 + 22 + 35 + \ldots$ to n terms = $\frac{n^2(n+1)}{2}$, n^{th} term of series is
    (A) $\frac{n(4n-1)}{3}$ (B) $\frac{n(3n-1)}{2}$
    (C) $\frac{n(3n+1)}{2}$ (D) $\frac{n(4n+1)}{3}$

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1. [AIEEE-2011]
   Statement-1: For each natural number n, $(n + 1)^7 - n^7 - 1$ is divisible by 7.
   Statement-2: For each natural number n, $n^7 - n$ is divisible by 7.
   (A) Statement-1 is false, Statement-2 is true
   (B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
   (C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
   (D) Statement-1 is true, Statement-2 is false

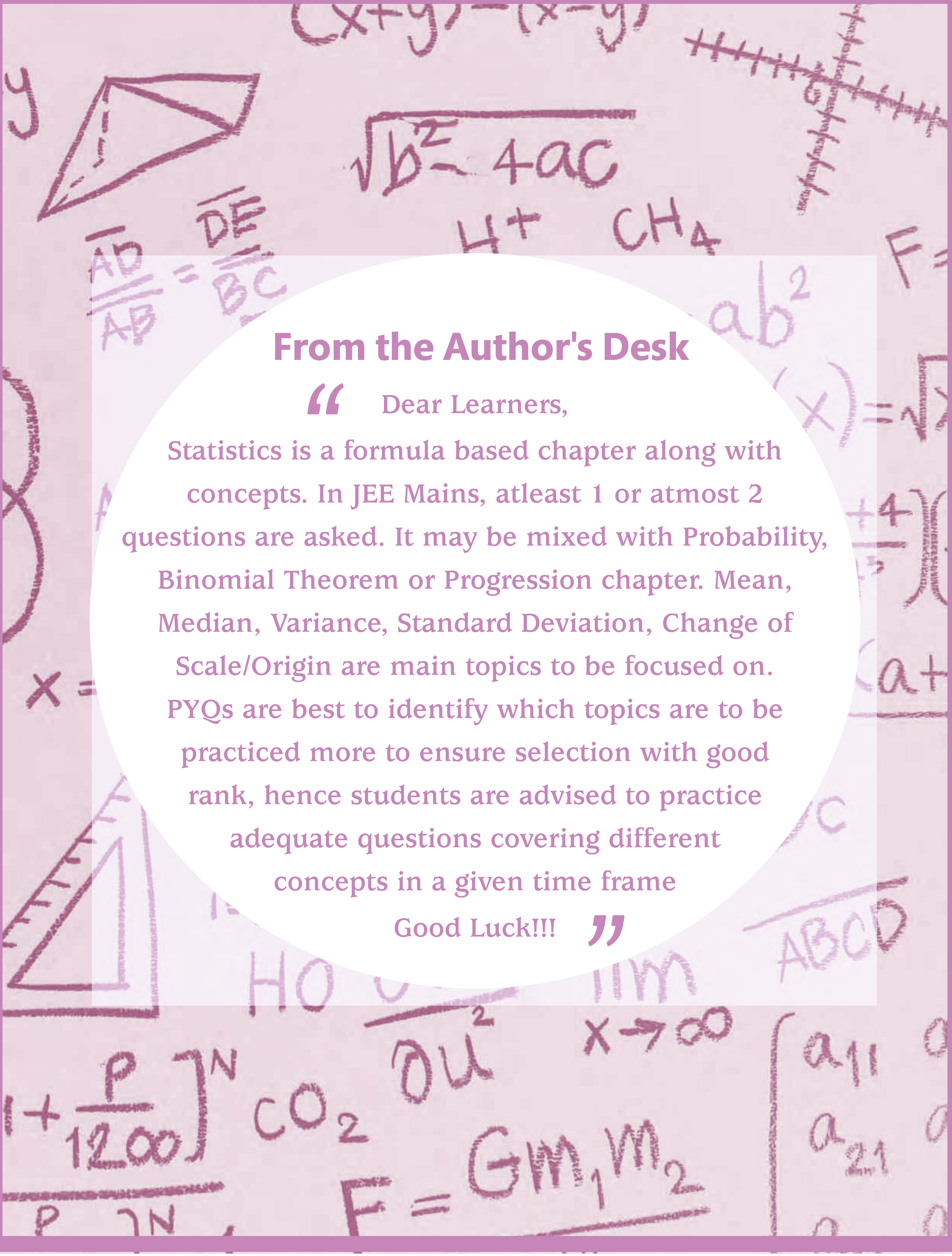
ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** A	**3.** C	**4.** C	**5.** A	**6.** B	**7.** A	**8.** B	**9.** A	**10.** C
11. A	**12.** B	**13.** A	**14.** B	**15.** C	**16.** B				

ARCHIVE: JEE MAIN

1. B



From the Author's Desk

> “ Dear Learners,
>
> Statistics is a formula based chapter along with concepts. In JEE Mains, atleast 1 or atmost 2 questions are asked. It may be mixed with Probability, Binomial Theorem or Progression chapter. Mean, Median, Variance, Standard Deviation, Change of Scale/Origin are main topics to be focused on. PYQs are best to identify which topics are to be practiced more to ensure selection with good rank, hence students are advised to practice adequate questions covering different concepts in a given time frame
>
> Good Luck!!! ”

CHAPTER

9 Statistics

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Mean, median, mode
(b) Mean deviation
(c) Standard deviation & variance

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

Statistics is branch of mathematics where we discuss numerous observations and their suitability of observations.

Representation of various observations by a single data which can represent central value of given observations, is known as measures of central value

There are mainly two types of measures of central tendency

1. Mathematical averages (Mean)
   - **(a)** Arithmetic mean
   - **(b)** Geometric mean
   - **(c)** Harmonic mean
2. Positional averages
   - **(a)** Median
   - **(b)** Mode

MEASURES OF CENTRAL VALUE

Mathematical Mean ($\bar{x}$)

The mean is the most common measure of central tendency and the one that can be mathematically manipulated. It is defined as the **average** of a distribution

Arithmetic Mean

(i) Arithmetic mean of individual series (Ungrouped data): If the series in this case be $x_1, x_2, x_3, \ldots, x_n$; then arithmetic mean $\bar{x}$ is given by

i.e. $\bar{x} = \dfrac{\text{Sum of the series}}{\text{Number of terms}}$

$$= \frac{x_1 + x_2 + x_3 + \ldots + x_n}{N} = \frac{1}{N}\sum_{i=1}^{n} x_i$$

(ii) Arithmetic mean for discrete frequency distribution: If the terms of the given series be $x_1, x_2, \ldots, x_n$ and the corresponding frequencies be $f_1, f_2, \ldots, f_n$, then the arithmetic mean $\bar{x}$ is given by,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{\sum_{i=1}^{n} f_i}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_ix_i$$

where cumulative frequency $N = \sum_{i=1}^{n} f_i$

(iii) Arithmetic mean for grouped or continuous frequency distribution:

Arithmetic mean $\bar{x} = A + \dfrac{1}{N}\sum_{i=1}^{n} f_i(x_i - A)$,

where A = assumed mean, f = frequency and $x_i - A$ = deviation of each item from the assumed mean.

(iv) Combined Arithmetic mean: If $\bar{x}_i$ (i = 1, 2, …, k) are the means of k-component series of sizes n_i, (i = 1, 2, … k) respectively, then the mean $\bar{x}$ of the composite series obtained on combining the component series is given by the formula

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots + n_k} = \frac{\sum_{i=1}^{n} n_i\bar{x}_i}{\sum_{i=1}^{n} n_i}$$

(v) Weighted Arithmetic Mean: Weighted arithmetic mean refers to the arithmetic mean calculated after assigning weights to different values of variable. It is suitable where the relative importance of different items of variable is not same.

Weighted Arithmetic Mean is given by

$$\bar{x}_w = \frac{\sum_{i=1}^{n} W_i X_i}{\sum_{i=1}^{n} W_i}$$

Remark(s)

(i) Algebraic sum of the deviation of a set of values from their arithmetic mean is zero. If x_i/f_i, $i = 1, 2, \ldots, n$ is the frequency distribution, then $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$, $\bar{x}$ being the mean of the distribution.

(ii) The sum of the squares of the deviations of a set of values is minimum when taken about mean.

(iii) If each of the values of a variable '*X*' is increased or decreased by some constant *k*, then arithmetic mean also increased or decreased by *k*. Similarly when the value of the variable '*X*' are multiplied/divided by constant say *k*, arithmetic mean also multiplied/divided by the same quantity *k*.

Geometric Mean (G.M.)

(i) Geometric mean is the n^{th} root of the product of n terms in a given series. $x_1, x_2, x_3, \ldots, x_n$ are n items then $\quad \text{G.M.} = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$

(ii) For a discrete frequency distribution, $\text{GM} = \sqrt[N]{x_1^{f_1} x_2^{f_2} x_3^{f_3} \ldots x_n^{f_n}}$ where $x_1, x_2, \ldots, x_n$ are mid values of n classes and $f_1, f_2, \ldots, f_n$ are corresponding frequencies and $N = \sum_{i=1}^{n} f_i$

Harmonic Mean (H.M.)

Harmonic mean of a given series is the reciprocal of the arithmetic mean of the reciprocal values of its various items in the variable. $x_1, x_2, x_3, \ldots, x_n$ are n items then

$$\text{H.M.} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}}$$

For discrete frequency distribution $\text{H.M.} = \dfrac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} \frac{f_i}{x_i}}$

ILLUSTRATION 1

If the mean of the numbers 27 + x, 31 + x, 89 + x, 107 + x, 156 + x is 82, then find the mean of 130 + x, 126 + x, 68 + x, 50 + x, 1 + x.

SOLUTION

Given

$$\frac{(27+x)+(31+x)+(89+x)+(107+x)+(156+x)}{5} = 82$$

$\Rightarrow \quad 82 \times 5 = 410 + 5x$

$\Rightarrow \quad 410 - 410 = 5x$

$\Rightarrow \quad x = 0$

$\therefore$ Required mean is

$$\bar{x} = \frac{130 + x + 126 + x + 68 + x + 50 + x + 1 + x}{5}$$

$$\bar{x} = \frac{375 + 5x}{5} = \frac{375 + 0}{5} = \frac{375}{5} = 75$$

ILLUSTRATION 2

The mean weight of 150 persons in a group is 60 kg. The mean weight of men in the group is 70 kg and that of the women is 55 kg. Find the number of men and women.

SOLUTION

Number of person = 150; their mean weight = 60 kg; mean weight of men ($\bar{x}_1$) = 70 kg and mean weight of women ($\bar{x}_2$) = 55 kg.

Let n_1 and n_2 be the number of men and number of women respectively.

We know that the total number of persons ($n_1 + n_2$) = 150 or $n_2 = 150 - n_1$.

We also know that the mean weight of all persons

$$(\bar{x}) = \left(\frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}\right)$$

or $\quad 60 = \dfrac{70n_1 + 55n_2}{150} \Rightarrow 60 = \dfrac{14n_1 + 11n_2}{30}$

or $\quad 3n_1 = (1800 - 1650) = 150$

or $\quad n_1 = 50$ and $n_2 = 100$

ILLUSTRATION 3

In the frequency distribution of the discrete data given below, the frequency k against value 0 is missing.

Variable x:	0	1	2	3	4	5
Frequency f:	k	20	40	40	20	4

If the mean is 2.5, then find the missing frequency k

SOLUTION

Given: Mean ($\bar{x}$) = 2.5 and discrete data

Variable x:	0	1	2	3	4	5
Frequency f:	k	20	40	40	20	4

We know that

$$\sum f = k + 20 + 40 + 40 + 20 + 4 = 124 + k$$

We also know that

$$\sum f_i x_i = (0 \times k) + (1 \times 20) + (2 \times 40) + (3 \times 40) + (4 \times 20) + (5 \times 4) = 0 + 20 + 80 + 120 + 80 + 20 = 320$$

Therefore, mean ($\bar{x}$) $= \dfrac{\sum f(x)}{\sum f} = \dfrac{320}{124+k} = 2.5$

or $\quad 2.5 = \dfrac{320}{124+k}$

or $\quad 124 + k = \dfrac{320}{2.5} = 128$

or $\quad k = 128 - 124 = 4$

ILLUSTRATION 4

Find the mean of the values 0, 1, 2, …, n having corresponding weight ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ respectively

SOLUTION

The required mean is

$$\bar{x} = \frac{0.1 + 1.{}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + \ldots + n.{}^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n}$$

$$= \frac{\sum_{r=0}^{n} r.{}^nC_r}{\sum_{r=0}^{n} {}^nC_r} = \frac{\sum_{r=0}^{n} r \cdot \frac{n}{r}\,{}^{n-1}C_{r-1}}{\sum_{r=0}^{n} {}^nC_r} = \frac{n\sum_{r=1}^{n} {}^{n-1}C_{r-1}}{\sum_{r=0}^{n} {}^nC_r}$$

$$= \frac{n.2^{n-1}}{2^n} = \frac{n}{2}.$$

ILLUSTRATION 5

If the combined mean of two groups is $\dfrac{40}{3}$ and if the mean of one group with 10 observations is 15 then find the mean of the other group with 8 observations

SOLUTION

$\because \quad \bar{x} = \dfrac{\sum x}{n_1}, \ \bar{y} = \dfrac{\sum y}{n_2}$

There are total 18 observations

$\therefore \quad$ Combined mean $= \dfrac{(\bar{x} \times 10) + (\bar{y} \times 8)}{18}$

where $\bar{x}$ and $\bar{y}$ denotes the means but $\bar{x} = 15$.

$$\Rightarrow \quad \frac{40}{3} = \frac{150 + 8\bar{y}}{18} \quad \Rightarrow \quad \bar{y} = \frac{45}{4}.$$

Positional Averages

Median

Median is the central value of the set of observations after all the observations are arranged in the ascending or descending order of their magnitudes.

(i) Individual series: If the data is raw, arrange in ascending or descending order,

Let n be the number of observations.

$$\text{Median} = \begin{cases} \left(\dfrac{n+1}{2}\right)^{th} \text{ observation when } n \text{ is odd} \\ \dfrac{1}{2}\left[\left(\dfrac{n}{2}\right)^{th} + \left(\dfrac{n}{2}+1\right)^{th}\right] \text{ observation when } n \text{ is even} \end{cases}$$

(ii) Discrete series: In this case, we first find the cumulative frequencies of the variable arranged in ascending or descending order and the median is given by

Median $= \left(\dfrac{N+1}{2}\right)^{\text{th}}$ observation, where $N = \sum f_i$ is the cumulative frequency.

(iii) For grouped or continuous distribution: In this case, following formula can be used.

$$\text{Median} = l + \frac{\left(\frac{N}{2} - C\right)}{f} \times i$$

Where l = Lower limit of the median class

f = Frequency of the median class

N = The sum of all frequencies

i = The width of the median class

C = The cumulative frequency of the class preceding to median class.

Note:

Class interval corresponding to cumulative frequency just more than $N/2$ is median class

Mode

Mode is the most frequent observation in the distribution. A distribution where a single score is most frequent has one mode and is called unimodal. When there are ties for the most frequent score, the distribution is bimodal if two scores tie or multimodal if more than two scores tie.

Mode for continuous series:

$$\text{Mode} = l_1 + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] \times i$$

Where l_1 = The lower limit of the model class

f_1 = The frequency of the model class

f_0 = The frequency of the class preceding the model class

f_2 = The frequency of the class succeeding the model class

i = The size of the model class.

Note:

Class interval having maximum frequency is called modal class

- **(i)** It is not based on all items of the series.
- **(ii)** As compared to other averages, mode is affected to a large extent by fluctuations of sampling.
- **(iii)** It is not suitable on a case where the relative importance of items have to be considered.

A distribution is a symmetric distribution if the values of mean, mode and median coincide. In a symmetric distribution frequencies are symmetrically distributed on both sides of the centre point of the frequency curve.

A distribution which is not symmetric is called a skewed distribution. In a moderately asymmetric distribution, the interval between the mean and the median is approximately one-third of the interval between the mean and the mode i.e., when have the following empirical relation between them,

$$\text{Mean} - \text{Mode} = 3(\text{Mean - Median})$$

$$\Rightarrow \quad \textbf{Mode} = \textbf{3 Median} - \textbf{2 Mean}.$$

It is known as Empirical relation.

Limitations of central values: An average, such as the mean or the median only locates the centre of the data and does not tell us anything about the spread of the data.

ILLUSTRATION 6

The marks obtained by 10 students is an examination are 22, 26, 14, 30, 18, 11, 35, 41, 12, 32. What is the median mark?

SOLUTION

Number of students (n) = 10 and marks obtained by the them = 22, 26, 14, 30, 18, 11, 35, 41, 12, 32.
Arranging the given marks in the ascending order, we get 11, 12, 14, 18, 22, 26, 30, 32, 35, 41.
Since the number of students is even, therefore median of their marks

$$= \text{Arithmetic mean of } 5^{\text{th}} \text{ and } 6^{\text{th}} \text{ marks}$$

$$= \frac{22+26}{2} = 24$$

ILLUSTRATION 7

Find the mean value of the median and mean of the odd divisors of 360.

SOLUTION

Odd divisors of 360 are

$$360 = 2^3 \times 3^2 \times 5$$

$$\Rightarrow \quad 1, 3, 5, 9, 15, 45$$

$$\text{so mean} = \frac{1+3+5+9+15+45}{6} = 13$$

$$\text{median} = \frac{5+9}{2} = 7$$

$$\text{mean of 13 and 7} = \frac{13+7}{2} = 10$$

Test Your Concepts-I

(Solutions on page H.263)

1. In a class of 100 students there are 70 boys whose average marks in a subject are 75. If the average marks of the complete class is 72, then what is the average of the girls?
2. Find median of ${}^{2n}C_0, {}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3, \ldots, {}^{2n}C_n$ (where n is even).
3. If the average of the numbers 1, 2, 3, ..., 98, 99, x is $100x$, then find the value of x.
4. If in a frequency distribution, the mean and median are 21 and 22 respectively, then its Mode is approximately.
5. Find the A.M. mean of $1, 3, 3^2, \ldots, 3^{n-1}$.
6. If mean of n item is $\bar{x}$. If each r^{th} item is increased by $2r$, then find new mean.

Answer keys

1. 65 **2.** ${}^{2n}C_{n/2}$ **3.** $\frac{50}{101}$ **4.** 24.0 **5.** $\frac{3^n - 1}{2n}$ **6.** $(\bar{x} + n + 1)$

MEASURES OF SPREAD OR DISPERSION

Measures of variability provide information about the degree to which individual scores are clustered about or deviate from the average value in a distribution i.e., The degree to which numerical data tend to spread about an average value is called the dispersion of the data. The four measures of dispersion are

1. Range
2. Mean deviation
3. Standard deviation

Implications

(a) A small value for a measure of dispersion indicate that the data are clustered closely (the mean is therefore representative of the data).

(b) A large value of dispersion indicates that the mean is not reliable (it is not representative of the data).

Range

The simplest measure of variability to compute and understand is the range. The range is the difference between the highest and lowest score in a distribution. Because it is based solely on the most extreme scores in the distribution and does not fully reflect the pattern of variation within a distribution, the range is a very limited measure of variability.

$$\text{Coefficient of range}: \frac{L-S}{L+S}$$

L = Largest value

S = Smallest value

MEAN DEVIATION

The arithmetic average of the deviations (all taking positive) from the mean, median or mode is known as mean deviation.

(i) If $\bar{x}$ is the mean of n observations $x_1, x_2, x_3, \ldots, x_n$

then $\text{Mean Deviation} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$

(ii) $\text{Mean Deviation} = \frac{1}{n}\sum|x_i - m|$ Where m is the median.

(iii) $\text{Mean Deviation} = \frac{1}{n}\sum|x_i - z|$ Where z is the mode.

(i) Let $x_1, x_2, x_3, \ldots, x_n$ occurs with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively and M can be either Mean, Median or Mode then Mean Deviation

$$= \frac{\sum f_i |x_i - M|}{\sum f_i}$$

(ii) Coefficient of Mean Deviation $= \dfrac{\text{Mean Deviation}}{M}$

where M is the Mean, Median or Mode

Note:

The mean deviation from the median is < that measured for any other value.

ILLUSTRATION 8

If the mean deviation about the median of the numbers a, $2a$, …, $50a$ is 50, then find $|a|$

SOLUTION

Mean of $a, 2a, 3a, \ldots, 50a$

$$\text{Median} = \frac{25a + 26a}{2} = \frac{51}{2}a$$

$$\frac{\sum |(x - M)|}{50} = 50$$

$$\Rightarrow \left|a - \frac{51}{2}a\right| + \left|2a - \frac{51}{2}a\right| + \ldots \left|25a - \frac{51}{2}a\right| + \left|26a - \frac{51}{2}a\right| + \ldots + \left|50a - \frac{51}{2}a\right| = 2500$$

$$\Rightarrow 2\left(\frac{49|a|}{2} + \frac{47}{2}|a| + \frac{45|a|}{2} + \ldots + \frac{|a|}{2}\right) = 2500$$

$$\Rightarrow (1 + 3 + \ldots + 49)|a| = 2500$$

$$\Rightarrow \frac{25}{2}(1 + 49)|a| = 2500$$

$$\Rightarrow 25 \times 25|a| = 2500$$

$$\Rightarrow |a| = 4$$

ILLUSTRATION 9

The scores of a batsman in ten innings are: 38, 70, 48, 34, 42, 55, 63, 46, 54, 44. Find the mean deviation about the median.

SOLUTION

Arranging the data in ascending order, we have 34, 38, 42, 44, 46, 48, 54, 55, 63, 70. Here $n = 10$. So, median is the A.M. of 5th and 6th observations.

$$\therefore \quad \text{Median}, M = \left(\frac{46 + 48}{2}\right) = 47$$

Calculation of Mean Deviation

x_i	38	70	48	34	42	55	63	46	54	44	Total
$\lvert d_i \rvert = \lvert x_i - 47 \rvert$	9	23	1	13	5	8	16	1	7	3	$\sum \lvert d_i \rvert = 86$

$$\therefore \quad \text{M.D.} = \frac{1}{n}\sum |d_i| = \frac{86}{10} = 8.6$$

ILLUSTRATION 10

Calculate the mean deviation from the median of the following data:

Wages Per Week in (₹)	10–20	20–30	30–40	40–50	50–60	60–70	70–80
No. of Worker	4	6	10	20	10	6	4

SOLUTION

Calculation of Mean Deviation from Median

Wages Per Week (in ₹)	Mid-Values (x_i)	Frequency (f_i)	Cumulative Frequency	$\lvert d_i \rvert = \lvert x_i - 45 \rvert$	$f_i \lvert d_i \rvert$
10–20	15	4	4	30	120
20–30	25	6	10	20	120
30–40	35	10	20	10	100
40–50	45	20	40	0	0
50–60	55	10	50	10	100
60–70	65	6	56	20	120
70–80	75	4	60	30	120
		$N = \sum f_i = 60$			$\sum f_i \lvert d_i \rvert = 680$

Here, $N = 60$, So $\frac{N}{2} = 30$

The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40–50.

So, 40–50 is the median class.

$$\therefore \quad \ell = 40, f = 20, h = 10, C = 20$$

$$\text{Now, Median} = \ell + \frac{\frac{N}{2} - C}{f} \times h = 40 + \frac{30-20}{20} \times 10 = 45$$

$$\text{Mean Deviation from median} = \frac{\sum f_i |d_i|}{N}$$

$$= \frac{680}{60} = 11.33$$

STANDARD DEVIATION

It is another approach to understand the dispersion on measures. The standard deviation (s or σ) is defined as the positive square root of the variance

standard deviation (s or σ) = $\sqrt{\text{variance}}$

Variance: The variance is a measure based on the deviations of individual scores from the mean.

(a) Variance of individual observations: If $x_1, x_2, \ldots, x_n$, are n values of a variable X, then

$$\text{Var}(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i^2 + \bar{x}^2 - 2\bar{x}x_i)$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \bar{x}^2 - 2\sum_{i=1}^{n} \bar{x}x_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 + \frac{n\bar{x}^2}{n} - 2\bar{x}\,\frac{\sum_{i=1}^{n} x_i}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 + \bar{x}^2 - 2\bar{x}^2$$

$$= \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$\text{Var}(x) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \bar{x}^2$ where mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

variance = Mean of squares – square of Mean

$$\sigma^2 \le \frac{1}{4}(M-m)^2$$

Where *M* and *m* are maximum & minimum value of given observation.

(b) Variance of discrete frequency distribution: If $x_1, x_2, \ldots, x_n$, are *n* values of a variable *X* and corresponding frequencies of them $f_1, f_2, \ldots f_n$

$$\text{Var}(x) = \frac{1}{N}\left|\sum_{i=1}^{n} f_i(x_i - \bar{x})^2\right|$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i x_i\right)^2$$

$$\text{Var}\ (x) = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$$

where cumulative frequency $N = \sum_{i=1}^{n} f_i$ and $\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i$

(c) Variance of a grouped or continuous frequency distribution:

$$\text{Var}(X) = h^2\left[\frac{1}{N}\sum f_i u_i^2 - \left(\frac{1}{N}\sum f_i u_i\right)^2\right]$$

$$u_i = \frac{x_i - \bar{x}}{h}$$

where h = Class width

The variance is a measure in squared units and has little meaning with respect to the data. Thus, the standard deviation is a measure of variability expressed in the same units as the data. The standard deviation is very much like a mean or an "average" of these deviations.

Combined Standard Deviation: If there are two sets of observations containing n_1 and n_2 items with respective mean $\bar{x}_1$ and $\bar{x}_2$ and standards deviations σ_1 and σ_2, then the mean $\bar{x}$ and the standard deviations of $n_1 + n_2$ observations, taken together, are

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$\sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

where $d_1 = \bar{x} - \bar{x}_1$, $d_2 = \bar{x} - \bar{x}_2$

ANALYSIS OF FREQUENCY DISTRIBUTIONS

(i) Coefficient of Standard Deviation = $\dfrac{\sigma}{\bar{x}}$

(ii) Coefficient of Variation = $\dfrac{\sigma}{\bar{x}} \times 100$

(i) Mean is dependent on change of scale and shift of origin

(ii) Variance and Standard deviation and Mean Deviation are independent of shift of origin but dependent on change of scale

Lets we chance data by

Replacing $x_i \to ax_i + b$

$$\bar{x}_{\text{new}} = a\left(\bar{x}_{\text{old}}\right) + b$$

$$\text{Variance}_{\text{new}} = a^2\left(\text{Variance}_{\text{old}}\right)$$

$$\text{Standard Deviation}_{\text{new}} = a\left(\text{Standard Deviation}_{\text{old}}\right)$$

(i) $v(x+a) = v(x)$

(ii) $v(ax) = a^2 v(x)$

(iii) $v(ax+b) = a^2 v(x)$ {$v(x)$ is variance of x}

Let's understand the changes made to the input and its impact on mean, variance , standard deviation and mean deviation

Variable (x_i)	Mean ($\bar{x}$)	Variance ($v(x)$)	Standard Deviation (σ)	Mean Deviation (ρ) About Mean
$x_i+\alpha$	$\bar{x}+\alpha$	$v(x)$	σ	ρ
$x_i-\alpha$	$\bar{x}-\alpha$	$v(x)$	σ	ρ
αx_i	$\alpha\bar{x}$	$\alpha^2 v(x)$	$\lvert\alpha\rvert\sigma$	$\lvert\alpha\rvert\rho$
x_i/α	$\bar{x}/\alpha$	$v(x)/\alpha^2$	$\sigma/\lvert\alpha\rvert$	$\rho/\lvert\alpha\rvert$
$\alpha x_i+\beta$	$\alpha\bar{x}+\beta$	$\alpha^2 v(x)$	$\lvert\alpha\rvert\sigma$	$\lvert\alpha\rvert\rho$
$\alpha x_i-\beta$	$\alpha\bar{x}-\beta$	$\alpha^2 v(x)$	$\lvert\alpha\rvert\sigma$	$\lvert\alpha\rvert\rho$

For series $a, a+d, a+2d, \ldots, a+(n-1)d$.

1. Arithmetic mean $(\bar{x}) = a+\dfrac{(n-1)d}{2}$;
2. Variance $\left(\sigma^2\right) = \dfrac{n^2-1}{12}d^2$
3. Standard deviation $= \sqrt{\dfrac{n^2-1}{12}}.d$
4. Standard deviation of n natural numbers $= \sqrt{\dfrac{n^2-1}{12}}$ $(\because\ d=1)$

BINOMIAL PROBABILITY DISTRIBUTION

In an unbiased experiment whose outcomes are independent of number of times experiment is being conducted
In a single trial, probability of success is p and probability of failure is q, hence $p+q=1$
Let there are n trials are conducted in this experiment
then **Mean = np** and **Variance = npq**
This topic is covered in detailed in probability chapter.

ILLUSTRATION 11

Calculate the mean and standard deviation of first n natural numbers.

SOLUTION

Here $x_i = i = 1, 2, \ldots, n$. Let $\bar{x}$ be the mean and σ be the S.D. Then,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}(1+2+3+\ldots+n)$$

$$\Rightarrow \quad \text{Mean } \bar{x} = \frac{n(n+1)}{2n} = \frac{n+1}{2} \text{ and}$$

$$\sigma^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

$$\Rightarrow \quad \sigma^2 = \frac{1}{n}(1^2+2^2+\ldots+n^2) - \left(\frac{n+1}{2}\right)^2$$

$$\Rightarrow \quad \sigma^2 = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2$$

$$\sigma^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

ILLUSTRATION 12

If the variance of the random variable X is 4, then find the variance of the random variable $5X+10$

SOLUTION

$$V(ax \pm b) = a^2 V(X) = 25{*}4 = 100$$

ILLUSTRATION 13

For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. Find the variance of the combined data set

SOLUTION

$$\sigma^2 = \frac{n_1\left(\sigma_1^2+d_1^2\right)+n_2\left(\sigma_2^2+d_2^2\right)}{n_1+n_2}$$

$$\Rightarrow \quad n_1 = n_2 = 5,\ \sigma_1^2 = 4,\ \sigma_2^2 = 5,\ \bar{x}_1 = 2,\ \bar{x}_2 = 4$$

Combined Mean $\bar{x} = \dfrac{5\times 2+5\times 4}{10} = 3$

$$\therefore \quad d_1 = \bar{x}_1 - \bar{x} = -1 \text{ and } d_2 = \bar{x}_2 - \bar{x} = 1$$

$$\sigma^2 = \frac{5(4+1)+5(5+1)}{5+5}$$

$$\therefore \quad \sigma^2 = \frac{11}{2}$$

ILLUSTRATION 14

If the coefficient of variation of two distributions are 50, 60 and their arithmetic means are 30 and 25 respectively, then find the difference of their standard deviation

SOLUTION

Here, $CV_1 = 50, CV_2 = 60, \bar{x}_1 = 30$ and $\bar{x}_2 = 25$

$$\therefore \quad CV_1 = \frac{\sigma_1}{\bar{x}_1} \times 100 \quad \Rightarrow \quad 50 = \frac{\sigma_1}{30} \times 100$$

$$\therefore \quad \sigma_1 = \frac{30 \times 50}{100} = 15 \text{ and}$$

$$CV_2 = \frac{\sigma_2}{\bar{x}_2} \times 100 = 60$$

$$\therefore \quad \sigma_2 = \frac{60 \times 25}{100} = 15$$

Now $\sigma_1 - \sigma_2 = 15 - 15 = 0$

ILLUSTRATION 15

The mean and S.D. of the marks of 200 candidates were found to be 40 and 15 respectively. Later, it was discovered that a score of 40 was wrongly read as 50. Then find correct mean and S.D.

SOLUTION

Number of candidates = 200
Mean = 40; s.d. = 15
Correct data = 40
Incorrect data = 50

(i) Incorrect mean = $\dfrac{\sum x_i}{200} = 40$

Incorrect $\sum x_i = 8000$

Correct sum $\sum x_i = 8000 - 50 + 40 = 7990$

Correct mean = $\dfrac{\sum x_i}{200} = 39.95$

(ii) Incorrect s.d. = $\sqrt{\dfrac{1}{n}\sum x_i^2 - \left(\dfrac{1}{n}\sum x_i\right)^2}$

$$\Rightarrow \left[(s.d.)^2 + \left(\frac{1}{n}\sum x_i\right)^2\right] n = \sum x_i^2$$

$\Rightarrow (225 + 1600) \times 200 = \sum x_i^2$ = Incorrected sum

$$\Rightarrow \sum x_i^2 = 365000$$

$\Rightarrow$ corrected sum = $\sum x_i^2 - (50)^2 + (40)^2$

Corrected $\sigma^2 = \left[\dfrac{364100}{200} - (39.95)^2\right]$

$= 1820.5 - 1596.25 = 224.49$; s.d. $= 14.98$

ILLUSTRATION 16

The mean and variance of 7 observations are 8 and 16 respectively. If 5 of the observations are 2, 4, 10, 12, 14, find the remaining two observations.

SOLUTION

Let x and y be the remaining two observation. Then,

Mean = 8

$$\Rightarrow \quad \frac{2 + 4 + 10 + 12 + 14 + x + y}{7} = 8$$

$$\Rightarrow \quad 42 + x + y = 56$$

$$\Rightarrow \quad x + y = 14 \qquad \ldots(1)$$

Variance = 16

$$\Rightarrow \quad \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x^2 + y^2) - (\text{Mean})^2 = 16$$

$$\Rightarrow \quad \frac{1}{7}(4 + 16 + 100 + 144 + 196 + x^2 + y^2) - 64 = 16$$

$$\Rightarrow \quad 460 + x^2 + y^2 = 7 \times 80 \quad \Rightarrow \quad x^2 + y^2 = 100$$

Now, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$

$$\Rightarrow \quad 196 + (x - y)^2 = 2 \times 100$$

$$\Rightarrow \quad (x - y)^2 = 4 \quad \Rightarrow \quad x - y = \pm 2$$

If $x - y = 2$, and $x + y = 14 \quad \Rightarrow \quad x = 8, y = 6$

If $x - y = -2$ and $x + y = 14 \quad \Rightarrow \quad x = 6, y = 8$

Hence, the remaining two observations are 6 and 8.

ILLUSTRATION 17

In a experiment with 15 observations on x, the following results were available $\sum x^2 = 2830$, $\sum x = 170$. One observation that was 20, we found to be wrong and was replaced by the correct value 30. Then find the corrected variance

SOLUTION

Given: $N = 15$

$$\Sigma x^2 = 2830, \Sigma x = 170$$

One observation 20 was replaced by 30, then

$$\Sigma x^2 = 2830 - 400 + 900 = 3330$$

$$\Sigma x = 170 - 20 + 30 = 180$$

$$\therefore \quad \text{Variance, } \sigma^2 = \frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2$$

$$= \frac{3330}{15} - \left(\frac{180}{15}\right)^2 = 222 - 144 = 78$$

ILLUSTRATION 18

Find the variance and standard deviation for the following distribution:

Classes	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	3	7	12	15	8	3	2

SOLUTION

Calculation of Variance and Standard Deviation

Class	Frequency (f_i)	Mid-point (x_i)	$y_i = \frac{x_i - 65}{10}$	y_i^2	$f_i y_i$	$f_i y_i^2$
30–40	3	35	–3	9	–9	27
40–50	7	45	–2	4	–14	28
50–60	12	55	–1	1	–12	12
60–70	15	65	0	0	0	0
70–80	8	75	1	1	8	8
80–90	3	85	2	4	6	12
90–100	2	95	3	9	6	18
	$N = 50$				–15	105

Therefore $\bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

Variance $\sigma^2 = \frac{h^2}{N^2}\left[N\sum f_i y_1^2 - \left(\sum f_i y_i\right)^2\right]$

$$= \frac{(10)^2}{(50)^2}[50 \times 105 - (-15)^2] = \frac{1}{25}[5250 - 225] = 201$$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$

ILLUSTRATION 19

The mean and standard deviation of 20 observations are found to be 10 and 2 respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:

(i) If the wrong item is omitted.
(ii) If it is replaced by 12.

SOLUTION

We have, $n = 20$, $\bar{X} = 10$ and $= \sigma = 2$

$\therefore \quad \bar{X} = \frac{1}{n}\sum x_i = 10$

$\Rightarrow \quad \sum x_i = n\bar{X} = 20 \times 10 = 200$

$\Rightarrow$ Incorrected $\sum x_i = 200$ and, $\sigma = 2 \quad \Rightarrow \quad \sigma^2 = 4$

$$\frac{1}{n}\sum x_i^2 - (\text{Mean})^2 = 4 \quad \Rightarrow \quad \frac{1}{20}\sum x_i^2 - 100 = 4$$

$\Rightarrow \quad \sum x_i^2 = 104 \times 20$

$\Rightarrow$ Incorrected $\sum x_i^2 = 2080$

(i) When 8 is omitted from the data. If 8 is omitted from the data, then 19 observations are left.

Now Incorrected $\sum x_i = 200$

$\Rightarrow$ Corrected $\sum x_i + 8 = 200$

$\Rightarrow$ Corrected $\sum x_i = 192$

and Incorrected $\sum x_i^2 = 2080$

$\Rightarrow$ Corrected $\sum x_i^2 + 8^2 = 2080$

$\Rightarrow$ Corrected $\sum x_i^2 = 2016$

$\therefore$ Corrected mean $= \frac{192}{19} = 10.10$

Corrected variance

$$= \frac{1}{19}\left(\text{Corrected} \sum x_i^2\right) - (\text{Corrected mean})^2$$

$$= \frac{2016}{19} - \left(\frac{192}{19}\right)^2 = \frac{38304 - 36304}{361} = \frac{1440}{361}$$

$\therefore$ Corrected standard derivation $= \sqrt{\frac{1440}{361}}$

$$= \frac{12\sqrt{10}}{19} = 1.997$$

(ii) When the incorrect observation 8 is replaced by 12: We have,

$$\text{Incorrected } \sum x_i = 200$$

$$\therefore \text{ Corrected } \sum x_i = 200 - 8 + 12 = 204$$

$$\text{and Incorrected } \sum x_i^2 = 2080$$

$$\Rightarrow \text{ Corrected } \sum x_i^2 = 2080 - 8^2 + 12^2 = 2160$$

$$\text{Now, Corrected mean} = \frac{204}{20} = 10.2$$

$\Rightarrow$ Corrected variance

$$= \frac{1}{20}\left(\text{Corrected } \sum x_1^2\right) - (\text{Corrected mean})^2$$

$$\Rightarrow \text{ Corrected variance} = \frac{2160}{20} - \left(\frac{204}{20}\right)^2$$

$$= \frac{2160 \times 20 - (204)^2}{(20)^2}$$

$$\Rightarrow \text{ Corrected variance} = \frac{43200 - 41616}{400} = \frac{1584}{400}$$

$$\therefore \text{ Corrected standard deviation} = \sqrt{\frac{1584}{400}}$$

$$= \frac{\sqrt{396}}{10} = 1.9899$$

ILLUSTRATION 20

The following values are calculated in respect of heights and weights of the students of a section of Class XI:

	Height	Weight
Mean	162.6 cm	52.36 kg
Variance	127.69 cm^2	23.1361 kg^2

Can we say that the weights show greater variation than the heights?

SOLUTION

To compare the variability, we have to calculate their coefficients of variation

Given

$$\text{Variance of height} = 127.69 \text{ cm}^2$$

Therefore

$$\text{Standard deviation of height} = \sqrt{127.69} \text{ cm}$$

$$= 11.3 \text{ cm}$$

Also

$$\text{Variance of weight} = 23.1361 \text{ kg}^2$$

Therefore

$$\text{Standard deviation of weight} = \sqrt{23.1361} \text{ kg}$$

$$= 4.81 \text{ kg}$$

Now, the coefficient of variations (C.V.) are given by

$$\text{(C.V.) in heights} = \frac{\text{Standard Deviation}}{\text{mean}} \times 100$$

$$= \frac{11.3}{162.6} \times 100 = 6.95$$

$$\text{and (C.V.) in weight} = \frac{4.81}{52.36} \times 100 = 9.18$$

Clearly C.V. in weights is greater that the C.V. in heights Therefore, we can say that weights show more variability than heights.

Test Your Concepts-II

(Solutions on page H.263)

1. The SD of 15 items is 6 and if each item is decreases by 1, then find standard deviation.
2. If the mean deviation of number 1, 1 + d, 1 + 2d, ..., 1 + 100 d from their mean is 255, then find d.
3. The mean and variance of n observations $x_1, x_2, x_3, \ldots, x_n$ are 5 and 0 respectively. If $\sum_{i=1}^{n} x_i^2 = 400$, then find the maximum value of n.
4. Find the variance of first 20 natural numbers.
5. Find the mean deviation from the mean for the set of observations – 1, 0, 4.
6. Find the standard deviations for first 10 natural numbers.
7. Find the standard deviation of data 6, 5, 9, 13, 12, 8 and 10.
8. The coefficient of variation of two distributions are 60 and 70. The standard deviation are 21 and 16 respectively, then find their mean.
9. Find the mean deviation from the data 3, 10, 10, 4, 7, 10, 5.

Answer keys

1. 6
2. 10.1
3. 16
4. $\frac{133}{4}$
5. 2
6. 2.87
7. $\sqrt{\frac{52}{7}}$
8. 35,22.85
9. 3

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If the standard deviation of the observations –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5 is $\sqrt{10}$. The standard deviation of observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be
(A) $\sqrt{10}+20$ (B) $\sqrt{10}+10$
(C) $\sqrt{10}$ (D) None of these

2. The arithmetic mean of the data given by

Variate (x)	0	1	2	3	…	n
Frequency (f)	${}^nC_0$	${}^nC_1$	${}^nC_2$	${}^nC_3$		${}^nC_n$

(A) $\frac{1}{2}(n+1)$ (B) $\frac{1}{2}n$
(C) $\frac{2^n}{n}$ (D) $\frac{2^{n-1}}{n}$

3. The mean of the numbers a, b, 8, 5, 10, is 6 and the variance is 6.80. Then, which one of the following gives possible values of a and b?
(A) $a=3, b=4$ (B) $a=0, b=7$
(C) $a=5, b=2$ (D) $a=1, b=6$

4. If the variable takes the values 0, 1, 2, …, n with frequencies proportional to the binomial coefficients $C(n, 0)$, $C(n, 1)$, $C(n, 2)$, …, $C(n, n)$ respectively, then the variance of the distribution is
(A) n (B) $\frac{\sqrt{n}}{2}$
(C) $\frac{n}{2}$ (D) $\frac{n}{4}$

5. The mean of 10 observations is 16.3 by an error one observation is registered as 32 instead of 23. Then the correct mean is
(A) 15.6 (B) 15.4
(C) 15.7 (D) 15.8

6. The mean of n items is $\bar{x}$. If the first item is increased by 1, second by 2 and so on, then the new mean is
(A) $\bar{x}+n$ (B) $\bar{x}+\frac{n}{2}$
(C) $\bar{x}+\frac{n+1}{2}$ (D) $\bar{x}-\frac{n}{2}$

7. 10 is the mean of a set of 7 observations and 5 is the mean of a set of 3 observations. The mean of the combined set is given by
(A) 15 (B) 10
(C) 8.5 (D) 7.5

8. The median of a set of 9 distinct observations is 20.5. If each of the largest 4 observations of the set is increased by 2, then the median of the new set
(A) is decreased by 2
(B) is two time the original median
(C) remains the same as that of the original set
(D) is increased by 2

9. If $\sum_{i=1}^{18}(x_i-8)=9$ and $\sum_{i=1}^{18}(x_i-8)^2=45$ then the standard deviation of $x_1, x_2 \ldots x_{18}$ is
(A) 4/9 (B) 9/4
(C) 3/2 (D) $\frac{2}{3}$

10. If the standard deviation of $x_1, x_2, \ldots x_n$ is 3.5, then the standard deviation of $-2x_1-3, -2x_2-3, \ldots, -2x_n-3$ is
(A) –7 (B) –4
(C) 7 (D) 1.75

11. Consider the following statements:
(1) Mode can be computed from histogram.
(2) Median is not independent of change of scale
(3) Variance is independent of change of origin and scale.
Which of these is/are correct?
(A) Only (1)
(B) Only (2)
(C) Only (1) and (2)
(D) (1), (2) and (3)

12. Let $x_1, x_2, \ldots, x_n$ be n observations such that $\Sigma x_i^2 = 400$ and $\Sigma x_i = 80$. Then a possible value of n among the following is
(A) 15 (B) 18
(C) 9 (D) 12

13. Marks scored by 100 students in a 25 marks unit test of Mathematics is given below. Their median is

Marks	0–5	5–10	10–15	15–20	20–25
Students	10	18	42	23	7

(A) 12 (B) 12.62
(C) 12.3 (D) 12.7

14. A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 gm and a standard deviation of 2 gm. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2 gm. The correct mean and standard deviation (in gm) of fishes are respectively
(A) 32, 2 (B) 32, 4
(C) 28, 2 (D) 28, 4

15. The starting value of the model class of a distribution is 20. The frequency of the model class is 18. The frequencies of the classes preceding and succeeding are 8, 10 and the width of the model class is 5, then mode =
(A) 18.5 (B) 20.5
(C) 21.4 (D) 22.78

16. In the series of $2n$ observations, half of them each equal to a and remaining half each equal to $-a$. If the standard deviation of the observations is 2, then $|a|$ equals to:

(A) $\frac{1}{n}$
(B) $\sqrt{2}$
(C) 2
(D) $\frac{\sqrt{2}}{n}$

17. Consider any set of observations $x_1, x_2, x_3, \ldots, x_{101}$; it being given that $x_1 < x_2 < x_3 < \ldots < x_{100} < x_{101}$; then the mean deviation of this set of observations about a point k is minimum when k equals

(A) x_1
(B) x_{51}
(C) $\frac{x_1 + x_2 + \ldots + x_{101}}{101}$
(D) x_{50}

18. Mean of the numbers 1, 2, 3, …, n with respective weights $1^2+1, 2^2+2, 3^2+3, \ldots, n^2+n$ is

(A) $\frac{3n(n+1)}{2(2n+1)}$
(B) $\frac{2n+1}{3}$
(C) $\frac{3n+1}{4}$
(D) $\frac{3n+1}{2}$

19. The sum of square of deviations for 10 observations taken from mean 50 is 250. The coefficient of variation is

(A) 10%
(B) 40%
(C) 50%
(D) None of these

20. Mean of n items is x. If these n items are increased by 1^2, 2^2, 3^2, … n^2 successively, then mean gets increased by

(A) $\frac{(n+1)(2n+1)}{6}$
(B) $\frac{n(n+1)(2n+1)}{6}$
(C) $\frac{n^2}{2}$
(D) remains same

21. Suppose two groups of scores A and B are such that $A = (x, x+2, x+4)$ and $B = (x-2, x+2, x+6)$.

Statement-1: Group B has more variability than group A.

Statement-2: The value of mean for group b is more than that of group A.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(C) Statement-1 is true, Statement-2 is false.
(D) Statement-1 is false, Statement-2 is true.

22. **Statement-1:** The standard deviation of variable $\frac{ax+b}{c}$ is $\left|\frac{a}{c}\right| \sigma$

Statement-2: The standard deviation of a linear equation is $\sigma \times |\text{coefficient of } x|$

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(C) Statement-1 is true, Statement-2 is false.
(D) Statement-1 is false, Statement-2 is true.

23. The A.M. of ${}^{2n+1}C_0, {}^{2n+1}C_1, {}^{2n+1}C_2, \ldots {}^{2n+1}C_n$ is

(A) $\frac{2^n}{n}$
(B) $\frac{2^n}{n+1}$
(C) $\frac{2^{2n}}{n}$
(D) $\frac{2^{2n}}{(n+1)}$

24. If a variable takes the discrete values $\alpha+4$, $\alpha-\frac{7}{2}$, $\alpha-\frac{5}{2}$, $\alpha-3$, $\alpha-2$, $\alpha+\frac{1}{2}$, $\alpha-\frac{1}{2}$, $\alpha+5$, then the median is

(A) $\alpha-\frac{5}{4}$
(B) $\alpha-\frac{1}{2}$
(C) $\alpha-2$
(D) $\alpha+\frac{5}{4}$

25. Mean deviation of the series $a, a+d, a+2d, \ldots a+2nd$ from its mean is

(A) $\frac{(n+1)d}{(2n+1)}$
(B) $\frac{nd}{2n+1}$
(C) $\frac{n(n+1)d}{(2n+1)}$
(D) $\frac{(2n+1)d}{n(n+1)}$

26. A sample of 35 observations has mean 80 and standard deviation as 4. A second sample of 65 observation from the same population has mean 60 and standard deviation 3. The standard deviation of the combined sample is

(A) 15.85
(B) 5.85
(C) 10.12
(D) None of these

27. The standard deviation of 25 numbers is 40. If each of the numbers is increased by 5, then the new standard deviation will be

(A) 40
(B) 45
(C) $40+\frac{21}{25}$
(D) none of these

28. The average of n numbers $x_1, x_2, x_3, \ldots x_n$ is M. If x_1 is replaced by x, then the new average is

(A) $M - x_1 + x$
(B) $\frac{(n-1)M - x_1 + x}{n}$
(C) $\frac{(n+1)M - x_1 + x}{n}$
(D) $\frac{nM - x_1 + x}{n}$

29. The mean of n items is $\bar{x}$. If each item is successively increased by $3, 3^2, 3^3, \ldots, 3^n$, then new mean equals

(A) $\bar{x} + \frac{3^{n+1}}{n}$
(B) $\bar{x} + 3\frac{(3^n+1)}{2n}$
(C) $\bar{x} + \frac{3^n}{n}$
(D) $\bar{x} + 3\frac{(3^n-1)}{2n}$

30. An automobile driver travels from plane to a hill station 120 km distant at an average speed of 30 km per hour. He then makes the return trip at an average speed of 25 km per hour. He covers another 120 km distance on plane at an average speed of 50 km per hour. His average speed over the entire distance of 360 km will be

(A) $\frac{30+25+50}{3}$ km/hr
(B) $(30+25+50)^{1/3}$ km/hr
(C) $\frac{3}{\frac{1}{30}+\frac{1}{25}+\frac{1}{50}}$ km/hr
(D) None of these

31. If the arithmetic mean of the numbers $x_1, x_2, x_3, \ldots, x_n$ is $\bar{x}$, then the arithmetic mean of numbers $ax_1 + b, ax_2 + b, ax_3 + b, \ldots, ax_n + b$, where a, b are two constants would be

(A) $\bar{x}$

(B) $na\bar{x} + nb$

(C) $a\bar{x}$

(D) $a\bar{x} + b$

32. The weighted mean of first n natural numbers whose weight are equal to the squares of corresponding numbers is

(A) $\dfrac{n+1}{2}$

(B) $\dfrac{3n(n+1)}{2(2n+1)}$

(C) $\dfrac{(n+1)(2n+1)}{6}$

(D) $\dfrac{n(n+1)}{2}$

33. If the values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}$ occur at frequencies 1, 2, 3, 4, 5, …, n in a distribution, then the mean is

(A) 1

(B) n

(C) $\dfrac{1}{n}$

(D) $\dfrac{2}{n+1}$

34. The number of observations in a group is 40. If the average of first 10 is 4.5 and that of the remaining 30 is 3.5, then the average of the whole group is

(A) $\dfrac{1}{5}$

(B) $\dfrac{15}{4}$

(C) 4

(D) 8

35. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is

(A) 16.8

(B) 16.0

(C) 15.8

(D) 14.0

36. If the median of $\frac{x}{5}, x, \frac{x}{4}, \frac{x}{2}, \frac{x}{3}$ $(x > 0)$ is 8, then the value of x is

(A) 24

(B) 32

(C) 8

(D) 16

37. If the standard deviation of 0, 1, 2, 3, …, 9 is K, then the standard deviation of 10, 11, 12, 13, …, 19 is

(A) K

(B) $K + 10$

(C) $K + \sqrt{10}$

(D) $10K$

38. If the standard deviation of the number 2, 3 a and 11 is 3.5, then which of the following is true.

(A) $3a^2 - 32a + 84 = 0$

(B) $3a^2 - 34a + 91 = 0$

(C) $3a^2 - 23a + 44 = 0$

(D) $3a^2 - 26a + 55 = 0$

39. The mean deviation of the data 3, 10, 10, 4, 7, 10, 5 from the mean is

(A) 2

(B) 2.57

(C) 3

(D) 3.75

40. Following are the marks obtained by 9 students in a mathematics test 50, 69, 20, 33, 53, 39, 40, 65, 59. The mean deviation from the median is

(A) 9

(B) 10.5

(C) 12.67

(D) 14.76

41. If a, b, c, d and e be the observations with mean m and standard deviation s, then the standard deviation of the observations $a + k, b + k, c + k, d + k$ and $e + k$ is

(A) s

(B) ks

(C) $s + k$

(D) $\dfrac{s}{k}$

42. If x_1, x_2, x_3, x_4 and x_5 and x_5 be the observations with mean m and standard deviation s then, the standard deviation of the observations kx_1, kx_2, kx_3, kx_4 and kx_5 is

(A) $k + s$

(B) $\dfrac{s}{k}$

(C) ks

(D) s

43. Let $x_1, x_2, \ldots, x_n$ be n observations. Let $w_i = \ell x_i + k$ for $i = 1, 2, 3, \ldots, n$, where ℓ and k are constants. If the mean of x_i's is 48 and their standard deviation is 12, the mean of w_i's is 55 and standard deviation of w_i's is 15, then the value of ℓ and k should be

(A) $\ell = 1.25, k = -5$

(B) $\ell = -1.25, k = 5$

(C) $\ell = 2.5, k = -5$

(D) $\ell = 2.5, k = 5$

44. Consider the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. If 1 is added to each number the variance of the numbers, so obtained is

(A) 6.5

(B) 2.87

(C) 3.87

(D) 8.25

45. Consider the first 10 positive integers. If we multiply each number by –1 and, then add 1 to each number, the variance of the numbers, so obtained is

(A) 8.25

(B) 6.5

(C) 3.87

(D) 2.87

46. The A.M. of a 50 set of numbers is 38. If two numbers of the set, namely 55 and 45 are discarded, the A.M. of the remaining set of numbers is

(A) 38.5

(B) 37.5

(C) 36.5

(D) 36

47. The mean of five observations is 4 and their variance is 5.2. If three of these observations are 2, 4 and 6, then the other two observations are

(A) 3 and 5

(B) 2 and 6

(C) 4 and 4

(D) 1 and 7

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. In a class, in an examination in Mathematics, 10 students scored 100 marks each, 2 students scored zero and the average of the remaining students is 72 marks. If the class average is 76, then find the number of students in the class.

2. The standard deviation of some temperature data in °C is 5. If the data were converted into °F, then find the variance.

3. The mean weight of 9 items is 15. If one more item is added to the series, the mean becomes 16. Find the value of 10th item.

4. If the variate takes the values 0, 2, 4, 8, … 2^n with frequencies $^nC_0, {}^nC_1, \ldots {}^nC_n$ and if the mean is $\frac{91\times 8}{2^n}$, then find n.

5. The variance of α, β and γ is 9, find the variance of 5α, 5β and 5γ.

6. Find the median from the following distribution.

Marks (x)	**No. of Students**
0–10	10
10–20	20
20–30	30
30–40	50
40–50	40
50–60	30

7. When tested, the lives (in hours) of 5 bulbs were noted as follows 1357, 1090, 1666, 1694, 1623. Find the mean deviations (in hours) from their mean.

8. If the mean of 100 observations is 50 and their standard deviation is 5, then find the sum of all squares of all the observations.

9. The following information relates to a sample of size 60, $\Sigma x^2 = 18000$, and $\Sigma x = 960$. Then, find the variance.

10. If the mean of the numbers a, b, 8, 5, 10, is 6 and their variance is 6.8, then find ab.

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1. [JEE Mains 2020]
Consider the data on x taking the values 0, 2, 4, 8, …, 2^n with frequencies $^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ respectively. If the mean of this data is $\frac{728}{2^n}$. Then n is equal to _______.

2. [JEE Mains 2020]
If the variance of the following frequency distribution:

Class	10–20	20–30	30–40
Frequency	2	x	2

is 50, then x is equal to ____.

3. [JEE Mains 2020]
If the variance of the terms in an increasing A.P., $b_1, b_2, b_3, \ldots b_{11}$ is 90, then the common difference of this A.P. is _____.

4. [JEE Mains 2020]
If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then $m + n$ is equal to ______

5. [JEE Mains 2020]
If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then xy is equal to ________.

6. [JEE Mains 2020]
The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to:
(A) –10 (B) 10
(C) –5 (D) –20

7. [JEE Mains 2020]
The mean and variance of 20 observations are found to be 10 and 4, respectively, On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is:
(A) 4.02 (B) 3.98
(C) 3.99 (D) 4.01

8. [JEE Mains 2020]
Let the observations $x_i (1 \le i \le 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$. If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, \ldots, x_{10} - 3$, then the ordered pair (μ, λ) is equal to:
(A) (6, 3) (B) (3, 6)
(C) (3, 3) (D) (6, 6)

9. **[JEE Mains 2020]**
Let $X = \{x \in N : 1 \le x \le 17\}$ and $Y = \{ax + b; x \in X$ and $a, b \in R, a > 0\}$. If mean and variance of elements of Y are 17 and 216 respectively then $a + b$ is equal to:
(A) -7 (B) 7
(C) 9 (D) -27

10. **[JEE Mains 2020]**
For the frequency distribution:

Variate (x):	x_1	x_2	$x_3 \ldots x_{15}$
Frequency (f):	f_1	f_2	$f_3 \ldots f_{15}$

where $0 < x_1 < x_2 < x_3 < \ldots < x_{15} = 10$ and $\sum_{i=1}^{15} f_i > 0$, the standard deviation cannot be:
(A) 2 (B) 1
(C) 4 (D) 6

11. **[JEE Mains 2020]**
Let $x_i (1 \le i \le 10)$ be ten observations of a random variable X. If $\sum_{i=1}^{10} (x_i - p) = 3$ and $\sum_{i=1}^{10} (x_i - p)^2 = 9$ where $0 \ne p \in R$, then the standard deviation of these observations is:
(A) $\sqrt{\frac{3}{5}}$ (B) $\frac{7}{10}$
(C) $\frac{9}{10}$ (D) $\frac{4}{5}$

12. **[JEE Mains 2020]**
The mean and variance of 8 observations are 10 and 13.5, respectively. If 6 of these observations are 5, 7, 10, 12, 14, 15, then the absolute difference of the remaining two observations is:
(A) 7 (B) 3
(C) 5 (D) 9

13. **[JEE Mains 2020]**
If the mean and the standard deviation of the data 3, 5, 7, a, b are 5 and 2 respectively, then a and b are the roots of the equation:
(A) $2x^2 - 20x + 19 = 0$ (B) $x^2 - 10x + 19 = 0$
(C) $x^2 - 10x + 18 = 0$ (D) $x^2 - 20x + 18 = 0$

14. **[JEE Mains 2020]**
The mean and variance of 7 observations are 8 and 16, respectively. If five observations are 2, 4, 10, 12, 14, then the absolute difference of the remaining two observation is:
(A) 2 (B) 4
(C) 3 (D) 1

15. **[JEE Mains 2020]**
If $\sum_{i-1}^{n} (x_i - a) = n$ and $\sum_{i-1}^{n} (x_i - a)^2 = na$, $(n.a > 1)$ then the standard deviation of n observations $x_1, x_2, \ldots, x_0$ is
(A) $n\sqrt{a-1}$ (B) $\sqrt{a-1}$
(C) $a - 1$ (D) $\sqrt{n(a-1)}$

16. **[JEE Mains 2019]**
5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is:
(A) 20 (B) 18
(C) 16 (D) 22

17. **[JEE Mains 2019]**
A data consists of n observation $x_1, x_2, \ldots, x_n$. If $\sum_{i=1}^{n} (x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n} (x_i - 1)^2 = 5n$, then the standard deviation of this data is:
(A) 2 (B) $\sqrt{5}$
(C) 5 (D) $\sqrt{7}$

18. **[JEE Mains 2019]**
The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is:
(A) 6 : 7 (B) 10 : 3
(C) 4 : 9 (D) 5 : 8

19. **[JEE Mains 2019]**
If mean and standard deviation of 5 observations x_1, x_2, x_3, x_4, x_5 are 10 and 3, respectively, then the variance of 6 observations $x_1, x_2, \ldots, x_5$ and -50 is equal to
(A) 582.5 (B) 507.5
(C) 586.5 (D) 509.5

20. **[JEE Mains 2019]**
The outcome of each of 30 items was observed; 10 items gave an outcome $\frac{1}{2} - d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2} + d$ each. If the variance of this outcome data is $\frac{4}{3}$ then $|d|$ equals:
(A) $\frac{2}{3}$ (B) 2
(C) $\frac{\sqrt{5}}{2}$ (D) $\sqrt{2}$

21. **[JEE Mains 2019]**
A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X be the number of white balls drawn, then $\left(\frac{\text{mean of } X}{\text{standard deviation of } X}\right)$ is equal to:
(A) 4 (B) $3\sqrt{2}$
(C) $\frac{4\sqrt{3}}{3}$ (D) $4\sqrt{3}$

22. **[JEE Mains 2019]**
If the sum of the deviations of 50 observations from 30 is 50, then the mean of these observations is:
(A) 31 (B) 50
(C) 51 (D) 30

23. **[JEE Mains 2019]**
The mean and the variance of five observations are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4; then the absolute value of the difference of the other two observations, is:
(A) 1 (B) 7
(C) 3 (D) 5

24. **[JEE Mains 2019]**
The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is:
(A) 45 (B) 48
(C) 49 (D) 40

25. **[JEE Mains 2019]**
A student scores the following marks in five tests: 45, 54, 41, 57, 43. His score is not known for the sixth test. If the mean score is 48 in the six tests, then the standard deviation of the marks in six tests is:
(A) $\frac{10}{\sqrt{3}}$ (B) $\frac{100}{3}$
(C) $\frac{100}{\sqrt{3}}$ (D) $\frac{100}{3}$

26. **[JEE Mains 2019]**
If the standard deviation of the numbers –1, 0, 1, k is $\sqrt{5}$ where $k > 0$, then k is equal to:
(A) $\sqrt{6}$ (B) $2\sqrt{\frac{10}{3}}$
(C) $4\sqrt{\frac{5}{3}}$ (D) $2\sqrt{6}$

27. **[JEE Mains 2019]**
The mean and the median of the following ten numbers in increasing order 10, 22, 26, 29, 34, x, 42, 67, 70, y are 42 and 35 respectively, then $\frac{y}{x}$ is equal to:
(A) 7/2 (B) 9/4
(C) 8/3 (D) 7/3

28. **[JEE Mains 2019]**
If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is:

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is
(A) 3.0 (B) 2.8
(C) 2.5 (D) 3.2

29. **[JEE Mains 2019]**
If both the mean and the standard deviation of 50 observations $x_1, x_2, \ldots, x_{50}$ are equal to 16, then the mean of $(x_1-4)^2$, $(x_2-4)^2, \ldots, (x_{50}-4)^2$ is:
(A) 400 (B) 480
(C) 380 (D) 525

30. **[JEE Mains 2019]**
If the data $x_1, x_2, \ldots x_{10}$ is such that the mean of first four of these is 11, the mean of the remaining six is 16 and the sum of squares of all of these is 2,000; then the standard deviation of this data is:
(A) $\sqrt{2}$ (B) 2
(C) $2\sqrt{2}$ (D) 4

31. **[JEE Mains 2018]**
If $\sum_{i=1}^{9}(x_i-5)=9$ and $\sum_{i=1}^{9}(x_i-5)^2=45$, then the standard deviation of the 9 items $x_1, x_2, \ldots, x_9$ is:
(A) 4 (B) 2
(C) 3 (D) 9

32. **[JEE Mains 2016]**
If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?
(A) $3a^2-34a+91=0$
(B) $3a^2-23a+44=0$
(C) $3a^2-26a+55=0$
(D) $3a^2-32a+84=0$

33. **[JEE Mains 2015]**
The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is:
(A) 15.8 (B) 14.0
(C) 16.8 (D) 16.0

34. **[JEE Mains 2014]**
The variance of first 50 even natural numbers is
(A) 437 (B) $\frac{437}{4}$
(C) $\frac{833}{4}$ (D) 833

35. **[JEE Mains 2013]**
All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given?
(A) mean
(B) median
(C) mode
(D) variance

36. **[JEE Mains 2012]**
Let $x_1, x_2, \ldots, x_n$ be n observations, and let $\bar{x}$ be their arithmetic mean and σ^2 be the variance.
Statement-1: Variance of $2x_1, 2x_2, \ldots, 2x_n$ is $4\sigma^2$.
Statement-2: Arithmetic mean $2x_1, 2x_2, \ldots, 2x_n$ is $4\bar{x}$.
(A) Statement-1 is false, Statement-2 is true
(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(D) Statement-1 is true, Statement-2 is false

37. **[JEE Mains 2011]**
If the mean deviation about the median of the numbers $a, 2a, \ldots, 50a$ is 50, then $|a|$ equals

(A) 3
(B) 4
(C) 5
(D) 2

38. **[JEE Mains 2010]**
For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is

(A) $\frac{11}{2}$
(B) 6
(C) $\frac{13}{2}$
(D) $\frac{5}{2}$

39. **[JEE Mains 2009]**
Statement-1: The variance of first n even natural numbers is $\frac{n^2-1}{4}$.

Statement-2: The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$.

(A) Statement-1 is true, Statement-2 is true Statement-2 is not a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is false
(C) Statement-1 is false, Statement-2 is true
(D) Statement-1 is true, Statement-2 is true Statement-2 is a correct explanation for Statement-1

40. **[JEE Mains 2009]**
If the mean deviation of the numbers $1, 1 + d, 1 + 2d, \ldots 1 + 100d$ from their mean is 255, then d is equal to:

(A) 20.0
(B) 10.1
(C) 20.2
(D) 10.0

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** B	**3.** A	**4.** D	**5.** B	**6.** C	**7.** C	**8.** C	**9.** C	**10.** C
11. D	**12.** B	**13.** B	**14.** A	**15.** D	**16.** C	**17.** B	**18.** C	**19.** A	**20.** A
21. C	**22.** A	**23.** D	**24.** A	**25.** C	**26.** C	**27.** A	**28.** D	**29.** D	**30.** C
31. D	**32.** B	**33.** D	**34.** B	**35.** D	**36.** A	**37.** A	**38.** A	**39.** B	**40.** C
41. A	**42.** C	**43.** A	**44.** D	**45.** A	**46.** B	**47.** D			

Integer/Numerical Answer Type Questions

1. 34	**2.** 81	**3.** 25	**4.** 6	**5.** 225	**6.** 36	**7.** 218	**8.** 252500	**9.** 44	**10.** 12

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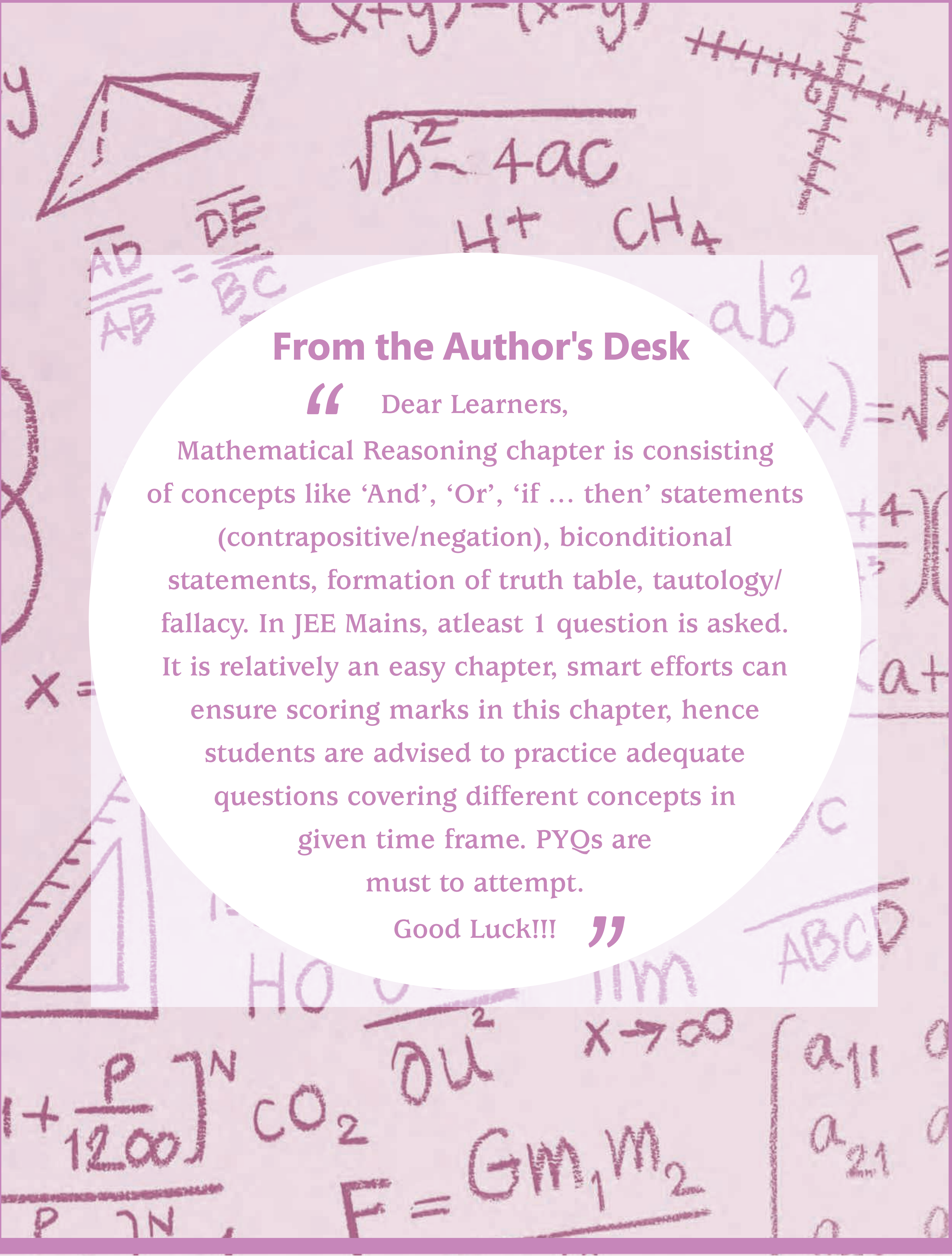
1. 6.00	**2.** 4	**3.** 3.00	**4.** 18	**5.** 54.00	**6.** D	**7.** C	**8.** C	**9.** A	**10.** D
11. C	**12.** A	**13.** B	**14.** A	**15.** B	**16.** A	**17.** B	**18.** C	**19.** B	**20.** D
21. B	**22.** A	**23.** B	**24.** B	**25.** A	**26.** D	**27.** D	**28.** B	**29.** A	**30.** B
31. B	**32.** D	**33.** B	**34.** D	**35.** D	**36.** D	**37.** B	**38.** A	**39.** C	**40.** B



From the Author's Desk

“ Dear Learners,

Mathematical Reasoning chapter is consisting of concepts like ‘And’, ‘Or’, ‘if … then’ statements (contrapositive/negation), biconditional statements, formation of truth table, tautology/ fallacy. In JEE Mains, atleast 1 question is asked. It is relatively an easy chapter, smart efforts can ensure scoring marks in this chapter, hence students are advised to practice adequate questions covering different concepts in given time frame. PYQs are must to attempt.

Good Luck!!! ”

CHAPTER

10

Mathematical Reasoning

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Statement & their negation
(b) Formation of truth table
(c) 'AND', 'OR' statements
(d) Conditional and biconditional statements
(e) Tautology and fallacy

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

STATEMENT

A sentence is called a mathematically acceptable statement if it is either true or false but not both.

Imperative, interrogative and exclamatory sentences are not considered as statement

A sentence which is a request, or a command is also not a statement.

Scientifically and Mathematically proven facts are considered as statement

Sentences involving variable time as such 'Now', Today, 'Yesterday', 'Tomorrow' are not considered as statements

ILLUSTRATION 1

Which of the following are statements?

(a) 4 is less than 6
(b) 2 is an odd number
(c) Every square is a rectangle
(d) KOTA is in India
(e) Please open the door
(f) Today is Monday
(g) Aishwarya is beautiful
(h) Ganga is a river

SOLUTION

(a) 4 is less than 6
It is always True, hence it is a mathematically acceptable statement

(b) 2 is an odd number
It is always False, hence it is a mathematically acceptable statement

(c) Every square is a rectangle
It is always true, hence it is a mathematically acceptable statement

(d) KOTA is in India
It is always true, hence it is a mathematically acceptable statement

(e) Please open the door
It is a request/order, hence it is not a mathematically acceptable statement

(f) Today is Monday
It is not always true, hence it is not a mathematically acceptable statement

(g) Aishwarya is beautiful
It is a sentence explaining beauty which can not be classified as true or false always, hence it is not a mathematically acceptable statement

(h) Ganga is a river
It is always true, hence it is a mathematically acceptable statement

TRUTH TABLE

A table which shows the relationship between the truth value of statement $S(p, q, r \ldots)$ and the truth values of its sub statements $p, q, r, \ldots$ is said to be truth table.

NEGATION (DENIAL) OF A STATEMENT (~*p*)

The denial of a statement is called the negation of the statement denoted as ~.

If p is true then $\sim p$ must be false and if p is false then $\sim p$ must be true.

Truth Table (~*p*)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

It may be noticed that $\boldsymbol{\sim(\sim p) = p}$. Also p and $\sim p$ are contrary.

1. $\sim p$ is true iff p is false.
2. $\sim p$ is false iff p is true.

While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.

ILLUSTRATION 2

Write negation of '5 is a rational number'.

SOLUTION

p : 5 is a rational number
$\sim p$: 5 is not a rational number

ILLUSTRATION 3

Write negation of 'The number 2 is greater than 7'.

SOLUTION

p : The number 2 is greater than 7.
The negation of this statement is -
$\sim p$: It is not the case that the number 2 is greater than 7.
or
$\sim p$: It is false that the number 2 is greater than 7.
or
$\sim p$: The number 2 is not greater than 7.

ILLUSTRATION 4

Write negation of statement '$7 > 9$'.

SOLUTION

$p : 7 > 9$ The negation of this statement is

$\sim p : 7 \ngtr 9$
or
$\sim p : 7 \leq 9$

ILLUSTRATION 5

Write negation of statement 'Everyone in Germany speaks German'.

SOLUTION

p : Everyone in Germany speaks German.
$\sim p$: It is false that everyone in Germany speaks German.
$\sim p$: There exist some person in Germany who do not speak German

ILLUSTRATION 6

Write negation of statement 'All integers are rational numbers'.

SOLUTION

p : All integers are rational numbers
$\sim p$: There exists an integer which is not a rational number.

SIMPLE STATEMENT

Any statement whose truth value does not depend on other statement are called simple statement

EXAMPLE:

(i) "$\sqrt{2}$ is an irrational number"
(ii) "The set of real number is an infinite set"

COMPOUND STATEMENT

A compound statement is a statement which is made up of two or more statements. In this case, each statement is called a component statement.

EXAMPLE:

All rational numbers are real and all real numbers are complex.
The component statement are
p : all rational numbers are real
q : all real numbers are complex numbers

CONNECTIVES

Compound statement are combined by using connectives such as the word "or ", "not", "and ", "if …then…", "…if and only if …"

'OR' STATEMENT

Compound statement when two statements are combined by using connectives "or "
The resulting statement is denoted as $p \vee q$.

(i) The compound statement with 'or' is true if any one of the component statements are true.
(ii) The compound statement with 'or' is false if both of its component statements are false

The following truth table shows the truth values of $p \vee q$ (p or q)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

INCLUSIVE OR

$p \vee q$ is inclusive OR if both statement p and q both can be True together

EXAMPLE:

A student who has taken biology or chemistry can apply for M.Sc. microbiology program. This means that student who have taken both biology and chemistry or only biology or only chemistry can apply for the microbiology program. This is example of **"inclusive Or"**.

EXCLUSIVE OR

$p \vee q$ is exclusive OR if both statement p and q both can not be True together

EXAMPLE:

A student who can opt for NIT or IIT college after clearing IIT JEE exam.
A student can do graduation only from one category of college NIT or IIT but not from both. This is example of "exclusive Or"

'AND' STATEMENT

Compound statement when two statements are combined by using connectives "and "
The resulting statement is denoted as $p \wedge q$.

- **(i)** The compound statement with 'And' is true if all its component statements are true.
- **(ii)** The compound statement with 'And' is false if any of its component statements is false

The following truth table shows the truth values of $p \wedge q$ (p and q)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE:

A point occupies a position and its location can be determined.
The component statement are

p : A point occupies a position
q : Its location can be determined

NEGATION OF 'OR' STATEMENT

$$\sim(p \vee q) = \sim p \wedge \sim q$$

Proof:

$$\sim(p \vee q) = \sim p \wedge \sim q$$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

NEGATION OF AND

$$\sim(p \wedge q) = \sim p \vee \sim q$$

Proof:

$$\sim(p \wedge q) = \sim p \vee \sim q$$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

ILLUSTRATION 7

Write the negation of "Jyoti is in class X or Ashu is in class XII".

SOLUTION

Using $\sim(p \vee q) = \sim p \wedge \sim q$

p : Jyoti is in class X
q : Ashu is in class XII

required negation statement is ' Jyoti is not in class X and Ashu is not in class XII'.

ILLUSTRATION 8

Write negation of "$2 + 3 = 5$ and $8 < 10$".

SOLUTION

Using $\sim(p \wedge q) = \sim p \vee \sim q$

$p : 2 + 3 = 5$
$q : 8 < 10$

Negation of '$2 + 3 = 5$ and $8 < 10$' is '$2 + 3 \neq 5$ or $8 \nless 10$'

ILLUSTRATION 9

Write the negation of $q \vee \sim (p \wedge r)$.

SOLUTION

$$\sim(q \vee \sim (p \wedge r)) = \sim q \wedge (\sim(\sim(p \wedge r))) = \sim q \wedge (p \wedge r)$$

ILLUSTRATION 10

If p : It is snowing q : I am cold, then represent the compound statement "It is snowing and it is not that I am cold".

SOLUTION

'It is snowing and it is not that I am cold' can be represented as $p \wedge \sim q$.

Test Your Concepts-I

(Solutions on page H.275)

1. Mark following as statement YES or NO.
  (i) Three plus two equals five.
  (ii) The sum of x and y is greater than 0
  (iii) Every relation is a function.
  (iv) The sum of two negative number is negative.
  (v) Three plus four equals six.
  (vi) All prime numbers are odd.
  (vii) The square of a number is even.
  (viii) Every square is a rectangle.

2. If $(p \wedge \sim q) \vee (q \wedge r)$ is true and q and r both true then p is
(A) True
(B) False
(C) may be true or false
(D) none

3. The negation of the statement "101 is not a multiple of 3" is
(A) 101 is a multiple of 3
(B) 101 is a multiple of 2
(C) 101 is an odd number
(D) 101 is an even number

4. Which of the following are mathematically acceptable statements?
  (i) All prime numbers are odd numbers
  (ii) Every set is a finite set
  (iii) $\sqrt{2}$ is a rational number or an irrational number
(A) Only (i), (ii)
(B) Only (ii), (iii)
(C) Only (i), (iii)
(D) All (i), (ii), (iii)

5. Which of the following is not a negation of the statement.
"There exists a rational number x such that $x^2 = 2$".
  (i) There does not exists a rational number x such that $x^2 = 2^n$
  (ii) For all rational numbers x, $x^2 \neq 2^n$
  (iii) For no rational number x, $x^2 \neq 2^n$
(A) Only (i)
(B) Only (ii)
(C) Only (iii)
(D) (ii), (iii)

6. The statement $(p \wedge \sim q) \vee p$ is logically equivalent to
(A) p
(B) $\sim p$
(C) q
(D) $\sim q$

Answer keys

1. (i) yes (ii) no (iii) yes (iv) yes (v) yes (vi) yes (vii) no (viii) yes **2.** (C) **3.** (A) **4.** (D) **5.** (C) **6.** (A)

IMPLICATION

There are two types of implications

(i) "If…, then…" is known as Conditional statement
(ii) "If and only if" is known as biconditional statement

CONDITIONAL STATEMENT

The statement 'if p then q' is denoted by $p \rightarrow q$ (to be read as 'p implies q') or by $p \Rightarrow q$.
Note that $p \rightarrow q$ also means

(i) p is sufficient for q
(ii) q is necessary for p
(iii) q only if p
(iv) p lead to q
(v) q if p
(vi) q when p
(vii) if p, then q

EXAMPLE:

p : a number is a multiple of 9
q : a number is a multiple of 3.
then $p \rightarrow q$ or $p \Rightarrow q$ can be written as 'if a number is a multiple of 9 then a number is a multiple of 3'

EXAMPLES:

1. If you work hard then you can succeed.
2. You can go on leave only if your boss permits.

$p \to q$ is false only when p is true and q is false.

Truth table for $p \to q$ is as follows.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

ILLUSTRATION 11

Discuss truth value of 'If $2+2=5$ then Kota is in England'.

SOLUTION

Let $p : 2+2=5$ has truth value False

q : Kota is in England has truth value False

$p \to q$ means False implies false which means True

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements

$(p \to q)$ and $(\sim p \vee q)$ are equivalent

i.e. $\boxed{p \to q \equiv \sim p \vee q}$

NEGATION OF CONDITIONAL IMPLICATION

If p and q are two statements, then $\sim(p \to q) = p \wedge \sim q$

Proof:

p	q	$\sim q$	$p \to q$	$p \wedge \sim q$
T	T	F	T	F
T	F	T	F	T
F	T	F	T	F
F	F	T	T	F

CONTRAPOSITIVE STATEMENT

Contrapositive of the conditional statement $p \to q$ is $\sim q \to \sim p$.

EXAMPLE:

If a number is multiple of 6 then it is multiple of 2.

Contrapositive If a number is not multiple of 2 then it is not multiple of 6.

INVERSE STATEMENT

The inverse of the conditional statement $p \to q$ is defined as $\sim p \to \sim q$.

CONVERSE STATEMENT

Converse of the conditional statement $p \to q$ is $q \to p$.

EXAMPLE:

If the angles of a triangle are equal then it is equilateral triangle.

Converse is if triangle is equilateral then angles of triangle are equal.

Truth Table ($p \to q$)

p	q	$p \to q$	**Contrapositive ($\sim q \to \sim p$)**	**Converse ($q \to p$)**
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Note:

Truth table for $p \to q$ is same as its contrapositive.

ILLUSTRATION 12

Write the negation of each of the following statements:

(i) if Ayush is tall then he is dancer

(ii) If it is raining, then the game is cancelled.

SOLUTION

(i) Let

p : Ayush is tall.

q : Ayush is dancer.

$\sim q$: Ayush is not slim

Now $\sim(p \to q) = p \wedge \sim q$

$\sim(p \to q)$ = Ayush is tall and he is not dancer

(ii) Let p : It is raining

q : The game is cancelled. Then,

$\sim q$: The game is not cancelled.

$\sim(p \to q) = p \wedge \sim q$

$\sim(p \to q)$ = It is raining and the game is not cancelled.

ILLUSTRATION 13

Write the contrapositive of the statement "If 7 is greater than 5, then 8 is greater than 6".

SOLUTION

Let p : 7 is greater than 5.

q : 8 is greater than 6.

given statement is $p \to q$

$\sim p$: 7 is not greater than 5.

$\sim q$: 8 is not greater than 6.

Contrapositive $\sim q \to \sim p$ i.e., If 8 is not greater than 6, then 7 is not greater than 5.

ILLUSTRATION 14

Write the converse of the statement "If sun is not shining, then sky is filled with clouds".

SOLUTION

Let p : Sun is not sinning.

q : Sky is filled with clods.

Converse of the above statement $p \to q$ is $q \to p$.

If sky is filled with clouds, then the Sun is not shining.

ILLUSTRATION 15

Write the contrapositive of the following. statements:

(i) If x is non-negative, then x is zero.
(ii) If the weather is cold, then it will snow.

SOLUTION

(i) Let p : x is non negative
q : x is zero
Now $\sim p$: x is negative
$\sim q$: x is not zero.
Contrapositive $\sim q \to \sim p$: If x is not zero, then x is negative.

(ii) Let p : The weather is cold.
q : It will snow
Now $\sim p$: The weather is not cold.
$\sim q$: It will not snow
$\therefore$ $\sim q \to \sim p$: If it will not snow, then the weather is not cold.

BICONDITIONAL STATEMENT

"If and only if" type of compound statement is called **Biconditional or equivalence** or **'double implication'**. Symbolically 'p iff q' is represented by $p \leftrightarrow q$ or by $p \Leftrightarrow q$.

(i) p is a necessary and sufficient condition for q.
(ii) q is necessary and sufficient condition for p.
(iii) If p then q and if q then p
(iv) q if and only if p.

EXAMPLE:

$p \to q$: If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

$q \to p$: If a number is divisible by 3, then the sum of its digits is divisible by 3.

$p \leftrightarrow q$: A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

The truth table for biconditional is as follows:

Truth Table ($p \leftrightarrow q$, $q \leftrightarrow p$)

p	q	$p \leftrightarrow q$	$q \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

RULE:

$p \leftrightarrow q$ is true only when both p and q have the same truth value.

NEGATION OF BICONDITIONAL STATEMENT OR EQUIVALENCE THEOREM

If p and q are two statement, then

$$\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (q \wedge \sim p)$$

Proof:

$$p \leftrightarrow q = (p \to q) \wedge (q \to p)$$

$$\therefore \quad \sim(p \leftrightarrow q) = \sim\{(p \to q) \wedge (q \to p)\}$$

$$= [\sim(p \to q)] \vee [\sim(q \to p)] = (p \wedge \sim q) \vee (q \wedge \sim p)$$

ILLUSTRATION 16

Write the negation of each of the following statements:

(i) ABC is an equilateral triangle if and only if it is equiangular.
(ii) Two lines are parallel if and only if they have the same slope.

SOLUTION

(i) Let p : ABC is an equilateral triangle
q : It is equiangular
$\therefore$ $\sim(p \leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$
$\sim(p \leftrightarrow q)$: Either ABC is an equilateral triangle and it is not equiangular or ABC is not an equilateral triangle and it is equiangular.

(ii) Let p : Two lines are parallel
q : They have the same slope
$\therefore$ $\sim(p \leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$

$\sim(p \leftrightarrow q)$ = Either two lines are parallel and they not have the same slopes or two lines are not parallel and they have the same slope.

TAUTOLOGY (*t*)

The compound statements (or propositions) which are true for any truth value of their components are called **'Tautology'**.

EXAMPLE:

'$p \vee \sim p$' is a tautology, p being logical statement. This is illustrated by the truth table given below with shows only Ts in the last column.

Truth Table $(p \vee \sim p)$		
p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$\therefore \quad \sim p \vee p = t$

FALLACY (CONTRADICTION)

The negation of tautology is called a **fallacy** or a **contradiction** i.e. a proposition which is false for any truth value of their components is called a fallacy. For example '$p \wedge \sim p$' is a fallacy, p being any logical statement. This is illustrated by the truth table given below which shows only Fs in the last column.

Truth Table $(p \wedge \sim p)$		
p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$\therefore \quad p \wedge \sim p = c$

Note:

(i) A tautology is always true.

(ii) A fallacy is always false.

(iii) The negation of a tautology is a contradiction and negation of a contradiction is a tautology

$\therefore \quad \sim t = c, \sim c = t$

ALGEBRA OF STATEMENTS

Statements satisfy many laws some of which are given below

1. **Idempotent Laws:** If p is any statement then
   (i) $p \vee p = p$
   (ii) $p \wedge p = p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

2. **Associative Laws:** If p, q, r are any three statements, then
   (i) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
   (ii) $p \vee (q \vee r) = (p \vee q) \vee r$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can prove result (ii)

3. **Commutative Laws:** If p, q are any two statement, then
   (i) $p \vee q = q \vee p$
   (ii) $p \wedge q = q \wedge p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

4. **Distributive Laws:** If p, q, r are any three statements, then
   (i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
   (ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can prove result (ii)

5. **Identity Laws:** If p is any statement, t is tautology and c is a contradiction, then
   (i) $p \wedge t = p$
   (ii) $p \vee t = t$
   (iii) $p \wedge c = c$
   (iv) $p \vee c = p$

p	**t**	**c**	**$(p \wedge t)$**	**$(p \vee t)$**	**$(p \wedge c)$**	**$(p \vee c)$**
T	T	F	T	T	F	T
F	T	F	F	T	F	F

6. **Complement Laws:** If t is a tautology, c is a contradiction and p is any statement, then
   (i) $p \vee (\sim p) = t$
   (ii) $p \wedge (\sim p) = c$
   (iii) $\sim t = c$
   (iv) $\sim c = t$
7. **Involution Law:** If p is any statement, then $\sim(\sim p) = p$.

p	**$\sim p$**	**$\sim(\sim p)$**
T	F	T
F	T	F

8. **De-morgan's Law:** If p and q are two statements, then
   (i) $\sim(p \wedge q) = \sim p \vee \sim q$
   (ii) $\sim(p \vee q) = \sim p \wedge \sim q$

p	**q**	**$\sim p$**	**$\sim q$**	**$(p \wedge q)$**	**$\sim(p \wedge q)$**	**$(\sim p \vee \sim q)$**
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can proved result (ii).

ILLUSTRATION 17

Simplify $\sim(p \vee q) \vee (\sim p \wedge q)$.

SOLUTION

$\because \quad \sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (By Demorgan Law)

$\equiv \sim p \wedge (\sim q \vee q)$ (By distributive laws)

$\equiv \sim p \wedge t$ (By complement laws)

$\equiv \sim p$ (By Identity Laws)

ILLUSTRATION 18

Prove by construction of truth table that $p \vee \sim(p \wedge q)$ is a tautology.

SOLUTION

Truth table

p	**q**	**$p \wedge q$**	**$\sim(p \wedge q)$**	**$p \vee \sim(p \wedge q)$**
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since the last column shows T's only,
therefore $p \vee \sim (p \wedge q)$ is a tautology.

ILLUSTRATION 19

Prove by constructing truth table that $(p \wedge q) \wedge \sim (p \vee q)$ is a fallacy (contradiction).

SOLUTION

Truth table

p	**q**	**$p \wedge q$**	**$p \vee q$**	**$\sim(p \vee q)$**	**$(p \wedge q) \wedge \sim(p \vee q)$**
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the last column shows F's only,
therefore $(p \wedge q) \wedge \sim(p \vee q)$ is a fallacy.

ILLUSTRATION 20

Show that, the statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

SOLUTION

p	**q**	**$\sim p$**	**$\sim q$**	**$p \vee q$**	**$(\sim p \wedge \sim q)$**	**$(p \vee q) \wedge (\sim p \wedge \sim q)$**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Clearly, then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Alternate

$$(p \vee q) \wedge (\sim p \wedge \sim q) = (p \vee q) \wedge (\sim(p \vee q)) = c$$

ILLUSTRATION 21

Show that $[(p \vee q) \wedge \sim p] \rightarrow q$ is tautology.

SOLUTION

Truth table

p	q	$p \vee q$	$\sim p$	$(p \vee q) \wedge \sim p$	$(p \vee q) \wedge \sim p \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

ILLUSTRATION 22

Show that $[(p \rightarrow q) \wedge \sim q] \rightarrow (\sim p)$ is tautology.

SOLUTION

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$(p \rightarrow q) \wedge \sim q$	$[(p \rightarrow q) \wedge \sim q] \rightarrow (\sim p)$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

ILLUSTRATION 23

Show that the statement $p \rightarrow (q \rightarrow p)$ is equivalent to $p \rightarrow p \vee q$.

SOLUTION

q	p	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	T	F	T

$\therefore$ Statement $p \rightarrow (q \rightarrow p)$ is equivalent to $p \rightarrow (p \vee q)$.

ILLUSTRATION 24

Write the negation of $\sim s \vee (\sim r \wedge s)$.

SOLUTION

$$\sim s \vee (\sim r \wedge s) = (\sim s \vee \sim r) \wedge (\sim s \vee s)$$

$$= \sim(s \wedge r) \wedge t = \sim(s \wedge r)$$

So negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to $s \wedge r$.

Test Your Concepts-II

(Solutions on page H.275)

1. The contrapositive of $p \Rightarrow (\sim p \wedge q)$ is
(A) $\sim p \Rightarrow (p \vee \sim q)$
(B) $(p \wedge \sim q) \Rightarrow \sim p$
(C) $(p \vee \sim q) \Rightarrow p$
(D) $(p \vee \sim q) \Rightarrow \sim p$

2. The statement $p \Leftrightarrow q$ is equal to
(A) $(\sim p \vee q) \vee (p \vee q)$
(B) $(p \wedge q) \vee (\sim p \wedge \sim q)$
(C) $(\sim p \vee q) \wedge (p \vee \sim q)$
(D) $(p \wedge q) \vee (p \vee q)$

3. The statement $(p \wedge q) \Leftrightarrow \sim p$ is a
(A) Tautology
(B) contradiction
(C) Neither tautology nor contradiction
(D) None of these

4. The statement $p \rightarrow p \vee q$ is a
(A) Tautology
(B) Contradiction
(C) Neither tautology nor contradiction
(D) None of these

5. The statement $(p \rightarrow \sim q) \Leftrightarrow (p \wedge q)$ is a
(A) Tautology
(B) contradiction
(C) Neither tautology nor contradiction
(D) None

6. Which of the following is correct for the statements p and q?
(A) $p \wedge q$ is true when at least one from p and q is true
(B) $p \rightarrow q$ is true when p is true and q is false
(C) $p \leftrightarrow q$ is true only when both p and q are true
(D) $\sim(p \vee q)$ is true only when both p and q are false

7. Negation of the statement $p \rightarrow (q \wedge r)$ is
(A) $\sim p \rightarrow \sim(q \wedge r)$ (B) $\sim p \vee (q \wedge r)$
(C) $(q \wedge r) \rightarrow p$ (D) $p \wedge (\sim q \vee \sim r)$

8. The negation of the statement "If a quadrilateral is a square then it is a rhombus" is
(A) If a quadrilateral is not a square then is a rhombus it
(B) If a quadrilateral is a square then it is not a rhombus
(C) a quadrilateral is a square and it is not a rhombus
(D) a quadrilateral is not a square and it is a rhombus

9. If p, q, r are three propositions then the negation of $p \rightarrow (q \wedge r)$ is logically equivalent to
(A) $(p \vee \sim q) \wedge (p \vee \sim r)$
(B) $(p \wedge \sim q) \vee (p \wedge \sim r)$
(C) $(\sim p \vee q) \wedge (\sim p \vee r)$
(D) $(\sim p \wedge q) \vee (\sim p \wedge r)$

10. If the inverse of the conditional $p \to (\sim q \wedge \sim r)$ is false, then the truth values of the proportions p, q, r are respectively

(A) T, T, T (B) T, F, F

(C) F, T, T (D) F, F, F

11. $S - I : \sim(\sim p \leftrightarrow \sim r) \equiv p \leftrightarrow q$

$S - II : \sim p \leftrightarrow \sim q \equiv (pv \sim q) \wedge (qv \sim p)$

Which of the following is true about above two statements S – I and S – II.

(A) Both S – I, S – II are true and S – II is a correct explanation of S – I.

(B) Both S – I, S – II are true, but S – II is not a correct explanation of S – I.

(C) S – I is true and S – II is false

(D) S – I is false and S – II is true

12. $((p \to q) \wedge (\sim p \to q)) \to q$ is logically equivalent to

(A) a tautology

(B) a contradiction

(C) $(\sim pvp) \to q$

(D) $(p \wedge \sim p) \to q$

13. Which of the following is a contradiction?

(A) $p \to (q \to p)$

(B) $p \to (pvq)$

(C) $(pvq) \to (\sim p \wedge \sim q)$

(D) $(pv \sim p) \to (q \wedge \sim q)$

Answer keys

1. (D) **2.** (C) **3.** (C) **4.** (A) **5.** (B) **6.** (D) **7.** (D) **8.** (C)

9. (B) **10.** (D) **11.** (D) **12.** (A) **13.** (D)

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Which of the following is not a statement?
   (A) Every set is a finite set
   (B) 8 is less than 6
   (C) Where are you going?
   (D) The sum of interior angles of a triangle is 180 degrees

2. Which of the following is a statement?
   (A) May you live long!
   (B) May God bless you!
   (C) The sun is a star
   (D) Hurrah! We have won the match

3. If p, q, r are simple propositions, then $(p \wedge p) \wedge (q \wedge r)$ is true, then
   (A) p, q, r are all false
   (B) p, q, r are all true
   (C) p, q are true and r is false
   (D) p is true and q and r are false

4. $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
   (A) $\sim p$ (B) p
   (C) q (D) $\sim q$

5. If p : The earth is round, $q : 3 + 4 = 7$, then $(\sim p) \vee (\sim q)$ is
   (A) It is not that the earth is round or $3 + 4 = 7$
   (B) The earth is round and $3 + 4 = 7$
   (C) It is not that the earth is round or it is not that $3 + 4 = 7$
   (D) The earth is round or $3 + 4 = 7$

6. The negation of statement $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is
   (A) $(p \vee \sim q) \wedge (p \vee q)$ (B) $(p \wedge \sim q) \vee (p \vee q)$
   (C) $(\sim p \vee q) \vee (\sim p \wedge \sim q)$ (D) $(p \wedge \sim q) \wedge (p \vee q)$

7. Which of the following is not a statement
   (A) Smoking is injurious to health
   (B) $2 + 2 = 4$
   (C) 2 is the only even prime number
   (D) come here

8. The negative of the statement “A circle is an ellipse” is
   (A) An ellipse is a circle
   (B) An ellipse is not a circle
   (C) A circle is not an ellipse
   (D) A circle is an ellipse

9. The negation of the statement “72 is divisible by 2 and 3” is
   (A) 72 is not divisible by 2 or 72 is not divisible by 3
   (B) 72 is not divisible by 2 and 72 is not divisible by 3
   (C) 72 is divisible by 2 and 72 is not divisible by 3
   (D) 72 is not divisible by 2 and 72 is divisible by 3

10. The statement “If x^2 is not even, then x is not even” is converse of the statement
    (A) If x^2 is odd, then x is even
    (B) If x is not even, then x^2 is not even
    (C) is x is even, then x^2 is even
    (D) Is x is odd, then x^2 is even

11. Which of the following is the conditional $p \rightarrow q$?
    (A) q is sufficient for p (B) p is necessary for q
    (C) p only if q (D) if q then p

12. Let p : roses are red and q : the sun is a star. Then the verbal translation is $(\sim p) \vee q$ is
    (A) Roses are not red and the sun is not a star
    (B) It is not true that roses are red or the sun is not a star
    (C) It is not true that roses are red and the sun is not a star
    (D) Roses are not red or the sun is a star

13. Negation of ʹʹ 3 is an odd number and 7 is a rational number is -
    (A) 3 is not an odd number and 7 is not a rational number
    (B) 3 is an odd number or 7 is a rational number
    (C) 3 is an odd number or 7 is not a rational number
    (D) 3 is not an odd number or 7 is not a rational number.

14. The negation of statement $(p \wedge q) \vee (q \vee \sim r)$ is
    (A) $(p \wedge q) \vee (\sim q \vee \sim r)$
    (B) $(\sim p \wedge \sim q) \wedge (\sim q \wedge r)$
    (C) $(\sim p \vee \sim q) \wedge (\sim q \wedge r)$
    (D) None of these

15. Truth value of the statement “if p then q” is false when
    (A) p is true, q is true
    (B) p is true, q is false
    (C) p is false, q is true
    (D) p is false, q is false

16. Let p and q be two statement. Then, $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is a
    (A) tautology
    (B) contradiction
    (C) neither tautology nor contradiction
    (D) either tautology or contradiction

17. Let $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. Then, this law is known as
    (A) Commutative law (B) Associate law
    (C) De-Morgan’s law (D) Distribute law

18. The false statement in the following is
    (A) $p \wedge (\sim p)$ is a contradiction
    (B) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
    (C) $\sim(\sim p) \Leftrightarrow p$ is a tautology
    (D) $p \vee (\sim p)$ is a tautology

19. If p : Ram is smart.
    q : Ram is intelligent.
    Then, the symbolic form Ram is smart and intelligent, is
    (A) $(p \wedge q)$ (B) $(p \vee q)$
    (C) $(p \wedge \sim q)$ (D) $(p \vee \sim q)$

20. If p = He is intelligent
q = He is strong.
Then, symbolic form of statement. "It is wrong that he is intelligent or strong," is
(A) $\sim p \vee \sim p$
(B) $\sim(p \wedge \sim q)$
(C) $\sim p \wedge \sim q$
(D) $p \vee \sim q$

21. If p and q are two statement, then statement $p \Rightarrow q \wedge \sim q$ is
(A) tautology
(B) contradiction
(C) neither tautology nor contradiction
(D) None of the above

22. If p and q are two statement, then $\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$ is
(A) tautology
(B) contradiction
(C) neither tautology nor contradiction
(D) either tautology or contradiction

23. $\sim[\sim p \wedge (p \Leftrightarrow q)] \equiv$
(A) $p \vee q$
(B) $q \wedge p$
(C) T
(D) F

24. $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
(A) $\sim p$
(B) p
(C) q
(D) $\sim q$

25. The only statement among the following that is a tautology is
(A) $A \wedge (A \vee B)$
(B) $A \vee (A \wedge B)$
(C) $[A \wedge (A \rightarrow B)] \rightarrow B$
(D) $B \rightarrow [A \wedge (A \rightarrow B)]$

26. Negation of the statement 'If a number is prime then it is odd' is.
(A) A number is not prime but odd.
(B) A number is prime and it is not odd.
(C) A number is neither primes nor odd.
(D) None of these

27. If p, q, r are substatements with truth values. T, T, F then the Statement $r \rightarrow (p \wedge \sim q) \vee (\sim q \wedge \sim r)$ will be
(A) True
(B) False
(C) may be true or false
(D) None of these

28. The contra positive of $(p \vee q) \Rightarrow r$ is
(A) $r \Rightarrow (p \vee q)$
(B) $\sim r \Rightarrow (p \vee q)$
(C) $\sim r \Rightarrow \sim p \wedge \sim q$
(D) $p \Rightarrow (q \vee r)$

29. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively
(A) T, F, F
(B) F, F, F
(C) F, T, T
(D) T, T, F

30. The Boolean expression $[(p \wedge \sim q) \vee q] \vee (\sim p \wedge q)$ is equivalent to
(A) $p \wedge q$
(B) $p \vee q$
(C) $p \vee \sim q$
(D) $\sim p \wedge q$

31. The negation of the statement "Plants take in CO_2 and given out O_2" is
(A) Plants do not take in CO_2 and do not give out O_2
(B) Plants do not take in CO_2 or do not give out O_2
(C) Plants take is CO_2 and do not give out O_2
(D) Plants take in CO_2 or do not give out O_2

32. The contra-positive of statement "If Jaipur is capital of Rajasthan, then Jaipur is in India" is
(A) If Jaipur is not in India, then Jaipur is not the capital of Rajasthan.
(B) If Jaipur is in India, then Jaipur is Capital of Rajasthan.
(C) If Jaipur is not capital of Rajasthan, then Jaipur is not capital of India.
(D) If Jaipur is capital of Rajasthan, then Jaipur is not is India.

33. The negation of the statement "The product of 3 and 4 is 9" is
(A) If is false that the product of 3 and 4 is 9
(B) The product of 3 and 4 is 12
(C) The product of 3 and 4 is not 12
(D) It is false that the product of 3 and 4 is not 9

34. Which of the following is not a negation of "A natural number is greater than zero"?
(A) A natural number if not greater than zero
(B) It is false that a natural number is greater than zero
(C) It is a false that a natural number is not greater than zero
(D) None of these

35. Which of the following is the inverse of the proposition. "If a number is prime then it is odd".
(A) If a number is not a prime then it is odd
(B) If a number is not a prime then it is not odd
(C) If a number is not odd then it is not a prime
(D) If a number is not odd then it s a prime

36. The Negation of the statement $(p \wedge q) \rightarrow r$ is
(A) $(\sim p \vee \sim q) \rightarrow r$
(B) $(\sim p \wedge \sim q) \wedge \sim r$
(C) $(p \wedge q) \wedge \sim r$
(D) $(\sim p \vee \sim q) \wedge r$

37. The statement $p \rightarrow (\sim q)$ is equivalent to
(A) $q \rightarrow p$
(B) $\sim q \vee \sim p$
(C) $p \wedge \sim q$
(D) $\sim q \rightarrow p$

38. Which of the following statement is a tautology?
(A) $(\sim q \wedge p) \wedge q$
(B) $(\sim q \wedge p) \wedge (p \wedge \sim q)$
(C) $(\sim q \wedge p) \vee (p \vee \sim p)$
(D) $(p \wedge q) \wedge (\sim(p \wedge q))$

39. The negation of $(p \vee \sim q) \wedge q$ is
(A) $(\sim p \vee q) \wedge \sim q$
(B) $(p \wedge q) \wedge q$
(C) $(\sim p \wedge q) \vee \sim q$
(D) $(p \wedge \sim q) \vee \sim q$

40. If it is raining then I will not come, Given its contrapositive
(A) If I will came, then it is not raining
(B) If I will not came, then it is raining
(C) If will not came, then it is not raining
(D) If I will came, then it is raining

41. Ram secures 100 marks in math, then he will get a mobile. The converse is
(A) If Ram gets a mobile, then he will not secures 100 marks
(B) If Ram does not get a mobile, then he will secures 100 marks
(C) If Ram will get a mobile, then he secures 100 marks in math
(D) None of these

42. If p : A man is happy.
q : A man is rich.
Then, the statement, “If a man is not happy, then he is not rich” is written as
(A) $\sim p \rightarrow \sim q$
(B) $\sim q \rightarrow p$
(C) $\sim q \rightarrow \sim p$
(D) $q \rightarrow \sim p$

43. The compound statement $p \rightarrow (\sim p \vee q)$ is false, then the truth values of p and q are respectively
(A) T, T
(B) T, F
(C) F, T
(D) F, F

44. The negation of the statement “Rajesh or Rajni lived in Bangaluru” is
(A) Rajesh did not live in Bengaluru or Rajni lives in Bengaluru
(B) Rajesh lives in Bengaluru and Rajni did not live in Bengaluru
(C) Rajesh did not live in Bengaluru and Rajni did not live in Bengaluru
(D) Raesh did not live in Bengaluru or Rajni lives in Bengaluru

45. The converse of the statement “If $x > y$, then $x + a > y + a$” is
(A) If $x < y$, then $x + a < y + a$
(B) If $x + a > y + a$, then $x > y$
(C) If $x < y$, then $x + a < y + a$
(D) If $x > y$, then $x + a < y + a$

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1. **[JEE Mains 2020]**
The logical statement $(p \rightarrow q) \wedge (q \rightarrow \sim p)$ is equivalent to:
(A) p
(B) q
(C) $\sim p$
(D) $\sim q$

2. **[JEE Mains 2020]**
Let A, B, C and D be four non-empty sets. The contrapositive statement of “If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$” is:
(A) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$
(B) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$
(C) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \not\subseteq D$
(D) If $A \subseteq C$, then $B \subset A$ and $D \subset B$

3. **[JEE Mains 2020]**
Which one of the following is a tautology?
(A) $P \vee (P \wedge Q)$
(B) $P \wedge (P \vee Q)$
(C) $Q \rightarrow (P \wedge (P \wedge Q))$
(D) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

4. **[JEE Mains 2020]**
Which of the following statements is a tautology?
(A) $\sim(p \vee \sim q) \rightarrow p \vee q$
(B) $\sim(p \wedge \sim q) \rightarrow p \vee q$
(C) $p \vee (\sim q) \rightarrow p \wedge q$
(D) $\sim(p \vee q) \rightarrow \sim p \vee q$

5. **[JEE Mains 2020]**
Negation of the statement: ‘ $\sqrt{5}$ is an integer or 5 is irrational’ is:
(A) $\sqrt{5}$ is an integer and 5 is irrational
(B) $\sqrt{5}$ is not an integer and 5 is not irrational
(C) $\sqrt{5}$ is not an integer or 5 is not irrational
(D) $\sqrt{5}$ is an irrational or 5 is an integer

6. **[JEE Mains 2020]**
If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively:
(A) F, F
(B) F, T
(C) T, F
(D) T, T

7. **[JEE Mains 2020]**
Which of the following is a tautology?
(A) $(\sim p) \wedge (p \vee q) \rightarrow q$
(B) $(q \rightarrow p) \vee \sim(p \rightarrow q)$
(C) $(p \rightarrow q) \wedge (q \rightarrow p)$
(D) $(\sim q) \vee (p \wedge q) \rightarrow q$

8. **[JEE Mains 2020]**
The contrapositive of the statement “If I reach the station in time, then I will catch the train” is:
(A) If I will catch the train, then 1 reach the station in time.
(B) If I do not reach the station in time, then 1 will not catch the train.
(C) If I will not catch the train, then 1 do not reach the station in time.
(D) If I do not reach the station in time, then 1 will catch the train.

9. **[JEE Mains 2020]**
The proposition $p \rightarrow \sim(p \wedge \sim q)$ is equivalent to:
(A) $(\sim p) \vee q$
(B) q
(C) $(\sim p) \wedge q$
(D) $(\sim p) \vee (\sim q)$

10. **[JEE Mains 2020]**
Contrapositive of the statement:
If a function f is differentiable at a, then it is also continuous at a’, is
(A) If a function f is continuous at a, then it is not differentiable at a.
(B) If a function f is not continuous at a, then it is differentiable at a.
(C) If a function f is not continuous at a, then it is not differentiable at a.
(D) If a function f is continuous at a, then it is differentiable at a.

11. **[JEE Mains 2020]**
Let p, q, r be three statements such that the truth value of $(p \wedge q) \rightarrow (\sim q \vee r)$ is F. Then the truth values of p, q, r are respectively:
(A) T, F, T
(B) F, T, F
(C) T, T, F
(D) T, T, T

12. **[JEE Mains 2020]**
Given the following two statements
$(S_1) : (q \vee p) \rightarrow (p \leftrightarrow \sim q)$ is a tautology.
$(S_2) : \sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy then:

(A) only (S_1) is correct
(B) both (S_1) and (S_2) are correct
(C) both (S_1) and (S_2) are not correct.
(D) only (S_2) is correct.

13. [JEE Mains 2020]
The statement $(p \to (q \to p)) \to (p \to (q \vee p))$ is:
(A) a contradiction
(B) equivalent to $(p \wedge q) \vee (\sim q)$
(C) a tautology
(D) equivalent to $(p \vee q) \wedge (\sim p)$

14. [JEE Mains 2020]
The negation of the Boolean expression $x \leftrightarrow \sim y$ is equivalent to:
(A) $(\sim x \wedge y) \vee (\sim x \wedge \sim y)$
(B) $(x \wedge \sim y) \vee (\sim x \wedge y)$
(C) $(x \wedge y) \vee (\sim x \wedge \sim y)$
(D) $(x \wedge y) \wedge (\sim x \vee \sim y)$

15. [JEE Mains 2020]
Consider the statement: "For an integer n, if $n^3 - 1$ is even, then n is odd. "The contrapositive statement of this statements is:
(A) For an integer n, if $n^3 - 1$ is not even, then n is not odd.
(B) For an integer n, if n is even, then $n^3 - 1$ is odd.
(C) For an integer n, if n is odd, then $n^3 - 1$ is even.
(D) For an integer n, if n is even, then $n^3 - 1$ is even.

16. [JEE Mains 2020]
The negation of the Boolean expression $p \vee (\sim p \wedge q)$ is equivalent to:
(A) $\sim p \vee \sim q$
(B) $\sim p \vee q$
(C) $\sim p \wedge \sim q$
(D) $p \wedge \sim q$

17. [JEE Mains 2019]
If the Boolean expression $(p \oplus q) \wedge (\sim p \odot q)$ is equivalent to $p \wedge q$, when $\oplus, \odot \in \{\wedge, \vee\}$, then the ordered pair $(\oplus, \odot)$ is
(A) $(\vee, \vee)$
(B) $(\wedge, \wedge)$
(C) $(\wedge, \vee)$
(D) $(\vee, \wedge)$

18. [JEE Mains 2019]
The logical statement $[\sim(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)]$ is equivalent to:
(A) $(\sim p \wedge \sim q) \wedge r$
(B) $(p \wedge \sim q) \vee r$
(C) $(p \wedge r) \wedge \sim q$
(D) $\sim p \wedge r$

19. [JEE Mains 2019]
Consider the statement: "$P(n) : n^2 - n + 41$ is prime". Then which one of the following is true?
(A) $P(5)$ is false but $P(3)$ is true
(B) Both $P(3)$ and $P(5)$ are true
(C) $P(3)$ is false but $P(5)$ is true
(D) Both $P(3)$ and $P(5)$ are false

20. [JEE Mains 2019]
Consider the following three statements:
P : 5 is a prime number
Q : 7 is a factor of 192
R : L.C.M. of 5 and 7 is 35
Then the truth value of which one of the following statement is true?
(A) $(P \wedge Q) \vee (\sim R)$
(B) $P \vee (\sim Q \wedge R)$
(C) $(\sim P) \wedge (\sim Q \wedge R)$
(D) $(\sim P) \vee (Q \wedge R)$

21. [JEE Mains 2019]
If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology?
(A) $p \wedge r$
(B) $(p \vee r) \to (p \wedge r)$
(C) $p \vee r$
(D) $(p \wedge r) \to (p \vee r)$

22. [JEE Mains 2019]
The Boolean expression $((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to:
(A) $p \wedge q$
(B) $p \wedge (\sim q)$
(C) $p \vee (\sim q)$
(D) $(\sim p) \wedge (\sim q)$

23. [JEE Mains 2019]
The expression $\sim(\sim p \to q)$ is logically equivalent to:
(A) $p \wedge q$
(B) $\sim p \wedge -q$
(C) $p \wedge \sim q$
(D) $\sim p \wedge q$

24. [JEE Mains 2019]
The contrapositive of the statement "If you are born in India, then you are a citizen of India", is
(A) If you are a citizen of India, then you are born in India
(B) If you are born in India, then you are not a citizen of India
(C) If you are not a citizen of India, then you are not born in India
(D) If you are not born in India, then you are not a citizen of India

25. [JEE Mains 2019]
Which one of the following statements is not a tautology?
(A) $p \to (p \vee q)$
(B) $(p \wedge q) \to (\sim p) \vee q$
(C) $(p \wedge q) \to p$
(D) $(p \vee q) \to (p) \vee (\sim q))$

26. [JEE Mains 2019]
For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is:
(A) $\sim p \vee \sim q$
(B) $p \leftrightarrow q$
(C) $p \wedge q$
(D) $\sim p \wedge \sim q$

27. [JEE Mains 2019]
If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively:
(A) F, F, F
(B) T, F, F
(C) F, T, T
(D) T, T, F

28. [JEE Mains 2019]
Which one of the following Boolean expressions is a tautology?
(A) $(p \vee q) \wedge (\sim p \vee \sim q)$
(B) $(p \vee q) \vee (p \vee \sim q)$
(C) $(p \wedge q) \vee (p \wedge \sim q)$
(D) $(p \vee q) \wedge (p \vee \sim q)$

29. [JEE Mains 2019]
The negation of the Boolean expression $\sim s \vee (\sim r \wedge s)$ is equivalent to:

(A) $\sim s \vee \sim r$ (B) r
(C) $s \wedge r$ (D) $s \vee r$

30. [JEE Mains 2019]
If the truth value of the statement $p \to (\sim q \vee r)$ is false (F), then the truth values of the statements p, q, r are respectively:
(A) T, F, T
(B) F, T, T
(C) T, T, F
(D) T, F, F

31. [JEE Mains 2019]
The Boolean expression $\sim(p \Rightarrow (\sim q))$ is equivalent to:
(A) $p \wedge q$ (B) $(\sim p) \Rightarrow q$
(C) $q \Rightarrow \sim p$ (D) $p \vee q$

32. [JEE Mains 2019]
Contrapositive of the statement "If two numbers are not equal, then their squares are not equal." Is:
(A) If the squares of two numbers are equal, then the numbers are not equal
(B) If the squares of two numbers are equal, then the numbers are equal
(C) If the squares of two numbers are not equal, then the numbers are equal
(D) If the squares of two numbers are not equal, then the numbers are not equal

33. [JEE Mains 2018]
The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to:
(A) p (B) q
(C) $\sim q$ (D) $\sim p$

34. [JEE Mains 2017]
The following statement $(p \to q) \to [(\sim p \to q) \to q]$ is
(A) a fallacy
(B) a tautology
(C) equivalent to $\sim p \to q$
(D) equivalent to $p \to \sim q$

35. [JEE Mains 2016]
The Boolean Expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to:
(A) $p \vee q$ (B) $p \vee \sim q$
(C) $\sim p \wedge q$ (D) $p \wedge q$

36. [JEE Mains 2015]
The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to:
(A) $s \vee (r \wedge \sim s)$ (B) $s \wedge r$
(C) $s \wedge \sim r$ (D) $s \wedge (r \wedge \sim s)$

37. [JEE Mains 2014]
The statement $\sim(p \leftrightarrow \sim q)$ is:
(A) a tautology (B) a fallacy
(C) equivalent to $p \leftrightarrow q$ (D) equivalent to $\sim p \leftrightarrow q$

38. [JEE Mains 2013]
Consider
Statement-1: $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.
Statement-2: $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.
(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

39. [JEE Mains 2012]
The negation of the statement "If I become a teacher, then I will open a school", is:
(A) I will become a teacher and I will not open a school.
(B) Either I will not become a teacher or I will not open a school.
(C) Neither I will become a teacher nor I will open a school.
(D) I will not become a teacher or I will open a school.

40. [JEE Mains 2011]
Consider the following statements

P : Suman is brilliant

Q : Suman is rich

R : Suman is honest

The negation of the statement "Suman is brilliant and dishonest if and only if Suman is rich" can be expressed as
(A) $\sim(Q \leftrightarrow (P \wedge \sim R))$
(B) $\sim Q \leftrightarrow \sim P \wedge R$
(C) $\sim(P \wedge \sim R) \leftrightarrow Q$
(D) $\sim(P \wedge (Q \leftrightarrow \sim R)$

41. [JEE Mains 2010]
Let S be a non-empty subset of R. Consider the following statement:
P : There is a rational number $x \in S$ such that $x > 0$.
Which of the following statements is the negation of the statement P?
(A) There is no rational number $x \in S$ such than $x \le 0$.
(B) Every rational number $x \in S$ satisfies $x \le 0$.
(C) $x \in S$ and x is not rational.
(D) There is a rational number $x \in S$ such that $x \le 0$.

42. [JEE Mains 2009]
Statement-1: $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.
Statement-2: $\sim (p \leftrightarrow \sim q)$ is a tautology
(A) Statement-1 is true, Statement-2 is true Statement-2 is not a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is false
(C) Statement-1 is false, Statement-2 is true
(D) Statement-1 is true, Statement-2 is true Statement-2 is a correct explanation for Statement-1

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** C	**3.** B	**4.** A	**5.** C	**6.** B	**7.** D	**8.** C	**9.** A	**10.** B
11. C	**12.** D	**13.** D	**14.** C	**15.** B	**16.** C	**17.** D	**18.** B	**19.** A	**20.** C
21. C	**22.** C	**23.** A	**24.** A	**25.** C	**26.** B	**27.** A	**28.** C	**29.** A	**30.** B
31. B	**32.** A	**33.** A	**34.** C	**35.** B	**36.** C	**37.** B	**38.** C	**39.** C	**40.** A
41. C	**42.** A	**43.** B	**44.** C	**45.** B					

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1. C	**2.** B	**3.** D	**4.** A	**5.** B	**6.** D	**7.** A	**8.** C	**9.** A	**10.** C
11. C	**12.** C	**13.** C	**14.** C	**15.** B	**16.** C	**17.** C	**18.** C	**19.** B	**20.** B
21. D	**22.** D	**23.** B	**24.** C	**25.** D	**26.** D	**27.** B	**28.** B	**29.** C	**30.** C
31. A	**32.** B	**33.** D	**34.** B	**35.** A	**36.** B	**37.** C	**38.** B	**39.** A	**40.** A
41. B	**42.** B								

From the Author's Desk

"Dear Learners,

Determinant is an important chapter as far as Algebra is concerned. It is often combined with Matrices, Probability, Complex Number, Vector 3-D. In both exams, JEE Mains and Advanced, questions are asked from it. Value of Determinant, Properties of determinant, Elementary Row transformation, Solving System of linear equation using Cramer Rule are key concepts. There are ample of illustration followed by solved exercises which will surely boost your concepts.

Students are advised to solve adequate questions covering various concepts in prescribed time limit.

Do not forget to attempt PYQs from JEE Mains & Advanced.

Good Luck !!"

CHAPTER 11

Determinant

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Minors and cofactors
(b) Value of determinant
(c) Sarrus rule
(d) Properties of determinant
(e) Elementary row transformation
(f) Multiplication of determinants
(g) Special determinants
(h) Cramer rule

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

DETERMINANT

Representation of numbers or variables in square form involving rows and columns is known as determinant. Determinant is a square array of real or complex numbers in rows and columns. In a determinant, horizontal lines are known as **rows** and vertical lines known as **columns**. If a determinant is of $n \times n$ order then it contains n rows and n columns. For determinant number of rows and columns are always equal

$$\Delta = \begin{array}{|cccc|l}
a_{11} & a_{12} & a_{13}\,\ldots\ldots\ldots & a_{1n} & \rightarrow \text{First row } (R_1)\\
a_{21} & a_{22} & a_{23}\,\ldots\ldots\ldots & a_{2n} & \rightarrow \text{Second row } (R_2)\\
a_{31} & a_{32} & a_{33}\,\ldots\ldots\ldots & a_{3n} & \rightarrow \text{Third row } (R_3)\\
\vdots & \vdots & \vdots\,\ldots\ldots\ldots & \vdots & \\
a_{n1} & a_{n2} & a_{n3}\,\ldots\ldots\ldots & a_{nn} & \rightarrow n^{th}\text{ row } (R_n)\\
\multicolumn{1}{c}{\downarrow} & \downarrow & \downarrow & \multicolumn{1}{c}{\downarrow} & n \times n\\
\multicolumn{1}{c}{\text{First}} & \text{Second} & \text{Third} & \multicolumn{1}{c}{n^{th}} & \\
\multicolumn{1}{c}{\text{column}} & \text{column} & \text{column} & \multicolumn{1}{c}{\text{column}} & \\
\multicolumn{1}{c}{(C_1)} & (C_2) & (C_3) & \multicolumn{1}{c}{(C_n)} &
\end{array}$$

We can always represent any element present in determinant with the notation a_{ij}

$$\boldsymbol{a_{ij} = \textbf{element present at } i^{th} \textbf{ row and } j^{th} \textbf{ column}}$$

for example a_{23} = element present at 2^{nd} row and 3^{rd} column.

Note: Number of elements in determinant of order $n \times n = n^2$.

e.g. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2x2}$ is a second order determinant

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3}$ is a third order determinant

Determinant of order 3×3 is most frequently used in this chapter. It has three rows and three columns as shown.

$$\Delta = \begin{array}{|ccc|l}
a_{11} & a_{12} & a_{13} & \rightarrow \text{First row } (R_1)\\
a_{21} & a_{22} & a_{23} & \rightarrow \text{Second row } (R_2)\\
a_{31} & a_{32} & a_{33} & \rightarrow \text{Third row } (R_3)\\
\downarrow & \downarrow & \downarrow & \\
\text{First} & \text{Second} & \text{Third} & \\
\text{column} & \text{column} & \text{column} & \\
(C_1) & (C_2) & (C_3) &
\end{array}$$

ILLUSTRATION 1

Δ is a determinant of 2×2 order with element $a_{ij} = i^j$ $1 \le i, j \le 2$, find it.

SOLUTION

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2\times 2} = \begin{vmatrix} 1^1 & 1^2 \\ 2^1 & 2^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

ILLUSTRATION 2

Find determinant of order 3 × 3 with element

$$a_{ij} \text{ defined as } \begin{cases} 2i+j & i \neq j \\ (i+2j)^i & i = j \end{cases}$$

SOLUTION

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3} = \begin{vmatrix} 3 & 2+2 & 2+3 \\ 4+1 & 6^2 & 4+3 \\ 6+1 & 6+2 & 9^3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 & 5 \\ 5 & 36 & 7 \\ 7 & 8 & 729 \end{vmatrix}$$

ILLUSTRATION 3

Write determinant of order 3 × 3 whose element

$$a_{ij} = \begin{cases} i.j & i < j \\ i^j + j^i & i = j \\ j/i & i > j \end{cases}$$

SOLUTION

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1+1 & 1\times2 & 1\times3 \\ 1/2 & 2^2+2^2 & 2\times3 \\ 1/3 & 2/3 & 3^3+3^3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 & 3 \\ 1/2 & 8 & 6 \\ 1/3 & 2/3 & 54 \end{vmatrix}$$

ILLUSTRATION 4

If element a_{ij} belong to determinant $\begin{vmatrix} 5 & -1 & 4 \\ 3 & 0 & -7 \\ 2 & 1 & 5 \end{vmatrix}$ then find $\dfrac{(a_{11} \times a_{23} + a_{31})^{a_{32}}}{a_{13} - a_{33}}$.

SOLUTION

$$a_{11} = 5, a_{23} = -7, a_{31} = 2, a_{13} = 4, a_{33} = 5$$

$$\frac{(a_{11} \times a_{23} + a_{31})^{a_{32}}}{a_{13} - a_{33}} = \frac{(5\times(-7)+2)^1}{4-5} = \frac{-35+2}{-1} = 33$$

ILLUSTRATION 5

If element a_{ij} belong to determinant $\begin{vmatrix} x & 3 & 7 \\ x & -5 & 2 \\ 4 & 3 & 0 \end{vmatrix}$ find equation whose roots are double of equation

$$a_{21}a_{11} + a_{22}a_{11} + a_{31} = 0.$$

SOLUTION

$$a_{21}a_{11} + a_{22}a_{11} + a_{31} = 0$$

$$\Rightarrow \quad x(x) - 5x + 4 = 0 \quad \Rightarrow \quad x^2 - 5x + 4 = 0$$

$$\Rightarrow \quad (x-1)(x-4) = 0 \quad \Rightarrow \quad x = 1, 4$$

New quadratic equation roots are $x = 2, 8$

$$(x-2)(x-8) = 0 \quad \Rightarrow \quad x^2 - 10x + 16 = 0$$

VALUE OF DETERMINANT OF ORDER 2 × 2

Expansion of determinant of order 2 × 2

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

ILLUSTRATION 6

If $\begin{vmatrix} |x| & 6 \\ 2 & 4 \end{vmatrix} = 0$. Find x.

SOLUTION

$$\begin{vmatrix} |x| & 6 \\ 2 & 4 \end{vmatrix} = 0 \quad \Rightarrow \quad 4|x| - 12 = 0$$

$$\Rightarrow \quad |x| = 3 \quad \Rightarrow \quad x = \pm 3$$

MINORS AND COFACTORS

Minor (M_{ij}) = Minor of element a_{ij}

= It is a sub-determinant after removing i^{th} row and j^{th} column from determinant

$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3\times3}$ has 9 elements and each element has unique minor

Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

Minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

Minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$,

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$,

Minor of $a_{22} = M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$,

$$\text{Minor of } a_{23} = M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{Minor of } a_{31} = M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},$$

$$\text{Minor of } a_{32} = M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix},$$

$$\text{Minor of } a_{33} = M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

ILLUSTRATION 7

For determinant $\begin{vmatrix} 1 & -2 & 4 \\ 3 & 1 & 7 \\ 10 & -3 & 9 \end{vmatrix}$ find minors of each elements.

SOLUTION

$$M_{11} = \begin{vmatrix} 1 & 7 \\ -3 & 9 \end{vmatrix} = 9 - (-21) = 30;$$

$$M_{12} = \begin{vmatrix} 3 & 7 \\ 10 & 9 \end{vmatrix} = 27 - 70 = -43;$$

$$M_{13} = \begin{vmatrix} 3 & 1 \\ 10 & -3 \end{vmatrix} = -9 - 10 = -19$$

$$M_{21} = \begin{vmatrix} -2 & 4 \\ -3 & 9 \end{vmatrix} = -18 + 12 = -6;$$

$$M_{22} = \begin{vmatrix} 1 & 4 \\ 10 & 9 \end{vmatrix} = 9 - 40 = -31$$

$$M_{23} = \begin{vmatrix} 1 & -2 \\ 10 & -3 \end{vmatrix} = -3 + 20 = 17;$$

$$M_{31} = \begin{vmatrix} -2 & 4 \\ 1 & 7 \end{vmatrix} = -14 - 4 = -18$$

$$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = 7 - 12 = -5;$$

$$M_{33} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 1 + 6 = 7$$

ILLUSTRATION 8

$\Delta = \begin{vmatrix} 4 & -1 & 3 \\ 7 & 2 & 0 \\ 1 & -4 & 5 \end{vmatrix}$ find $\left(\dfrac{M_{22}}{M_{31}}\right) + \left(\dfrac{M_{13}}{M_{21}}\right)$ where M_{ij} is minor of element a_{ij}.

SOLUTION

$$M_{13} = \begin{vmatrix} 7 & 2 \\ 1 & -4 \end{vmatrix} = -28 - 2 = -30$$

$$M_{21} = \begin{vmatrix} -1 & 3 \\ -4 & 5 \end{vmatrix} = -5 + 12 = 7$$

$$M_{22} = \begin{vmatrix} 4 & 3 \\ 1 & 5 \end{vmatrix} = 20 - 3 = 17$$

$$M_{31} = \begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} = 0 - 6 = -6$$

$$\left(\frac{M_{22}}{M_{31}}\right) + \left(\frac{M_{13}}{M_{21}}\right) = \left(\frac{17}{-6}\right) + \left(\frac{-30}{7}\right)$$

$$= \frac{-17 \times 7 - 180}{42} = \frac{-299}{42}$$

COFACTOR

Cofactor (C_{ij}) = Cofactor of element a_{ij}
Numeric vale of cofactor and minor are equal

$$C_{ij} = (-1)^{i+j}\, M_{ij}$$

Sign of cofactors are dependent on position of elements

$$\text{Cofactor of element } a_{11} = C_{11} = (-1)^{1+1} M_{11}$$

$$= M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{Cofactor of element } a_{12} = C_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$\text{Cofactor of element } a_{13} = C_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{Cofactor of element } a_{21} = C_{21} = -M_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},$$

$$\text{Cofactor of element } a_{22} = C_{22} = M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},$$

$$\text{Cofactor of element } a_{23} = C_{23} = -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$\text{Cofactor of element } a_{31} = C_{31} = M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},$$

$$\text{Cofactor of element } a_{32} = C_{32} = -M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix},$$

$$\text{Cofactor of element } a_{33} = C_{33} = M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

SIGN OF COFACTOR

For determinant of order 2×2 $\begin{vmatrix} +() & -() \\ -() & +() \end{vmatrix}_{2\times 2}$

For determinant of order 3×3 $\begin{vmatrix} +() & -() & +() \\ -() & +() & -() \\ +() & -() & +() \end{vmatrix}_{3\times 3}$

where $(\cdot)$ is corresponding Minor (M_{ij})

Determinant formed by minors

$$M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix},$$

Determinant formed by cofactors $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

Value of Determinant (Δ) = Sum of product of elements with corresponding cofactors along any row or column

along any row $\Delta = \sum_{j=1}^{n} a_{ij}C_{ij}$ for $i = 1, 2, ..., n$

along any column $\Delta = \sum_{j=1}^{n} a_{ji}C_{ji}$ for $i = 1, 2, ..., n$

Expansion of determinant of order 3×3

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3}$$

Expansion along R_1

$$\Delta = a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

VALUE OF DETERMINANT CAN BE OBTAINED BY EXPANDING ALONG ANY ROW OR ANY COLUMN

Along any row $\because \Delta = \sum_{j=1}^{3} a_{ij}C_{ij}$ for $i = 1, 2,3$

$\Delta = a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13}$

$= a_{11}\,M_{11} - a_{12}\,M_{12} + a_{13}\,M_{13}$ (along R_1, $i = 1$)

$\Delta = a_{21}\,C_{21} + a_{22}\,C_{22} + a_{23}\,C_{23}$

$= -a_{21}\,M_{21} + a_{22}\,M_{22} - a_{23}\,M_{23}$ (along R_2, $i = 2$)

$\Delta = a_{31}\,C_{31} + a_{32}\,C_{32} + a_{33}\,C_{33}$

$= a_{31}\,M_{31} - a_{32}\,M_{32} + a_{33}\,M_{33}$ (along R_3, $i = 3$)

along any column $\because \Delta = \sum_{j=1}^{3} a_{ji}C_{ji}$ for $i = 1, 2,3$

$\Delta = a_{11}\,C_{11} + a_{21}\,C_{21} + a_{31}\,C_{31}$

$= a_{11}\,M_{11} - a_{21}\,M_{21} + a_{31}\,M_{31}$ (along C_1, $i = 1$)

$\Delta = a_{12}\,C_{12} + a_{22}\,C_{22} + a_{32}\,C_{32}$

$= -\,a_{12}\,M_{12} + a_{22}\,M_{22} - a_{32}\,M_{32}$ (along C_2, $i = 2$)

$\Delta = a_{13}\,C_{13} + a_{23}\,C_{23} + a_{33}\,C_{33}$

$= a_{13}\,M_{13} - a_{23}\,M_{23} + a_{33}\,M_{33}$ (along C_3, $i = 3$)

If we take sum of products of elements from one row (or column) and cofactors from other row (or column) then it becomes zero

Along row $\sum_{k=1}^{n} a_{ik}C_{jk} = 0, i \neq j$ and

along column $\sum_{k=1}^{n} a_{ki}C_{kj} = 0, i \neq j$

$a_{11}\,C_{21} + a_{12}\,C_{22} + a_{13}\,C_{23} = 0$ (elements from row R_1, Cofactor from row R_2)

$a_{21}\,C_{31} + a_{22}\,C_{32} + a_{23}\,C_{33} = 0$ (elements from row R_2, Cofactor from row R_3)

$a_{11}\,C_{12} + a_{21}\,C_{22} + a_{31}\,C_{32} = 0$ (elements from column C_1, Cofactor from column C_2)

$a_{13}\,C_{11} + a_{23}\,C_{21} + a_{33}\,C_{31} = 0$ (elements from column C_3, Cofactor from column C_1)

ILLUSTRATION 9

Find cofactor of determinant $\begin{vmatrix} 1 & -1 & 2 \\ 4 & 1 & 7 \\ -2 & 0 & 5 \end{vmatrix}$ and also compute

(i) value of determinant

(ii) $a_{11}\,C_{21} + a_{12}\,C_{22} + a_{13}\,C_{23}$

(iii) $a_{11}\,C_{12} + a_{21}\,C_{22} + a_{31}\,C_{32}$

SOLUTION

$$C_{11} = +M_{11} = \begin{vmatrix} 1 & 7 \\ 0 & 5 \end{vmatrix} = 5;$$

$$C_{12} = -M_{12} = -\begin{vmatrix} 4 & 7 \\ -2 & 5 \end{vmatrix} = -\,(20 + 14) = -\,34$$

$$C_{13} = +M_{13} = \begin{vmatrix} 4 & 1 \\ -2 & 0 \end{vmatrix} = 2;$$

$$C_{21} = -M_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 5 \end{vmatrix} = 5$$

$$C_{22} = +M_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 9;$$

$$C_{23} = -M_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$C_{31} = +M_{31} = \begin{vmatrix} -1 & 2 \\ 1 & 7 \end{vmatrix} = -9;$$

$$C_{32} = -M_{32} = -\begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} = 1$$

$$C_{33} = +M_{33} = \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} = 5$$

(i) value of determinant

Along row R_1

$$\Delta = a_{11}\,C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13}$$

$$= 1\begin{vmatrix} 1 & 7 \\ 0 & 5 \end{vmatrix} - 1\left(-\begin{vmatrix} 4 & 7 \\ -2 & 5 \end{vmatrix}\right) + 2\begin{vmatrix} 4 & 1 \\ -2 & 0 \end{vmatrix}$$

$$= 5 + 34 + 4 = 43$$

Along row R_2

$$\Delta = a_{21}\,C_{21} + a_{22}\,C_{22} + a_{23}\,C_{23}$$

$$= 4(5) + 1(9) + 7(2) = 43$$

We can compute value of determinant along any column also

Along Column C_2

$$\Delta = a_{12}\,C_{12} + a_{22}\,C_{22} + a_{32}\,C_{32}$$

$$= -1(-34) + 1\,(9) + 0(1) = 43$$

(ii) $a_{11}\,C_{21} + a_{12}\,C_{22} + a_{13}\,C_{23}$ = sum of product of elements (from row R_1) with cofactor (from row R_2)

$= 1(5) - 1(9) + 2(2) = 5 - 9 + 4 = 0$

(iii) $a_{11}\,C_{12} + a_{21}\,C_{22} + a_{31}\,C_{32}$= sum of product of elements (from column C_1) with cofactor (from column C_2)

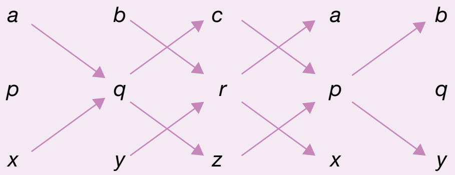
$= 1(-34) + 4\,(9) - 2(1) = -34 + 36 - 2 = 0$

SARRUS RULE FOR 3X3 ORDER DETERMINANT

The rule was discovered by a French mathematician Pierre Frederic Sarrus. Using this rule we can solve third-order determinants more efficiently

Consider a third-order determinant, $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$.

According to Sarrus rule, it can be expanded as follows



The downward diagonal gives the positive product of three elements, and the upward diagonal gives the negative product of three elements.

Sum of all the products is the value of determinant.

$$\Delta = aqz + brx + cpy - xqc - yra - zpb$$

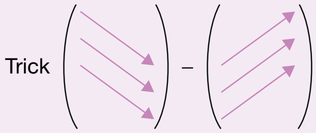


ILLUSTRATION 10

Compute following determinant using sarrus rule

(i) $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ 2 & 1 & -2 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 4 & 1 & 5 \end{vmatrix}$ **(iii)** $\begin{vmatrix} 2 & 1 & 2 \\ -2 & 4 & 3 \\ 1 & 0 & 1 \end{vmatrix}$

SOLUTION

(i) $$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 4 \\ 2 & 1 & -2 \end{vmatrix}_{3\times3} = [-2 + 0 - 8] - [4 + 4 + 0]$$

$$= -10 - 8 = -18$$

(ii) $$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 4 & 1 & 5 \end{vmatrix} = (0 + 1 + 4 \cdot 2 \cdot 3) - (0 + 3 - 10)$$

$$= 25 + 7 = 32$$

(iii) $$\begin{vmatrix} 2 & 1 & 2 \\ -2 & 4 & 3 \\ 1 & 0 & 1 \end{vmatrix} = (8 + 0 + 3) - (8 + 0 - 2)$$

$$= 11 - 6 = 5$$

ILLUSTRATION 11

Let a, b, c be positive and not all equal. Show that the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative.

SOLUTION

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

As given a, b, c are distinct positive A.M. > GM

$$\frac{a^3 + b^3 + c^3}{3} > (a^3 b^3 c^3)^{1/3}$$

$$\Rightarrow \quad a^3 + b^3 + c^3 > 3abc \Rightarrow 3abc - a^3 - b^3 - c^3 < 0$$

Hence $\Delta < 0$

Alternate

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$$= -(a^3 + b^3 + c^3 - 3abc)$$

$$= -(a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$

$$= -\frac{1}{2}(a + b + c)(2a^2 + 2b^2 + 2c^2 - 2bc - 2ca - 2ab)$$

$$= -\frac{1}{2}(a + b + c)\,[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)]$$

$$= -\frac{1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2] \qquad \ldots(1)$$

$\because \quad a, b, c$ are positive $\quad \therefore \quad a + b + c > 0$

since a, b, c are unequal

$\therefore \quad (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$

$\therefore$ From (i), we can surely say that $\Delta < 0$

Hence proved

ILLUSTRATION 12

If α, β, γ are the roots of $x^3 + ax^2 + b = 0$, then find the value of $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$.

SOLUTION

Since α, β, γ are the roots of the given equation, therefore $\alpha + \beta + \gamma = -a$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$ and $\alpha\beta\gamma = -b$.

$$\text{Now} \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = -(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$= -(\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\}$$

$$= -(-a)\{(a^2 - 0)\} = a^3.$$

ILLUSTRATION 13

If x is an imaginary root of the equation

$$\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} + \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{vmatrix} = 0 \text{ find } x^{2007} + x^{-2007}.$$

SOLUTION

Expanding the two determinants, using Sarrus Rule

$$(1 - 3x^2 + 2x^3) + (1 - x)^3 + 2 - 3(1 - x)$$

$$\Rightarrow \quad (1 - 3x^2 + 2x^3) + (3x^2 - x^3) = 0$$

$$\Rightarrow \quad x^3 + 1 = 0$$

$$\Rightarrow \quad x = -\omega, -\omega^2 \text{ as } x \text{ is non -real}$$

Now $x^{2007} + x^{-2007} = -1 - 1 = -2$.

PROPERTIES OF DETERMINANTS

P-I:

If all the elements of any row or columns are zero then the value of determinant is zero.

$$\begin{vmatrix} 0 & 0 & 0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0; \begin{vmatrix} a_2 & b_2 & 0 \\ a_1 & b_1 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

As we know that we can open determinant along any row or column, it is best practice to open determinant along the row or column which has more zeroes to have least calculation

P-II:

If any two rows (or columns) of a determinant be interchanged, then value of new determinant becomes negative of previous value.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{(Interchanging } R_1 \text{ with } R_2\text{)}$$

$$k = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{then } k = -\Delta$$

P-III:

If all rows are interchanged with columns then value of the determinant remains unchanged.

This is denoted by ′ or T and is also called transpose determinant of given determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \Delta' = \Delta^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta$$

P– IV:

If a determinant has two rows (or columns) identical or proportional then, then its value is zero.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$ then $\Delta = 0$ (where Row R_2 and R_3 are identical, expand along R_1, all cofactors become zeros)

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then $\Delta = 0$ (where Row R_1 and R_2 are proportional, expand along R_3, all cofactors become zeros)

P– V:

If a determinant is multiplied by a scalar *k* then we can multiply any one row or column by *k*, we should not multiply all rows by *k*

In other words if any row or column is multiplied by a scalar k then we can take out k as common outside determinant

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $S = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then $S = k\Delta$

P- VI:

If each element of any one row (column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of two determinants.

e.g. $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If each constituent in any row or column consists of r terms then the determinant can be expressed as the sum of r determinants.

Thus $\begin{vmatrix} a_1 & b_1 & c_1+\alpha_1+\beta_1 \\ a_2 & b_2 & c_2+\alpha_2+\beta_2 \\ a_3 & b_3 & c_3+\alpha_3+\beta_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \alpha_1 \\ a_2 & b_2 & \alpha_2 \\ a_3 & b_3 & \alpha_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \beta_1 \\ a_2 & b_2 & \beta_2 \\ a_3 & b_3 & \beta_3 \end{vmatrix}$$

$\begin{vmatrix} a_1+x & b_1+l & c_1+\alpha \\ a_2+y & b_2+m & c_2+\beta \\ a_3+z & b_3+n & c_3+\gamma \end{vmatrix}$ splitting into 8 determinants

$$= \begin{vmatrix} a_1 & b_1+l & c_1+\alpha \\ a_2 & b_2+m & c_2+\beta \\ a_3 & b_3+n & c_3+\gamma \end{vmatrix} + \begin{vmatrix} x & b_1+l & c_1+\alpha \\ y & b_2+m & c_2+\beta \\ z & b_3+n & c_3+\gamma \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1+\alpha \\ a_2 & b_2 & c_2+\beta \\ a_3 & b_3 & c_3+\gamma \end{vmatrix} + \begin{vmatrix} a_1 & l & c_1+\alpha \\ a_2 & m & c_2+\beta \\ a_3 & n & c_3+\gamma \end{vmatrix} +$$

$$\begin{vmatrix} x & b_1+l & c_1 \\ y & b_2+m & c_2 \\ z & b_3+n & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1+l & \alpha \\ y & b_2+m & \beta \\ z & b_3+n & \gamma \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \alpha \\ a_2 & b_2 & \beta \\ a_3 & b_3 & \gamma \end{vmatrix} + \begin{vmatrix} a_1 & l & c_1 \\ a_2 & m & c_2 \\ a_3 & n & c_3 \end{vmatrix} +$$

$$\begin{vmatrix} a_1 & l & \alpha \\ a_2 & m & \beta \\ a_3 & n & \gamma \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & l & c_1 \\ y & m & c_2 \\ z & n & c_3 \end{vmatrix} +$$

$$\begin{vmatrix} x & b_1 & \alpha \\ y & b_2 & \beta \\ z & b_3 & \gamma \end{vmatrix} + \begin{vmatrix} x & l & \alpha \\ y & m & \beta \\ z & n & \gamma \end{vmatrix}$$

We can conclude it as if a determinant of order 3×3, having each element as sum of two terms in first row, sum of three terms in second row and sum of four terms in third row then we can split this determinant in maximum $2.3.4 = 24$ determinants

P- VII:

If elements of all the rows/columns are in A.P. then the value of determinant is zero

P -VIII:

If the elements of a Determinant that involves *x* are polynomials in *x* and if *a* determinant Δ vanishes for *x* = *a* then (*x* – *a*) is a factor of Δ.

In other words if two rows (or two column) become identical for $x = a$ then $(x - a)$ is a factor of Δ.

$\therefore$ If r rows (or r columns) become identical when a is substituted for x then $(x - a)^{r-1}$ is a factor of Δ.

e.g. If $\Delta = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$

at $x = 2$, $\Delta = 0$. It means $(x - 2)$ is a factor of Δ

ILLUSTRATION 14

If $\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = 125$, then find $\begin{vmatrix} 3a & 3b & c \\ x & 2y & z \\ p & 5 & 5 \end{vmatrix}$ without expanding it.

SOLUTION

$$\begin{vmatrix}3a&3b&c\\x&2y&z\\p&5&5\end{vmatrix}=\begin{vmatrix}3a&x&p\\3b&2y&5\\c&z&5\end{vmatrix}$$

(changing rows into columns)

$$=\frac{1}{3}\begin{vmatrix}3a&x&p\\3b&2y&5\\3c&3z&15\end{vmatrix}=\frac{3}{3}\times\frac{1}{5}\begin{vmatrix}a&5x&p\\b&10y&5\\c&15z&15\end{vmatrix}$$

$$=\frac{1}{5}(125)=25.$$

ILLUSTRATION 15

$$\begin{vmatrix}2&1\\3&2\end{vmatrix}+\begin{vmatrix}2&1/2\\3&1\end{vmatrix}+\begin{vmatrix}2&1/4\\3&1/2\end{vmatrix}+\begin{vmatrix}2&1/8\\3&1/4\end{vmatrix}+\ldots\infty$$

SOLUTION

$$=(4-3)+(2-3/2)+(1-3/4)+(1/2-3/8)+\ldots$$

$=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ Using the formula for sum of infinite G.P.

$$=\frac{A}{1-r}=\frac{1}{1-\frac{1}{2}}=2$$

Alternate

$$\begin{vmatrix}2&1\\3&2\end{vmatrix}+\begin{vmatrix}2&1/2\\3&1\end{vmatrix}+\begin{vmatrix}2&1/4\\3&1/2\end{vmatrix}+\begin{vmatrix}2&1/8\\3&1/4\end{vmatrix}+\ldots\infty$$

As we observe first column is same in all the determinants

$$=\begin{vmatrix}2&1+1/2+1/4+1/8+\ldots\infty\\3&2+1+1/2+1/4+\ldots\infty\end{vmatrix}=\begin{vmatrix}2&\frac{1}{1-1/2}\\3&\frac{2}{1-1/2}\end{vmatrix}$$

$$=\begin{vmatrix}2&2\\3&4\end{vmatrix}=8-6=2$$

ILLUSTRATION 16

Let a determinant is given by $A=\begin{vmatrix}a&b&c\\p&q&r\\x&y&z\end{vmatrix}$ and suppose that det. $(A)=6$. If $B=\begin{vmatrix}p+x&q+y&r+z\\a+x&b+y&c+z\\a+p&b+q&c+r\end{vmatrix}$ then find det. B.

SOLUTION

$$B=\begin{vmatrix}p+x&q+y&r+z\\a+x&b+y&c+z\\a&b&c\end{vmatrix}+\begin{vmatrix}p+x&q+y&r+z\\a+x&b+y&c+z\\p&q&r\end{vmatrix}$$

$$=\begin{vmatrix}p+x&q+y&r+z\\x&y&z\\a&b&c\end{vmatrix}+\begin{vmatrix}p+x&q+y&r+z\\a&b&c\\a&b&c\end{vmatrix}$$

$$+\begin{vmatrix}p&q&r\\a+x&b+y&c+z\\p&q&r\end{vmatrix}+\begin{vmatrix}x&y&z\\a+x&b+y&c+z\\p&q&r\end{vmatrix}$$

$$=\begin{vmatrix}p&q&r\\x&y&z\\a&b&c\end{vmatrix}+\begin{vmatrix}x&y&z\\x&y&z\\a&b&c\end{vmatrix}+0+0$$

$$+\begin{vmatrix}x&y&z\\a&b&c\\p&q&r\end{vmatrix}+\begin{vmatrix}x&y&z\\x&y&z\\p&q&r\end{vmatrix}=\begin{vmatrix}p&q&r\\x&y&z\\a&b&c\end{vmatrix}+\begin{vmatrix}x&y&z\\a&b&c\\p&q&r\end{vmatrix}$$

using $R_1\leftrightarrow R_2$; $R_1\leftrightarrow R_3$ in second determinant

$$=6+6=12$$

ILLUSTRATION 17

If a, b, c are p^{th}, q^{th} and r^{th} terms respectively of a H.P., then find the value of the determinant $\begin{vmatrix}bc&ca&ab\\p&q&r\\1&1&1\end{vmatrix}$.

SOLUTION

Let A and d be the first term and common difference of the corresponding A.P., then

$$\frac{1}{a}=A+(p-1)d;\ \frac{1}{b}=A+(q-1)d;\ \frac{1}{c}=A+(r-1)d$$

Now det. $=abc\begin{vmatrix}1/a&1/b&1/c\\p&q&r\\1&1&1\end{vmatrix}$

$$=abc\begin{vmatrix}A+(p-1)d&A+(q-1)d&A+(r-1)d\\p&q&r\\1&1&1\end{vmatrix}$$

$$=abc\begin{vmatrix}A&A&A\\p&q&r\\1&1&1\end{vmatrix}+abcd\begin{vmatrix}p-1&q-1&r-1\\p&q&r\\1&1&1\end{vmatrix}$$

$$=0+0=0$$

ILLUSTRATION 18

$\Delta_r=\begin{vmatrix}2r-1&{}^nC_r&1\\n^2-1&2^n&n+1\\\cos^2 n^2&\cos^2 n&\sin n\end{vmatrix}$ find $\sum\limits_{r=a}^{n}\Delta_r$.

SOLUTION

As we observe only first row has variable r in it and second and third row do not have any variable r in them

$$\sum_{r=0}^{n}\Delta_r = \Delta_0+\Delta_1+\Delta_2+\Delta_3+\ldots+\Delta_n$$

$$= \begin{vmatrix} \sum\limits_{r=0}^{n}(2r-1) & \sum\limits_{r=0}^{n}{}^nC_r & \sum\limits_{r=0}^{n}1 \\ n^2-1 & 2^n & n+1 \\ \cos^2 n^2 & \cos^2 n & \sin n \end{vmatrix}$$

$$\sum_{r=0}^{n}1 = 1+1+\ldots+1 = n+1$$

$$\sum_{r=0}^{n}{}^nC_r = {}^nC_0+{}^nC_1+{}^nC_2+\ldots+{}^nC_n = 2^n$$

$$\sum_{r=0}^{n}(2r-1) = 2\sum_{r=0}^{n}r-\sum_{r=0}^{n}1$$

$$= 2[0+1+2+3+\ldots+n]-(n+1)$$

$$= 2\left[\frac{n(n+1)}{2}\right]-(n+1)$$

$$= (n+1)\,(n-1) = n^2-1$$

Now required sum $\sum\limits_{r=0}^{n}\Delta_r = \begin{vmatrix} n^2-1 & 2^n & n+1 \\ n^2-1 & 2^n & n+1 \\ \cos^2 n^2 & \cos^2 n & \sin n \end{vmatrix} = 0$

(as first and second row are identical)

ILLUSTRATION 19

If $S_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$ find $\sum\limits_{r=1}^{n}S_r$.

SOLUTION

As we observe only first column has variable r in it and second and third column do not have any variable r in them

$$\sum_{r=1}^{n}S_r = \begin{vmatrix} \sum\limits_{r=1}^{n}r & x & \frac{n(n+1)}{2} \\ 2\sum\limits_{r=1}^{n}r-\sum\limits_{r=1}^{n}1 & y & n^2 \\ 3\sum\limits_{r=1}^{n}r-\sum\limits_{r=1}^{n}2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ 2\left(\frac{n(n+1)}{2}\right)-n & y & n^2 \\ 3\left(\frac{n(n+1)}{2}\right)-2n & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix} = 0$$

(As first column and third column are identical)

ILLUSTRATION 20

Let a, b, c be three real numbers such that $a \in \left[0, \frac{\pi}{2}\right]$, $b \in R-\{0\}$ and $c \in (-\infty, \infty)$.

If $|(\sin a+\cos a)-\sqrt{2}| + \left|b+\frac{1}{b}-2\right| + |c| \le 0$, then find the value of $\begin{vmatrix} \sin(4a) & \cos^2 a & 1 \\ b+2 & b^3 & \frac{1}{b} \\ \ln(c+1) & c+5 & c^3+2 \end{vmatrix}$.

SOLUTION

$$|(\sin a+\cos a)-\sqrt{2}| + \left|b+\frac{1}{b}-2\right| + |c| \le 0,$$

Each term is zero independently for solution to exist

$$\sin a+\cos a = \sqrt{2},\ b+\frac{1}{b} = 2,\ c = 0$$

$$\Rightarrow \quad a = \frac{\pi}{4},\ b = 1,\ c = 0$$

$$\therefore \text{ Required determinant} = \begin{vmatrix} \sin(4a) & \cos^2 a & 1 \\ b+2 & b^3 & \frac{1}{b} \\ \ln(c+1) & c+5 & c^3+2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1/2 & 1 \\ 3 & 1 & 1 \\ 0 & 5 & 2 \end{vmatrix} = 12$$

ILLUSTRATION 21

If $\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$, then find the number of values of θ in $[0, 2\pi]$.

SOLUTION

$\sin 3\theta\,(28-21)+1(7\cos 2\theta-6)+7\cos 2\theta-8=0$

$\Rightarrow\quad 7[\sin 3\theta+2\cos 2\theta-2]=0$

$\Rightarrow\quad \sin 3\theta+2\cos 2\theta=2$

$\Rightarrow\quad 3\sin\theta-4\sin^3\theta+2(1-2\sin^2\theta)-2=0$

$\Rightarrow\quad -4\sin^3\theta-4\sin^2\theta+3\sin\theta=0$

$\Rightarrow\quad \sin\theta[4\sin^2\theta+4\sin\theta-3]=0$

$\Rightarrow\quad \sin\theta\,(2\sin\theta-1)\,(2\sin\theta+3)=0;$

$\sin\theta=0$ then $\theta=0,\pi,2\pi$

$\sin\theta=\dfrac{1}{2}$ then $\theta=\pi/6, 5\pi/6$

$\sin\theta=-\dfrac{3}{2}$ (not possible)

5 solution $\theta=\{0, \pi, 2\pi, \pi/6, 5\pi/6\}$

ILLUSTRATION 22

Find real distinct x for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix}=10.$$

SOLUTION

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix}=10$$

$$\Rightarrow\quad \begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix}=0$$

$$\Rightarrow\quad x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix}+x^3\cdot x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix}=10$$

$$\Rightarrow\quad x^3(25-23)+6x^6\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix}=10$$

$\Rightarrow\quad 2x^3+6x^6.\,2=10$

$\Rightarrow\quad 6x^6+x^3-5=0\quad\Rightarrow\quad x^3=\dfrac{5}{6},-1$

$\therefore\quad$ Two distinct real solutions

Test Your Concepts-I

(Solutions on page H.282)

1. If element $a_{ij}=\begin{cases} 2i-j & i>j \\ \dfrac{i\cdot j}{3} & i\le j \end{cases}$ are determinant of order 3×3. Write determinant.

2. Find the minor and cofactor of each element of the determinant $\begin{vmatrix} 2 & -2 & 3 \\ 1 & 4 & 5 \\ 2 & 1 & -3 \end{vmatrix}$.

3. Show that $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \Psi \\ \lambda & \mu & v \end{vmatrix}=\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & v & \Psi \end{vmatrix}$.

4. Compute using sarrus rule $\begin{vmatrix} 1 & -2 & 1 \\ 4 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$.

5. Using properties prove that $\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ x & y & z \end{vmatrix}$.

6. If $\Delta_r=\begin{vmatrix} r & 1 \\ r^2 & 2 \end{vmatrix}$ find $\displaystyle\sum_{r=1}^{3}\Delta_r$.

7. Let $\Delta_a=\begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$ show that $\displaystyle\sum_{a=1}^{n}\Delta_a$ is a constant. **[IIT - 89]**

Answer keys

1. $\begin{vmatrix} 1/3 & 2/3 & 1 \\ 3 & 4/3 & 2 \\ 5 & 4 & 3 \end{vmatrix}$ **2.** $C_{11}=-17, C_{12}=13, C_{13}=-7, C_{21}=-3, C_{22}=-12, C_{23}=-6, C_{31}=-22, C_{32}=-7, C_{33}=10$ **4.** 11 **6.** -2

ELEMENTORY ROW/COLUMN TRANSFORMATION

The value of a determinant is unchanged by using elementary row/column transformation i.e. adding the elements of any row (or column) to the some multiple of the corresponding elements of any other row (or column).

For determinant of order $n \times n$, we can take one row (or column) as reference and change other rows (or columns) using it .

e.g. $R_i \to R_i + m\,R_j$ keeping one row fixed and we can change other $(n-1)$ rows using it

For determinant of order 3×3

(i) $R_1 \to R_1 + mR_3$

$R_2 \to R_2 - kR_3$ where R_3 works as a reference and m, k are non-zero scalars

These transformations are known as ‘operations’

(ii) $R_1 \to R_1 + m\,R_2 - \lambda R_3$ where R_2 and R_3 are references and m , λ are non-zero scalars

e.g. Let $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ using $C_1 \to C_1 + m\,C_2 - C_3,$

we can rewrite it as without changing the value of determinant

$$= \begin{vmatrix} a_1 + mb_1 - c_1 & b_1 & c_1 \\ a_2 + mb_2 - c_2 & b_2 & c_2 \\ a_3 + mb_3 - c_3 & b_3 & c_3 \end{vmatrix}$$

(iii) $R_1 \to \alpha R_1 + mR_2 - \lambda R_3$ then value of new determinant becomes α times the previous one.

(iv) Following operations are not allowed

$R_1 \to R_1 \times R_2$, $R_2 \to R_2 / R_1$, $R_3 \to (R_3)^2$, $R_1 \to \ln R_1$, $R_1 \to e^{R_1}$, $R_1 \to R_1 + \alpha R_2 R_3$

All the operations which are applicable on rows are also valid for columns.

USEFUL IMPORTANT DETERMINANTS

(i) The elements of the rows (or columns) are in cyclic arrangement.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

$$= -\frac{1}{2}(a + b + c) \times \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

$$= -(a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega),$$

where ω, ω^2 are cube roots of unity

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= (a-b)(b-c)(c-a)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(a^2 + b^2 + c^2 - ab - bc - ca)$

(vi) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

(vii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$

$= abc(a-b)(b-c)(c-a)$

ILLUSTRATION 23

Simplify $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

SOLUTION

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

using $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$]

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \text{ using } R_3 \to R_3 - 3R_2$$

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix} = a\begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} = a^3$$

ILLUSTRATION 24

Simplify $\begin{vmatrix} (5^x-5^{-x})^2 & (5^x+5^{-x})^2 & \ell n2 \\ (6^x-6^{-x})^2 & (6^x+6^{-x})^2 & \ell n2 \\ (7^x-7^{-x})^2 & (7^x+7^{-x})^2 & \ell n2 \end{vmatrix}$

SOLUTION

$$\Delta = \begin{vmatrix} (5^x-5^{-x})^2 & (5^x+5^{-x})^2 & \ell n2 \\ (6^x-6^{-x})^2 & (6^x+6^{-x})^2 & \ell n2 \\ (7^x-7^{-x})^2 & (7^x+7^{-x})^2 & \ell n2 \end{vmatrix}$$

$$= \begin{vmatrix} 5^{2x}+5^{-2x}-2 & 5^{2x}+5^{-2x}+2 & \ell n2 \\ 6^{2x}+6^{-2x}-2 & 6^{2x}+6^{-2x}+2 & \ell n2 \\ 7^{2x}+7^{-2x}-2 & 7^{2x}+7^{-2x}+2 & \ell n2 \end{vmatrix}$$

using $C_2 \to C_2 - C_1$

$$= \begin{vmatrix} 5^{2x}+5^{-2x}-2 & 4 & \ell n2 \\ 6^{2x}+6^{-2x}-2 & 4 & \ell n2 \\ 7^{2x}+7^{-2x}-2 & 4 & \ell n2 \end{vmatrix}$$

column C_2 and C_3 are proportional

$= 0$

ILLUSTRATION 25

If $a_1, a_2, a_3, \ldots, a_n, a_{n+1} \ldots$ are in G.P. then

prove that $\begin{vmatrix} \ell na_n & \ell na_{n+3} & \ell na_{n+6} \\ \ell na_{n+1} & \ell na_{n+4} & \ell na_{n+7} \\ \ell na_{n+2} & \ell na_{n+5} & \ell na_{n+8} \end{vmatrix} = 0.$

SOLUTION

$a_1, a_2, a_3, \ldots, a_n, a_{n+1} \ldots$ are in G.P.

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_{n+1}}{a_n} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+3}}{a_{n+2}} = \ldots = \frac{a_{n+8}}{a_{n+7}} = r$$

$$\begin{vmatrix} \ell na_n & \ell na_{n+3} & \ell na_{n+6} \\ \ell na_{n+1} & \ell na_{n+4} & \ell na_{n+7} \\ \ell na_{n+2} & \ell na_{n+5} & \ell na_{n+8} \end{vmatrix}$$

Using $R_1 \to R_1 - R_2$; $R_3 \to R_3 - R_2$]

$$= \begin{vmatrix} \ell na_n - \ell na_{n+1} & \ell na_{n+3} - \ell na_{n+4} & \ell na_{n+6} - \ell na_{n+7} \\ \ell na_{n+1} & \ell na_{n+4} & \ell na_{n+7} \\ \ell na_{n+2} - \ell na_{n+1} & \ell na_{n+5} - \ell na_{n+4} & \ell na_{n+8} - \ell na_{n+7} \end{vmatrix}$$

$$\left(\because \ell n\, x - \ell n\, y = \ell n\left({}^{x}\!/\!_{y}\right)\right)$$

$$= \begin{vmatrix} \ell n\dfrac{a_n}{a_{n+1}} & \ell n\dfrac{a_{n+3}}{a_{n+4}} & \ell n\dfrac{a_{n+6}}{a_{n+7}} \\ \ell na_{n+1} & \ell na_{n+4} & \ell na_{n+7} \\ \ell n\dfrac{a_{n+2}}{a_{n+1}} & \ell n\dfrac{a_{n+5}}{a_{n+4}} & \ell n\dfrac{a_{n+8}}{a_{n+7}} \end{vmatrix}$$

$$= \begin{vmatrix} \ell n\dfrac{1}{r} & \ell n\dfrac{1}{r} & \ell n\dfrac{1}{r} \\ \ell na_{n+1} & \ell na_{n+4} & \ell na_{n+7} \\ \ell nr & \ell nr & \ell nr \end{vmatrix}$$

[Row R_1 and R_3 are proportional]

$= 0$

ILLUSTRATION 26

If x, y, z are distinct and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then find xyz.

SOLUTION

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

using $C_1 \leftrightarrow C_3$; $C_2 \leftrightarrow C_3$ in first determinant

$$\Rightarrow \quad -(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$\Rightarrow \quad (1+xyz)\,(x-y)\,(y-z)\,(z-x) = 0$ given $x \neq y \neq z$

$\Rightarrow \quad 1 + xyz = 0 \quad \Rightarrow \quad xyz = -1$

ILLUSTRATION 27

Evaluate $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

SOLUTION

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

using $R_1 \to aR_1$; $R_2 \to bR_2$; $R_3 \to cR_3$

$$= \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & cab & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{(abc)^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix} \qquad \text{using } C_3 \to C_3 + C_1$$

$$= abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$

(column C_2 and C_3 are proportional)

$= 0$

ILLUSTRATION 28

Evaluate the determinant $\Delta = \begin{vmatrix} \sqrt{p}+\sqrt{q} & 2\sqrt{r} & \sqrt{r} \\ \sqrt{qr}+\sqrt{2p} & r & \sqrt{2r} \\ q+\sqrt{pr} & \sqrt{qr} & r \end{vmatrix}$

where p, q and r are positive real numbers.

SOLUTION

Taking $\sqrt{r}$ common from C_2 and C_3 we get

$$\Delta = r\begin{vmatrix} \sqrt{p}+\sqrt{q} & 2 & 1 \\ \sqrt{qr}+\sqrt{2p} & \sqrt{r} & \sqrt{2} \\ q+\sqrt{pr} & \sqrt{q} & \sqrt{r} \end{vmatrix}$$

Applying $C_1 \to C_1 - \sqrt{q}\,C_2 - \sqrt{p}\,C_3$ we get

$$\Delta = r\begin{vmatrix} -\sqrt{q} & 2 & 1 \\ 0 & \sqrt{r} & \sqrt{2} \\ 0 & \sqrt{q} & \sqrt{r} \end{vmatrix} = -r\sqrt{q}\,(r-\sqrt{2q})$$

$$= r(\sqrt{2}\,q - r\sqrt{q})$$

ILLUSTRATION 29

If ω is an imaginary cube root of unity, then

find the value of $\begin{vmatrix} 1+\omega & w^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$

SOLUTION

Since $1+\omega+\omega^2 = 0$. Therefore $\begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2$

$$= \begin{vmatrix} 1+\omega+\omega^2 & \omega^2 & -\omega \\ 1+\omega^2+\omega & \omega & -\omega^2 \\ \omega^2+2\omega & \omega & -\omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \omega^2 & -\omega \\ 0 & \omega & -\omega^2 \\ \omega-1 & \omega & -\omega^2 \end{vmatrix} = (\omega-1)\begin{vmatrix} \omega^2 & -\omega \\ \omega & -\omega^2 \end{vmatrix}$$

$$= (\omega-1)(-\omega^4+\omega^2) = (\omega-1)(-\omega+\omega^2) \qquad [\because \omega^3 = 1]$$

$$= -\omega^2+\omega^3+\omega-\omega^2 = -\omega^2+(1+\omega)-\omega^2 = -3\omega^2$$

ILLUSTRATION 30

Three-digit numbers $x17$, $3y6$ and $12z$ where x, y, z are integers from 0 to 9, are divisible by a fixed constant k.

Then show that the determinant $\begin{vmatrix} x & 3 & 1 \\ 7 & 6 & z \\ 1 & y & 2 \end{vmatrix}$ is divisible by k.

SOLUTION

$$x17 = x.100 + 10 + 7 = \alpha k$$

$$3y6 = 300 + y.10 + 6 = \beta k$$

$$12z = 100 + 20 + z = \gamma k$$

Given det. $= \begin{vmatrix} x & 3 & 1 \\ 7 & 6 & z \\ 1 & y & 2 \end{vmatrix}$ using $R_2 \to R_2 + 10R_3 + 100R_1$

$$= \begin{vmatrix} x & 3 & 1 \\ x17 & 3y6 & 12z \\ 1 & y & 2 \end{vmatrix} = \begin{vmatrix} x & 3 & 1 \\ \alpha k & \beta k & \gamma k \\ 1 & y & 2 \end{vmatrix} = k\begin{vmatrix} x & 3 & 1 \\ \alpha & \beta & \gamma \\ 1 & y & 2 \end{vmatrix}$$

$\therefore$ Given determinant is divisible by k

ILLUSTRATION 31

Let $N = \begin{vmatrix} 28 & 25 & 38 \\ 42 & 38 & 65 \\ 56 & 47 & 83 \end{vmatrix}$, then find the number of ways in which N can be resolved as a product of two divisors which are relatively prime.

SOLUTION

$$N = 14\begin{vmatrix} 2 & 25 & 38 \\ 3 & 38 & 65 \\ 4 & 47 & 83 \end{vmatrix}$$

using $C_2 \to C_2 - 12C_1$; $C_3 \to C_3 - 19C_1$

$$= 14\begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 8 \\ 4 & -1 & 7 \end{vmatrix}$$

$$= 14(28 + 32 + 16 - 21)$$

$$= 14(55)$$

$$= 2 \times 7 \times 5 \times 11$$

Number of prime numbers present in $N = n = 4$

N can be resolved as a product of two divisors which are relatively prime $= 2^{n-1} = 8$

ILLUSTRATION 32

If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then find $f(2x) - f(x)$.

SOLUTION

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$$

Applying $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - xR_2$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a\,(a+x)^2$$

Now $f(2x) = a(a+2x)^2$

$\therefore \quad f(2x) - f(x) = a\,(a+2x)^2 - a\,(a+x)^2 = ax\,(2a+3x).$

ILLUSTRATION 33

Let $f(x) = \begin{vmatrix} x^2 & kx & 4+kx \\ kx & 4+kx & x^2 \\ 4+kx & x^2 & kx \end{vmatrix}$ If $f(x)$ is positive for all $x \in R$, then find the number of integral values in the range of k.

SOLUTION

$$f(x) = \begin{vmatrix} x^2 & kx & 4+kx \\ kx & 4+kx & x^2 \\ 4+kx & x^2 & kx \end{vmatrix} \quad \text{using } R_1 \to R_1 + R_2 + R_3$$

$$= (x^2 + 2kx + 4)\begin{vmatrix} 1 & 1 & 1 \\ kx & 4+kx & x^2 \\ 4+kx & x^2 & kx \end{vmatrix}$$

(expand along R_1)

$$= (x^2 + 2kx + 4)[4kx + k^2x^2 - x^4 - (k^2x^2 - 4x^2 - kx^3) + kx^3 - 16 - k^2x^2 - 8kx]$$

$$= (x^2 + 2kx + 4)(-x^4 + 2kx^3 + (4 - k^2)\,x^2 - 4kx - 16)$$

$$= -(x^2 + 2k + 4)\,[(x^2 - kx - 2)^2 + 12] > 0$$

for all real x

$(x^2 - kx - 2)^2 + 12$ is always positive

then $x^2 + 2kx + 4 < 0$ but it is not possible for all real x, hence $k \in \phi$

ILLUSTRATION 34

If α, β and γ are the roots of the equation $x^3 + px + q = 0$ (with $p \neq 0$ and $q \neq 0$), the find the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$

SOLUTION

Since α, β, γ are the roots of the given equation, therefore $\alpha + \beta + \gamma = 0$.

Now,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix}$$

[Using $C_1 \to C_1 + C_2 + C_3$]

$$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} \qquad [\because \alpha + \beta + \gamma = 0]$$

$$= 0$$

ILLUSTRATION 35

Find the value of the determinant $\begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$

SOLUTION

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have Given determinant

$$= \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 - R_2 \text{ and } R_3 \to R_3 - R_2]$$

$$= \begin{vmatrix} x & x & x+1 \\ 2 & 3 & 4 \\ 2 & 2 & 2 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_2]$$

$$= \begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} \, [\text{Expanding along } R_3]$$

$$= -2$$

ILLUSTRATION 36

If x, y, z are in A.P., then find the value of the determinant

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$

SOLUTION

Since x, y, z are in A.P. Therefore, $x + z - 2y = 0$.

Now, $\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$ Applying $R_1 \to R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 0 & 0 & 2(x+z-2y) \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix} \qquad [\because x+z-2y=0]$$

$$= 0$$

ILLUSTRATION 37

Evaluate the determinant $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$.

SOLUTION

We have,

$$\Delta = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} \qquad [\text{using } C_4 \to C_4 - C_3, \; C_3 \to C_3 - C_2, C_2 \to C_2 - C_1]$$

$$= \begin{vmatrix} 1 & 3 & 5 & 7 \\ 4 & 5 & 7 & 9 \\ 9 & 7 & 9 & 11 \\ 16 & 9 & 11 & 13 \end{vmatrix} \qquad \text{Applying } C_4 \to C_4 - C_3, \; C_3 \to C_3 - C_2$$

$$= \begin{vmatrix} 1 & 3 & 2 & 2 \\ 4 & 5 & 2 & 2 \\ 9 & 7 & 2 & 2 \\ 16 & 9 & 2 & 2 \end{vmatrix} \qquad \text{Column } C_3 \text{ ad } C_4 \text{ are identical}$$

$$= 0$$

ILLUSTRATION 38

Show that the roots of the equation $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ are independent of m and n.

SOLUTION

Using $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, R_3 \to R_3 - R_4$

$$\Delta = \begin{vmatrix} x-a & m-x & 0 & 0 \\ 0 & x-b & n-x & 0 \\ 0 & 0 & x-c & 0 \\ a & b & c & 1 \end{vmatrix},$$

Expand along column 4

$$= \begin{vmatrix} x-a & m-x & 0 \\ 0 & x-b & n-x \\ 0 & 0 & x-c \end{vmatrix} = (x-a)(x-b)(x-c)$$

$\therefore \quad \Delta = 0 \Rightarrow (x-a)(x-b)(x-c) = 0$

whose roots are $x = a, b, c$ and they are independent of m, n.

ILLUSTRATION 39

Show that determinant $\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$ is independent of α.

SOLUTION

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 (\cos\beta) + R_2 (\sin\beta)$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix} \quad (\text{Expand along } R_3)$$

$$= (1 + \sin\beta - \cos\beta)(\cos^2\alpha + \sin^2\alpha)$$

$= 1 + \sin\beta - \cos\beta$, which is independent of α

ILLUSTRATION 40

If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$ then find the value of $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c}$

SOLUTION

$$\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} \text{ using } R_3 \to R_3 - R_2; R_2 \to R_2 - R_1$$

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ 0 & b-q & r-c \end{vmatrix},$$

taking $p - a$, $q - b$ and $r - c$ common from column C_1, C_2, C_3 respectively

$$= (p-a)(q-b)(r-c)\begin{vmatrix} \dfrac{p}{p-a} & \dfrac{b}{q-b} & \dfrac{c}{r-c} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

using $C_1 \to C_1 + C_2 + C_3$

$$= (p-a)(q-b)(r-c)\times\begin{vmatrix} \dfrac{p}{p-a}+\dfrac{b}{q-b}+\dfrac{c}{r-c} & \dfrac{b}{q-b} & \dfrac{c}{r-c} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

(Expand along C_1)

$$\therefore \quad \Delta = (p-a)(q-b)(r-c)\left(\frac{p}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}\right)=0$$

(given)

But $a \neq p, b \neq q, c \neq r$. So $\dfrac{p}{p-a}+\dfrac{b}{q-b}+\dfrac{c}{r-c}=0$

$$\therefore \quad \frac{p}{p-a}+\left(\frac{b}{q-b}+1\right)+\left(\frac{c}{r-c}+1\right)=1+1$$

or $$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

ILLUSTRATION 41

Prove that

$$\begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix} = a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right).$$

SOLUTION

Let $\Delta = \begin{vmatrix} 1+a_1 & 1 & 1 \\ 1 & 1+a_2 & 1 \\ 1 & 1 & 1+a_3 \end{vmatrix}$

Now $\Delta = a_1\, a_2\, a_3 \begin{vmatrix} \dfrac{1}{a_1}+1 & \dfrac{1}{a_2} & \dfrac{1}{a_3} \\ \dfrac{1}{a_1} & \dfrac{1}{a_2}+1 & \dfrac{1}{a_3} \\ \dfrac{1}{a_1} & \dfrac{1}{a_2} & \dfrac{1}{a_3}+1 \end{vmatrix}$

Using $[C_1 \to C_1 + C_2 + C_3]$

$$= a_1a_2a_3\begin{vmatrix} 1+\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3} & \dfrac{1}{a_2} & \dfrac{1}{a_3} \\ 1+\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3} & \dfrac{1}{a_2}+1 & \dfrac{1}{a_3} \\ 1+\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3} & \dfrac{1}{a_2} & \dfrac{1}{a_3}+1 \end{vmatrix}$$

[taking $\left(1+\dfrac{1}{a_1}+\dfrac{1}{a_2}+\dfrac{1}{a_3}\right)$ common from first column]

$$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)\begin{vmatrix} 1 & \dfrac{1}{a_2} & \dfrac{1}{a_3} \\ 1 & \dfrac{1}{a_2}+1 & \dfrac{1}{a_3} \\ 1 & \dfrac{1}{a_2} & \dfrac{1}{a_3}+1 \end{vmatrix}$$

$[R_2 \to R_2 - R_1\,, R_3 \to R_3 - R_1]$

$$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)\begin{vmatrix} 1 & \dfrac{1}{a_2} & \dfrac{1}{a_3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[expanding along the first column]

$$= a_1a_2a_3\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)$$

ILLUSTRATION 42

X, Y and Z are positive numbers greater than 10 such that Y and Z have respectively 1 and 0 in their unit places and Δ is the value of determinant $\begin{vmatrix} X & 5 & 1 \\ Y & 0 & 2 \\ Z & 1 & 0 \end{vmatrix}$. If $(\Delta + 1)$ is divisible by 10, then find unit place in X.

SOLUTION

Let $X = 10x + t$, $Y = 10y + 1$, $Z = 10z$, where x, y, z are positive integers and t is a non-negative integer.

Then $\Delta = \begin{vmatrix} 10x+t & 5 & 1 \\ 10y+1 & 0 & 2 \\ 10z & 1 & 0 \end{vmatrix} = \begin{vmatrix} 10x & 5 & 1 \\ 10y & 0 & 2 \\ 10z & 1 & 0 \end{vmatrix} + \begin{vmatrix} t & 5 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}$

$\Delta = 10k + (1 - 2t)$ where $k = \begin{vmatrix} x & 5 & 1 \\ y & 0 & 2 \\ z & 1 & 0 \end{vmatrix}$

or $\Delta + 1 = 10k + (2 - 2t)$

as given $\Delta + 1$ is divisible by 10 then $2 - 2t = 0 \Rightarrow t = 1$

ILLUSTRATION 43

If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$,

find the degree of polynomial $f(x)$.

SOLUTION

$$f(x)=\begin{vmatrix}1+a^2x & (1+b^2)x & (1+c^2)x\\ (1+a^2)x & 1+b^2x & (1+c^2)x\\ (1+a^2)x & (1+b^2)x & 1+c^2x\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$f(x)=\begin{vmatrix}1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x\\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x\\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+c^2x\end{vmatrix}$$

$(\because a^2+b^2+c^2+2=0)$

$$f(x)=\begin{vmatrix}1 & (1+b^2)x & (1+c^2)x\\ 1 & 1+b^2x & (1+c^2)x\\ 1 & (1+b^2)x & 1+c^2x\end{vmatrix}$$

Now, applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$f(x)=\begin{vmatrix}1 & (1+b^2)x & (1+c^2)x\\ 0 & 1-x & 0\\ 0 & 0 & 1-x\end{vmatrix}=(1-x)^2$$

Hence, degree of $f(x) = 2$

ILLUSTRATION 44

If p, q, r, s are in A.P. and

$$f(x)=\begin{vmatrix}p+\sin x & q+\sin x & p-r+\sin x\\ q+\sin x & r+\sin x & -1+\sin x\\ r+\sin x & s+\sin x & s-q+\sin x\end{vmatrix}$$ such that

$\int_0^2 f(x)\,dx = -4$ then the find common difference of the A.P.

SOLUTION

p, q, r, s are in A.P. and let common difference (α)

$$q-p=r-q=s-r=\alpha$$

Using $C_2 \to C_2 - C_1\,;\, C_3 \to C_3 - C_1$

$$f(x)=\begin{vmatrix}p+\sin x & q-p & -r\\ q+\sin x & r-q & -1-q\\ r+\sin x & s-r & s-q-r\end{vmatrix}$$

$$f(x)=\begin{vmatrix}p+\sin x & \alpha & -r\\ q+\sin x & \alpha & -1-q\\ r+\sin x & \alpha & s-q-r\end{vmatrix}$$

using $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$f(x)=\begin{vmatrix}p+\sin x & \alpha & -r\\ \alpha & 0 & -1+\alpha\\ 2\alpha & 0 & 2\alpha\end{vmatrix}$$

$$= -\alpha(2\alpha^2 - 2\alpha(-1+\alpha))$$

$$= -\alpha(2\alpha^2 + 2\alpha - 2\alpha^2) = -2\alpha^2$$

$$\int_0^2 f(x)\,dx = -4 \quad\Rightarrow\quad -2\alpha^2\cdot 2 = -4$$

$$\Rightarrow\quad \alpha^2 = 1 \quad\Rightarrow\quad \alpha = \pm 1$$

$\therefore$ Required common difference $= \pm 1$

ILLUSTRATION 45

Evaluate $\begin{vmatrix}{}^xC_1 & {}^xC_2 & {}^xC_3\\ {}^yC_1 & {}^yC_2 & {}^yC_3\\ {}^zC_1 & {}^zC_2 & {}^zC_3\end{vmatrix}$

SOLUTION

$$\text{Det.} = \begin{vmatrix}{}^xC_1 & {}^xC_2 & {}^xC_3\\ {}^yC_1 & {}^yC_2 & {}^yC_3\\ {}^zC_1 & {}^zC_2 & {}^zC_3\end{vmatrix}$$

$$=\begin{vmatrix}x & \dfrac{x(x-1)}{2} & \dfrac{x(x-1)(x-2)}{6}\\ y & \dfrac{y(y-1)}{2} & \dfrac{y(y-1)(y-2)}{6}\\ z & \dfrac{z(z-1)}{2} & \dfrac{z(z-1)(z-2)}{6}\end{vmatrix}$$

(by $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_1 + 3C_2$)

$$=\frac{xyz}{12}\begin{vmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{vmatrix}$$

$= \dfrac{xyz}{12}(x-y)\,(y-z)\,(z-x)$ using standardt result

ILLUSTRATION 46

If $x_1 \neq 0; r = 1,2,3$ then prove that

$$\begin{vmatrix}x_1+a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & x_2+a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & x_3+a_3b_3\end{vmatrix}$$

$$= x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{x_3}\right)$$

SOLUTION

$$\Delta = \frac{1}{b_1^2}\begin{vmatrix}x_1+a_1b_1 & a_1b_1b_2 & a_1b_1b_3\\ a_2b_1 & b_1x_2+a_2b_1b_2 & a_2b_1b_3\\ a_3b_1 & a_3b_1b_2 & b_1x_3+a_3b_1b_3\end{vmatrix}$$

$$=\frac{1}{b_1^2}\begin{vmatrix} x_1+a_1b_1 & -b_2x_1 & -b_3x_1 \\ a_2b_1 & b_1x_2 & 0 \\ a_3b_1 & 0 & b_1x_3 \end{vmatrix},$$

$C_2 \to C_2-b_2C_1; C_3 \to C_3-b_3C_1$

$$=\frac{x_1x_2x_3}{b_1^2}\begin{vmatrix} 1+\frac{a_1b_1}{x_1} & -b_2 & -b_3 \\ \frac{a_2b_1}{x_2} & b_1 & 0 \\ \frac{a_3b_1}{x_3} & 0 & b_1 \end{vmatrix},$$

taking common x_1, x_2, x_3 from C_1, C_2, C_3 respectively and dividing R_2 and R_3 by b_1

$$=x_1x_2x_3\begin{vmatrix} 1+\frac{a_1b_1}{x_1} & -b_2 & -b_3 \\ \frac{a_2}{x_2} & 1 & 0 \\ \frac{a_3}{x_3} & 0 & 1 \end{vmatrix},$$

Now $R_1 \to R_1+b_2R_2+b_3R_3$

$$=x_1x_2x_3\begin{vmatrix} 1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{x_3} & 0 & 0 \\ \frac{a_2}{x_2} & 1 & 0 \\ \frac{a_3}{x_3} & 0 & 1 \end{vmatrix},$$

expand it along C_3

$$=x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{x_3}\right)$$

Hence proved.

ILLUSTRATION 47

Prove that

$$\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} =(x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

SOLUTION

$$\Delta=\frac{1}{a}\begin{vmatrix} a^2x-aby-caz & ay+bx & cx+az \\ a^2y+abx & by-cz-ax & bz+cy \\ acx+a^2z & bz+cy & cz-ax-by \end{vmatrix}$$

$C_1 \to C_1+bC_2+cC_3$

$$=\frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+az \\ \left(a^2+b^2+c^2\right)y & by-cz-ax & bz+cy \\ \left(a^2+b^2+c^2\right)z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=(a^2+b^2+c^2)\frac{1}{a}\begin{vmatrix} x & ay+bx & cx+az \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=(a^2+b^2+c^2)\frac{1}{ax}\begin{vmatrix} x^2 & axy+bx^2 & cx^2+azx \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$R_1 \to R_1+yR_2+zR_3$

$$=(a^2+b^2+c^2)\frac{1}{ax}\begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=(a^2+b^2+c^2)(x^2+y^2+z^2)\frac{1}{ax}\begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}, \quad R_2 \to R_2-yR_1,\ R_3 \to R_3-zR_1$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\begin{vmatrix} -cz-ax & bz \\ cy & -ax-by \end{vmatrix}$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\left\{acxz+bcyz+a^2x^2+abxy-bcyz\right\}$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}\left\{acxz+a^2x^2+abxy\right\}$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax}ax(cz+ax+by)$$

$$=(x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

MULTIPLICATION OF DETERMINANTS

(i) The two determinants to be multiplied must be of the same order.

(ii) As determinants have fixed values so we can use any of following methods for multiplication in determinant form

Row by Row multiply Method (Preferred)
Row by column multiply Method
Column by Row multiply Method
Column by Column multiply Method

(a) Multiplication of two determinant of order 2×2

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & \ell_2 \\ m_1 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

(b) Multiplication of two determinant of order 3×3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 + c_1\ell_3 & a_1m_1 + b_1m_2 + c_1m_3 & a_1n_1 + b_1n_2 + c_1n_3 \\ a_2\ell_1 + b_2\ell_2 + c_2\ell_3 & a_2m_1 + b_2m_2 + c_2m_3 & a_2n_1 + b_2n_2 + c_2n_3 \\ a_3\ell_1 + b_3\ell_2 + c_3\ell_3 & a_3m_1 + b_3m_2 + c_3m_3 & a_3n_1 + b_3n_2 + c_3n_3 \end{vmatrix}$$

(iii) If order of determinant Δ is $n \times n$ then value of determinant formed by cofactors $= \Delta^{n-1}$

Proof:

Let consider determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and determinant formed by cofactors

$$C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \begin{vmatrix} a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} & a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} & a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} \\ a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} & a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} & a_{21}C_{31} + a_{22}C_{32} + a_{23}C_{33} \\ a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13} & a_{31}C_{21} + a_{32}C_{22} + a_{33}C_{23} & a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \end{vmatrix}$$

$$\Rightarrow \quad \Delta C = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$$\Rightarrow \quad C = \Delta^2$$

Similarly for determinant of order $n \times n$ then value of determinant formed by cofactors $= \Delta^{n-1}$

ILLUSTRATION 48

If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 2$, find $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$

SOLUTION

If we closely observe that we conclude that given determinant $\begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$ is cofactor determinant of first determinant Δ of order 3×3

Hence $C = \Delta^{n-1} = \Delta^2 = 4$

ILLUSTRATION 49

Let $\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \Delta_1 \neq 0; \Delta_2 = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}$ where b_{ij} is cofactor of $a_{ij}\ \forall\ i, j = 1, 2, 3$

and $\Delta_3 = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$ where c_{ij} is cofactor of $b_{ij}\ \forall\ i, j = 1, 2, 3$.then show that $\Delta_1^2 = \dfrac{\Delta_3}{\Delta_2}$.

SOLUTION

Determinant formed by cofactors of determinant (Δ) of order $n \times n = \Delta^{n-1}$

In given question $\Delta_2 = \Delta_1^2$

and $\Delta_3 = \Delta_2^2 = \Delta_1^4 = \Delta_1^2 \cdot \Delta_1^2 = \Delta_2 \cdot \Delta_1^2$

$$\Delta_1^2 = \frac{\Delta_3}{\Delta_2} \text{ hence proved}$$

ILLUSTRATION 50

Let $\Delta_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and let Δ_1 denote the determinant formed by the cofactors of elements of Δ_0 and Δ_2 denote the determinant formed by the cofactors of Δ_1 similarly Δ_n denotes then the determinant formed by the cofactors of Δ_{n-1} then find the determinant value of Δ_n.

SOLUTION

Determinant formed by cofactors of determinant (Δ) of order $n \times n = \Delta^{n-1}$
according to given question $\Delta_1 = \Delta_0^2$,

$$\Delta_2 = \Delta_1^2 = \Delta_0^4 = \Delta_0^{2^2}$$

$$\Delta_3 = \Delta_2^2 = \Delta_0^8 = \Delta_0^{2^3}$$

Similarly we can say $\Delta_n = \Delta_0^{2^n}$

ILLUSTRATION 51

Find the value of

$$\begin{vmatrix} a_1\ell_1 + b_1m_1 & a_1\ell_2 + b_1m_2 & a_1\ell_3 + b_1m_3 \\ a_2\ell_1 + b_2m_1 & a_2\ell_2 + b_2m_2 & a_2\ell_3 + b_2m_3 \\ a_3\ell_1 + b_3m_1 & a_3\ell_2 + b_3m_2 & a_3\ell_3 + b_3m_3 \end{vmatrix}$$

SOLUTION

$$\begin{vmatrix} a_1\ell_1 + b_1m_1 & a_1\ell_2 + b_1m_2 & a_1\ell_3 + b_1m_3 \\ a_2\ell_1 + b_2m_1 & a_2\ell_2 + b_2m_2 & a_2\ell_3 + b_2m_3 \\ a_3\ell_1 + b_3m_1 & a_3\ell_2 + b_3m_2 & a_3\ell_3 + b_3m_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} \ell_1 & m_1 & 0 \\ \ell_2 & m_2 & 0 \\ \ell_3 & m_3 & 0 \end{vmatrix} = 0$$

ILLUSTRATION 52

Evaluate

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

SOLUTION

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

We can closely observe the product of two determinant in row by row multiplication

$$= \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} = 0$$

or

$$= \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 & a \\ \alpha+\beta & \gamma+\delta & b \\ \alpha\beta & \gamma\delta & c \end{vmatrix} = 0$$

In both cases product is zero.

ILLUSTRATION 53

Find the value of

$$\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \cos(\alpha_1-\beta_3) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \cos(\alpha_2-\beta_3) \\ \cos(\alpha_3-\beta_1) & \cos(\alpha_3-\beta_2) & \cos(\alpha_3-\beta_3) \end{vmatrix}$$

SOLUTION

We have $\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \cos(\alpha_1-\beta_3) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \cos(\alpha_2-\beta_3) \\ \cos(\alpha_3-\beta_1) & \cos(\alpha_3-\beta_2) & \cos(\alpha_3-\beta_3) \end{vmatrix}$

$$= \begin{vmatrix} \cos\alpha_1\cos\beta_1 + \sin\alpha_1\sin\beta_1 & \cos\alpha_1\cos\beta_2 + \sin\alpha_1\sin\beta_2 & \cos\alpha_1\cos\beta_3 + \sin\alpha_1\sin\beta_3 \\ \cos\alpha_2\cos\beta_1 + \sin\alpha_2\sin\beta_1 & \cos\alpha_2\cos\beta_2 + \sin\alpha_2\sin\beta_2 & \cos\alpha_2\cos\beta_3 + \sin\alpha_2\sin\beta_3 \\ \cos\alpha_3\cos\beta_1 + \sin\alpha_3\sin\beta_1 & \cos\alpha_3\cos\beta_2 + \sin\alpha_3\sin\beta_2 & \cos\alpha_3\cos\beta_3 + \sin\alpha_3\sin\beta_3 \end{vmatrix}$$

We can closely observe the product of two determinant in row by row multiplication

$$\Delta = \begin{vmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ \cos\alpha_2 & \sin\alpha_2 & 0 \\ \cos\alpha_3 & \sin\alpha_3 & 0 \end{vmatrix}\begin{vmatrix} \cos\beta_1 & \sin\beta_1 & 0 \\ \cos\beta_2 & \sin\beta_2 & 0 \\ \cos\beta_3 & \sin\beta_3 & 0 \end{vmatrix} = 0\times 0 = 0$$

ILLUSTRATION 54

Prove that in a $\Delta ABC, \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$

SOLUTION

For ΔABC, we have $A+B+C=\pi$

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = \begin{vmatrix} \sin 2A & \sin(\pi-(A+B)) & \sin(\pi-(A+C)) \\ \sin(\pi-(A+B)) & \sin 2B & \sin(\pi-(C+B)) \\ \sin(\pi-(A+C)) & \sin(\pi-(C+B)) & \sin 2C \end{vmatrix}$$

$$= \begin{vmatrix} \sin 2A & \sin(A+B) & \sin(C+A)) \\ \sin(A+B) & \sin 2B & \sin(B+C) \\ \sin(A+C) & \sin(B+C) & \sin 2C \end{vmatrix}$$

$$= \begin{vmatrix} 2\sin A\cos A & \sin A\cos B+\cos A\sin B & \sin A\cos C+\cos A\sin C \\ \sin A\cos B+\cos A\sin B & 2\sin B\cos B & \sin C\cos B+\cos C\sin B \\ \sin A\cos C+\cos A\sin C & \sin C\cos B+\cos C\sin B & 2\sin C\cos C \end{vmatrix}$$

$$= \begin{vmatrix} \sin A & \cos A & 0 \\ \sin B & \cos B & 0 \\ \sin C & \cos C & 0 \end{vmatrix}\begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} = 0$$

DIFFERENTIATION OF A DETERMINANT

If $f(x) = \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}$ where $f_1, f_2, f_3\,;\, g_1, g_2, g_3\,;\, h_1, h_2, h_3$ are functions of x then.

Differentiation of $f(x)$ is same as sum of determinants with differentiation of one row at a time while keeping other rows constant (Same thing can be done with columns also)

$$f'(x) = \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1' & h_2' & h_3' \end{vmatrix}$$

or

$$f'(x) = \begin{vmatrix} f_1' & f_2 & f_3 \\ g_1' & g_2 & g_3 \\ h_1' & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2' & f_3 \\ g_1 & g_2' & g_3 \\ h_1 & h_2' & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3' \\ g_1 & g_2 & g_3' \\ h_1 & h_2 & h_3' \end{vmatrix}$$

ILLUSTRATION 55

$\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix} = ax^5+bx^4+cx^3+dx^2+ex+f$. Find e, f.

SOLUTION

As given relation is an identity so it is true for $\forall\, x \in R$

Put $x=0$

We get $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0+f \quad\Rightarrow\quad f=1$

Differentiate it along column

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & x^2 \\ 2x & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & 1 & x^2 \\ x & 1 & x^2 \\ x^2 & 1 & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & 2x \\ x & 1+x & 2x \\ x^2 & x & 1 \end{vmatrix}$$

$$= 5ax^4+4bx^3+3cx^2+2dx+e$$

Put $x = 0$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 + e$$

$$1 + 1 + 1 = e$$

We get $e = 3, f = 1$

ILLUSTRATION 56

Let $f(x) = \begin{vmatrix} 1 & 1+\sin x & 1+\sin x+\cos x \\ 2 & 3+2\sin x & 4+3\sin x+2\cos x \\ 3 & 6+3\sin x & 10+6\sin x+3\cos x \end{vmatrix}$

then find $f'\left(\dfrac{\pi}{2}\right)$

SOLUTION

$$f(x) = \begin{vmatrix} 1 & 1+\sin x & 1+\sin x+\cos x \\ 2 & 3+2\sin x & 4+3\sin x+2\cos x \\ 3 & 6+3\sin x & 10+6\sin x+3\cos x \end{vmatrix}$$

using $R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$

$$f(x) = \begin{vmatrix} 1 & 1+\sin x & 1+\sin + \cos x \\ 0 & 1 & 2+\sin x \\ 0 & 3 & 7+3\sin x \end{vmatrix} \quad \text{Expand along } C_1$$

$$= 1(7 + 3\sin x - 6 - 3\sin x)$$

$$f(x) = 1$$

$$\Rightarrow \quad f'(x) = 0 \quad \Rightarrow \quad f'\left(\frac{\pi}{2}\right) = 0$$

ILLUSTRATION 57

Find the coefficient of x in the expansion of $\begin{vmatrix} (1+x)^{22} & (1+x)^{44} & (1+x)^{66} \\ (1+x)^{33} & (1+x)^{66} & (1+x)^{99} \\ (1+x)^{44} & (1+x)^{88} & (1+x)^{144} \end{vmatrix}$

SOLUTION

Let $f(x) = \begin{vmatrix} (1+x)^{22} & (1+x)^{44} & (1+x)^{66} \\ (1+x)^{33} & (1+x)^{66} & (1+x)^{99} \\ (1+x)^{44} & (1+x)^{88} & (1+x)^{144} \end{vmatrix}$

Since $f(x)$ is a polynomial.

$$\Rightarrow \quad f(x) = A_0 + A_1 x + A_2 x^2 + \ldots$$

Thus, coefficient of $x = f'(0)$

Now, differentiating both sides w.r.t. x and then put $x = 0$ in both sides.

$$f'(0) = \begin{vmatrix} 22 & 44 & 66 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 33 & 66 & 99 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 44 & 88 & 144 \end{vmatrix}$$

$$\Rightarrow \quad f'(0) = 0 + 0 + 0$$

$$\Rightarrow \quad f'(0) = 0$$

Hence, coefficient of x in the given determinant = 0.

SPECIAL DETERMINANT

(a) Symmetric Determinant

(i) If the elements of a determinant are such that

$a_{ij} = a_{ji}$ for all i,j

(ii) Determinant becomes symmetric relative to the diagonal.

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(b) Skew Symmetric Determinant

(i) If the elements of a determinant are such that

$a_{ij} = -\, a_{ji}$ for all i,j

(ii) All the elements of diagonal, i.e. $a_{ii} = 0$

For diagonal $i = j$

$a_{ii} = -a_{ji} \quad \Rightarrow \quad 2a_{ii} = 0$

i.e. $a_{ii} = 0 \quad \Rightarrow \quad a_{11} = a_{22} = a_{33} = a_{44} = 0$

(c) The value of an odd order skew symmetric determinant is always zero

e.g. $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$

(d) The value of an even order skew symmetric determinant is perfect square.

e.g. $\begin{vmatrix} 0 & -a \\ a & 0 \end{vmatrix} = a^2$ = perfect square

ILLUSTRATION 58

If determinant $\begin{vmatrix} 6 & \sin x & \ell n y \\ 1 & n & \{z\} \\ e & 0 & 7 \end{vmatrix}$ is symmetric determinant.

find x, y, z where $\{\cdot\}$ is fractional part function.

SOLUTION

For symmetric determinant

$$a_{ij} = a_{ji} \quad \text{for } \forall\, i, j$$

$$\sin x = 1 \quad \Rightarrow \quad x = 2n\pi + \frac{\pi}{2} \text{ where } n \in I$$

$$\ln y = e \quad \Rightarrow \quad y = e^e$$

$$\{z\} = 0 \quad \Rightarrow \quad z \in I$$

ILLUSTRATION 59

Δ is skew symmetric determinant of order 3×3,

$a_{12} = [x]$, $a_{21} = -1$, $a_{23} = e^y$; $a_{32} = 1$; $a_{13} = \sin 3z$; $a_{31} = \log_2 \sqrt{2}$ find x, y, z where $[\cdot]$ is GIF.

SOLUTION

$$\Delta = \begin{vmatrix} 0 & [x] & \sin 3z \\ -1 & 0 & e^y \\ \log_2 \sqrt{2} & 1 & 0 \end{vmatrix}$$

For skew symmetric determinant

$$a_{ij} = -a_{ji} \quad \text{for all } i, j$$

$$[x] = 1 \quad \Rightarrow \quad x \in [1, 2)$$

$$e^y = -1 \quad \Rightarrow \quad y \in \phi$$

$$\sin 3z = -\log_2 \sqrt{2} = -1/2$$

$$\Rightarrow \quad 3z = n\pi + (-1)^n \left(-\frac{\pi}{6}\right)$$

$$\Rightarrow \quad z = \frac{n\pi}{3} + (-1)^{n+1} \frac{\pi}{18} \text{ where } n \in I$$

Test Your Concepts-II

(Solutions on page H.282)

1. Simplify $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

2. Simplify $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$

3. Simplify $\begin{vmatrix} 1 & a^2 & b^2+c^2 \\ 1 & b^2 & c^2+a^2 \\ 1 & c^2 & a^2+b^2 \end{vmatrix}$

4. Simplify $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

5. Simplify $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$

6. If $\begin{vmatrix} p & q-y & r-z \\ p-x & q & r-z \\ p-x & q-y & r \end{vmatrix} = 0$, then

find the value of $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} + 2$.

7. If $\Delta = \begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$, then

find the value of $\frac{\Delta}{abc}$.

8. Prove that $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$

9. Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

10. Prove the following identity

$$\begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix} = 4\alpha^2\beta^2\gamma^2$$

11. Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

12. If $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$ find x.

13. $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$ find $\lim_{x \to 0} \frac{f(x)}{x}$.

14. Evaluate $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

15. For a fixed $n \in N$, if $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$ then show that $\left\{ \dfrac{D}{(n!)^3} - 4 \right\}$ is divisible by n.

Answer keys

1. $(x-y)(y-z)(z-x)$
2. $(x-y)(y-z)(z-x)(x+y+z)$
3. 0
4. 0
5. $(a+b+c)[a^2+b^2+c^2-ab-bc-ca)$
6. 4
7. 4
12. 4
13. 0
14. $5\sqrt{3}(\sqrt{6}-5)$

CRAMER RULE FOR SOLVING SYSTEM OF LINEAR EQUATION

System of Equation Involving Two Variables (*x*, *y*)

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

These two equations represent two straight lines and we are looking for their point of intersection as solution of these equations but cramer rule provides us a different approach using determinant to find out the solution of above equations.
Let understand few terms related to system of equations and their solution

Consistent System

If the solution set of system of equations exists then it is considered as consistent system

Inconsistent System

If system of equations does not possess any solution then it is considered as inconsistent system

Trivial Solution

In the solution set of system of equation if all the variables assume zero, then such solution set is called trivial solution otherwise the solution are called **non-trivial** solution.

$(x, y) = (0, 0)$ and $(x, y, z) = (0,0,0)$ are trivial solution

Non- zero solutions are non-trivial solution

Unique Solution

If the system of equation has exactly one solution set then it is unique solution

Now we can try them solving using determinant (Cramer rule)
For solving system of linear equations in two variables x and y

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (Determinant made by coefficients of equations)

$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ (Replace coefficient of x by constant terms in determinant Δ)

$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ (Replace coefficient of y by constant terms in determinant Δ)

$$x\Delta = x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$$

now using $C_1 \to C_1 + yC_2$

or $\quad x\Delta = \begin{vmatrix} a_1x+b_1y & b_1 \\ a_2x+b_2y & b_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

or $\quad x\Delta = \Delta_x$

similarly $y\Delta = y\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y \\ a_2 & b_2y \end{vmatrix}$

now using $C_2 \to C_2 + xC_1$

or $\quad y\Delta = \begin{vmatrix} a_1 & a_1x+b_1y \\ a_2 & a_2x+b_2y \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

or $\quad y\Delta = \Delta_y$

we have $x\Delta = \Delta_x$ and $y\Delta = \Delta_y$

then $x = \dfrac{\Delta_x}{\Delta}, y = \dfrac{\Delta_y}{\Delta}$ if $\Delta \neq 0$

or $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$; $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

For infinite solution: $\Delta = \Delta_x = \Delta_y = 0$
For no solution: $\Delta = 0$ and atleast one of them $\Delta_x, \Delta_y \neq 0$
For trivial solution: $\Delta \neq 0$ and $\Delta_x = \Delta_y = 0$
For non-trivial solution: $\Delta \neq 0$ and atleast one of them Δ_x, $\Delta_y \neq 0$
For unique solution: $\Delta \neq 0$
We can also understand different cases depending upon the values of Δ, Δ_x and Δ_y and their possible solutions

	$\Delta_x = \Delta_y = 0$	**Atleast one them** $\Delta_x, \Delta_y \neq 0$
$\Delta = 0$	Infinite Solution Consistent System	No Solution Inconsistent System
	Both Straight lines are coincident lines $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Both Straight lines are parallel $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
$\Delta \neq 0$	Trivial Solution Unique Solution Consistent System	Non-Trivial Solution Unique Solution Consistent System
	Both Straight lines are intersecting at origin $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Both Straight lines are intersecting at other than origin $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

ILLUSTRATION 60

Solve the following linear equations

$$\frac{4}{x+5} + \frac{3}{y+7} = -1 \text{ and } \frac{6}{x+5} - \frac{6}{y+7} = -5$$

SOLUTION

Let us put $\frac{1}{x+5} = a$ and $\frac{1}{y+7} = b$

the 2 linear equations become $4a + 3b = -1$ ...(1)

and $6a - 6b = -5$ ...(2)

Using Cramer's Rule, we get

$$\frac{a}{\Delta_a} = \frac{b}{\Delta_b} = \frac{1}{\Delta} \Rightarrow \frac{a}{\begin{vmatrix} -1 & 3 \\ -5 & -6 \end{vmatrix}} = \frac{b}{\begin{vmatrix} 4 & -1 \\ 6 & -5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 6 & -6 \end{vmatrix}}$$

$$\frac{a}{6+15} = \frac{b}{-20+6} = \frac{1}{-24-18}$$

$$\therefore \quad \frac{a}{21} = \frac{b}{-14} = \frac{1}{-42}$$

$$\Rightarrow \quad a = \frac{-21}{42} = \frac{-1}{2} = \frac{1}{x+5} \quad \Rightarrow \quad x = -7$$

$$\Rightarrow \quad b = \frac{14}{42} = \frac{1}{3} = \frac{1}{y+7} \quad \Rightarrow \quad y = -4$$

CRAMER RULE FOR SOLVING LINEAR-EQUATIONS IN THREE VARIABLES

The solution of the system of linear equations.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

all these represents equation of planes and we are looking for point of intersections of these planes.

For using Crammer rule we need four determinants Δ is the determinant of the coefficient determinant. The determinant Δ_x is obtained by replacing the elements in first column of Δ by constant term d_1, d_2, d_3. D_y is obtained by replacing the elements in the second column of Δ by constant terms d_1, d_2, d_3 and to obtain D_z, replace elements in the third column of Δ by constant terms d_1, d_2, d_3

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

We have

$$x\Delta = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

[Applying $C_1 \to C_1 + yC_2 + zC_3$]

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_x$$

$$x\Delta = \Delta_x$$

Similarly

$$y\Delta = \Delta_y \text{ and } z\Delta = \Delta_z.$$

$$x\Delta = \Delta_x, y\Delta = \Delta_y, z\Delta = \Delta_z$$

$$\therefore \quad x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta} \text{ provided that } \Delta \neq 0.$$

	$\Delta_x = \Delta_y = \Delta_z = 0$	**Atleast one them** $\Delta_x, \Delta_y, \Delta_z \neq 0$
$\Delta = 0$	Case I Infinite Solution Consistent System Case II No Solution Inconsistent System	No Solution Inconsistent System
$\Delta \neq 0$	Trivial Solution Unique Solution Consistent System	Non-Trivial Solution Unique Solution Consistent System

Let discuss the first case where $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, then system of equation may or may not be consistent:

(a) Assume any one variable x, y or z as t and get other two variable in terms of t from first two equation, If the values of x, y, z in terms of t satisfy third equation then system is said to be consistent and will have infinite solutions

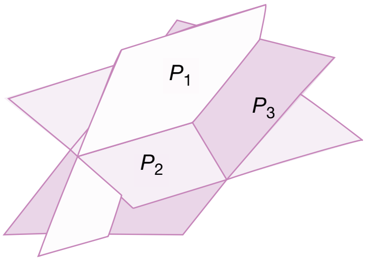
(b) If the values of x, y, z in terms of t doesn't satisfy third equation, then system is said to be inconsistent and will have no solution

Analyze Solutions Relative to Positions of Planes

By analyzing the position of planes in three-dimensional geometry

Plane $P_1 : a_1x + b_1y + c_1z = d_1$
Plane $P_2 : a_2x + b_2y + c_2z = d_2$
Plane $P_3 : a_3x + b_3y + c_3z = d_3$

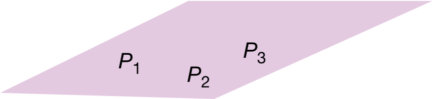
(a) Unique solution



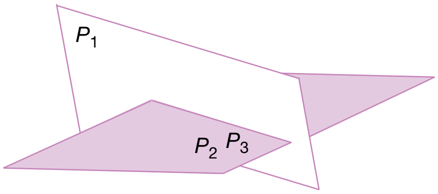
If point of intersection is origin then trivial solution and for other than origin it is non-trivial solution

(b) Infinite solution

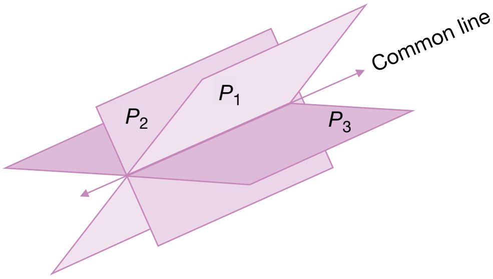
CASE 1: All three planes are coincident



CASE 2: Two planes are coincident and one oblique plane cuts them

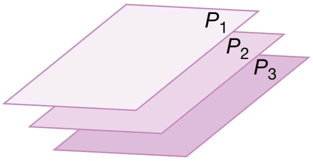


CASE 3: All three planes pass through a common line of intersection

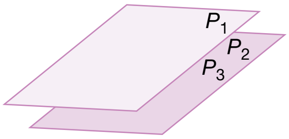


(c) No solution

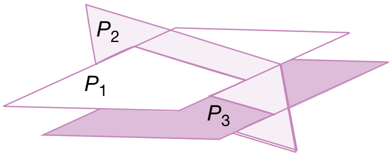
CASE 1: All three planes are parallel



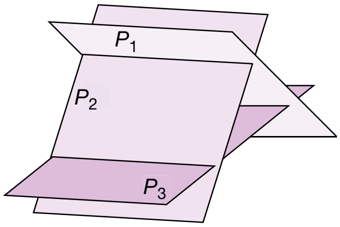
CASE 2: Two planes are coincident and third plane is parallel



CASE 3: two planes are parallel and third plane cuts these two planes



CASE 4: Three planes intersect in three parallel lines (planes are forming triangular prism sort of shape)



Homogeneous System

If all constant are zero then system of linear equation are known as system of homogeneous linear equation which always possess atleast one solution (0, 0, 0)

If system of homogeneous linear equation possess non-zero/non-trivial solution then $\Delta = 0$. In such case given system has infinite solutions.

Remark(s)

The above method of solving a system of three linear equations in three unknown can be used exactly the same way to solve a system of n equations in n unknown as stated below. (Cramer's Rule) Let there be a system of n simultaneous linear equations n unknown given by

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$\vdots \; \vdots \; \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. then,

$x_1 = \dfrac{D_1}{D}, x_2 = \dfrac{D_2}{D}, ..., x_n = \dfrac{D_n}{D}$, provided that $D \neq 0$

ILLUSTRATION 61

Solve system of linear equations

$3x - 2y + 3z = 8;\ 2x + y - z = 1;\ 4x - 3y + 2z = 4$

SOLUTION

$$\Delta = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = (6 - 18 + 8) - (12 + 9 - 8) = -17$$

$$\Delta_x = \begin{vmatrix} 8 & -2 & 3 \\ 1 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = -17;$$

$$\Delta_y = \begin{vmatrix} 3 & 8 & 3 \\ 2 & 1 & -1 \\ 4 & 4 & 2 \end{vmatrix} = -34;$$

$$\Delta_z = \begin{vmatrix} 3 & -2 & 8 \\ 2 & 1 & 1 \\ 4 & -3 & 4 \end{vmatrix} = -51$$

For using Cramer rule, we have

$$\Delta = -17; \quad \Delta_x = -17; \quad \Delta_y = -34; \quad \Delta_7 = -51$$

$$x = \frac{\Delta x}{\Delta} = \frac{-17}{-17} = 1; \quad y = \frac{\Delta y}{\Delta} = \frac{-34}{-17} = 2;$$

$$z = \frac{\Delta z}{\Delta} = \frac{-51}{-17} = 3;$$

$$x = 1, y = 2, z = 3$$

$$(y, y, z) = (1, 2, 3)$$

As we can say that given system of equation is consistent with unique non- trivial solution.

ILLUSTRATION 62

Show that solution of system of equations

$3x - 7y + 5z = 3;\ 3x + y + 5z = 7$ and $2x + 3y + 5z = 5$ is unique and non-trivial.

SOLUTION

Using Sarrus Rule

$$\Delta = \begin{vmatrix} 3 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 15 + 45 - 70 - (10 + 45 - 105)$$

$$= 15 - 80 + 105 = 120 - 80 = 40 \neq 0$$

$$\Delta_x = \begin{vmatrix} 3 & -7 & 5 \\ 7 & 1 & 5 \\ 5 & 3 & 5 \end{vmatrix} = 5\begin{vmatrix} 3 & -7 & 1 \\ 7 & 1 & 1 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= 5(3 + 21 - 35 - (5 + 9 - 49)) = 120 \neq 0$$

$$\Delta \neq 0, \Delta_x \neq 0$$

Given system of equations is consistent with unique non trivial solution

ILLUSTRATION 63

For what value of k the following system of equations possess non-trivial solution,

$$x + y - kz = 0;\ 3x - y - 2z = 0;\ x - y + 2z = 0$$

Also find all the solutions of the system for that value of k.

SOLUTION

For non-trivial solution, $\Delta = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -k \\ 3 & -1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} 2 & 0 & -k+2 \\ 2 & 0 & -4 \\ 1 & -1 & 2 \end{vmatrix} = 0$$

$$[R_1 \to R_1 + R_3, R_2 \to R_2 - R_3]$$

$$\Rightarrow \quad -(-1)\,[-8 - 2(2 - k)] = 0$$

$$\Rightarrow \quad 2k - 12 = 0$$

$$\therefore \quad k = 6$$

Putting the value of k in given equations, we get

$$x + y - 6z = 0 \qquad \ldots\text{(i)}$$

$$3x - y - 2z = 0 \qquad \ldots\text{(ii)}$$

$$x - y + 2z = 0 \qquad \ldots\text{(iii)}$$

(i) + (ii) $\Rightarrow \quad 4x - 8z = 0 \quad \therefore z = \frac{x}{2}$

Putting the value of z in (i),

we get $x + y - 3x = 0 \quad \therefore y = 2x$

Thus when $k = 6$, solution of given system of equation will be $x = t, y = 2t, z = \frac{t}{2}$, where t is an arbitrary number.

It is satisfied by (iii) also.

ILLUSTRATION 64

If the equations $x + y = 1$, $(c + 2)x + (c + 4)y = 6$, $(c + 2)^2x + (c + 4)^2y = 36$ are consistent, then find c.

SOLUTION

Rewriting given equation

$$x + y + 0.z = 1$$
$$(c + 2)x + (c + 4)y + 0z = 6$$
$$(c + 2)^2x + (c + 4)^2y + 0z = 36$$

We have $\Delta = \Delta_x = \Delta_y = 0$, for system to be consistent

$$\Delta_z = \begin{vmatrix} 1 & 1 & 1 \\ c+2 & c+4 & 6 \\ (c+2)^2 & (c+4)^2 & 36 \end{vmatrix} = 0$$

using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ c+2 & 2 & 4-c \\ (c+2)^2 & 12+4c & 32-c^2-4c \end{vmatrix}$$

$$= 64 - 2c^2 - 8c - (4 - c)(12 + 4c)$$
$$= 64 - 2c^2 - 8c - (48 - 12c + 16c - 4c^2)$$
$$= 2c^2 - 12c + 16 = 2\,(c - 4)\,(c - 2) = 0$$

$\Rightarrow \quad c = 2, 4$

ILLUSTRATION 65

Find the value of a for which the system of equations; $a^3x + (a + 1)^3y + (a + 2)^3\, z = 0$, $ax + (a + 1)\, y + (a + 2)z = 0$ and $x + y + z = 0$ has a non-zero solution.

SOLUTION

For a homogenous system, we have $\Delta_x = \Delta_y = \Delta_z = 0$

Now for system of equation to have non zero solution

Then $\Delta = \begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & a+1 & a+2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\begin{vmatrix} a^3 & 1+3a+3a^2 & 8+6a^2+12a \\ a & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (2 + 6a + 6a^2 - 8 - 6a^2 - 12a) = 0$

$\Rightarrow \quad (-6 - 6a) = 0 \quad \Rightarrow \quad a = -1$

ILLUSTRATION 66

If (x_0, y_0, z_0) is any solutions of the system of equations $1 + x + y = 2z$, $1 + y + z = 2x$, $x + z = 2y + 2$, then find the value of $\frac{z_0^2 - y_0^2 + 1}{x_0}$ (where $x_0 \neq 0$).

SOLUTION

Given planes are $P_1 : 1 + x + y = 2z$, $P_2 : 1 + y + z = 2x$, $P_3 : x + z = 2y + 2$

We can observe that

$P_1 + P_2 = P_3$ infinite solutions (Family of planes, it has common straight line)

Let difference of first two equations we get $x = z$

Solution set is $(x_0, y_0, z_0) \equiv (\lambda, \lambda - 1, \lambda)$

$$\frac{z_0^2 - y_0^2 + 1}{x_0} = \frac{\lambda^2 - (\lambda - 1)^2 + 1}{\lambda} = \frac{2\lambda}{\lambda} = 2$$

ILLUSTRATION 67

If the system of linear equations

$(\cos\theta)x + (\sin\theta)y + \cos\theta = 0$; $(\sin\theta)x + (\cos\theta)y + \sin\theta = 0$; $(\cos\theta)x + (\sin\theta)y - \cos\theta = 0$ is consistent, then find the number of possible values of $\theta \in [0, 2\pi]$.

SOLUTION

We have $\Delta = \Delta_x = \Delta_y = 0$,

for system to be consistent $\Delta_z = 0$

$$\Rightarrow \begin{vmatrix} \cos\theta & \sin\theta & \cos\theta \\ \sin\theta & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & -\cos\theta \end{vmatrix} = 0$$

$\Rightarrow \quad -\cos\theta - \sin\theta\,(-2\sin\theta\cos\theta) + \cos\theta(-\cos 2\theta) = 0$

$\Rightarrow \quad -\cos\theta - \cos 3\theta = 0$

$\Rightarrow \quad 2\cos\theta\cos 2\theta = 0$ where $\theta \in [0, 2\pi]$

$\cos\theta = 0$ or $\cos 2\theta = 0$

$\therefore \quad \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

But $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (rejected)

As lines are parallel. So, system is inconsistent.

For $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ system is consistent (2 solutions)

ILLUSTRATION 68

Find the values of θ, λ for which the following equations $\sin\theta\, x - \cos\theta\, y + (\lambda+1)z = 0$; $\cos\theta\, x + \sin\theta\, y - \lambda z = 0$; $\lambda x + (\lambda+1)y + \cos\theta\, z = 0$ have non trivial solution.

SOLUTION

For a homogenous system, we have $\Delta_x = \Delta_y = \Delta_z = 0$

Now for system of equation to have non zero solution

$\Delta = 0$

$$\Rightarrow \begin{vmatrix} \sin\theta & -\cos\theta & \lambda+1 \\ \cos\theta & \sin\theta & -\lambda \\ \lambda & \lambda+1 & \cos\theta \end{vmatrix} = 0;$$

$\Rightarrow$ $\lambda(\lambda\cos\theta - \lambda\sin\theta - \sin\theta) - (\lambda+1)(-\lambda\sin\theta - \lambda\cos\theta - \cos\theta) + \cos\theta = 0$

$\Rightarrow$ $\lambda^2\cos\theta - \lambda(\lambda+1)\sin\theta + \lambda(\lambda+1)\sin\theta + (\lambda+1)^2\cos\theta + \cos\theta = 0$

$\Rightarrow$ $2\lambda^2\cos\theta + 2\lambda\cos\theta + 2\cos\theta = 0$

$\Rightarrow$ $2\cos\theta\,[\lambda^2 + \lambda + 1] = 0$

$\lambda^2 + \lambda + 1$ can not be zero as $D = -3 < 0$

$\Rightarrow$ $\theta = (2n+1)\dfrac{\pi}{2}$ where $\lambda \in R, n \in I$

ILLUSTRATION 69

Show that the system of equations $x - y\cos\theta + z\cos 2\theta = 0$; $-x\cos\theta + y - z\cos\theta = 0$; $x\cos 2\theta - y\cos\theta + z = 0$ has non trivial solution for all real values of θ.

SOLUTION

For a homogenous system, we have $\Delta_x = \Delta_y = \Delta_z = 0$

Now for system of equation to have non zero solution

$$\Delta = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & -\cos\theta & \cos 2\theta \\ -\cos\theta & 1 & -\cos\theta \\ \cos 2\theta & -\cos\theta & 1 \end{vmatrix} = 0$$

$\Rightarrow$ $(1 - \cos^2\theta) + \cos\theta(-\cos\theta + \cos\theta\cos 2\theta) + \cos 2\theta(\cos^2\theta - \cos 2\theta) = 0$

$\Rightarrow$ $1 - 2\cos^2\theta + 2\cos^2\theta\cos 2\theta - \cos^2 2\theta$

$\Rightarrow$ $\cos 2\theta(2\cos^2\theta - 1) - \cos^2 2\theta$

$\Rightarrow$ $\cos 2\theta[\cos 2\theta - \cos 2\theta] = 0$

$\Delta = 0$ for all real values of θ

ILLUSTRATION 70

The set of homogeneous equations

$tx + (t+1)y + (t-1)z = 0$; $(t+1)x + ty + (t+2)z = 0$; $(t-1)x + (t+2)y + tz = 0$

has non-trivial solutions, find t.

SOLUTION

For homogenous system of equation, $\Delta_x = \Delta_y = \Delta_z = 0$

$$\text{For non-trivial solutions } \Delta = \begin{vmatrix} t & t+1 & t-1 \\ t+1 & t & t+2 \\ t-1 & t+2 & t \end{vmatrix} = 0$$

using $C_2 \to C_2 - C_1 : C_3 \to C_3 - C_1$

$$\text{or} \quad \begin{vmatrix} t & 1 & -1 \\ t+1 & -1 & 1 \\ t-1 & 3 & 1 \end{vmatrix} = 0 \text{ using } R_1 \to R_1 + R_2$$

$$\text{or} \quad \begin{vmatrix} 2t+1 & 0 & 0 \\ t+1 & -1 & 1 \\ t-1 & 3 & 1 \end{vmatrix} = (2t+1)(-1-3) = 0$$

we get $t = -\dfrac{1}{2}$

ILLUSTRATION 71

If the system of equations $x - 2y + z = a$; $2x + y - 2z = b$ and $x + 3y - 3z = c$ have atleast one solution, then find condition in a, b and c.

SOLUTION

Let plane $P_1 : x - 2y + z = a$; plane $P_2 : 2x + y - 2z = b$ and plane P_3: $x + 3y - 3z = c$

As for consistent system, we can observe that

$P_1 + P_3 = P_2$, system has infinite solution

$a + c = b$

ILLUSTRATION 72

Prove that the system of equation $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ has at least one solution for any real λ. Find the set of solutions when $\lambda = -5$.

SOLUTION

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$$

$$= 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) = 7(\lambda + 5)$$

CASE 1: When $\lambda \neq -5$. In this case $\Delta \neq 0$, therefore, given system of equations has unique solution.

CASE 2: When $\lambda = -5$. In this case $\Delta = 0$

$$\text{Also } \Delta_x = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0,$$

$$\Delta_y = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0 \text{ and } \Delta_z = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$$

Hence in this case given system of equations has infinitely many solutions. Thus the given system of equations has at least one solution.

Let plane $P_1 : 3x - y + 4z = 3$

$$P_2 : x + 2y - 3z = -2$$

$$P_3 : 6x + 5y + \lambda z = -3$$

We can have $P_1 + 3P_2 = P_3$

It means its a family of planes hence infinite solution

Now we are looking for infinite solutions

Putting the value of λ, given equations become

$$3x - y + 4z = 3 \qquad \ldots(1)$$

$$x + 2y - 3z = -2 \qquad \ldots(2)$$

$$6x + 5y - 5z = -3 \qquad \ldots(3)$$

Multiplying equations (1) by 2 and adding it to (2), we get

$$7x + 5z = 4 \text{ or } z = \frac{4-7x}{5}$$

From (1), $y = 3x + 4z - 3$

$$= 3x + \frac{16-28x}{5} - 3 = \frac{1-13x}{5}$$

Thus solutions is given by $x = t, y = \dfrac{1-13t}{5}, z = \dfrac{4-7t}{5}$ for any real t

We get infinite number of solutions

ILLUSTRATION 73

If system of linear equations $(a-1)x + z = \alpha$, $x + (b-1)\,y = \beta$ and $y + (c-1)\,z = \gamma$ where $a, b, c \in I$ does not have a unique solution, then find maximum possible value of $|a + b + c|$.

SOLUTION

Rewriting the given system of equations

$$(a-1)\,x + 0y + z = \alpha \qquad \ldots(1)$$

$$x + (b-1)\,y + 0z = \beta \qquad \ldots(2)$$

$$0x + y + (c-1)\,z = \gamma \qquad \ldots(3)$$

For no unique solution $\Delta = 0$

$$\begin{vmatrix} (a-1) & 0 & 1 \\ 1 & (b-1) & 0 \\ 0 & 1 & (c-1) \end{vmatrix} = 0$$

$$(a-1)\,(b-1)\,(c-1) + 1 = 0$$

$$(a-1)\,(b-1)\,(c-1) = -1 \text{ where } a, b, c \in I$$

For maximum possible value of $|a + b + \text{c}|$

$\therefore \quad a = 2;\ b = 2;\ c = 0$

Hence, $|a + b + c| = 4$

ILLUSTRATION 74

If the system of equations $px + (p+1)y + (p-1)z = 0$, $(p+1)x + py + (p+2)z = 0$, $(p-1)x + (p+2)y + pz = 0$ has a non-trivial solutions then find p.

SOLUTION

For homogenous system of equation, $\Delta_x = \Delta_y = \Delta_z = 0$

For non-trivial solutions $\therefore\ \Delta = 0$

$$\Delta = \begin{vmatrix} p & p+1 & p-1 \\ p+1 & p & p+2 \\ p-1 & p+2 & p \end{vmatrix} = 0$$

Using $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} p & p+1 & p-1 \\ 1 & -1 & 3 \\ -1 & 1 & 1 \end{vmatrix} = 0 \qquad \text{Using } R_3 \to R_3 + R_2$$

$$\Delta = \begin{vmatrix} p & p+1 & p-1 \\ 1 & -1 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad 4(-p - p - 1) = 0 \quad \Rightarrow \quad p = \frac{-1}{2}$$

$\therefore \quad$ Exactly one real value of p.

ILLUSTRATION 75

If system of equations $(\tan\alpha)x + (\cot\alpha)y + (8\cos 2\alpha)\,z = 0$, $(\cot\alpha)x + (8\cos 2\alpha)y + (\tan\alpha)\,z = 0$, $(8\cos 2\alpha)x + (\tan\alpha)\,y + (\cot\alpha)\,z = 0$ have non-trivial solution, then find $\sin 4\alpha$.

SOLUTION

$$\Delta = \begin{vmatrix} \tan\alpha & \cot\alpha & 8\cos 2\alpha \\ \cot\alpha & 8\cos 2\alpha & \tan\alpha \\ 8\cos 2\alpha & \tan\alpha & \cot\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{\sin^3\alpha\cos^3\alpha}\begin{vmatrix} \sin^2\alpha & \cos^2\alpha & 2\sin 4\alpha \\ \cos^2\alpha & 2\sin 4\alpha & \sin^2\alpha \\ 2\sin 4\alpha & \sin^2\alpha & \cos^2\alpha \end{vmatrix} = 0$$

using $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \quad \begin{vmatrix} 1+2\sin 4\alpha & 1+2\sin 4\alpha & 1+2\sin 4\alpha \\ \cos^2\alpha & 2\sin 4\alpha & \sin^2\alpha \\ 2\sin 4\alpha & \sin^2\alpha & \cos^2\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad (1+2\sin 4\alpha)\begin{vmatrix} 1 & 1 & 1 \\ \cos^2\alpha & 2\sin 4\alpha & \sin^2\alpha \\ 2\sin 4\alpha & \sin^2\alpha & \cos^2\alpha \end{vmatrix} = 0$$

$1 + 2\sin 4\alpha = 0$ we get $\sin 4\alpha = -1/2$

ILLUSTRATION 76

$2x + py + 6z = 8$; $x + 2y + qz = 5$; $x + y + 3z = 4$ find p, q for which system has

(i) unique solution **(ii)** infinite solution

(iii) no solution

SOLUTION

First we should compute

$$\Delta = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = 12 + 6 + pq - 12 - 2q - 3p = (p-2)\,(q-3)$$

$$\Delta y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix} = 0; \text{ (R_1 and R_3 are proportional)}$$

$$\Delta x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (2-p)(15-4q);$$

$$\Delta z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = p - 2$$

Now we have all required determinant for using Cramer rule

$$\Delta = (p-2)(q-3);\ \Delta_x = (2-p)(15-4q);$$

$$\Delta_y = 0;\ \Delta_z = p - 2$$

(i) unique solution $\Delta \neq 0$

$\Rightarrow (p-2)(q-3) \neq 0 \quad \Rightarrow \quad p \neq 2, q \neq 3$

(ii) infinite solution

$\Delta = 0;\ \Delta_x = \Delta_y = \Delta_z = 0$

We have for $p = 2$, q for any real value

(iii) No solution

We have $\Delta = 0$ and $\Delta_z = p - 2 \neq 0$

Required values $p \neq 2, q = 3$

ILLUSTRATION 77

If the equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have a unique solution, then find λ ($\lambda \in$ whole number and $00 \leq \lambda \leq 09$).

SOLUTION

The given equations are

$$x + y + z = 6 \qquad \ldots(1)$$

$$x + 2y + 3z = 10 \qquad \ldots(2)$$

$$x + 2y + \lambda z = \mu \qquad \ldots(3)$$

If $\lambda = 3$, $\mu = 10$, (2) and (3) are same and hence there are infinite number of solutions.

If $\lambda = 3$, $\mu \neq 10$, then the equations are inconsistent as second and third plane become parallel.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = \lambda - 3 \neq 0$$

Hence, $\lambda \neq 3$ for the equation to have a unique solution.

ILLUSTRATION 78

If $x = \dfrac{a}{b-c}$, $y = \dfrac{b}{c-a}$, $z = \dfrac{c}{a-b}$, then find $xy + yz + zx$.

SOLUTION

Rewriting given data as

$$-a + bx - cx = 0, -ay - b + cy = 0, az - bz - c = 0.$$

Considering these equations as system of equation in variable a, b, c then

Eliminating a, b, c; $\Delta = \begin{vmatrix} -1 & x & -x \\ -y & -1 & y \\ z & -z & -1 \end{vmatrix} = 0$

$\Rightarrow \quad -(1 + yz) - x(y - yz) - x(yz + z) = 0$

$\Rightarrow \quad xy + yz + zx = -1$

Application of Determinant in Coordinate Geometry

(a) The lines : $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent

If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$

It is also a condition for the consistency of three simultaneous linear equations in 2 variables

(b) General two degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines

If $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

and $h^2 - ab \geq 0$.

(c) Area of a triangle whose vertices are (x_r, y_r); $r = 1, 2, 3$ is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If $\Delta = 0$ then the three points are collinear.

(d) Equation of a straight line passing through (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(e) Suppose the sides of a triangle lie along the lines whose equations are given by $a_1x+b_1y+c_1=0, a_2x+b_2y+c_2=0$ and $a_3x + b_3y + c_3 = 0$ then the area of the triangle

$$\frac{1}{2C_1C_2C_3}\cdot\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where C_1, C_2, C_3 are cofactors of the elements c_1, c_2, c_3, respectively in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

ILLUSTRATION 79

Three distinct points $P(3u^2, 2u^3)$; $Q(3v^2, 2v^3)$ and $R(3w^2, 2w^3)$ are collinear then find $uv + vw + wu$.

SOLUTION

Here, points P, Q, R are distinct so $u \neq v \neq w$. We have

$$\begin{vmatrix} 3u^2 & 2u^3 & 1 \\ 3v^2 & 2v^3 & 1 \\ 3w^2 & 2w^3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} u^2 & u^3 & 1 \\ v^2 & v^3 & 1 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

Apply, $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} u^2-v^2 & u^3-v^3 & 0 \\ v^2-w^2 & v^3-w^3 & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (u-v)(v-w)\begin{vmatrix} v+u & u^2+v^2+uv & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

Apply, $R_1 \to R_1 - R_2$

$$\Rightarrow \begin{vmatrix} u-w & (u^2-w^2)+v(u-w) & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (u-w)\begin{vmatrix} 1 & u+w+v & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$\therefore$ Expanding along C_3, we get

$$[(v^2+w^2+vw)-(v+w)(w+v+u)] = 0$$

$$\Rightarrow (v^2+w^2+vw) = (v+w)^2 + u(v+w)$$

$$\Rightarrow v^2+w^2+vw = v^2+w^2+2vw+uv+uw$$

$$\Rightarrow uv+vw+wu = 0$$

ILLUSTRATION 80

If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$ then the two triangles with vertices (x_1, y_1) (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent. State whether True/False. **[IIT - 95]**

SOLUTION

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Rightarrow \text{Area of } \Delta_1 = \text{Area of } \Delta_2$$

But two Δ's of same area may or may not be congruent.
$\therefore$ Given statement is false.

Test Your Concepts-III

(Solutions on page H.285)

1. If the equations $2x - y + z = 0$, $x - 2y + z = 0$, $\lambda x - y + 2x = 0$ have non trivial solution then find λ.
2. Show that system of equations $3x - 7y + 5z = 3$; $3x + y + 5z = 7$ and $2x + 3y + 5z = 5$ consistent with unique non trivial solution.
3. If the system of linear equations $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has a non-zero solution, then show that a, b, c (taken in that order) are in H.P.
4. Find number of positive integral value of p for which the system of equations $x + 2y + z = px$, $-3x + py = 2z$ and $2x - 2y + z = -pz$ has a non-trivial solutions.
5. If $a + b + c \neq 0$ and the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$ has a non-trivial solution, then show that the roots of the equation $at^2 + bt + c = 0$ are imaginary.
6. Solve the following equations by Cramer's rule
$x + y + z = 9$; $2x + 5y + 7z = 52$; $2x + y - z = 0$

Answer keys

1. 5 **4.** 1 **6.** $x = 1, y = 3, z = 5$

SOLVED PROBLEMS

PROBLEM 1

If $\begin{vmatrix} x^2+x+2 & 2x^2+3x+1 & 3x^2+5x+3 \\ 20x^2+20x+59 & 40x^2+60x+20 & 60x^2+100x+70 \\ 2x^2+2x+6 & 4x^2+6x+2 & 6x^2+10x+7 \end{vmatrix} =$ Ax^2+Bx+C, where A and B are determinant of order 3 not involving x, then find A, B and C.

SOLUTION

By applying $R_2 \to R_2 - 2R_1 - 9R_3$, $R_3 \to R_3 - 2R_1$ we get

$$\Delta = \begin{vmatrix} x^2+x+2 & 2x^2+3x+1 & 3x^2+5x+3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2+x & 2x^2+3x & 3x^2+5x \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + x \begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

where $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$; $B = \begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ and $C = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

$\therefore\ A = 2, B = 3, C = 1.$

PROBLEM 2

If f, g and h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$$

Prove $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$

SOLUTION

We have,

$$(x^2 f)'' = [x\,(xf)]'' = [x\,(xf)' + 1(xf)]'$$

$$= 1(xf)' + x(xf)'' + (xf)' = 2(xf)' + x(xf)''$$

Write each element of the third row in Δ in the above form and apply

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$$

$R_3 \to R_3 - 2R_2$, to get

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ x(xf)'' & x(xg)'' & x(xh)'' \end{vmatrix}$$

Taking x common from R_3 and multiplying R_1 by x we get

$$\Delta = \begin{vmatrix} xf & xg & xh \\ (xf)' & (xg)' & (xh)' \\ (xf)'' & (xg)'' & (xh)'' \end{vmatrix}$$

Differentiating w.r.t. x, we get

$$\Delta' = \begin{vmatrix} (xf)' & (xg)' & (xh)' \\ (xf)' & (xg)' & (xh)' \\ (xf)'' & (xg)'' & (xh)'' \end{vmatrix} + \begin{vmatrix} xf & xg & xh \\ (xf)'' & (xg)'' & (xh)'' \\ (xf)'' & (xg)'' & (xh)'' \end{vmatrix}$$

$$+ \begin{vmatrix} xf & xg & xh \\ (xf)' & (xg)' & (xh)' \\ (xf)''' & (xg)''' & (xh)''' \end{vmatrix}$$

The first two determinants are equal to zero, because in each of them two row are identical. Thus, we can write.

$$\Delta' = \begin{vmatrix} xf & xg & xh \\ (xf)' & (xg)' & (xh)' \\ (xf)''' & (xg)''' & (xh)''' \end{vmatrix} = x \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (xf)''' & (xg)''' & (xh)''' \end{vmatrix}$$

We know that $(xf)' = xf' + f$.

Applying $R_2 \to R_2 - R_1$, we get

$$\Delta' = x \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ (xf)''' & (xg)''' & (xh)''' \end{vmatrix}$$

$$= x^2 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (xf)''' & (xg)''' & (xh)''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^2(xf)''' & x^2(xg)''' & x^2(xh)''' \end{vmatrix}$$

We have $x^2\,(xf)''' = x^2\,(xf' + f)'' = x^2\,(xf'' + 2f')'$

$$= x^2\,(xf''' + 3f'') = x^3 f''' + 3x^2 f'' = (x^3 f'')'$$

Hence $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$ Hence proved.

PROBLEM 3

Let a, b, c be the real numbers then show that following system of equations in x, y and z

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ has finitely many solutions. **[IIT - 95]**

SOLUTION

Let $\frac{x^2}{a^2}=X$, $\frac{y^2}{b^2}=Y$ and $\frac{z^2}{c^2}=Z$. Then given system of equation is

$X+Y-Z=1; \quad X-Y+Z=1; \quad -X+Y+Z=1$

$$\Delta=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{vmatrix}=1(-1-1)-1\,(1+1)-1\,(1-1)=-4\neq 0$$

$\therefore$ Given system of equation has a unique solution.

$$\Delta_X=\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}=-4;\ \Delta_Y=\begin{vmatrix}1 & 1 & -1\\ 1 & 1 & 1\\ -1 & 1 & 1\end{vmatrix}=-4;$$

$$\Delta_Z=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{vmatrix}=-4;$$

Using $\frac{X}{\Delta_X}=\frac{Y}{\Delta_Y}=\frac{Z}{\Delta_Z}=\frac{1}{\Delta}$

We get $X=1 \Rightarrow \frac{x^2}{a^2}=1 \Rightarrow x=\pm a$

$Y=1 \Rightarrow \frac{y^2}{b^2}=1 \Rightarrow y=\pm b$

$Z=1 \Rightarrow \frac{z^2}{c^2}=1 \Rightarrow z=\pm c$

Required number of solutions are (x, y, z) are 8

PROBLEM 4

Suppose $f(x)$ is function satisfying the following conditions

(a) $f(0)=2, f(1)=1$

(b) f has minimum value at $x=5/2$ and

(c) for all x, $f'(x)=\begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 2(ax+b) & 2ax+2b+1 & 2ax+b\end{vmatrix}$

where a, b are some constants. Determine the constants a, b and the function $f(x)$. **[IIT - 98]**

SOLUTION

$$f'(x)=\begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 2(ax+b) & 2ax+2b+1 & 2ax+b\end{vmatrix}$$

Applying $R_3\to R_3-R_1-2R_2$, we get

$$f'(x)=\begin{vmatrix}2ax & 2ax-1 & 2ax+b+1\\ b & b+1 & -1\\ 0 & 0 & 1\end{vmatrix}=\begin{vmatrix}2ax & 2ax-1\\ b & b+1\end{vmatrix}$$

Applying $[C_2\to C_2-C_1]$

$$=\begin{vmatrix}2ax & -1\\ b & 1\end{vmatrix}$$

$\Rightarrow \quad f'(x)=2ax+b$

Integrating, we get, $f(x)=ax^2+bx+c$

Where c is an arbitrary constant since f has minimum at $x=5/2$

$f'(5/2)=0 \quad \Rightarrow \quad 5a+b=0$ ...(1)

Also $f(0)=2 \quad \Rightarrow \quad c=2$ and

$f(1)=1 \quad \Rightarrow \quad a+b+c=1$

$\therefore \quad a+b=-1$ ...(2)

From (1) and (2) we get $a=\frac{1}{4}$, $b=-\frac{5}{4}$

Thus, $f(x)=\frac{x^2}{4}-\frac{5x}{4}+2$

PROBLEM 5

Let a, b, c be real number with $a^2+b^2+c^2=1$ then show that $\begin{vmatrix}ax-by-c & bx-ay & cx+a\\ bx+ay & -ax+by-c & cy+b\\ cx+a & cx+b & -ax-by+c\end{vmatrix}=0$ represent a straight line. **[IIT - 2001]**

SOLUTION

Applying $C_1\to aC_1$

$$\Delta=\frac{1}{a}\begin{vmatrix}a^2x-aby-ac & bx+ay & cx+a\\ abx+a^2y & -ax+by-c & cy+b\\ acx+a^2 & cy+b & -ax-by+c\end{vmatrix}$$

Applying $C_1\to C_1+bC_2+cC_3$

$$\Delta=\frac{1}{a}\begin{vmatrix}(a^2+b^2+c^2)x & ay+bx & cx+a\\ (a^2+b^2+c^2)y & by-c-ax & cy+b\\ (a^2+b^2+c^2) & b+cy & -ax-by+c\end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b+cy & c-ax-by \end{vmatrix}$$

As $a^2+b^2+c^2=1$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

Then $\Delta = \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$

$$= \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

Applying $R_1 \to R_1 + yR_2 + R_3$

$$\Delta = \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix}$$

On expanding along R_1

$$\Delta = \frac{(x^2+y^2+1)}{ax}\, ax\,(ax+by+c)$$

$$= (x^2+y^2+1)\,(ax+by+c) = 0$$

$x^2+y^2+1 \neq 0$ hence $ax+by+c=0$, which represents a straight line.

PROBLEM 6

Prove that the determinant $\begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix}$ vanishes if any two of x, y, z are equal or $x+y+z=(2n+1)\frac{\pi}{2}$ where $n \in I$.

SOLUTION

$$\Delta = \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y & \tan z \\ 1 & 1 & 1 \\ \cos^2 x & \cos^2 y & \cos^2 z \end{vmatrix}$$

$$= \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y - \tan x & \tan z - \tan y \\ 1 & 0 & 0 \\ \cos^2 x & \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix}$$

Using $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$

$$= -\cos x \cos y \cos z \begin{vmatrix} \tan y - \tan x & \tan z - \tan y \\ \cos^2 y - \cos^2 x & \cos^2 z - \cos^2 y \end{vmatrix}$$

$$= -\cos x \cos y \cos z \begin{vmatrix} \dfrac{\sin(y-x)}{\cos x \cos y} & \dfrac{\sin(z-y)}{\cos y . \cos z} \\ \sin^2 x - \sin^2 y & \sin^2 y - \sin^2 z \end{vmatrix}$$

$$= \begin{vmatrix} \cos z \cdot \sin(x-y) & \cos x \cdot \sin(y-z) \\ \sin(x+y)\cdot\sin(x-y) & \sin(y+z)\cdot\sin(y-z) \end{vmatrix}$$

$$= \sin(x-y)\sin(y-z)\begin{vmatrix} \cos z & \cos x \\ \sin(x+y) & \sin(y+z) \end{vmatrix}$$

$= \sin(x-y)\sin(y-z) \times [\sin(y+z)\cos z - \sin(x+y)\cos x)$

$= \frac{1}{2}\sin(x-y)\sin(y-z)[\{\sin(y+2z)+\sin y\} - \{\sin(y+2x)+\sin y\}]$

$= \frac{1}{2}\sin(x-y)\sin(y-z)\,[\sin(y+2z) - \sin(y+2x\}]$

$= \frac{1}{2}\sin(x-y)\sin(y-z).\ 2\cos(x+y+z)\sin(z-x)$

$= \sin(x-y)\sin(y-z)\sin(z-x)\cos(x+y+z)$

Clearly, Δ is zero when any two of x, y, z are equal or $x+y+z=$ odd multiple of $\frac{\pi}{2}$

$= (2n+1)\,\frac{\pi}{2}$

PROBLEM 7

If $u_n = \int_0^{\pi/2} \frac{1-\cos 2nx}{1-\cos 2x}dx$, then find the value of the determinant $\Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix}$.

SOLUTION

We have, $u_1 = \int_0^{\pi/2} \frac{1-\cos 2x}{1-\cos 2x}\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

Now $2u_{n+1} - (u_n + u_{n+2})$

$$= \int_0^{\pi/2} \frac{-2\cos(2n+2)x + [\cos(2n+4)x + \cos(2nx)]}{1-\cos 2x}\,dx$$

$$= \int_0^{\pi/2} \frac{-2\cos(2n+2)x + 2\cos(2n+2)x\cos 2x}{1-\cos 2x}\,dx$$

$$= \int_0^{\pi/2} -2\cos(2n+2)x\,dx = -\left[\frac{\sin(2n+2)x}{n+1}\right]_0^{\pi/2} = 0$$

Hence $u_n + u_{n+2} - 2u_{n+1} = 0$

$\therefore \quad u_n, u_{n+1}, u_{n+2}$ are in AP

$$\text{Now } \Delta = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{vmatrix} = \begin{vmatrix} u_1 - 2u_2 + u_3 & u_2 & u_3 \\ u_4 - 2u_5 + u_6 & u_5 & u_6 \\ u_7 - 2u_8 + u_9 & u_8 & u_9 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2 + C_3$, we obtain

$$= \begin{vmatrix} 0 & u_2 & u_3 \\ 0 & u_5 & u_6 \\ 0 & u_8 & u_9 \end{vmatrix} = 0$$

PROBLEM 8

Show that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$

SOLUTION

Using $[C_1 \to C_1 - C_2, C_2 \to C_2 - C_3]$

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ b^2 - (c+a)^2 & (c+a)^2 - b^2 & b^2 \\ 0 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ (b+c+a)(b-c-a) & (c+a+b)(c+a-b) & b^2 \\ 0 & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 0 & c-a-b & (a+b)^2 \end{vmatrix}$$

Using $[R_3 \to R_3 - R_1 - R_2]$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ b-c-a & c+a-b & b^2 \\ 2a-2b & -2a & 2ab \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad [C_1 \to C_1 + C_2]$$

$$= \frac{(a+b+c)^2}{ab} \begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(c+a) & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Using $[C_1 \to aC_1 + C_3, C_2 \to bC_2 + C_3]$

$$= \frac{(a+b+c)^2}{ab}.\ ab\ .\ 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2ab\,(a+b+c)^2\,[(b+c)\,(c+a) - ab]$$

$$= 2ab\,(a+b+c)^2\,.\,[bc + c^2 + ab + ac - ab]$$

$$= 2abc\,(a+b+c)^3$$

PROBLEM 9

Let α, β, γ are the real roots of the equation $x^3 + ax^2 + bx + c = 0$ ($a, b, c \in R$ and $a \neq 0$). If the system of equations (in u, v and w) given by $\alpha u + \beta v + \gamma w = 0$, $\beta u + \gamma v + \alpha w = 0$, $\gamma u + \alpha v + \beta w = 0$ has non-trivial solutions, then find a^2.

SOLUTION

If $\alpha\,\beta, \gamma$ are the real roots of the equation $x^3 + ax^2 + bx + c = 0$

$\therefore \quad \alpha + \beta + \gamma = -a;\ \alpha\beta + \beta\gamma + \gamma\alpha = b;\ \alpha\beta\gamma = -c$

Since the given system of equations (in u, v and w) has non-trivial solutions,

$$\text{So, } \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0 \quad \Rightarrow \quad \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$$

$$\Rightarrow \quad (\alpha + \beta + \gamma)\,(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = 0$$

$$\Rightarrow \quad (\alpha + \beta + \gamma)\,((\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)) = 0$$

$$\Rightarrow \quad -a(a^2 - 3b) = 0$$

As $a \neq 0$ (given) $\quad \Rightarrow \quad a^2 = 3b$

PROBLEM 10

Find the value of the determinant

$$\begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin^2\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x + \frac{3\pi}{2}\right) & \sin\left(x + \frac{5\pi}{2}\right) & \sin\left(x + \frac{7\pi}{2}\right) \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & \sin\left(x - \frac{7\pi}{2}\right) \end{vmatrix}.$$

SOLUTION

Applying $R_2 \to R_2 + R_3$, then

$$\begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin^2\left(x + \frac{7\pi}{2}\right) \\ 2\sin x \cos\left(\frac{3\pi}{2}\right) & 2\sin x \cos\left(\frac{5\pi}{2}\right) & 2\sin x \cos\left(\frac{7\pi}{2}\right) \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & \sin\left(x - \frac{7\pi}{2}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2\left(x + \frac{3\pi}{2}\right) & \sin^2\left(x + \frac{5\pi}{2}\right) & \sin^2\left(x + \frac{7\pi}{2}\right) \\ 0 & 0 & 0 \\ \sin\left(x - \frac{3\pi}{2}\right) & \sin\left(x - \frac{5\pi}{2}\right) & \sin\left(x - \frac{7\pi}{2}\right) \end{vmatrix} = 0$$

PROBLEM 11

Find the coefficient of x in the determinant
$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

SOLUTION

We know that if $f(x)$ be a polynomial in x, then coefficient of x^n in $f(x) = \frac{1}{n!} f^n(0)$.

$$\text{Let } f(x) = \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$

$$= m_0 + m_1x + m_2x^2 + \ldots$$

$\therefore \quad f'(x) =$

$$\begin{vmatrix} a_1b_1(1+x)^{a_1b_1-1} & a_1b_2(1+x)^{a_1b_2-1} & a_1b_3(1+x)^{a_1b_3-1} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} +$$

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ a_2b_1(1+x)^{a_2b_1-1} & a_2b_2(1+x)^{a_2b_2-1} & a_2b_3(1+x)^{a_2b_3-1} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} +$$

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ a_3b_1(1+x)^{a_3b_1-1} & a_3b_2(1+x)^{a_3b_2-1} & a_3b_3(1+x)^{a_3b_3-1} \end{vmatrix}$$

$$= m_1 + 2m_2x + \ldots$$

$$\therefore \quad f'(0) = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix} = m_1$$

$$= 0 + 0 + 0 =$$

$\therefore$ Coefficient of x in $f(x) = m_1 = 0$

PROBLEM 12

If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that
$$f_r(a) = g_r(a) = h_r(a),\ r = 1, 2, 3 \text{ and } F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$
then find $F'(x)$ at $x = a$. **[IIT – 85]**

SOLUTION

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$\therefore \quad F'(x) = \begin{vmatrix} f_1'(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2'(x) & f_3(x) \\ g_1(x) & g_2'(x) & g_3(x) \\ h_1(x) & h_2'(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3'(x) \end{vmatrix}$$

$$\therefore \quad F'(a) = \begin{vmatrix} f_1'(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2'(a) & f_3(a) \\ g_1(a) & g_2'(a) & g_3(a) \\ h_1(a) & h_2'(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3'(a) \end{vmatrix}$$

$\because \quad f_r(a) = g_r(a) = h_r(a),\ r = 1, 2, 3$

$\therefore \quad F'(a) = 0$

PROBLEM 13

$$\text{Let } f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \cos ec x \\ \cos^2 x & \cos^2 x & \cos ec^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix} \text{ then prove}$$

that $\int_0^{\pi/2} f(x)dx = -\left(\frac{\pi}{4} + \frac{8}{15}\right)$. **[IIT – 87]**

SOLUTION

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 \to R_1 - \sec x\, R_3$, we get

$$\begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \cos ec x - \cos x \\ \cos^2 x & \cos^2 x & \cos ec^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1, we get

$$= (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(\cos^4 x - \cos^2 x)$$

$$= \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x\right)\cos^2 (\cos^2 x - 1)$$

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\cos^2 x \sin^2 x}\right]\cos^2 x \sin^2 x$$

$$= -\sin^2 x - \cos^3 x(1 - \sin^2 x) = -\sin^2 x - \cos^5 x$$

$$\therefore \quad \int_0^{\pi/2} f(x)\,dx = -\int_0^{\pi/2} (\sin^2 x + \cos^5 x)\,dx$$

$$= -\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{4}{5}\cdot\frac{2}{3}\right] = -\left[\frac{\pi}{4} + \frac{8}{15}\right]$$

[Use Wallis formula

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{(n-1)(n-3)\ldots 2 \text{ or } 1}{(n)(n-2)\ldots 2 \text{ or } 1} \text{ and}$$

multiply above by $\pi/2$ when n is even.]

PROBLEM 14

Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$; $x + (\cos\alpha)y + (\sin\alpha)z = 0$; $-x + (\sin\alpha)y - (\cos\alpha)z = 0$ has a non-trivial solution.

For $\lambda = 1$, find all values of α. **[IIT – 93]**

SOLUTION

Given that, $\lambda, \alpha \in R$ and system of linear equations

$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$; $x + (\cos\alpha)y + (\sin\alpha)z = 0$; $-x + (\sin\alpha)y - (\cos\alpha)z = 0$

has a non-trivial solution $\Rightarrow \quad D = 0$

$$\Rightarrow \quad \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha(-\cos\alpha + \sin\alpha) + \cos\alpha(\sin\alpha + \cos\alpha) = 0$$

$$\Rightarrow \quad -\lambda + \sin\alpha\cos\alpha - \sin^2\alpha + \sin\alpha\cos\alpha + \cos^2\alpha = 0$$

$$\Rightarrow \quad \lambda = \cos^2\alpha - \sin^2\alpha + 2\sin\alpha\cos\alpha$$

$$\Rightarrow \quad \lambda = \cos 2\alpha + 2\sin\alpha$$

For $\lambda = 1$, $\cos 2\alpha + \sin 2\alpha = 1$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos 2\alpha + \frac{1}{\sqrt{2}}\cos 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos 2\alpha \cos\frac{\pi}{4} + \text{sins } 2\alpha \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(2\alpha - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad 2\alpha - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow \quad 2\alpha = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4} \quad \Rightarrow \quad \alpha = n\pi + \frac{\pi}{4} \text{ or } n\pi$$

PROBLEM 15

Let $a > 0$, $d > 0$. Find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

[IIT – 96]

SOLUTION

Let us denote the given determinant by Δ and taking $\frac{1}{a(a+d)(a+2d)}$ as common from R_1 and $\frac{1}{(a+d)(a+2d)(a+3d)}$ from R_2 and $\frac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3,

we get $\Delta = \dfrac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta_1$

$$\text{where } \Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta_1 = (2d^2)\begin{vmatrix} a+2d & a \\ d & d \end{vmatrix} = (2d^2)\,(d)\,(a + 2d - a) = 4d^4$$

$$\Rightarrow \quad \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

PROBLEM 16

Find the parameter on which the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix} \text{ does not depend upon.}$$

[IIT – 97]

SOLUTION

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$\Delta = \begin{vmatrix} 1+a^2 & a & a \\ \cos(p-d)x+\cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x+\sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1+a^2 & a & a \\ 2\cos px\cos dx & \cos px & \cos(p+d)x \\ 2\sin px\cos dx & \sin px & \sin(p+d)x \end{vmatrix}$$

Applying $C_1 \to C_1 - 2\cos dx\, C_2$

$$\Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$$

Expanding along C_1, we get

$\Rightarrow$ $\Delta = (1 + a^2 - 2a \cos dx)[\sin(p + d)x \cos px - \sin px \cos (p + d)x]$

$\Rightarrow$ $\Delta = (1 + a^2 - 2a \cos dx) [\sin \{(p + d)x - px\}]$

$\Rightarrow$ $\Delta = (1 + a^2 - 2a \cos dx) [\sin dx]$

which is independent of p.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. For positive numbers x, y, z the value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is

(A) 0
(B) 1
(C) 2
(D) None of these

2. Find the roots of the equation $\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0$.

(A) α and β
(B) β and γ
(C) γ and α
(D) All of these.

3. If x, y, z are in A.P, then the value of the determinant $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$, is

(A) 2
(B) –2
(C) 0
(D) None of these

4. If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$ then

(A) $f(\pi/4) = 1$
(B) $f(\pi/4) = 2$
(C) $f(\pi/8) = 3$
(D) None of these

5. If $f(x) = \begin{vmatrix} x^2-4x+6 & 2x^2+4x+10 & 3x^2-2x+16 \\ x-2 & 2x+2 & 3x-1 \\ 1 & 2 & 3 \end{vmatrix}$ then the function $f(x)$ is

(A) Increasing when $x \in (3, 5)$
(B) Increasing when $x \in (-3, -5)$
(C) Decreasing when $x \in (3, 5)$
(D) constant for all real x

6. If $\Delta_2 = \begin{vmatrix} y^5z^6(z^3-y^3) & x^4z^6(x^3-z^3) & y^4z^5(y^3-z^3) \\ y^2z^3(y^6-z^6) & xz^3(z^6-x^6) & xy^2(x^6-y^6) \\ y^2z^3(z^3-y^3) & xz^3(x^3-z^3) & xy^2(y^3-x^3) \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} x & y^2 & z^3 \\ x^4 & y^5 & z^6 \\ x^7 & y^8 & z^9 \end{vmatrix}$. Then $\Delta_1\Delta_2$ is equal to

(A) Δ_1^3
(B) Δ_2^2
(C) Δ_2^4
(D) None of these

7. If $\Delta_r = \begin{vmatrix} 2^{r-1} & x & 2^n-1 \\ 2.3^{r-1} & y & 3^n-1 \\ 4.5^{r-1} & z & 5^n-1 \end{vmatrix}$, then $\sum_{r=1}^{n} \Delta_r$ is equal to

(A) 500 only if $n = 15$
(B) 500 only if $x = y = z$
(C) 500 only if $x + y + z = 0$
(D) None of these

8. If x, y and z are the positive integers then the sum of all the possible values of x for which $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 30$ is

(A) 5
(B) 3
(C) 6
(D) None of these

9. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x = 0$ is

(A) p
(B) $p + p^2$
(C) $p + p^3$
(D) independent of p

10. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(A) 0
(B) 10
(C) 50
(D) none

11. If $\begin{vmatrix} 3^2+k & 4^2 & 3^2+3+k \\ 4^2+k & 5^2 & 4^2+4+k \\ 5^2+k & 6^2 & 5^2+5+k \end{vmatrix} = 0$, then the value of k is

(A) 2
(B) 1
(C) – 1
(D) 0

12. The determinant $\begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-a)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix}$ is equal to

(A) $(a-b)^2 (b-c)^2 (c-a)^2$
(B) 0
(C) $2(a-b)^2 (b-c)^2 (c-a)^2$
(D) None of these

13. If $0 < \theta < \pi/2$ and $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$ then θ is equal to

(A) $\pi/24, 5\pi/24$ (B) $5\pi/24, 7\pi/24$
(C) $7\pi/24, 11\pi/24$ (D) None of these

14. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ then

(A) $\Delta_1 = 3\Delta_2^2$ (B) $\frac{d}{dx}(\Delta_1) = 3\Delta_2^2$
(C) $\frac{d}{dx}(\Delta_1) = 3\Delta_2$ (D) None of these

15. The value of the determinant (when $n \in N$) $\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix} =$

(A) $(n!)^3$ $((2n^3 + 8n^2 + 10n + 4)$
(B) $(n!)^3$ $(2n^2 + 8n + 10)$
(C) $(n!)^2$ $((2n^3 + 8n^2 + 10n + 4)$
(D) None of these

16. If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Px - 12$ then

(A) $P = 24$ (B) $P = -24$
(C) $P = 0$ (D) $P = 12$

17. Find the number of distinct real roots of the equation $\begin{vmatrix} x^2 & (x-1)^2 & (x-2)^2 \\ (x-1)^2 & (x-2)^2 & (x-3)^2 \\ (x-2)^2 & (x-3)^2 & (x-4)^2 \end{vmatrix} = 0.$

(A) 6 (B) 3
(C) 2 (D) zero

18. The determinant $\Delta = \begin{vmatrix} -1+x\sin\alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ 13+x\sin\beta & \cos(x+\beta) & \sin(x+\beta) \\ -12+x\sin\gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$ is independent of

(A) x (B) x if $x = \alpha = \beta = \gamma$
(C) x if $x = \alpha = \beta = \gamma$ (D) None of these

19. If $f(x) = \begin{vmatrix} x^5 & |\sin x| & 2x^4 \\ \tan^3 x & 1 & \sec 2x \\ \sin^3 x & x^4 & 5 \end{vmatrix}$, then $\int_{-\pi/2}^{\pi/2} f(x)\,dx$ is equal to

(A) 2 (B) –2
(C) 0 (D) None of these

20. If $U_n = \begin{vmatrix} 1 & k & k \\ 2n & k^2+k+1 & k^2+k \\ 2n-1 & k^2 & k^2+k+1 \end{vmatrix}$ and $\sum_{n=1}^{k} U_n = 110$, then k is equal to

(A) 10 (B) 9
(C) 6 (D) None of these

21. The value of the determinant

$$\begin{vmatrix} 2 & \tan A\cot B + \cot A\tan B & \tan A\cot C + \cot A\tan C \\ \tan B\cot A + \tan A\cot B & 2 & \tan B\cot C + \cot B\tan C \\ \tan C\cot A + \cot C\tan A & \tan C\cot B + \cot C\tan B & 2 \end{vmatrix}$$ is

(A) 1 (B) 2
(C) 0 (D) None of these

22. If $f(x) = \begin{vmatrix} 4x-4 & (x-2)^2 & x^3 \\ 8x-4\sqrt{2} & \left(x-2\sqrt{2}\right)^2 & (x+1)^3 \\ 12x-4\sqrt{3} & \left(x-2\sqrt{3}\right)^2 & (x-1)^3 \end{vmatrix}$ then the coefficient of x in $f(x)$ is

(A) $64(5 - \sqrt{2} - \sqrt{3})$ (B) $64(5 + \sqrt{2} - \sqrt{3})$
(C) $64(5 + \sqrt{2} + \sqrt{3})$ (D) None of these

23. Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $a_{pq} = (i^p)^q$, where $i = \sqrt{-1}$. The value of Δ is

(A) real and positive
(B) real and negative
(C) 0
(D) imaginary

24. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$and if the system of equations$\alpha u + \beta v + \gamma w = 0$, $\beta u + \gamma v + \alpha w = 0$ and $\gamma u + \alpha v + \beta w = 0$ possess non-zero solutions, then $a^3 - 6c = 0$then $2c$ is

(A) ab (B) $2ab$
(C) $3ab$ (D) none of these

25. The total number of roots of $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$

(A) 1 (B) 2
(C) 3 (D) 4

26. If $pqr \neq 0$ and the system of equation $(p + a)x + by + cz = 0$, $ax + (q + b)y + cz = 0$, $ax + by + (r + c)z = 0$ has a non-trivial solution then $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} =$

(A) 1 (B) 2
(C) 0 (D) –1

27. The value of the determinant $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$ is

(A) $2(a + b + c)$ (B) $2(a + b + c)^2$
(C) $2(a + b + c)^3$ (D) $(2a + 2b + 2c)^3$

28. $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$, then a, b, c are in

(A) A.P.
(B) G.P.
(C) H.P.
(D) None of these

29. The number of distinct positive values of a 2×2 determinant whose entries are from the set $\{-1, 0, 1\}$ is

(A) 2
(B) 3
(C) 5
(D) 6

30. $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$ then $\dfrac{p}{p-a} + \dfrac{q}{q-b} + \dfrac{r}{r-c} =$ ______

(A) 1
(B) 0
(C) 2
(D) 3

31. $\Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$, then the value of

$\Delta_0 = \begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ac+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix}$ is -

(A) 3Δ
(B) Δ^2
(C) Δ^3
(D) None of these

32. $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$ then the Maximum value of $f(x)$ is

(A) 2
(B) 4
(C) 6
(D) 8

33. If $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0$, then

(A) a/b is one of the cube root of unity
(B) a is one of the cube roots of unity
(C) b is one of the cube roots of unity
(D) a/b is one of the cube roots of -1

34. If the system of linear equations

$$x + 2ay + az = 0;\ x + 3by + bz = 0;\ x + 4cy + cz = 0$$

has a non-zero solution, then a, b, c

(A) are in G.P.
(B) are in H.P.
(C) satisfy $a + 2b + 3c = 0$
(D) are in A.P.

35. Let $\Delta_1 = \begin{vmatrix} ap^2 & 2ap & 1 \\ aq^2 & 2aq & 1 \\ ar^2 & 2ar & 1 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} apq & a(p+q) & 1 \\ aqr & a(q+r) & 1 \\ arp & a(r+p) & 1 \end{vmatrix}$ then

(A) $\Delta_1 = \Delta_2$
(B) $\Delta_2 = 2\Delta_1$
(C) $\Delta_1 = 2\Delta_2$
(D) $\Delta_1 = 2\Delta_2 = 0$

36. $\begin{vmatrix} 2xy & x^2 & y^2 \\ x^2 & y^2 & 2xy \\ y^2 & 2xy & x^2 \end{vmatrix} =$

(A) $(x^3+y^3)^2$
(B) $(x^2+y^2)^3$
(C) $-(x^2+y^2)^2$
(D) $-(x^3+y^3)^2$

37. $\Delta = \begin{vmatrix} x & 2y-z & -z \\ y & 2x-z & -z \\ z & 2y-z & 2x-2y-z \end{vmatrix}$ then

(A) $(x-y)$ is a factor of Δ
(B) $(x-y)^2$ is a factor of Δ
(C) $(x-y)^3$ is a factor of Δ
(D) Δ is independent of z

38. The number of positive integral solutions of the equation $\begin{vmatrix} y^3+1 & y^2z & y^2x \\ yz^2 & z^3+1 & z^2x \\ yx^2 & x^2z & x^3+1 \end{vmatrix} = 11$ is

(A) 1
(B) 2
(C) 3
(D) 4

39. Let $f(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2(ax+b)+1 & 2ax+b \end{vmatrix}$ where a and b are real constants and $a \neq 0$, then $f^{-1}(x)$ equal

(A) $\dfrac{x+b}{2a}$
(B) $\dfrac{x-b}{a}$
(C) $\dfrac{x-b}{2a}$
(D) $\dfrac{2x+b}{a}$

40. If α, β are the roots of the equation $x^2 - 5x + 10 = 0$, then $\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$ is equal to

(A) 540
(B) –540
(C) 360
(D) –360

41. If $\Delta_i = \begin{vmatrix} 2^{i-1} & \dfrac{1}{i(i+1)} & \sin i\theta \\ \displaystyle\sum_{r=1}^{i} r & \displaystyle\sum_{r=1}^{i} r^2 & \displaystyle\sum_{r=1}^{i} r^3 \\ 2^n - 1 & \dfrac{n}{n+1} & \dfrac{\sin\left(n+\frac{1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \end{vmatrix}$. Then $\displaystyle\sum_{i=1}^{n} \Delta_i$ is

(A) $n!$
(B) $n(n+1)$
(C) 0
(D) None of these

42. The system of equations $x - ky - z = 0, kx - y - z = 0, x + y - z = 0$ has a non-zero solution. Then possible values of k are

(A) –1, 2
(B) 1, 2
(C) 0, 1
(D) –1, 1

43. If $D_r = \begin{vmatrix} r & x & \dfrac{n((n+1))}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \dfrac{n((3n-1))}{2} \end{vmatrix}$ then $\sum_{r=1}^{n} D_r =$

(A) 0 (B) –1
(C) 1 (D) n

44. (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle. Then the equation $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ represents

(A) Altitude through A
(B) Median through A
(C) Equation of the $\perp^{lar}$ bisector of BC
(D) Angular bisector of A

45. If $\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} = 2010$ and if $\begin{vmatrix} c-a & c-b & ab \\ a-b & a-c & bc \\ b-c & b-a & ca \end{vmatrix} - \begin{vmatrix} c-a & c-b & c^2 \\ a-b & a-c & a^2 \\ b-c & b-a & b^2 \end{vmatrix} = p$ then the number of positive divisors of the number p is

(A) 36 (B) 49
(C) 64 (D) 81

46. If α, β, γ are the real roots of a cubical polynomial satisfying, $\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ (\alpha+1)^2 & (\beta+1)^2 & (\gamma+1)^2 \\ (\alpha-1)^2 & (\beta-1)^2 & (\gamma-1)^2 \end{vmatrix} = 0$ then

(A) all roots must be equal
(B) at least two roots are equal
(C) at least one root is non-zero
(D) none of these

47. The equations $(\lambda-1)x + (3\lambda+1)y + 2\lambda z = 0$, $(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$, $2x + (3\lambda+1)y + 3(\lambda-1)z = 0$. Gives non-trivial solution for some values of λ, find the ratio $x : y : z$ when λ has the smallest of these values:

(A) $3:2:1$ (B) $3:3:2$
(C) $1:3:1$ (D) $1:1:1$

48. If $\begin{vmatrix} a & l & m \\ l & b & n \\ m & n & c \end{vmatrix} \begin{vmatrix} bc-n^2 & mn-lc & ln-bm \\ mn-lc & ac-m^2 & lm-an \\ ln-bm & lm-an & ab-l^2 \end{vmatrix} = 64$ then the value of $\begin{vmatrix} 2a+3l & 3l+5m & 5m+4a \\ 2l+3b & 3b+5n & 5n+4l \\ 2m+3n & 3n+5c & 5c+4m \end{vmatrix}$ equals

(A) 120 (B) 240
(C) 360 (D) 480.

49. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value –1. Then

(A) C is empty (B) B has as many elements as C
(C) $A = B \cup C$ (D) B has twice as many elements as C

50. If

$$\begin{vmatrix} \binom{x}{r} & \binom{x+1}{r+1} & \binom{x+2}{r+2} \\ \binom{y}{r} & \binom{y+1}{r+1} & \binom{y+2}{r+2} \\ \binom{z}{r} & \binom{z+1}{r+1} & \binom{z+2}{r+2} \end{vmatrix} = \lambda \begin{vmatrix} \binom{x}{r} & \binom{x}{r+1} & \binom{x}{r+2} \\ \binom{y}{r} & \binom{y}{r+1} & \binom{y}{r+2} \\ \binom{z}{r} & \binom{z}{r+1} & \binom{z}{r+2} \end{vmatrix}$$ then λ

(A) 0 (B) 1
(C) –1 (D) 2

51. If $\begin{vmatrix} \alpha & \beta \\ \gamma & -\alpha \end{vmatrix}$ is to be square root of a determinant of the two rowed unit matrix, then α, β, γ should satisfy the relation

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 - \beta\gamma = 0$
(C) $1 - \alpha^2 + \beta\gamma = 0$ (D) $\alpha^2 + \beta\gamma = 2$

52. If $\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ \alpha & \beta & \lambda \end{vmatrix}$ then $\alpha + \beta + \lambda =$

(A) $x^3 + y^3 + z^3$ (B) $x^2 + y^2 + z^2$
(C) $xy + yz + zx$ (D) $x + y + z$

53. If p, q, r are in A.P. then the value of determinant $\begin{vmatrix} a^2 + 2^{n+1} + 2p & b^2 + 2^{n+2} + 3q & c^2 + p \\ 2^n + p & 2^{n+1} + q & 2q \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{vmatrix}$

(A) 1 (B) 0
(C) $a^2b^2c^2 - 2^n$ (D) $\left(a^2 + b^2 + c^2\right) - 2^n q$

54. If $x > m$, $y > n$, $z > r$ ($x, y, z > 0$) such that $\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$. The greatest value of $\dfrac{xyz}{(x-m)(y-n)(z-r)}$ is

(A) 27 (B) 8/27
(C) 64/27 (D) 128/27

55. If $a_1, a_2, a_3 \cdots a_9$ are in H.P and $a_4 = 5, a_5 = 4$ then the value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ is

(A) $\dfrac{31}{15}$ (B) $\dfrac{41}{18}$

(C) $\dfrac{50}{21}$ (D) $\dfrac{61}{27}$

56. If $x_i = a_ib_ic_i$, $i = 1, 2, 3$ are three - digit positive integer such that each x_i is a multiple of 19, then for some integer n, $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is given by

(A) $19n + 1$ (B) $19n + 2$
(C) $19n$ (D) $19n + 3$

57. If $\Delta(x) = \begin{vmatrix} x-2 & (x-1)^2 & x^3 \\ x-1 & x^2 & (x+1)^3 \\ x & (x+1)^2 & (x+2)^3 \end{vmatrix}$, then coefficient of x in $\Delta(x)$ is:

(A) –4 (B) –2
(C) –6 (D) 0

58. Consider the system of linear equations in x, y and z, $(\sin 3\theta)x - y + z = 0$, $(\cos 2\theta)x + 4y + 3z = 02x + 7y + 7z = 0$ then which of the following can be the values of for which the system has a non-trivial solution

(A) $n\pi + (-1)^n \dfrac{\pi}{6}, \forall n \in z$ (B) $n\pi + (-1)^n \dfrac{\pi}{3}, \forall n \in z$
(C) $n\pi + (-1)^n \dfrac{\pi}{9}, \forall n \in z$ (D) $2n\pi + \dfrac{\pi}{10}, \forall n \in Z$

59. If $a - 2b + c = 1$, then the value of $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ is

(A) x (B) $-x$
(C) –1 (D) 1

60. If $p + q + r = 0$ and $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = k \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, then $K =$

(A) 0 (B) abc
(C) pqr (D) $a + b + c$

61. $\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} - \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} =$

(A) 0 (B) 2^n
(C) ${}^{x+y+z}C_r$ (D) ${}^{x+y+z}C_{r+2}$

62. If $a, b, c, d > 0$; $x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$, then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} =$

(A) 1 (B) – 1
(C) 0 (D) none of these

63. If $Y = SX$, $Z = tX$ all the variables being differentiable functions of x and lower suffices denote the derivative with respect to x and $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} \div \begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix} = X^n$, then $n =$

(A) 1 (B) 2
(C) 3 (D) 4

64. If $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and the equation $f(x) - x = 0$ has imaginary roots α and β and γ and δ be the roots of $f(f(x)) - x = 0$, then $\begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$ is

(A) 0 (B) purely real
(C) purely imaginary (D) none of these

65. The digits A, B, C are such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72, then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by

(A) 76 (B) 144
(C) 216 (D) 276

66. If $f(x)$, $g(x)$, $h(x)$ are polynomials of degree 4 and $\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ p & q & r \end{vmatrix} = mx^4 + nx^3 + rx^2 + 5x + t$ be an identity in x, then the value of $\begin{vmatrix} f'''(0) - f''(0) & g'''(0) - g''(0) & h'''(0) - h''(0) \\ a & b & c \\ p & q & r \end{vmatrix}$ is

(A) $(3n - r)$ (B) $2(3n - r)$
(C) $3(3n - r)$ (D) $3n + r$

67. Let $x > 0, y > 0, z > 0$ are respectively the 2nd, 3rd, 4th, terms of a G.P and $\Delta = \begin{vmatrix} x^k & x^{k+1} & x^{k+2} \\ y^k & y^{k+1} & y^{k+2} \\ z^k & z^{k+1} & z^{k+2} \end{vmatrix} = (r-1)^2 \left(1 - \dfrac{1}{r^2}\right)$ (where r is the common ratio) then

(A) $k = -1$ (B) $k = 1$
(C) $k = 0$ (D) None of these

68. The value of determinant $\begin{vmatrix} 1 + a^2 - b^2 & 2ab & -2b \\ 2ab & 1 - ab^2 + b^2 & 2a \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix}$ is equal to

(A) $(1 - a^2 - b^2)^3$ (B) $(a + b + 1)^2\,(ab + b + a)$
(C) $(1 + a^2 + b^2)^3$ (D) $(1 - a^2 + b^2)^3$

69. If $\begin{vmatrix} a^2 + \lambda^2 & ab + c\lambda & ca - b\lambda \\ ab - c\lambda & b^2 + \lambda^2 & bc + a\lambda \\ ac + b\lambda & bc - a\lambda & c^2 + \lambda^2 \end{vmatrix} \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = (1 + a^2 + b^2 + c^2)^3$ then λ is equal to

(A) 0 (B) 1
(C) –1 (D) ±1

70. Constant term in $f(x) = \begin{vmatrix} x & (1+\sin x)^3 & \cos x \\ 1 & In(1+x) & 2 \\ x^2 & (1+x)^2 & 0 \end{vmatrix}$ when $f(x)$ is expressed polynomial in x, is

(A) 0 (B) –1
(C) 1 (D) 2

71. The system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ possess a non-trivial solution over the set of rational, then $2k$ is an integral element of the interval

(A) [10, 20] (B) (20, 30)
(C) [30, 40] (D) (40, 50)

72. If $p + q + r = 0 = a + b + c$, then the value of the determinant $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$ is

(A) 0 (B) $pq + qb + rc$
(C) 1 (D) none of these

73. If a determinant of order 3×3 is formed by using the numbers 1 or – 1 then minimum value of determinant is

(A) –2 (B) –4
(C) 0 (D) –8

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If $\begin{vmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & \cos 2\phi \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix} = 0$ then

(A) $\theta = \pi$ and $\phi = \dfrac{\pi}{2}$ (B) $\theta = -\pi$ and $\phi = \dfrac{\pi}{2}$
(C) $\theta = -\pi$ and $\phi = \pi$ (D) None of these

2. The value of θ lying between 0 and $\pi/2$ and satisfying the equation $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$ are

(A) $\dfrac{7\pi}{24}$ (B) $\dfrac{5\pi}{24}$
(C) $\dfrac{11\pi}{24}$ (D) $\dfrac{\pi}{24}$

3. Consider the system of linear equations $x + 2y + z = 1$, $2x + y + z = \alpha$, $4x + 5y + 3z = \alpha^2$ then the system has

(A) infinitely many solutions when $\alpha = -1$ or 2
(B) infinitely many solutions when $\alpha = -2$ or 1.
(C) no solutions when $\alpha \in R - \{-1, 2\}$
(D) no solutions when $\alpha \in R - \{-2, 1\}$

4. Consider the system of equations:
$x\sin\theta - 2y\cos\theta - az = 0, x + 2y + z = 0, -x + y + z = 0, \theta \in R$

(A) The given system will have infinite solutions for $a = 2$
(B) The number of integer values of a is 3 for the system to have nontrivial solutions.
(C) For $a = 1$ the system will have infinite solutions.
(D) For $a = 3$ the system will have unique solution.

5. Let $f(x) = \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$ and $\alpha < \beta < \gamma$ be the roots of $f(x) = 0$. then

(A) $\gamma = 2$ (B) $f(x) < 0$ for $\alpha < x < \beta$
(C) $f(x) > 0$ for $\beta < x < \gamma$ (D) $f(x) > 0$ for $\alpha < x < \beta$

6. If maximum and minimum value of the determinant $\begin{vmatrix} 1+\cos^2 x & \sin^2 x & \cos 2x \\ \cos^2 x & 1+\sin^2 x & \cos 2x \\ \cos^2 x & \sin^2 x & 1+\cos 2x \end{vmatrix}$ are α and β respectively then

(A) $\alpha^2 + \beta^{101} = 10$ (B) $\alpha^3 - \beta^{99} = 26$
(C) $2\alpha^2 - 18\beta^{11} = 0$ (D) $\alpha.\beta = 4$

7. The system of equations

$(a\alpha + b)x + ay + bz = 0$
$(b\alpha + c)x + by + cz = 0$ has a non -trivial solution, if
$(a\alpha + b)y + (b\alpha + c)z = 0$

(A) a, b, c are in A.P (B) a, b, c are in G.P
(C) a, b, c are in H.P (D) α is a root of $ax^2 + 2bx + c = 0$

8. $f(x) = \begin{vmatrix} 3 & 3x & 3x^2+2a^2 \\ 3x & 3x^2+2a^2 & 3x^3+6a^2x \\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$ then

(A) $f'(x) = 0$ (B) $f(x)$ is independent of x.

(C) $\int_0^1 f(x)dx = 4a^6$ (D) $y^2 = f(x)$ represents a pair of lines

9. If $\begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+4 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix} = ax^3 + bx^2 + cx + d$ then which of the following are correct?

(A) $a = 0$ (B) $b = 0$
(C) $c = 0$ (D) $d = 0$

10. If $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ which of the following are correct?

(A) $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^2$

(B) $\begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix} = \Delta^2$

(C) $\Delta = 0 \Rightarrow$ either $a = b = c$ or $a + b + c = 0$

(D) $\Delta = 3abc - a^3 - b^3 - c^3$

11. If $f(x)$ and $g(x)$ are functions such that $f(x + y) = f(x) \cdot g(y) + g(x) f(y)$ then $\begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha+\theta) \\ f(\beta) & g(\beta) & f(\beta+\theta) \\ f(\gamma) & g(\gamma) & f(\gamma+\theta) \end{vmatrix}$ is independent of ________

(A) α (B) β

(C) γ (D) θ

12. If the equations $x + ay - z = 0$, $2x - y + az = 0$, $ax + y + 2z = 0$ have nontrivial solutions, then $a =$

(A) 2 (B) –2

(C) $1+\sqrt{3}$ (D) $1-\sqrt{3}$

13. Let $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix} = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

(A) $f = 1$ (B) $e = 3$

(C) $a + c = -1$ (D) $b + d = 1$

14. If $\Delta = \begin{vmatrix} a & c & c-a & a+c \\ c & b & b-c & b+c \\ a-b & b-c & 0 & a-c \\ x & y & z & 1+x+y \end{vmatrix} = 0$, then

(A) a, b, c are in A.P

(B) b, a, c are in A.P

(C) a, c, b are in G.P

(D) a, c, b are in H.P

15. If

$$\Delta(x) = \begin{vmatrix} x^2+4x-3 & 2x+4 & 13 \\ 2x^2+5x-9 & 4x+5 & 26 \\ 8x^2-6x+1 & 16x-6 & 104 \end{vmatrix} = ax^3 + bx^2 + cx + d,$$ then

(A) $a = 3$ (B) $a = 0$

(C) $c = 0$ (D) $b = 2$

16. If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$ where a, b being positive integers, then

(A) Constant term of $f(x)$ is 0

(B) Coefficient of x in $f(x)$ is 0

(C) Constant term in $f(x)$ is $(a - b)$

(D) Coefficient of x in $f(x)$ is $(a - b)$

17. The determinant $\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos nx & \cos(n+1)x \\ \sin(n-1)x & \sin nx & \sin(n+1)x \end{vmatrix} = 0$, if

(A) $a = 0$ (B) $\cos x = 0$

(C) $\sin x = 0$ (D) $\cos x = \dfrac{1+a^2}{2a}$

18. The system of equation is $x - y\cos\theta + z\cos 2\theta = 0$, $x\cos 2\theta - y + z\cos\theta = 0$

$x\cos 2\theta - y\cos\theta + z = 0$ has non-trivial solution for θ equals to

(A) $\dfrac{8\pi}{3}$ (B) $\dfrac{\pi}{6}$

(C) $\dfrac{2\pi}{3}$ (D) $\dfrac{\pi}{12}$

19. If $A = \displaystyle\int_1^{\sin\theta} \frac{t}{1+t^2}dt$ and $B = \displaystyle\int_1^{\operatorname{cosec}\theta} \frac{dt}{t(1+t^2)}$, $\theta \in \left(0, \dfrac{\pi}{2}\right)$ then value of determinant $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix}$

(A) depends of θ, A but is independent from B

(B) depends on θ, B but is independent from A

(C) depends on A, B but is independent from θ

(D) independent from A, B and θ.

20. If p, q, r, s are in A.P. and $F(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$ such that $\displaystyle\int_0^2 F(x)dx = -4$ then the common difference of the A.P. can be:

(A) -1 (B) $\dfrac{1}{2}$

(C) 1 (D) 2

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let $\Delta \neq 0$ and Δ^c denotes the determinant of cofactors, then $\Delta^c = \Delta^{n-1}$, where n (>0) is the order of Δ.

On the basis of above information, answer the following questions:

1. If a, b, c are the roots of the equation $x^3 - px^2 + r = 0$, then the value of

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

(A) p^2 (B) p^4
(C) p^6 (D) p^9

2. If a, b, c are the roots of the equation $x^3 - 3x^2 + 3x + 7 = 0$, then the value of

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

(A) 9 (B) 27
(C) 81 (D) 0

3. Suppose $a, b, c \in R$, $a + b + c > 0$, $A = bc - a^2$, $B = ca - b^2$ and $C = ab - c^2$ and $\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = 49$ then $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ equals

(A) – 7 (B) 7
(C) – 2401 (D) 2401

Comprehension 2

$y = f(x)$ is a function satisfying

(a) $f(0) = 2, f(1) = 1$ **(b)** f has minimum at $x = 5/2$

(c) For all $x \in R$

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$ by the above information

4. $y = f(x)$ represents a parabola. Then equation of directrix is
(A) $4y + 9 = 0$ (B) $8y + 9 = 0$
(C) $12y + 9 = 0$ (D) $16y + 9 = 0$

5. The roots of $f(x) = 7/2$ are
(A) imaginary
(B) Two positive roots
(C) Two negative roots
(D) One positive and one negative

Comprehension 3

Consider, $$f(x) = \begin{vmatrix} 5 & 7 & 3 \\ 2 & 9g^2(x) & 2x^2 \\ x & 3g(x)-x^2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 4 & 5 \\ 2x^2 & 2 & 9g^2(x) \\ 1 & x & 3g(x)-x^2 \end{vmatrix} + \begin{vmatrix} 2 & x & 0 \\ 9g^2(x) & 3g(x)-x^2 & -2 \\ 2x^2 & 1 & -1 \end{vmatrix}$$

where $g(x)$ is a polynomial function such that $m = \int_0^1 f(x)dx$ is minimum.

6. The value of m equals :
(A) $\frac{4}{5}$ (B) $\frac{2}{5}$
(C) $\frac{1}{5}$ (D) 0

7. The number of points of inflection on the graph of the function $y = \phi(x)$ where $\phi(x) = \int_0^x \left(\int_0^x g(y)dy - (e^t - e^{-t}) \right) dt$ is (are):
(A) 0 (B) 1
(C) 2 (D) 3

Comprehension 4

Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

M_{ij} = Minor of the element of i^{th} row and j^{th} column

C_{ij} = Cofactor of the element of i^{th} row and j^{th} column

8. Value of $b_1 . C_{31} + b_2 . C_{32} + b_3 . C_{33}$ is
(A) 0 (B) Δ
(C) 2Δ (D) Δ^2

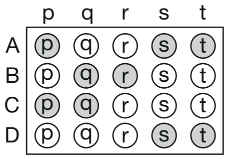
9. If all the elements of the determinant are multiplied by 2, then the value of new determinant is
(A) 0 (B) 8Δ
(C) 2Δ (D) $2^9\Delta$

10. $a_3 M_{13} - b_3 . M_{23} + d_3 . M_{33}$ is equal to
(A) 0 (B) 4Δ
(C) 2Δ (D) Δ

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Given system of equation $x + y + 2z = a, - 2x - z = b, x + 3y + 5z = c$, (a, b, c) is an ordered triplet then

COLUMN-I	COLUMN-II
(A) (0, 0, 0)	(p) no solution
(B) (–1, 1, 1)	(q) infinite solution
(C) (1, 1, 4)	(r) non-trivial solution
(D) (2, 1, – 1)	(s) trivial solution

2. If α,β,γ are the roots of $x^3-3x^2+3x-1=\begin{vmatrix}1^2 & 2^2 & 3^2\\ 2^2 & 3^2 & 4^2\\ 3^2 & 4^2 & 5^2\end{vmatrix}$, then

COLUMN-I	COLUMN-II
(A) $\alpha\beta\gamma$	(p) –8
(B) $\dfrac{\alpha-1}{\beta-1}+\dfrac{\beta-1}{\gamma-1}+\dfrac{\gamma-1}{\alpha-1}$	(q) 0
(C) $(\alpha-1)(\beta-1)(\gamma-1)$	(r) $\dfrac{3}{\omega}$
(D) $\begin{vmatrix}\alpha-1 & \beta-1 & \gamma-1\\ \beta-1 & \gamma-1 & \alpha-1\\ \gamma-1 & \alpha-1 & \beta-1\end{vmatrix}$	(s) –7
	(t) $3\omega^2$

3. Let $p(\theta)=\begin{vmatrix}-\sqrt{2} & \sin\theta & \cos\theta\\ 1 & \cos\theta & \sin\theta\\ -1 & \sin\theta & -\cos\theta\end{vmatrix}$, $q(\theta)=\begin{vmatrix}\sin 2\theta & -1 & 1\\ \cos 2\theta & 4 & -3\\ 2 & 7 & -5\end{vmatrix}$,

$r(\theta)=\begin{vmatrix}\cos\theta & \sin\theta & \cos\theta\\ -\sin\theta & \cos\theta & \sin\theta\\ -\cos\theta & -\sin\theta & \cos\theta\end{vmatrix}$ and

$s(\theta)=\begin{vmatrix}\sec^2\theta & 1 & 1\\ \cos^2\theta & \cos^2\theta & \operatorname{cosec}^2\theta\\ 1 & \cos^2\theta & \cot^2\theta\end{vmatrix}$

Match the functions on the left with their range on the right.

COLUMN-I	COLUMN-II
(A) $p(\theta)$	(p) [0, 1]
(B) $q(\theta)$	(q) $\left[0,2\sqrt{2}\right]$
(C) $r(\theta)$	(r) [–2, 2]
(D) $s(\theta)$	(s) $\left[-\sqrt{5}-2,\sqrt{5}-2\right]$

4. Match the parameter on which the value of the determinant does not depend upon

COLUMN-I	COLUMN-II
(A) $\begin{vmatrix}1 & x & x^2\\ \cos(a-b)y & \cos ay & \cos(a+b)y\\ \sin(a-b)y & \sin ay & \sin(a+b)y\end{vmatrix}$	(p) a
(B) $\begin{vmatrix}x^2+y^2 & ax+by & x+y\\ ax+by & a^2+b^2 & a+b\\ x+y & a+b & 2\end{vmatrix}$	(q) b

(Continued)

COLUMN-I	COLUMN-II
(C) $\begin{vmatrix} 1 & a & a^2+b \\ 1 & b & b^2+a \\ 1 & 1 & 1+ab \end{vmatrix}$	(r) x
(D) $\dfrac{1}{(a^2+b^2+x^2)^3}\begin{vmatrix} a^2+(b^2+x^2)\cos y & ab(1-\cos y) & ax(1-\cos y) \\ ab(1-\cos y) & b^2+(x^2+a^2)\cos y & bx(1-\cos y) \\ ax(1-\cos y) & bx(1^2-\cos y) & x^2+(a^2+b^2)\cos y \end{vmatrix}$	(s) y

5. Match the following: -

COLUMN-I	COLUMN-II
(A) Let $\lvert A\rvert = \lvert a_{ij}\rvert_{3\times3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $\lvert B\rvert$ the resulting determinant, where $k_1\lvert A\rvert + k_2\lvert B\rvert = 0$. Then $k_1 + k_2 =$	(p) 0
(B) The maximum value of a third order determinant each of its entries are ±1 equals	(q) 4
(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$	(r) 1
(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$ then $A + 2B =$	(s) 3

6. Match the following for system of linear equations.

$2x - 3y + 5z = 12$
$3x + y + \lambda z = \mu$
$x - 7y + 8z = 17$

COLUMN-I	COLUMN-II
(A) Unique solution	(p) $\lambda = 2, \mu = 7$
(B) Infinite solutions	(q) $\lambda \neq 2, \mu = 7$
(C) No solutions	(r) $\lambda \neq 2, \mu \neq 7$
(D) Consistent system of equation	(s) $\lambda = 2, \mu \neq 7$

7.

COLUMN-I	COLUMN-II
(A) Coefficient of x in $f(x) = \begin{vmatrix} x & (1+\sin x)^3 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$ is	(p) 10

(Continued)

COLUMN-I	COLUMN-II
(B) Value of $\begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$ is $(\theta \in R)$	(q) 0
(C) If a, b, c are in A.P. and $f(x) = \begin{vmatrix} x+a & x^2+1 & 1 \\ x+b & 2x^2-1 & 1 \\ x+c & 3x^2-2 & 1 \end{vmatrix}$ then $f'(0)$ is	(r) –12
(D) If $\begin{vmatrix} x & 2 & x \\ 1 & x & 6 \\ x & x & x+1 \end{vmatrix} = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then a_0 is	(s) –2

8.

COLUMN-I	COLUMN-II
(A) The value of the determinant $\begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$ is	(p) 1
(B) If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix} = 0$ is – 2. Then sum of all other five roots is	(q) – 6
(C) The value of $\begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$ is $(i=\sqrt{-1})$	(r) 2
(D) If $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$, then $f\left(\frac{\pi}{3}\right)$ is	(s) – 2

9.

COLUMN-I	COLUMN-II
(A) $\begin{vmatrix} \frac{1}{c} & \frac{1}{a} & \frac{-(a+b)}{c^2} \\ \frac{-(b+c)}{a^2} & \frac{1}{c} & \frac{1}{a} \\ \frac{-(b+c)}{a^2c} & \frac{(a+2b+c)}{ac} & \frac{-b(a+b)}{ac^2} \end{vmatrix}$	(p) independent of a.
(B) $\begin{vmatrix} \sin a\cos b & \sin a\sin b & \cos a \\ \cos a\cos b & \cos a\sin b & -\sin a \\ -\sin a\sin b & \sin a\cos b & 0 \end{vmatrix}$ is	(q) independent of b.
(C) $\begin{vmatrix} \frac{1}{\sin a\cos b} & \frac{1}{\sin a\cos b} & \frac{1}{\cos a} \\ \frac{-\cos a}{\sin^2 a\cos b} & \frac{-\cos a}{\sin^2 a\sin b} & \frac{\sin a}{\cos^2 a} \\ \frac{\sin b}{\sin a\cos^2 b} & \frac{-\cos b}{\sin a\sin^2 b} & 0 \end{vmatrix}$	(r) independent of c.
(D) If a, b and c are the sides of a triangle ABC and A, B and C are the angles opposite to a, b, and c, respectively, then $\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$ is	(s) dependent on a, b

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Consider the system of equations $\lambda x + (b-a)y + (c-a)z = 0$, $(a-b)x + \lambda y + (c-b)z = 0$ and $(a-c)x + (b-c)y + \lambda z = 0$.If it has a non-trivial solution, then the value of λ is

2. Find number of all the values of λ for which the system of equations

$(1+\lambda)x_1 + x_2 + x_3 = 1;$

$x_1 + (1+\lambda)x_2 + x_3 = \lambda;$

$x_1 + x_2 + (1+\lambda)x_3 = \lambda^2$ is inconsistent.

3. Find number of all values of λ for which the system of equation

$\lambda x_1 + x_2 + x_3 = 1;$

$x_1 + \lambda x_2 + x_3 = 1;$

$x_1 + x_2 + \lambda x_3 = 1$ is inconsistent.

4. Let x, y, z be positive integers greater than 10 such that y and z have 1 and 0 at their unit's places respectively and let $\Delta = \begin{vmatrix} x & 4 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}$. If $\Delta+1$ is divisible by 10 then the digit in the unit's place of x is

5. If A, B, C are the angles of an acute angled triangle ABC and

$$D = \begin{vmatrix} (\tan B+\tan C)^2 & \tan^2 A & \tan^2 A \\ \tan^2 B & (\tan A+\tan C)^2 & \tan^2 B \\ \tan^2 C & \tan^2 C & (\tan A+\tan B)^2 \end{vmatrix},$$

then the least integral value of $\frac{D}{1000}$ is equal to _______

6. If $f(x)$ satisfies the equation $\begin{vmatrix} f(x+1) & f(x+8) & f(x+1) \\ 1 & 2 & -5 \\ 2 & 3 & \lambda \end{vmatrix} = 0$ for all real x. If f is periodic with period 7, then find the value of $|\lambda|$

7. If $l_i^2+m_i^2+n_i^2=1,(i=1,2,3)$, and $l_1l_2+m_1m_2+n_1n_2=0$ $,l_2l_3+m_2m_3+n_2n_3=0,$ $l_3l_1+m_1m_3+n_1n_3=0,$ and $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ then absolute value of $\Delta$is _______.

8. If $x-2y+z=-4, 2x-y+2z=2, x+y+\lambda z=4$ are inconsistent then $\lambda =$

9. If $\Delta_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+1 & n^2+n \\ 2k-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{k=1}^{n} \Delta_k = 56$, then n is equal to

10. If $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0, \left(-\pi/4 \le x \le \pi/4\right)$ then $\tan x =$

11. If $f(\theta) = \begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}$ then number of solutions of $f(\theta)=0$ in $\left[0, \frac{\pi}{2}\right]$ is

12. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\beta) & \sin(\alpha-\gamma) \end{vmatrix}$ and $f(2) = \frac{1}{4}$ then $\left[\sum_{r=1}^{15} f(x_r)\right]$ is (where [.] denotes greatest integer function)

13. If the system of equations $3x-2y+z=0, \lambda x-14y+15z=0, x+2y-3z=0$ has a non - zero solution, then $\lambda =$

14. Given $2x-y+2z=2,\ x-2y+z=-4,\ x+y+\lambda z=4$ then the value of such that the given system of equations has no solution is _______

15. $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = ka^2b^2c^2$ then $k =$

16. If $-2 \le [x] < 2,\ -1 \le [y] < 1,\ -3 \le [z] < 3$, where $[x]$ denotes greatest integer function and the minimum value of $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is equals to K then $|K|$ equals to

17. If $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = ma^nb^nc^n$ then $m + n$ value is

18. The system of equation $2x + 3y - z = 0, 3x + 2y + kz = 0, 4x + y + z = 0$ have a set of nonzero integral solution then the least positive value of z is _______

19. If $a_1, a_2, a_3, 5, 4, a_6, a_7, a_8, a_9$ are in H.P., then the value of the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ can be expressed in the lowest form as $\frac{p}{q}$, find $(p-2q)$.

20. If $f(x) = \begin{vmatrix} (1+x)^{21} & (1+x)^{22} & (1+x)^{23} \\ (1+x)^{31} & (1+x)^{32} & (1+x)^{33} \\ (1+x)^{41} & (1+x)^{42} & (1+x)^{43} \end{vmatrix}$ then the coefficient of x in $f(x)$ is _______

21. If x, y, z are non-zero real numbers and $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 1+3z \end{vmatrix} = 0$ then $-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ is equal to _______

22. If $f(x) = \begin{vmatrix} x-1 & n & 6 \\ (x-1)^2 & 2n^2 & 4n-2 \\ (x-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$ then $\sum_{x=1}^{n} f(x) =$ _____

23. If α, β are the roots of $ax^2+bx+c=0$ and $S_n = \alpha^n + \beta^n$ then $\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix} = \frac{(a+b+c)^p(b^2-4ac)^q}{a^r}$ where $p + q + r =$ _______

24. If $\begin{vmatrix} x & 3 & x \\ x^2 & x & 5 \\ x & x & 5 \end{vmatrix} = Ax^4+Bx^3+Cx^2+Dx+E$, then the value of $5A + 4B + 3C + 2D + E + 19$ is equal to

25. If the integers a, b, c in order are in A.P., lying between 1 and 9 and $a23$, $b53$, and $c83$ are three-digit numbers, then the value of the determinant $\begin{vmatrix} 2 & 5 & 8 \\ a23 & b53 & c83 \\ a & b & c \end{vmatrix}$ is _______

26. Find the value of $f\left(\frac{\pi}{6}\right)$,

where $f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$

27. If x, y, z are cube roots of unity and $D = \begin{vmatrix} x^2+y^2 & z^2 & z^2 \\ x^2 & y^2+z^2 & x^2 \\ y^2 & y^2 & z^2+x^2 \end{vmatrix}$, then the real part of D is

28. If $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \ell(1+a^2+b^2)^m$ then $l + m =$

ARCHIVE: JEE MAIN

1. **[JEE Main 2020]**
The sum of distinct values of λ for which the system of equations $(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0, (\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$, $2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$, has non-zero solutions, is _________.

2. **[JEE Main 2020]**
If the system of equations $x - 2y + 3z = 9$, $2x + y + z = b$, $x - 7y + az = 24$ has infinitely many solutions, then $a - b$ is equal to ______.

3. **[JEE Main 2020]**
Let S be the set of all integer solutions (x, y, z) of the system of equations
$x - 2y + 5z = 0; -2x + 4y + z = 0, -7x + 14y + 9z = 0$ such that $15 \le x^2 + y^2 + z^2 \le 150$. Then, the number of elements in the set S is equal to ______.

4. **[JEE Main 2020]**
If the system of linear equations $x + y + z = 6$, $x + 2y + 3z = 10$, $3x + 2y + \lambda z = \mu$
has more than two solutions, then $\mu - \lambda^2$ is equal to ________.

5. **[JEE Main 2020]**
If the system of linear equations $2x + 2ay + az = 0$, $2x + 3by + bz = 0$, $2x + 4cy + cz = 0$ where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution, then :
(A) $a + b + c = 0$
(B) a, b, c are in G.P.
(C) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
(D) a, b, c are in A.P.

6. **[JEE Main 2020]**
For which of the following ordered pairs (μ, δ), the system of linear equations
$x + 2y + 3z = 1; 3x + 4y + 5z = \mu; 4x + 4y + 4z = \delta$ is inconsistent?
(A) $(1, 0)$
(B) $(4, 6)$
(C) $(3, 4)$
(D) $(4, 3)$

7. **[JEE Main 2020]**
The system of linear equations $\lambda x + 2y + 2z = 5$; $2\lambda x + 3y + 5z = 8$; $4x + \lambda y + 6z = 10$ has :
(A) no solution when $\lambda = 2$
(B) infinitely many solutions when $\lambda = 2$
(C) a unique solution when $\lambda = -8$
(D) no solution when $\lambda = 8$

8. **[JEE Main 2020]**
If for some α and β in R, the intersection of the following three planes
$x + 4y - 2z = 1$; $x + 7y - 5z = \beta$; $x + 5y + \alpha z = 5$ is a line in R^3, then $\alpha + \beta$ is equal to :
(A) 0
(B) 2
(C) 10
(D) –10

9. **[JEE Main 2020]**
Let $a - 2b + c = 1$, If $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$, then:
(A) $f(-50) = 501$
(B) $f(50) = 1$
(C) $f(-50) = -1$
(D) $f(50) = -501$

10. **[JEE Main 2020]**
The following system of linear equations $7x + 6y - 2z = 0$; $3x + 4y + 2z = 0$; $x - 2y - 6z = 0$, has
(A) infinitely many solutions, (x, y, z) satisfying $y = 2z$.
(B) infinitely many solutions, (x, y, z) satisfying $x = 2z$.
(C) no solution.
(D) only the trivial solution.

11. **[JEE Main 2020]**
Let S be the set of all $\lambda \in R$ for which the system of linear equations
$2x - y + 2z = 2; x - 2y + \lambda z = -4; x + \lambda y + z = 4$ has no solution. Then the set S
(A) contains more than two elements
(B) is a singleton
(C) contains exactly two elements
(D) is an empty set

12. **[JEE Main 2020]**
If $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 2x-3 & 3x-4 & 4x-5 \\ 3x-5 & 5x-8 & 10x-17 \end{vmatrix} = Ax^3 + Bx^2 + Cx + D$, then
$B + C$ is equal to:
(A) –1
(B) 1
(C) –3
(D) 9

13. **[JEE Main 2020]**
If the system of equations $x + y + z = 2$, $2x + 4y - z = 6$, $3x + 2y + \lambda z = \mu$ has infinitely many solutions, then
(A) $\lambda - 2\mu = -5$
(B) $2\lambda - \mu = 5$
(C) $2\lambda + \mu = 14$
(D) $\lambda + 2\mu = 14$

14. **[JEE Main 2020]**
If the system of linear equations $x + y + 3z = 0$, $x + 3y + k^2z = 0$, $3x + y + 3z = 0$ has a non-zero solution (x, y, z) for some $k \in R$, then $x + \left(\frac{y}{z}\right)$ is equal to:
(A) 9
(B) –3
(C) –9
(D) 3

15. **[JEE Main 2020]**
If $a + x = b + y = c + z + 1$, where a, b, c, x, y, z are non-zero distinct real numbers, then $\begin{vmatrix} x & a+y & x+a \\ y & b+y & y+b \\ z & c+y & z+c \end{vmatrix}$ is equal to :
(A) 0
(B) $y(a - b)$
(C) $y(b - a)$
(D) $y(a - c)$

16. **[JEE Main 2020]**
If the minimum and the maximum values of the function f : $\left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to R$. defined by:
$f(\theta) = \begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 1 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$ are m and M respectively,
then the ordered pair (m, M) is equal to:
(A) $(0, 4)$
(B) $(-4, 4)$
(C) $(0, 2\sqrt{2})$
(D) $(-4, 0)$

17. **[JEE Main 2020]**

Let $\lambda \in R$. The system of linear equations $2x_1 - 4x_2 + \lambda x_3 = 1$; $x_1 - 6x_2 + x_3 = 2$; $\lambda x_1 - 10x_2 + 4x_3 = 3$ is inconsistent for:

(A) exactly one negative value of λ.
(B) exactly one positive value of λ.
(C) every value of λ.
(D) exactly two values of λ.

18. **[JEE Main 2020]**

The values of λ and μ for which the system of linear equations $x + y + z = 2$, $x + 2y + 3z = 5$, $x + 3y + \lambda z = \mu$ has infinitely many solutions are, respectively

(A) 5 and 7 (B) 6 and 8
(C) 4 and 9 (D) 5 and 8

19. **[JEE Main 2020]**

Let m and M be respectively the minimum and maximum values of

$$\begin{vmatrix} \cos^2 x & 1+\sin^2 x & \sin 2x \\ 1+\cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

then the ordered pair (m, M) is equal to

(A) $(-3, -1)$ (B) $(-4, -1)$
(C) $(1, 3)$ (D) $(-3, 3)$

20. **[JEE Main 2019]**

The system of linear equations $x + y + z = 2$, $2x + 3y + 2z = 5$, $2x + 3y + (a^2 - 1)z = a + 1$

(A) has infinitely many solutions for $a = 4$
(B) has a unique solution for $|a| = \sqrt{3}$
(C) is inconsistent when $|a| = \sqrt{3}$
(D) is inconsistent when $a = 4$

21. **[JEE Main 2019]**

If the system of linear equation $x - 4y + 7z = g$, $3y - 5z = h$, $-2x + 5y - 9z = k$ is consistent, then:

(A) $g + 2h + k = 0$ (B) $g + h + 2k = 0$
(C) $g + h + k = 0$ (D) $2g + h + k = 0$

22. **[JEE Main 2019]**

If the system of equations $x + y + z = 5$, $x + 2y + 3z = 9$, $x + 3y + \alpha z = \beta$ has infinitely many solutions, then $\beta - \alpha$ equals-

(A) 8 (B) 21
(C) 18 (D) 5

23. **[JEE Main 2019]**

Let $a_1, a_2, a_3, \ldots, a_{10}$ be in G.P. with $a_i > 0$ for $i = 1, 2, \ldots, 10$ and S be the set of pairs (r, k), $r, k \in N$ (the set of natural numbers) for which

$$\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0.$$

Then the number of elements in S, is:

(A) 4 (B) infinitely many
(C) 2 (D) 10

24. **[JEE Main 2019]**

The number of values of $\theta \in (0, \pi)$ for which the system of linear equations $x + 3y + 7z = 0$, $-x + 4y + 7z = 0$, $(\sin 3\theta)x + (\cos 2\theta)y + 2z = 0$ has a non-trivial solution, is -

(A) two (B) one
(C) four (D) three

25. **[JEE Main 2019]**

If the system of linear equations $2x + 2y + 3z = a$, $3x - y + 5z = b$, $x - 3y + 2z = c$ where a, b, c are non zero real numbers, has more than one solution, then :

(A) $b - c - a = 0$ (B) $a + b + c = 0$
(C) $b - c + a = 0$ (D) $b + c - a = 0$

26. **[JEE Main 2019]**

If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)\,(x + a + b + c)^2$,

$x \neq 0$ and $a + b + c \neq 0$, then x is equal to:

(A) $-2(a + b + c)$ (B) $2(a + b + c)$
(C) abc (D) $-(a + b + c)$

27. **[JEE Main 2019]**

An ordered pair (α, β) for which the system of linear equations $(1 + \alpha)\,x + \beta y + z = 2$, $\alpha x + (1 + \beta)y + z = 3$, $\alpha x + \beta y + 2z = 2$ has a unique solution, is:

(A) $(-3, 1)$ (B) $(1, -3)$
(C) $(-4, 2)$ (D) $(2, 4)$

28. **[JEE Main 2019]**

The set of all values of λ for which the system of linear equations $x - 2y - 2z = \lambda x$, $x + 2y + z = \lambda y$, $-x - y = \lambda z$ has a non-trivial solutions:

(A) is an empty set
(B) contains more than two elements
(C) is a singleton
(D) contains exactly two elements

29. **[JEE Main 2019]**

The greatest value of $c \in R$ for which the system of linear equations $x - cy - cz = 0$, $cx - y + cz = 0$, $cx + cy - z = 0$ has a non-trivial solution, is -

(A) $\frac{1}{2}$ (B) 0
(C) 2 (D) -1

30. **[JEE Main 2019]**

Let α and β be the roots of the equation $x^2 + x + 1 = 0$. Then for $y \neq 0$ in R, $\begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$ is equal to

(A) $y(y^2 - 3)$ (B) y^3
(C) $y(y^2 - 1)$ (D) $y^3 - 1$

31. [JEE Main 2019]

If the system of equations $2x + 3y - z = 0$, $x + ky - 2z = 0$ and $2x - y + z = 0$ has a non-trivial solution (x, y, z), then $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + k$ is equal to :

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$

(C) $-\frac{1}{4}$ (D) -4

32. [JEE Main 2019]

If $\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}$, $x \neq 0$; then for all $\theta \in \left(0, \frac{\pi}{2}\right)$:

(A) $\Delta_1 - \Delta_2 = x(\cos 2\theta - \cos 4\theta)$

(B) $\Delta_1 - \Delta_2 = -2x^3$

(C) $\Delta_1 + \Delta_2 = -2(x^3 + x - 1)$

(D) $\Delta_1 + \Delta_2 = -2x^3$

33. [JEE Main 2019]

Let λ be a real number for which the system of linear equations $x + y + z = 6$, $4x + \lambda y - \lambda z = \lambda - 2$, $3x + 2y - 4z = -5$ has infinitely many solutions. Then λ is a root of the quadratic equation

(A) $\lambda^2 + \lambda - 6 = 0$ (B) $\lambda^2 - \lambda - 6 = 0$

(C) $\lambda^2 - 3\lambda - 4 = 0$ (D) $\lambda^2 + 3\lambda - 4 = 0$

34. [JEE Main 2019]

The sum of the real roots of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$, is equal to:

(A) -4 (B) 0

(C) 1 (D) 6

35. [JEE Main 2019]

A value of $\theta \in \left(0, \frac{\pi}{3}\right)$, for which

$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$, is

(A) $\frac{\pi}{18}$ (B) $\frac{\pi}{9}$

(C) $\frac{7\pi}{24}$ (D) $\frac{7\pi}{36}$

36. [JEE Main 2018]

If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A + Bx)(x - A)^2$. Then ordered pair (A, B) is equal to:

(A) $(-4, 3)$ (B) $(-4, 5)$

(C) $(4, 5)$ (D) $(-4, -5)$

37. [JEE Main 2018]

If the system of linear equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 4y - 3z = 0$ has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

(A) 10 (B) -30

(C) 30 (D) -10

38. [JEE Main 2017]

If S is the set of distinct values of 'b' for which the following system of linear equations

$x + y + z = 1$, $x + ay + z = 1$, $ax + by + z = 0$ has no solution, then S is

(A) a singleton

(B) an empty set

(C) an infinite set

(D) a finite set containing two or more elements

39. [JEE Main 2016]

The system of linear equations $x + \lambda y - z = 0$, $\lambda x - y - z = 0$, $x + y - \lambda z = 0$ has non-trivial solution.

(A) exactly three values of λ.

(B) infinitely many values of λ.

(C) exactly one value of λ.

(D) exactly two values of λ.

40. [JEE Main 2015]

The set of all values of λ for which the system of linear equations:

$2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$, $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution

(A) contains two elements

(B) contains more then two elements

(C) is an empty set

(D) is a singleton

41. [JEE Main 2014]

If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then

K is equal to:

(A) $\alpha\beta$ (B) $\frac{1}{\alpha\beta}$

(C) 1 (D) -1

42. [JEE Main 2013]

The number of values of k, for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has no solution, is -

(A) infinite (B) 1

(C) 2 (D) 3

43. [AIEEE - 2011]

The number of values of k for which the linear equations $4x + ky + 2z = 0$,

$kx + 4y + z = 0$, $2x + 2y + z = 0$ possess a non-zero solutions is -

(A) 1 (B) zero

(C) 3 (D) 2

44. [AIEEE - 2010]

Consider the system of linear equation $x_1 + 2x_2 + x_3 = 3$, $2x_1 + 3x_2 + x_3 = 3$, $3x_1 + 5x_2 + 2x_3 = 1$ The system has

(A) Infinite number of solutions

(B) Exactly 3 solutions

(C) A unique solution

(D) No solution

45. [AIEEE - 2009]

Let a, b, c be such that $b(a + c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0.$$

Then the value of n is

(A) Any odd integer (B) Any integer

(C) Zero (D) Any even integer

ARCHIVE: JEE ADVANCED

Multiple Correct Choice Type Problem

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. [JEE (Advanced)-2016]

Let $\alpha, \lambda, \mu \in R$. Consider the system of linear equation $\alpha x + 2y = \lambda$; $3x - 2y = \mu$

Which of the following statement(s) is(are) correct?

(A) If $\alpha = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $\alpha \neq -3$, then the system has a unique solutions for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $\alpha = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solutions for $\alpha = -3$

2. [JEE (Advanced)-2015]

Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$

(A) –4 (B) 9

(C) –9 (D) 4

Reasoning Based Questions

This section contains Reasoning type questions, each having four choices (A), (B), (C) and (D) out of which ONLY ONE is correct. Each question contains STATEMENT 1 and STATEMENT 2. You have to mark your answer as

Bubble (A) If both statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

Bubble (B) If both statements are TRUE but STATEMENT 2 is not the correct explanation of STATEMENT 1.

Bubble (C) If STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

Bubble (D) If STATEMENT 1 is FALSE but STATEMENT 2 is TRUE.

1. [IIT 2008]

Consider the system of equations $x - 2y + 3z = -1$, $-x + y - 2z = k$, $x - 3y + 4z = 1$

Statement-1 : The system of equations has no solution for $k \neq 3$.

Statement-2 : The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is not a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True.

Integer/Numerical Answer Type Questions

In this section, the answer to each question is a numerical value obtained after series of calculations based on the data provided in the question(s).

1. [JEE (Advanced)-2016]

The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is

2. [IIT- 2010]

The number of all possible values of θ, where $0 < \theta < \pi$, for which the system of equations

$$(y + z)\cos 3\theta = (xyz)\sin 3\theta,\ x \sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z},$$

$$(xyz)\sin 3\theta = (y + 2z)\cos 3\theta + y\sin 3\theta)$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** B	**3.** C	**4.** A	**5.** D	**6.** A	**7.** D	**8.** A	**9.** D	**10.** A
11. B	**12.** C	**13.** C	**14.** C	**15.** A	**16.** A	**17.** D	**18.** A	**19.** C	**20.** A
21. C	**22.** D	**23.** D	**24.** A	**25.** B	**26.** D	**27.** C	**28.** B	**29.** A	**30.** C
31. B	**32.** C	**33.** D	**34.** B	**35.** C	**36.** D	**37.** A	**38.** C	**39.** C	**40.** B
41. C	**42.** D	**43.** A	**44.** B	**45.** D	**46.** B	**47.** D	**48.** C	**49.** B	**50.** B
51. A	**52.** C	**53.** B	**54.** B	**55.** C	**56.** C	**57.** B	**58.** A	**59.** C	**60.** C
61. A	**62.** C	**63.** C	**64.** B	**65.** B	**66.** B	**67.** A	**68.** C	**69.** B	**70.** C
71. C	**72.** A	**73.** B							

Multiple Correct Choice Type Questions

1. A, B	**2.** A, C	**3.** A, C	**4.** B, C, D	**5.** A, B, C
6. A, B, C	**7.** B, D	**8.** A, B, C, D	**9.** A, B, C	**10.** A, B, C, D
11. A, B, C, D	**12.** B, C, D	**13.** A, B, C, D	**14.** A, C	**15.** B, C
16. A, B	**17.** C, D	**18.** A, C	**19.** D	**20.** A, C

Linked Comprehension Type Questions

1. C	**2.** D	**3.** A	**4.** D	**5.** D	**6.** C	**7.** A	**8.** A	**9.** B	**10.** D

Matrix Match/Column Match Type Questions

1. A → (q, r, s)	B → (p)	C → (q, r)	D → (p)
2. A → (s)	B → (t)	C → (p)	D → (q)
3. A → (q)	B → (s)	C → (r)	D → (p)
4. A → (p)	B → (p, q, r, s)	C → (r, s)	D → (p, q, r)
5. A → (p)	B → (q)	C → (r)	D → (p)
6. A → (q, r)	B → (p)	C → (s)	D → (p, q, r)
7. A → (s)	B → (p)	C → (q)	D → (s)
8. A → (s)	B → (r)	C → (q)	D → (p)
9. A → (p, q, r)	B → (q, r)	C → (r, s)	D → (p, q, r)

Integer/Numerical Answer Type Questions

1. 0	**2.** 2	**3.** 1	**4.** 2	**5.** 2	**6.** 4	**7.** 1	**8.** 1	**9.** 7	**10.** 1
11. 2	**12.** 3	**13.** 5	**14.** 1	**15.** 4	**16.** 5	**17.** 6	**18.** 5	**19.** 8	**20.** 0
21. 3	**22.** 0	**23.** 7	**24.** 5	**25.** 0	**26.** 1	**27.** 4	**28.** 4		

ARCHIVE: JEE MAIN

1. 3	**2.** 5	**3.** 8	**4.** 13	**5.** C	**6.** D	**7.** A	**8.** C	**9.** B	**10.** B
11. C	**12.** C	**13.** C	**14.** B	**15.** B	**16.** D	**17.** A	**18.** D	**19.** A	**20.** C
21. D	**22.** A	**23.** B	**24.** A	**25.** A	**26.** A	**27.** D	**28.** C	**29.** A	**30.** B
31. A	**32.** D	**33.** B	**34.** B	**35.** B	**36.** B	**37.** A	**38.** A	**39.** A	**40.** A
41. C	**42.** B	**43.** D	**44.** D	**45.** A					

ARCHIVE: JEE ADVANCED

Multiple Correct Choice Type Questions

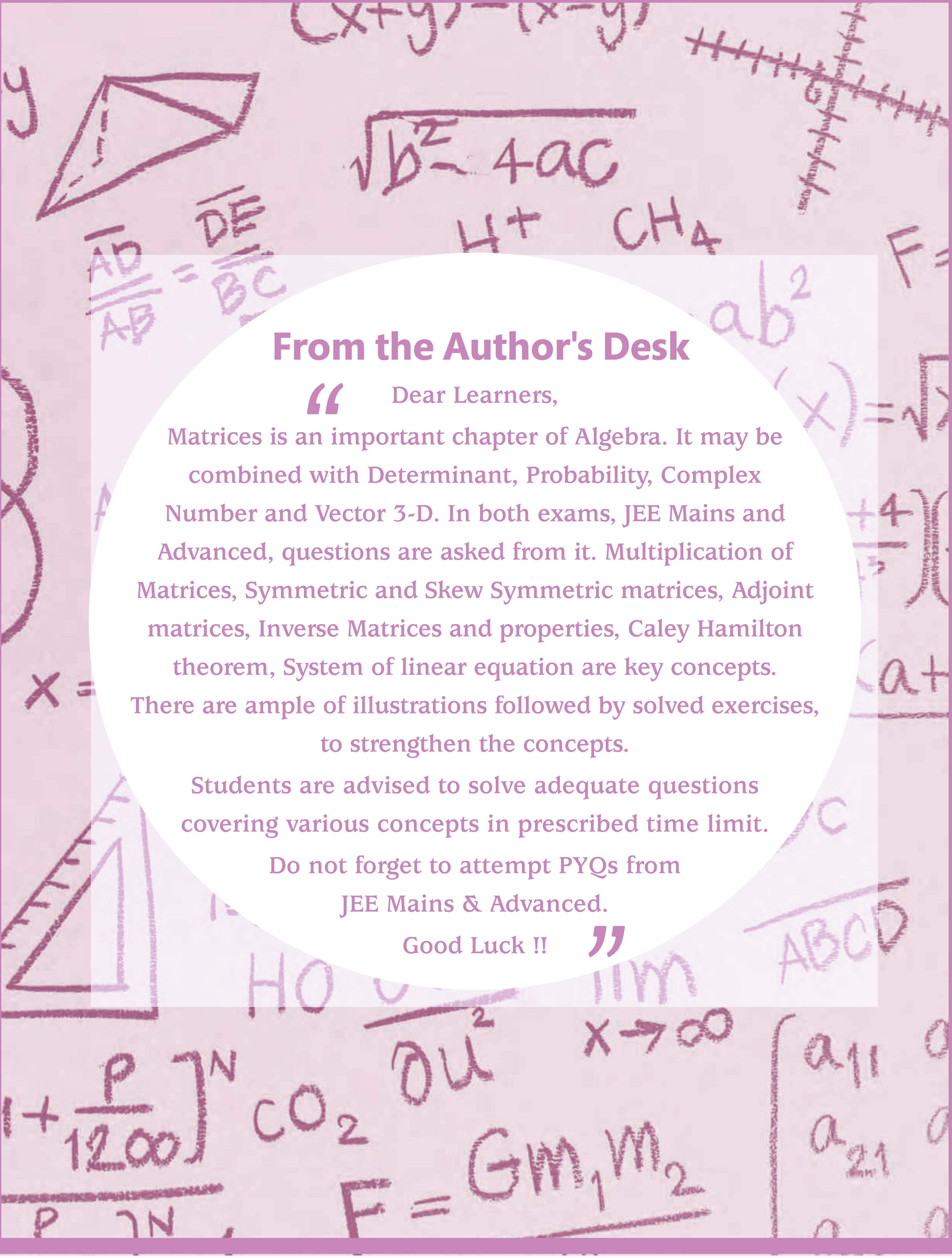
1. B, C, D	**2.** B, C

Reasoning Based Questions

1. A

Integer/Numerical Answer Type Questions

1. 2	**2.** 3



From the Author's Desk

“ Dear Learners,

Matrices is an important chapter of Algebra. It may be combined with Determinant, Probability, Complex Number and Vector 3-D. In both exams, JEE Mains and Advanced, questions are asked from it. Multiplication of Matrices, Symmetric and Skew Symmetric matrices, Adjoint matrices, Inverse Matrices and properties, Caley Hamilton theorem, System of linear equation are key concepts. There are ample of illustrations followed by solved exercises, to strengthen the concepts.

Students are advised to solve adequate questions covering various concepts in prescribed time limit. Do not forget to attempt PYQs from JEE Mains & Advanced.

Good Luck !! ”

CHAPTER

12 Matrices

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Type of matrices
(b) Algebra of matrices
(c) Trace of matrices
(d) Symmetric and skew symmetric matrices
(e) Determinant of square matrix
(f) Adjoint/Inverse matrices and properties
(g) Caley Hamilton theorem
(h) Special matrices

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE (Main and Advanced) are also given.

INTRODUCTION

Matrices is a rectangular array of real or complex numbers in rows and columns. A matrix is denoted by the capital letters A, B, C etc. A matrix is represented in following way

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} or \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

if there are m rows and n columns in a matrix, then the matrix is called of order $m \times n$

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & -4 \end{bmatrix}; B = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 10 & -4 & 7 \end{bmatrix}$$

Here A is a 2×3 order matrix as A has two rows and three columns and B is a 2×4 order matrix as B has 2 rows and 4 columns

Note:
Matrices is just an array of numbers and has no numerical value

$$A = \begin{array}{c} \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots & a_{3n} \\ \vdots & \vdots & \vdots \cdots\cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots\cdots & a_{mn} \end{array}\right]_{m \times n} \begin{array}{l} \rightarrow \text{First row } (R_1) \\ \rightarrow \text{Second row } (R_2) \\ \rightarrow \text{Third row } (R_3) \\ \\ \rightarrow m^{th} \text{ row } (R_m) \end{array} \\ \begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ \text{First column } (C_1) & \text{Second column } (C_2) & \text{Third column } (C_3) & n^{th} \text{ column } (C_n) \end{array} \end{array}$$

We can always represent any element present in matrices with the notation a_{ij}

$$\boldsymbol{a_{ij} = \textbf{element present at } i^{th} \textbf{ row and } j^{th} \textbf{ column}}$$

like a_{23} = element present at 2nd row and 3rd column

Note:
No. of elements in matrix of order $m \times n$ is = $m.n$

ILLUSTRATION 1

If order of matrix is 2×3. $a_{ij} = i - 2j$, then write matrix.

SOLUTION

It has 2 rows and 3 columns

$$a_{ij} = i - 2j \text{ where } i \in \{1,2\} \text{ and } j \in \{1,2,3\}$$

Total elements $= 2 \times 3 = 6$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1-2 & 1-4 & 1-6 \\ 2-2 & 2-4 & 2-6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & -5 \\ 0 & -2 & -4 \end{bmatrix}_{2\times 3}$$

ILLUSTRATION 2

Matrix is of order 5×2 where elements $a_{ij} = \begin{cases} i \cdot j & i < j \\ i + j & else \end{cases}$

SOLUTION

Rows = 5; column = 2

$$\text{Required matrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 4 \\ 4 & 5 \\ 5 & 6 \\ 6 & 7 \end{bmatrix}$$

ILLUSTRATION 3

In a matrix consisting of total elements as 12 then how many different order of matrices are possible?

SOLUTION

Order of matrices = no. of row × no. of columns $= m \times n$

Total elements $= mn = 12$

Possible different order of Matrices can be 1×12, 2×6, 3×4, 4×3, 6×2, 12×1
Six different orders of matrices are possible

ILLUSTRATION 4

Using 6 distinct numbers, how many different Matrices can be formed?

SOLUTION

Let distinct numbers are a, b, c, d, e and f
Possible order of matrices can be $1 \times 6, 2 \times 3, 3 \times 2, 6 \times 1 = 4$ orders
Total possible matrices = (possible orders of matrices) × (arrangement)

$$= 4 \times 6! = 4 \times 720 = 2880 \text{ Matrices}$$

ILLUSTRATION 5

Using letters a, a, a, b, b, c how many distinct matrices are possible?

SOLUTION

Using letters a, a, a, b, b, c
Possible order of matrices can be $1 \times 6, 2 \times 3, 3 \times 2, 6 \times 1 = 4$ orders

$$\text{Arrangement of given letters} = \left(\frac{6!}{3!\,2!}\right)$$

$$\text{Required number of matrices} = 4 \times \left(\frac{6!}{3!\,2!}\right) = \frac{4 \times 6 \cdot 5 \cdot 4}{2}$$

$$= 30 \times 8 = 240$$

TYPES OF MATRICES

Row Matrix

A matrix having only one row is called row matrices like $A = [1\ 2\ 3]$. Here A is a row matrices of order 1×3

Column Matrix

A matrix having only one column is called column matrices or column vector.

like $B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$. Here B is a column matrices of order 4×1

Zero or Null Matrix

A matrix which has each element as zero is called zero matrices or null matrices.

Example

$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where O are zero matrices of order 2×3 and 3×3

Square Matrix

A matrix having equal number of rows and columns is called square matrices. If the matrices A has n rows and n columns is said to be a square matrices of order n.

$P = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ where P and Q are square matrices of order 2 and 3 respectively

Horizontal Matrix

A $m \times n$ matrices is called horizontal matrices if number of rows < number of column ($m < n$).

$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix}$ where A is a horizontal matrix of order 2×3

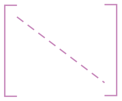
Vertical Matrix

A $m \times n$ matrices is called vertical matrices if number of rows > number of column ($m > n$)

$B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ where B is a vertical matrix of order 3×2

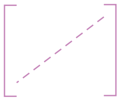
Diagonal Matrix

A square matrix (a_{ij}) is called the diagonal matrices if all the non-diagonal elements are zero.
i.e. $a_{ij} = 0$ for $i \neq j$.



Diagonal ($i = j$)

a_{ii}



Not diagonal

($i \neq j$)

In square matrices has only one diagonal which has $i = j$
Elements of diagonal are as $i = j$ and $a_{ij} = a_{11}, a_{22}, a_{33}, a_{44}, a_{55}$......

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{diag }(2,1,0);$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{diag }(1,0,2)$$

diag (a, b, c) is a another way of representing diagonal matrices
Minimum number of zeros in a diagonal matrix of order $n = n\,(n - 1)$.

Scalar Matrix

A diagonal matrix, whose all diagonal element are equal.

$$a_{ij} = \begin{cases} 0, & i \neq j \\ k, & i = j \end{cases} \text{ where } k \text{ is a constant.}$$

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \text{diag }[k \quad k \quad k]$$

Unit Matrix (Identity Matrix)

A square matrix is said to be a unit matrix if all the elements along the main diagonal are unity and all non-diagonal elements are zero. i.e. $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = 1$ for all $i = j$.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A unit matrices of order n is denoted by I_n

Every identity matrix is a scalar matrix but converse is not true

Triangular Matrix

(a) Lower Triangular Matrix

A square matrix (a_{ij}) is called a lower triangular matrix if $a_{ij} = 0$ for each $i < j$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Here A and B are lower triangular matrices

(b) Upper Triangular Matrix

A square matrix (a_{ij}) is called an upper triangular matrix if $a_{ij} = 0$ for each $i > j$.

$$C = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where C and D are upper triangular matrix

ILLUSTRATION 6

Using 0, 1, 2, 3, 4 how many

(i) diagonal matrices can be formed of order 4 × 4 if repetition of digits is not allowed
(ii) non-zero scalar matrix of order 4 × 4

SOLUTION

(i) $$\begin{bmatrix} - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{bmatrix}_{4\times 4} \quad 5 \times 4 \times 3 \times 2 = 120 \text{ diagonal matrices}$$

(ii) $$\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix} \text{ where } k = 1, 2, 3, 4,$$

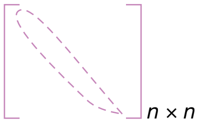
number of required non-zero scalar matrix = 4

ILLUSTRATION 7

In lower triangular matrix of order $n \times n$ find

(i) Minimum no. of zeroes
(ii) Maximum no. of zero
(iii) Maximum no. of non-zero elements
(iv) Maximum no. of distinct elements

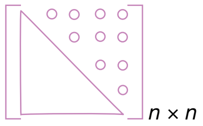
SOLUTION



Total elements = $n \times n = n^2$
no. of diagonal elements = n
no. of non-diagonal elements = $n^2 - n$
In case of square matrix, half of the non-diagonal elements are above diagonal and others are below diagonal

hence no. of elements below the diagonal = $\dfrac{n^2 - n}{2}$ = no. of elements above the diagonal
in case of lower triangular matrix all elements above the diagonal are zero

(i) minimum no. of zeroes



no. of non-diagonal elements above diagonal $= \dfrac{n^2 - n}{2}$

(ii) As it is given in question that it is an lower triangular matrix then below the diagonal there exists atleast one non- zero element

Maximum no. of zero $= n^2 - 1$

$= \text{total} - 1$

(iii) maximum no. of non-zero elements = diagonal elements + below-diagonal elements

$$n + \left(\frac{n^2 - n}{2}\right) = \frac{n^2 + n}{2}$$

(iv) maximum no. of distinct elements = maximum no. of non-zero elements + zero element

$$= \frac{n^2 + n}{2} + 1$$

ILLUSTRATION 8

A is a square matrix of order $n \times n$

$\ell =$ max. no. of distinct elements if A is triangular

$m =$ max. no. of distinct elements if A is a diagonal matrix

$p =$ minimum no. of zeroes if A is triangular matrix.

If $\ell + 5 = p + 2m$ then find n.

SOLUTION

max. no. of distinct elements if A is triangular

$\ell =$ diagonal elements + below/above diagonal

elements + zero $= n + \dfrac{n^2 - n}{2} + 1 = \dfrac{n^2 + n}{2} + 1$

max. no. of distinct elements if A is a diagonal matrix

$m =$ number of diagonal elements + zero $= n + 1$

minimum no. of zeroes if A is triangular matrix =

$p =$ no. of non-diagonal elements above/below

diagonal $= \dfrac{n^2 - n}{2}$:

given condition $\ell + 5 = p + 2m$

$$\Rightarrow \quad \frac{n^2 + n}{2} + 1 + 5 = \frac{n^2 - n}{2} + 2(n + 1)$$

$$\Rightarrow \quad n^2 + n + 12 = n^2 + 3n + 4$$

$$\Rightarrow \quad n = 4$$

ILLUSTRATION 9

Find no. of 2×2 order matrix whose a_{ij} is 1 or –1 and $a_{11}\,a_{21} + a_{12}\,a_{22} = 0$.

SOLUTION

Let matrix is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given conditions are a_{ij} is 1 or –1 and $a_{11}\,a_{21} + a_{12}\,a_{22} = 0$

8 matrices are possible

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

EQUALITY OF TWO MATRICES

Two $m \times n$ matrix A and B are said to be equal if each corresponding elements of the two matrix are equal

i.e., if $a_{ij} = b_{ij}$ for $i = 1, 2, \ldots\ldots m; j = 1, 2, 3, \ldots, n$

i.e., if $a_{11} = b_{11}, a_{12} = b_{12}$ etc.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & x & y \\ 3 & 2 & 4 \end{bmatrix}$

where A and B are equal matrices and we write $A = B$. then $x = 2, y = 0$

ILLUSTRATION 10

If $\begin{bmatrix} x - y & z \\ 2x - y & w \end{bmatrix}_{2\times 2} = \begin{bmatrix} -2 & y \\ 0 & 10 \end{bmatrix}_{2\times 2}$ then find x, y, z.

SOLUTION

Equating each corresponding element

$x - y = -2$ and $z = y$

$2x - y = 0 \quad \Rightarrow \quad y = 2x$ and $w = 10$

On solving

$x - 2x = -2 \quad \Rightarrow \quad x = 2, y = 4, z = 4, w = 10$

ILLUSTRATION 11

If $\begin{bmatrix} x + 3 & 2y + x \\ z - 1 & 4w - 8 \end{bmatrix} = \begin{bmatrix} -x + 1 & 0 \\ 3 & 2w \end{bmatrix}$ find x, y, z, w.

SOLUTION

Equating each corresponding element

$x + 3 = -x + 1 \quad \Rightarrow \quad 2x = -2 \quad \Rightarrow \quad x = -1$

$2y + x = 0 \quad \Rightarrow \quad 2y = -x = 1 \quad \Rightarrow \quad y = \dfrac{1}{2}$

$$z - 1 = 3 \quad \Rightarrow \quad z = 4$$

$$4w - 8 = 2w \quad \Rightarrow \quad 2w = 8 \quad \Rightarrow \quad w = 4$$

we get $x = -1, y = 1/2, z = 4, w = 4$

ILLUSTRATION 12

$\begin{bmatrix} x^2 - 4x & x^2 \\ x^2 & x^3 + 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -x + 2 & 2 \end{bmatrix}$ find x.

SOLUTION

Equating each corresponding element

$$x^2 - 4x = -3 \quad \Rightarrow \quad x^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 1)(x - 3) = 0 \quad \Rightarrow \quad x = 1, 3$$

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

$$x^2 = -x + 2 \quad \Rightarrow \quad (x + 2)(x - 1) = 0$$

$$\Rightarrow \quad x = 1, -2;$$

$$x^3 + 1 = 2 \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = 1$$

After taking common values of all four results we get $x = 1$ only

ADDITION OF MATRICES

Let A and B be two $m \times n$ matrices. The $m \times n$ matrices obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by $A + B$.

Thus if $A = (a_{ij})$, $B = (b_{ij})$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Then $A + B = (a_{ij} + b_{ij})$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is the sum of matrices A and B.

(i) Matrices addition is commutative i.e, if A and B be any two $m \times n$ matrices, then $A + B = B + A$.

(ii) Matrices addition is associative *i.e.*, if A, B and C be three $m \times n$ matrices, then $A + (B + C) = (A + B) + C$

(iii) Cancellation laws hold good for addition of matrices i.e., if A, B, C, be any three $m \times n$ matrices, then
  (*i*) $A + B = A + C \Rightarrow B = C$ (left cancellation law)
  (*ii*) $B + A = C + A \Rightarrow B = C$ (right cancellation law)

Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 & 5 \\ 6 & 2 & 9 \end{bmatrix}$

then

$$A + B = \begin{bmatrix} 2-3 & 3+4 & 4+5 \\ 0+6 & 2+2 & -1+9 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 9 \\ 6 & 4 & 8 \end{bmatrix}$$

SUBTRACTION OF TWO MATRICES

Let A and B be two $m \times n$ matrices. Then the difference of A and B is denoted by $A - B$ and is defined by $A - B = A + (-B)$. Where $A - B$ will also be a $m \times n$ matrices. In order to find $A - B$, the elements of B must be subtracted from the corresponding elements of A.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$ then

$$A - B = \begin{bmatrix} 1-2 & 2+1 & 3-4 \\ 3-5 & -1-0 & 0-6 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & -1 & -6 \end{bmatrix}$$

MULTIPLICATION BY A SCALAR

Let A be any $m \times n$ matrix and k be any scalar (real or complex number), then the scalar multiple of matrix A by k is denoted by kA or Ak and is defined as the $m \times n$ matrices obtained by multiplying each element of A by k.

Thus if $A = (a_{ij})$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

then $kA = (ka_{ij})$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 4 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 6 \\ 9 & 0 \\ 15 & 12 \end{bmatrix}$

(i) If A and B be any two $m \times n$ matrices and k is any scalar, then $k(A + B) = kA + kB$

(ii) If A is any $m \times n$ matrix and a and b are any two scalars, then $(a + b)A = aA + bA$

(iii) If A be any $m \times n$ matrix and k be any scalar, then $(-k)A = -(kA) = k(-A)$

TRANSPOSE MATRIX A' OR A^T

Let A be any matrix then the matrix obtained by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T.
If A is a $m \times n$ matrix then A' will be a $n \times m$ matrix.

If $A = [a_{ij}]$ where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

then $A' = A^T = [a_{ji}]$ where $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}, \text{ then } A^T = A' = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 5 \end{bmatrix}$$

Results

(i) If A is any matrix, then $(A^T)^T = A$

(ii) If k is any scalar and A be any matrix, then $(kA)' = kA'$ or $(kA)^T = kA^T$

(iii) $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$ where λ, μ are scalar
e.g. $(2A - 3B)^T = 2A^T - 3B^T$

(iv) $(-A)' = -A'$ or $(-A)^T = -A^T$

ILLUSTRATION 13

$A=\begin{bmatrix}4 & 6 & -1\\ 1 & -2 & 3\end{bmatrix}; B=\begin{bmatrix}0 & -2 & 3\\ 1 & -1 & 4\end{bmatrix}$

Compute (i) $3A+2B$ (ii) $2A-3B$

SOLUTION

(i) $$3A+2B=\begin{bmatrix}12 & 18 & -3\\ 3 & -6 & 9\end{bmatrix}+\begin{bmatrix}0 & -4 & 6\\ 2 & -2 & 8\end{bmatrix}$$

$$=\begin{bmatrix}12 & 14 & 3\\ 5 & -8 & 17\end{bmatrix}$$

(ii) $$2A-3B=\begin{bmatrix}8 & 12 & -2\\ 2 & -4 & 6\end{bmatrix}-\begin{bmatrix}0 & -6 & 9\\ 3 & -3 & 12\end{bmatrix}$$

$$=\begin{bmatrix}8 & 18 & -11\\ -1 & -1 & -6\end{bmatrix}$$

ILLUSTRATION 14

If $A=\begin{bmatrix}1 & -3 & 2\\ 2 & 0 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & 1\end{bmatrix}$, find the matrix C such that $A+B+C$ is null matrix.

SOLUTION

$$A+B+C=0$$

$$C=-(A+B)=-\begin{bmatrix}1 & -3 & 2\\ 2 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-3 & 4 & -1\\ -3 & 0 & -3\end{bmatrix}$$

ILLUSTRATION 15

Find matrices X, if $2X-Y=\begin{bmatrix}6 & -6 & 0\\ -4 & 2 & 1\end{bmatrix}$ and

$$X+2Y=\begin{bmatrix}3 & 2 & 5\\ -2 & 1 & -7\end{bmatrix}.$$

SOLUTION

$$2X-Y=\begin{bmatrix}6 & -6 & 0\\ -4 & 2 & 1\end{bmatrix}$$

$$\Rightarrow\quad 4X-2Y=2\begin{bmatrix}6 & -6 & 0\\ -4 & 2 & 1\end{bmatrix}=\begin{bmatrix}12 & -12 & 0\\ -8 & 4 & 2\end{bmatrix}\qquad\ldots(1)$$

and $$X+2Y=\begin{bmatrix}3 & 2 & 5\\ -2 & 1 & -7\end{bmatrix}\qquad\ldots(2)$$

Now, adding equations(1) and (2),

$$5X=\begin{bmatrix}15 & -10 & 5\\ -10 & 5 & -5\end{bmatrix}\text{ So, }X=\begin{bmatrix}3 & -2 & 1\\ -2 & 1 & -1\end{bmatrix}.$$

ILLUSTRATION 16

If $3A+4B^T=\begin{bmatrix}1 & 3 & 7\\ -5 & 4 & 6\end{bmatrix}$ and $2B-3A^T=\begin{bmatrix}-1 & 5\\ 3 & 6\\ 8 & 2\end{bmatrix}$ then find matrix A.

SOLUTION

$$3A+4B^T=\begin{bmatrix}1 & 3 & 7\\ -5 & 4 & 6\end{bmatrix}\qquad\ldots(1)$$

Taking transpose of second relation

$$(2B-3A^T)^T=\begin{bmatrix}-1 & 3 & 8\\ 5 & 6 & 2\end{bmatrix}=2B^T-3A\qquad\ldots(2)$$

Eq,(1) – 2(Eq.(2))

$$3A+4B^T-4B^T+6A=\begin{bmatrix}1 & 3 & 7\\ -5 & 4 & 6\end{bmatrix}-\begin{bmatrix}-2 & 6 & 16\\ 10 & 12 & 4\end{bmatrix}$$

$$9A=\begin{bmatrix}3 & -3 & -9\\ -15 & -8 & 2\end{bmatrix}$$

$$A=\frac{1}{9}\begin{bmatrix}3 & -3 & -9\\ -15 & -8 & 2\end{bmatrix}$$

Test Your Concepts-I

(Solutions on page H.312)

1. If number of elements is 20 then how many different order of matrices can be formed if number of rows is always even?

2. How many matrices can be formed with elements {1, 2, 3, 4}, if each element is used once?

3. If $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 5 & 0\\ 2 & 4 & 3\end{bmatrix}$ then find sum of elements of second row of $A+A^T$.

4. If $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\\ 1 & -3 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 5 & 6\\ -1 & 0 & 1\\ 2 & 1 & 2\end{bmatrix}$, find $4A-3B$.

5. If $A=\begin{bmatrix}1 & 0\\ 3 & -1\\ -5 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2\\ -2 & 2\\ 1 & 1\end{bmatrix}$, then find the matrix 'X' such that $3A+X=5B$.

6. If $A = \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix}$ then find X such that $A + 2X = B$.

Answer keys

1. 4 **2.** 72 **3.** 16 **4.** $\begin{bmatrix} -8 & -7 & -6 \\ -1 & 0 & 5 \\ -2 & -15 & -2 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & -10 \\ -19 & 13 \\ 20 & -1 \end{bmatrix}$ **6.** $\begin{bmatrix} -5/2 & 1/2 \\ 5/2 & 7/2 \end{bmatrix}$

MULTIPLICATION OF TWO MATRICES

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{ij}]$ be a $k \times p$ matrix Multiplication product of matrix AB is defined when

(i) Number of columns in A = Number of rows in B, i.e. $n = k$

(ii) for multiplication only **row by column** method is used

(iii) order of multiplication matrix AB is $m \times p$

As matrix $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{k \times p}$

Product $= AB = C = [c_{ij}]$ $[\because n = k]$

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

NOTE:

(i) In the product AB, A is called the pre-factor and B is called the post-factor.

(ii) If the product AB is possible then it is not necessary that the product BA also exists

(iii) If A be a $m \times n$ matrices and both AB and BA are defined then B will be a $n \times m$ matrices.

If A, B and C are three matrices such that their product is defined, then

(i) $AB \neq BA$ (Generally not commutative)

(ii) If A, B and C be $m \times n$, $n \times p$ and $p \times q$ matrices, then $A(BC) = (AB)C$ (Associative Law)

(iii) If A be a $m \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$

(iv) If A is a $m \times n$ matrix and B and C are both $n \times p$ matrices, then $A(B + C) = AB + AC$ (Distributive Law)

(v) **If $AB = AC \not\Rightarrow B = C$ (Cancellation Law is not applicable)**

For example, consider

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}.$$

Here, $B \neq C$, but $AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$

and $AC = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$

$\therefore$ Thus, $AB = AC$, but $B \neq C$

(vi) **If $AB = O$ (It does not mean that $A = O$ or $B = O$, product of two non- zero matrix may be zero matrix)**

For example, $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Here, A and B are not null matrices but their product is null matrix.

(vii) $(AB)^T = B^T A^T$

(viii) $(A_1 A_2 A_3 \ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$

Note:

(i) $I^n = I$ where I identity matrix (unit matrix)

(ii) The multiplication of two diagonal matrices is again a diagonal matrix.

(iii) The multiplication of two triangular matrices is again a triangular matrix.

(iv) The multiplication of two scalar matrices is also a scalar matrix.

(v) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) $A(-B) = (-A)B = -(AB)$

(vi) General maths formulas including binomial theorem are valid only when AB is commutative ($AB = BA$)

$(A + B)^2 = A^2 + 2AB + B^2$

$(A + B)^3 = A^3 + 3AB(A + B) + B^3$

$(A + B)^n = {}^nC_0 A^n + {}^nC_1 A^{n-1} B + {}^nC_2 A^{n-2} B^2 + \ldots + {}^nC_n B^n$

ILLUSTRATION 17

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -2 & -1 \end{bmatrix}$

Compute (i) AB (ii) BA (iii) $(AB)^T$ (iv) $B^T A^T$

SOLUTION

$$\textbf{(i)}\ AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -2 & -1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 1+6-6 & -2+2-3 \\ 0+3+8 & 0+1+4 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & -3 \\ 11 & 5 \end{bmatrix}_{2\times 2}$$

$$\textbf{(ii)}\ BA = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -2 & -1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 1-0 & 2-2 & 3+8 \\ 3+0 & 6+1 & 9-4 \\ -2+0 & -4-1 & -6+4 \end{bmatrix}_{3\times 3}$$

$$BA = \begin{bmatrix} 1 & 0 & 11 \\ 3 & 7 & 5 \\ -2 & -5 & -2 \end{bmatrix}_{3x3}$$

clearly we can say that $AB \neq BA$

$$\textbf{(iii)}\ (AB)^T = \begin{bmatrix} 1 & -3 \\ 11 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 & 11 \\ -3 & 5 \end{bmatrix}$$

$$\textbf{(iv)}\ B^T A^T = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -2 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 3 & -2 \\ -2 & 1 & -1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & -4 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 1+6-6 & 0+3+8 \\ -2+2-3 & 0+1+4 \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ -3 & 5 \end{bmatrix} = (AB)^T$$

We can conclude that $AB \neq BA$ and $B^T A^T = (AB)^T$

ILLUSTRATION 18

$\begin{bmatrix} 2x & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 3 \end{bmatrix} = O$, where O is zero matrix find x.

SOLUTION

$$\begin{bmatrix} 2x & 3 \end{bmatrix}_{1\times 2} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}_{2x2} \begin{bmatrix} x \\ 3 \end{bmatrix}_{2\times 1} = O$$

$$\Rightarrow \quad \begin{bmatrix} 2x-9 & 4x+0 \end{bmatrix}_{1\times 2} \begin{bmatrix} x \\ 3 \end{bmatrix}_{2\times 1} = O$$

$$\Rightarrow \quad \begin{bmatrix} x(2x-9)+4x(3) \end{bmatrix}_{1\times 1} = O$$

$$\Rightarrow \quad \begin{bmatrix} 2x^2+3x \end{bmatrix}_{1\times 1} = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1}$$

$$2x^2+3x = 0 \quad \Rightarrow \quad x(2x+3) = 0$$

$$\Rightarrow \quad x = 0,\ x = -3/2$$

ILLUSTRATION 19

Compute $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$

SOLUTION

Given product

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}_{3\times 1} \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}_{1\times 3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$$

$$= \begin{bmatrix} 4 & 1 & 0 \\ -8 & -2 & 0 \\ 12 & 3 & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} 2 & 4 \\ -1 & -2 \\ 2 & 4 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} 8-1+0 & 16-2+0 \\ -16+2+0 & -32+4+0 \\ 24-3+0 & 48-6+0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 7 & 14 \\ -14 & -28 \\ 21 & 42 \end{bmatrix}_{3\times 2}$$

ILLUSTRATION 20

If $f(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and if α, β, γ are angles of a triangle, then prove that $f(\alpha).\, f(\beta).\, f(\gamma) = -I_2$.

SOLUTION

$$f(\alpha)\, f(\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

Similarly

$$f(\alpha)\, f(\beta)\, f(\gamma) = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta)\cos\gamma - & \cos(\alpha+\beta)\sin\gamma + \\ \sin(\alpha+\beta)\sin\gamma & \sin(\alpha+\beta)\cos\gamma \\ -(\sin(\alpha+\beta)\cos\gamma + & -\sin(\alpha+\beta)\sin\gamma + \\ \cos(\alpha+\beta)\sin\gamma) & \cos(\alpha+\beta)\cos\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) \\ -\sin(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix} \text{ as } \alpha+\beta+\gamma=\pi$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_2.$$

ILLUSTRATION 21

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be two matrices such that there product is commutative to each other and $c \neq 3b$, find the value of $\left|\dfrac{a-d}{3b-c}\right|$.

SOLUTION

$$AB = BA$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

$$\Rightarrow 2c = 3b,\ 2d = 2a+3b,\ 3a+3c = 3d,\ 3b = 2c$$

Now $3b - c = c$; $a - d = -c$

$$\therefore \left|\frac{a-d}{3b-c}\right| = \left|\frac{-c}{c}\right| = 1$$

ILLUSTRATION 22

If matrix A and B are of same order such that $AB = A$ and $BA = B$ show that

(i) $A^2 = A$

(ii) $(A + B)(A - B) = O$, where O is null matrix

(iii) $A^5 + B^5 = A + B$

(iv) $(A + B)^5 = 16(A + B)$

(v) $(A + B)^n = 2^{n-1}(A + B)$

SOLUTION

(i) $AB = A$

$\Rightarrow$ $(AB)A = A \cdot A$ (multiply by A from right side)

$\Rightarrow$ $A(BA) = A^2 \ \Rightarrow\ AB = A^2 \Rightarrow A = A^2$

(ii) $BA = B$

$\Rightarrow$ $(BA)B = B \cdot B$ (multiply by B from right side)

$\Rightarrow$ $B(AB) = B^2$

$\Rightarrow$ $BA = B^2 \ \Rightarrow\ B = B^2$

(iii) $(A + B)(A - B) = AA - AB + BA - BB$

$(\because A^2 = A, B^2 = B, AB = A, BA = B)$

$= A^2 - AB + BA - B^2$

$= A - A + B - B = O$

(iv) $A^5 + B^5 = A^2A^2A + B^2B^2B \qquad (\because A^2 = A, B^2 = B)$

$= AAA + BBB$

$= A^2A + B^2B = AA + BB = A^2 + B^2 = A + B.$

(v) $(A + B)^2 = (A + B)(A + B)$

$= AA + AB + BA + BB$

$= A^2 + AB + BA + B^2 \qquad (\because A^2 = A, B^2 = B, AB = A, BA = B)$

$= A + A + B + B$

$(A + B)^2 = 2(A + B)$

$(A + B)^4 = ((A + B)^2)^2 = (2(A + B))^2 = 4(A + B)^2$

$= 4(2(A + B)) = 8(A + B) = 2^3(A + B)$

$(A + B)^5 = (A + B)^4(A + B)$

$= (8(A + B))(A + B) = 8(A + B)^2$

$= (8)(2(A + B)) = 16(A + B) = 2^4(A + B)$

We can also conclude that

$\therefore \quad (A + B)^n = 2^{n-1}(A + B)$

ILLUSTRATION 23

If $A^2 = A$, A is square matrix. Find $(I + A)^4$.

SOLUTION

$IA = A = AI$ commutative

so we can use Binomial Theorem

$$(I + A)^4 = {}^4C_0I^4 + {}^4C_1I^3A + {}^4C_2I^2A^2 + {}^4C_3IA^3 + {}^4C_4A^4$$

$$= I + 4IA + 6IA + 4IA + A^2$$

$$= I + 4A + 6A + 4A + A$$

$$= I + 15A$$

ILLUSTRATION 24

Matrix A satisfies $A^2 = 2A - I$ where I is the identity matrix, then, find A^{10} in terms on A and I.

SOLUTION

$A^2 = 2A - I \ \Rightarrow\ A^3 = 2A^2 - IA = 2(2A - I) - A$

$A^3 = 3A - 2I$

$A^4 = 3A^2 - 2A = 3(2A - I) - 2A$

$A^4 = 4A - 3I$

Similarly

$A^5 = 5A - 4I$

We can say

$A^n = nA - (n - 1)I$

So, $A^{10} = 10A - 9I$

ILLUSTRATION 25

If A is a diagonal matrix (not null matrix) of order 3 such that $A^2 = A$, then find the number of possible matrices A.

SOLUTION

$$A^2 = A$$

$$\Rightarrow \quad \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

On equating we get $a^2 = a,\ b^2 = b,\ c^2 = c$

Or $\quad a = 0, 1;\ b = 0, 1;\ c = 0, 1$

all a, b, c cannot be zero as it becomes null matrix the number of required possible matrices $A = 2 \cdot 2 \cdot 2 - 1 = 7$

ILLUSTRATION 26

Given the matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and X be the solution set of the equation $A^x = A$, where $x \in N - \{1\}$. Evaluate $\prod \left(\dfrac{x^3+1}{x^3-1}\right)$ where the continued product extends $\forall\, x \in X$.

SOLUTION

$$A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$A^2 = A \qquad \therefore x = 2$

$A^3 = A^2 . A = A . A = A^2 = A \qquad x = 3$

Solution set $X = \{2, 3, 4, 5, \ldots\ldots\}$

$$\prod\left(\frac{x^3+1}{x^3-1}\right) = \prod\left(\frac{(x+1)(x^2-x+1)}{(x-1)(x^2+x+1)}\right)$$

where $x \in \{2, 3, 4, 5, \ldots\, n\}$

$$= \left(\prod_{x=2}^{n}\left(\frac{x+1}{x-1}\right)\right)\left(\prod_{x=2}^{n}\left(\frac{x^2-x+1}{x^2+x+1}\right)\right)$$

$$= \left(\frac{3}{1}\cdot\frac{4}{2}\cdot\frac{5}{3}\cdot\frac{6}{4}\cdots\frac{n}{n-2}\cdot\frac{n+1}{n-1}\right)\left(\frac{3}{7}\cdot\frac{7}{13}\cdot\frac{13}{21}\cdots\frac{n^2-n+1}{n^2+n+1}\right)$$

$$= \left(\frac{n(n+1)}{1.2}\right)\left(\frac{3}{n^2+n+1}\right) = \frac{3}{2\left(1+\dfrac{1}{n^2+n}\right)}$$

$$= \frac{3}{2} \qquad \text{As } n \to \infty$$

ILLUSTRATION 27

Let $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, show that for $A^n = \begin{bmatrix} a^n & \dfrac{b(a^n-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$.

SOLUTION

We can use mathematical induction

For $n = 1$

$$A^1 = \begin{bmatrix} a & b\dfrac{(a-1)}{(a-1)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{Which is true for } n = 1$$

Assume it is true for $n = k \qquad \therefore A^k = \begin{bmatrix} a^k & b\dfrac{(a^k-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Now we need to prove it for $n = k + 1$

$$A^{k+1} = \begin{bmatrix} a^{k+1} & \dfrac{b(a^{k+1}-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$$

$$\text{LHS} = A^{k+1} = A^k . A$$

$$= \begin{bmatrix} a^k & \dfrac{b(a^k-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k.a+0 & a^k.b+\dfrac{b(a^k-1)}{(a-1)}.1 \\ 0+0 & 0+1.1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & b\left\{\dfrac{a^{k+1}-1}{(a-1)}\right\} \\ 0 & 1 \end{bmatrix} = \text{RHS}$$

Hence, it is true for all positive integral values of k

TRACE OF SQUARE MATRIX

$$A = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{33} \end{bmatrix}_{n\times n}$$

$Tr(A)$ = trace of square matrix (A) = sum of diagonal elements

$$Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \ldots + a_{n\,n}$$

(i) $Tr(kA) = k\,Tr(A)$ where k is a scalar
(ii) $Tr(A) = Tr(A^T)$
(iii) $Tr(A \pm B) = Tr(A) \pm Tr(B)$
(iv) $Tr(\lambda A \pm \mu B) = \lambda Tr(A) \pm \mu Tr(B)$
where λ and μ are scalars
(v) $Tr(AB) = Tr(BA)$ if matrix A and B are of same order

ILLUSTRATION 28

If trace of matrix $A = \begin{bmatrix} 2\sin x & 1 & 0 \\ 0 & 5\cos y & 3 \\ 7 & 4 & 3\sin z \end{bmatrix}$ is 10.

Find trace of matrix diag. $\left(8\tan\dfrac{x}{2}, 5\sin^2 y, -6\cos z\right)$ where x, y, z are in $[0, \pi]$.

SOLUTION

Trace of matrix A $= tr(A) = 2\sin x + 5\cos y + 3\sin z = 10$

$$2\sin x \le 2,\ 5\cos y \le 5,\ 3\sin z \le 3$$

For $2\sin x + 5\cos y + 3\sin z = 10$

We must have $\sin x = 1$, $\cos y = 1$, $\sin z = 1$

We have $x = \pi/2$, $y = 0$, $z = \pi/2$

Trace of diag. $\left(8\tan\dfrac{x}{2}, 5\sin^2 y, -6\cos z\right)$

$$= 8\tan 45° + 5\sin^2 0° - 6\cos 90°$$

$$= 8(1) + 0 - 0 = 8$$

ILLUSTRATION 29

Let $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ and $2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then find $Tr(A) - Tr(B)$.

SOLUTION

$$Tr.(A) + 2\,Tr.(B) = 1 - 3 + 1 = -1 \quad \ldots(1)$$

$$2Tr.(A) - Tr.(B) = 2 - 1 + 2 = 3 \quad \ldots(2)$$

∴ On solving (1) and (2), we get

$Tr.(A) = 1$ and $Tr.(B) = -1$

Hence, $Tr.(A) - Tr.(B)$

$$= 1 - (-1) = 2$$

ILLUSTRATION 30

If matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ then

find $tr(A) + tr(A^2) + tr(A^3) + \ldots. + tr(A^{100})$.

SOLUTION

Consider $A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$; $tr(A) = -1$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I;\ tr(A^2) = 3$$

So $A^3 = A$, $A^4 = I$, $A^5 = A$, $A^6 = I$ and so on

$$tr(A) + tr(A^2) + tr(A^3) + \ldots. + tr(A^{100})$$

$$= (-1) + (3) + (-1) + (3) + \ldots + (-1) + (3)$$

$$= (-1)50 + (3)50 = 100$$

ILLUSTRATION 31

The set of natural numbers is divided into arrays of rows and columns in the form of matrices as $A_1 = (1)$, $A_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $A_3 = \begin{pmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{pmatrix}$, …. so on. Find the value of $T_r(A_{10})$.

SOLUTION

First term of A_n is the n^{th} term of series

$$S = 1 + 2 + 6 + 15 + \ldots + T_n$$

(Using method of difference)

$$S = 0 + 1 + 2 + 6 + \ldots + T_{n-1} + T_n \qquad \text{(Subtracting)}$$

$$0 = 1 + 1^2 + 2^2 + 3^2 + \ldots + T_n - T_{n-1} - T_n$$

$$\Rightarrow \quad T_n = 1 + \sum_{n=1}^{n-1} n^2$$

$$\Rightarrow \quad T_n = 1 + \frac{(n-1)n(2n-1)}{6} = \frac{2n^3 - 3n^2 + n + 6}{6}$$

$$\text{So,} \quad T_{10} = \frac{2000 - 300 + 10 + 6}{6} = 286$$

Now common difference of the diagonal elements of A_{10} is 11 and number of diagonal elements is 10.

Hence $T_r(A_{10})$ = Sum of its diagonal elements

$$= \frac{10}{2}(2.286 + 9.11) = 3355$$

SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be a symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j

$$\text{i.e.}\quad A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 0 \\ 5 & 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$$

Cleary $A^T = A, B^T = B$

Note:

(i) A square matrices A is symmetric if and only if $A' = A$.

(ii) Every unit matrix and null square matrix are symmetric matrices.

(iii) Maximum number of different elements in a symmetric matrix is $\frac{n(n+1)}{2}$.

SKEW SYMMETRIC MATRIX

A square matrices $A = [a_{ij}]$ is said to be a skew symmetric matrix if $a_{ij} = -a_{ji}$ for all i and j.

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, A' = A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -A$$

$$B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}, B' = B^T = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = -B$$

Note:

(i) A square matrix A is skew symmetric if and only if $A^T = -A$.

(ii) in skew symmetric matrices all elements along the diagonal are zero

$\because\ a_{ij} = -a_{ji}$ for all i and j

$\Rightarrow a_{ii} = -a_{ii}$ [putting $j = i$]

or $2a_{ii} = 0$ or $a_{ii} = 0$

(iii) Trace of a skew symmetric matrix is always 0

A square matrices A is symmetric if and only if $A' = A$.

A square matrix A is skew symmetric if and only if $A' = -A$.

Remark(s)

(i) If A is a square matrix, then $A + A^T, AA^T, A^TA$ are symmetric matrices while $A - A^T$ is Skew Symmetric matrices.

(ii) If A is a Symmetric matrix, then $-A, kA, A^T, A^n, A^{-1}, B^TAB$ are also symmetric matrices where $n \in N$, $k \in R$ and B is a square matrix of order that of A.

(iii) If A is a skew symmetric matrix, then -

(a) A^{2n} is a symmetric matrix for $n \in N$

(b) A^{2n+1} is a skew symmetric matrices for $n \in N$.

(c) kA is also skew symmetric matrix where $k \in R$.

(d) B^TAB is also skew symmetric matrix where B is a square matrix of order that of A

(iv) If A, B are two symmetric matrices, then

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew symmetric matrix.

(c) AB is a symmetric matrix when $AB = BA$.

(v) If A, B are two skew symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(vi) If A is a skew symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix can be uniquely expressed as the sum or difference of a symmetric matrix and a skew symmetric matrix.

(a) $A = \left(\frac{A + A^T}{2}\right) + \left(\frac{A - A^T}{2}\right)$ where $\frac{A + A^T}{2}$ is symmetric matrix and $\frac{A - A^T}{2}$ is skew symmetric matrix

(b) $A = \left(\frac{A + A^T}{2}\right) - \left(\frac{A^T - A}{2}\right)$ where $\frac{A + A^T}{2}$ is symmetric matrix and $\frac{A^T - A}{2}$ is skew symmetric matrix

ILLUSTRATION 32

$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 3 & 4 & 7 \end{bmatrix}$ Express A as sum of symmetric and skew symmetric matrix.

SOLUTION

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 3 & 4 & 7 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 4 \\ -1 & 2 & 7 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & 6 \\ 2 & 6 & 14 \end{bmatrix} \text{ is a symmetric matrix}$$

$$A - A^T = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} \text{ is a skew symmetric matrix}$$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & 6 \\ 2 & 6 & 14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 3 \\ 1 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

First matrices is symmetric and second matrix is skew symmetric matrix

ILLUSTRATION 33

Show that positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.

SOLUTION

Let A be a skew-symmetric matrix then

$$A^T = -A.$$

we have $(A^n)^T = (A^T)^n$ for all $n \in N$.

$$\therefore \quad (A^n)^T = (-A)^n \qquad [\because A^T = -A]$$

$$\Rightarrow \quad (A^n)^T = (-1)^n A^n$$

$$\Rightarrow \quad (A^n)^T = \begin{cases} A^n & \text{if } n \text{ is even} \\ -A^n & \text{if } n \text{ is odd} \end{cases}$$

Hence A^n is symmetric if n is even and skew-symmetric if n is odd.

ILLUSTRATION 34

A, B, C are three matrices of the same order such that any two are symmetric and remaining 3rd one is skew symmetric. Now $X = ABC + CBA$ and $Y = ABC - CBA$, then find $(XY)^T$.

SOLUTION

$$X^T = (ABC + CBA)^T = (ABC)^T + (CBA)^T$$

$$= C^TB^TA^T + A^TB^TC^T = -CBA - ABC = -X, \ X \text{ is skew symmetric}$$

$\because$ Exactly one of matrices A, B, C is skew symmetric matrices

$$Y^T = (ABC - CBA)^T = (ABC)^T - (CBA)^T$$

$$= C^TB^TA^T - A^TB^TC^T = -CBA + ABC = Y, \ Y \text{ is symmetric}$$

$$\Rightarrow \quad (XY)^T = Y^TX^T = -YX$$

ILLUSTRATION 35

If 'A' is skew –symmetric matrix of order $n \times n$ and 'C' is column matrix, then show that $C^T AC$ is null matrix.

SOLUTION

A is of order $n \times n$, Column matrix $n \times 1$ and C^T is of order $1 \times n$

$$C^T AC \text{ is of order } 1 \times 1$$

'A' is skew –symmetric matrix then $A^T = -A$

Let $X = C^TAC$

$$X^T = (C^TAC)^T = C^TA^T(C^T)^T = C^T(-A)C = -C^TAC$$

So clearly C^TAC is a skew symmetric matrix of order 1×1

$$a_{11} = -a_{11} \text{ we get } a_{11} = 0$$

hence $C^TAC = O$ null matrix

ILLUSTRATION 36

If

$$\left\{\begin{pmatrix} 3 & 1 & 2 \\ 8 & 9 & 5 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 7 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 3 & 8 & 1 \\ 1 & 9 & 1 \\ 2 & 5 & 3 \end{pmatrix}\right\}^2 = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

then find the value of $|a_2 - b_1| + |a_3 - c_1| + |b_3 - c_2|$.

SOLUTION

Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 8 & 9 & 5 \\ 1 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 7 \\ 3 & 7 & 9 \end{pmatrix}$

and $X = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Clearly B is symmetric then $B^T = B$

$$(ABA^T)^T = (A^T)^TB^TA^T = AB^TA^T = ABA^T$$

Hence ABA^T is also a symmetric matrix

$$\text{Now given } X = \left\{\begin{pmatrix} 3 & 1 & 2 \\ 8 & 9 & 5 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 7 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 3 & 8 & 1 \\ 1 & 9 & 1 \\ 2 & 5 & 3 \end{pmatrix}\right\}^2$$

$$= (ABA^T)^2$$

$$\Rightarrow \quad X^T = X \quad \therefore X \text{ is symmetric matrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

symmetric hence required value

$$|a_2 - b_1| + |a_3 - c_1| + |b_3 - c_2| = 0$$

DETERMINANT OF A SQUARE MATRIX

Let A be a square matrix of order n then the determinant of the matrix A is the value of the determinant whose elements are the corresponding elements of the matrix A and is denoted by $|A|$ or Determinant of matrix A. Thus if $A = [a_{ij}]$ be a square matrix of order n, then the number

$$|A| = \text{Determinant of matrix } A = \text{Det } (A)$$

$$= \begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{vmatrix}$

$$= 1\,(-5-8) - 0\,(10-12) + 3\,(4+3) = 8$$

MATRIX VS DETERMINANT

(i) Matrix has no value but determinant has a value

(ii) Product with scalar (k)

if matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $kA = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{bmatrix}$

(k is multiplied to each element of matrix)

if det $(A) = |A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

then $k|A| = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{vmatrix}$

(either we multiply by k to only any one row or column)

Det $(kA) = \begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix} = k^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

Det $(kA) = k^3$ Det (A)

(iii) Let matrices $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} x & y & z \\ l & m & n \\ p & q & r \end{bmatrix}$

Addition of matrices $A + B = \begin{bmatrix} a+x & b+y & c+z \\ d+l & e+m & f+n \\ g+p & h+q & i+r \end{bmatrix}$

Addition of two determinants = Det(A) + Det(B) =

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & y & z \\ l & m & n \\ p & q & r \end{vmatrix} \neq \begin{vmatrix} a+x & b+y & c+z \\ d+l & e+m & f+n \\ g+p & h+q & i+r \end{vmatrix}$$

Let square matrices A and B are of $n \times n$ order

det$(A) = |A|$ = determinant of square matrix A

(i) $\det(A^T) = \det(A)$ or $|A^T| = |A|$

(ii) $\det(kA) = k^n \det(A)$ or $|kA| = k^n|A|$ where k is scalar

(iii) $\det(-A) = (-1)^n \det(A)$ or $|-A| = (-1)^n|A|$

(iv) $\det(AB) = \det(A)\det(B)$ or $|AB| = |A|\,|B|$

(v) $\det(A^mB^n) = (\det(A))^m (\det(B))^n$ or $|A^mB^n| = |A|^m\,|B|^n$

(vi) $\det(A+B) \neq \det(A) + \det(B)$ or $|A+B| \neq |A| + |B|$

(vii) det (diag $(a\ b\ c)$) = abc or $|\text{diag}(a\ b\ c)| = abc$

ILLUSTRATION 37

Let A and B are matrices of order 3×3 if $|A| = -2$, $|B| = 3$ then find (i) $\det(2AB)$ (ii) $\det(4A^2B^3)$

SOLUTION

(i) $\det(2AB) = 2^3\det(AB) = 8\det(A)\det(B)$

$= 8(-2)(3) = -48$

(ii) $\det(4A^2B^3) = 4^3\det(A^2B^3)$

$= 64\det(A))^2\,(\det(B))^3$

$= 64(-2)^2\,(3)^3$

$= 64 \times 4 \times 27 = 6912$

ILLUSTRATION 38

If matrix A and B are of order 3×3 each and

$\det(A) = -2$, $\det(B) = -\sqrt{3}$

Compute (i) $\det(-2AB)$ (ii) $\det(-A^2B^4)$ (iii) $\det(3A^2B^{-2})$

SOLUTION

(i) $\det(-2AB) = (-2)^3\det(A)\det(B)$

$= -8(-2)(-\sqrt{3}) = -16\sqrt{3}$

(ii) $\det(-A^2B^4) = (-1)^3\,(\det(A))^2\,(\det(B))^4$

$= -1(-2)^2\,(-\sqrt{3})^4$

$= -(4)\,(9) = -36$

(iii) $\det(3A^2B^{-2}) = 3^3(\det(A))^2\,(\det(B))^{-2}$

$= 27(-2)^2(-\sqrt{3})^{-2}$

$= 27(4)\left(\dfrac{1}{3}\right) = 36$

ILLUSTRATION 39

Let $\alpha = \dfrac{\pi}{5}$ and $A = \begin{bmatrix} \cos\dfrac{\pi}{5} & \sin\dfrac{\pi}{5} \\ -\sin\dfrac{\pi}{5} & \cos\dfrac{\pi}{5} \end{bmatrix}$ then prove that $\det(A + A^2 + A^3 + A^4) = 16\cos^2\dfrac{\pi}{5}\cos^2\dfrac{\pi}{10}$.

SOLUTION

Let $\alpha = \dfrac{\pi}{5}$ then $A = \begin{bmatrix} \cos\dfrac{\pi}{5} & \sin\dfrac{\pi}{5} \\ -\sin\dfrac{\pi}{5} & \cos\dfrac{\pi}{5} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$A^2 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

$$A^3 = \begin{pmatrix} \cos 3\alpha & \sin 3\alpha \\ -\sin 3\alpha & \cos 3\alpha \end{pmatrix} \text{ and } A^4 = \begin{pmatrix} \cos 4\alpha & \sin 4\alpha \\ -\sin 4\alpha & \cos 4\alpha \end{pmatrix}$$

We have $\cos\alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha$

$$= \cos\alpha + \cos 2\alpha + \cos(\pi - 2\alpha) + \cos(\pi - \alpha) \quad [\because 5\alpha = \pi]$$

$$= \cos\alpha + \cos 2\alpha - \cos 2\alpha - \cos\alpha = 0$$

Also $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha$

$$= \sin\alpha + \sin 2\alpha + \sin(\pi - 2\alpha) + \sin(\pi - \alpha)$$

$$= 2[\sin\alpha + \sin 2\alpha]$$

$$= 2\left\{2\sin\left(\frac{3\alpha}{2}\right)\cos\frac{\alpha}{2}\right\} = 4\sin\left(\frac{3\pi}{10}\right)\cos\frac{\pi}{10}$$

$$= 4\sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right)\cos\frac{\pi}{10} = 4\cos\frac{\pi}{5}\cos\frac{\pi}{10} = k \text{ (say)}$$

Thus, $B = A + A^2 + A^3 + A^4 = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$,

$$|B| = k^2 = 16\cos^2\frac{\pi}{5}\cos^2\frac{\pi}{10}$$

ILLUSTRATION 40

If A = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, then find the value of determinant $(A + A^2B^2 + A^3 + A^4B^4 + \ldots 100 \text{ terms})$.

SOLUTION

$$B^2 = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

and

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

also $\det(A) = 2(-9 + 8) + 1(6 - 8) + (-8 + 12) = 0$.

$$\det(A + A^2B^2 + A^3 + A^4 \cdot B^4 + \ldots + 100 \text{ terms})$$

$$= \det(A + A^2I + A^3 + A^4I + \ldots 100 \text{ terms})$$

$$= \det(A + A + \ldots 100 \text{ terms}) = \det(100A)$$

$$= 100^3 \det(A) = 0$$

ILLUSTRATION 41

A is a 4×4 matrix with $a_{11} = 1 + x_1$, $a_{22} = 1 + x_2$, $a_{33} = 1 + x_3$, $a_{44} = 1 + x_4$ and all other entries 1, where x_i are the roots of $n^4 - n^2 + 1 = 0$. Find the value of det(A).

SOLUTION

$n^4 - n^2 + 1 = 0$ has roots x_1, x_2, x_3, x_4

$$\sum x_1x_2x_3 = 0 \text{ and } x_1x_2x_3x_4 = 1 \qquad \ldots(1)$$

$$A = \begin{vmatrix} 1+x_1 & 1 & 1 & 1 \\ 1 & 1+x_2 & 1 & 1 \\ 1 & 1 & 1+x_3 & 1 \\ 1 & 1 & 1 & 1+x_4 \end{vmatrix}$$

using $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, $C_3 \to C_3 - C_4$

$$= \begin{vmatrix} x_1 & 0 & 0 & 1 \\ -x_2 & x_2 & 0 & 1 \\ 0 & -x_3 & x_3 & 1 \\ 0 & 0 & -x_4 & 1+x_4 \end{vmatrix}$$

Expand along R_1,

$$= x_1\begin{vmatrix} x_2 & 0 & 1 \\ -x_3 & x_3 & 1 \\ 0 & -x_4 & 1+x_4 \end{vmatrix} - 1\begin{vmatrix} -x_2 & x_2 & 0 \\ 0 & -x_3 & x_3 \\ 0 & 0 & -x_4 \end{vmatrix}.$$

$$= x_1\{x_2(x_3 + x_3x_4 + x_4)\} + x_3(x_4)\} - 1(-x_4)\{x_3x_2\}$$

$$= x_1\{x_2x_3 + x_2x_3x_4 + x_2x_4 + x_3x_4\} + x_4x_3x_2$$

$$= \sum x_1x_2x_3 + x_1x_2x_3x_4 \text{ using (1)}$$

$$= 0 + 1 = 1$$

Test Your Concepts-II

(Solutions on page H.312)

1. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ Find AB and BA and show that $AB \neq BA$.
2. $A = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 2 & -2 \\ 5 & 3 \end{bmatrix}$ compute AB and BA.
3. A square matrix P satisfies $P^2 = I - P$; where I is an identity matrix. If $P^n = 5I - 8P$, then find the value of n.
4. If $\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, find number of values of α.
5. If $A = \begin{bmatrix} 1 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$, then find $AB + CD$.

6. If $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ and $X = BC$, find X^n where n is natural number.

7. If $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$, then find x.

8. If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and AB = I_3, then find $x + y$.

9. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$ then find $|3AB|$.

10. $|A_{3\times3}| = 3, |B_{3\times3}| = -1$ and $|C_{2\times2}| = +2$ then find $|2ABC|$.

11. If A is a skew-symmetric matrix of order 3, then the matrix A^4 is symmetric. (True/False)

12. Let A and B be square matrices of same order satisfying $AB = A$ and $BA = B$ then find A^2B^2.

Answer keys

1. $AB = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$ **2.** $AB = \begin{bmatrix} 29 & 14 \\ 20 & 9 \\ 38 & 19 \end{bmatrix}$, BA does not exist **3.** 6

4. 0 **5.** [66] **6.** I **7.** $\pm 4\sqrt{3}$ **8.** 0

9. –81 **10**. Does not exist **11.** True **12.** A

ADJOINT OF A SQUARE MATRIX

Let $A = [a_{ij}]$ be a square matrix.

Let $B = [C_{ij}]$ where C_{ij} is the cofactor of the element a_{ij} in the matrix A. The transpose B^T of the matrix B is called the adjoint of the matrix A and is denoted by adj. A

$$\text{adj. } A = B^T = [C_{ij}]^T$$

Theorem: If A is any square matrix of order n, then

$$A.\,(\text{adj } A) = (\text{adj } A).\, A = |A|\, I_n$$

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ Then

$$\text{adj. } A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

$$\text{Now } A.\,(\text{adj. } A) = = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

[using the result that value of determinant is sum of product of elements with corresponding cofactor]

$$\left(\because \sum_{j=1}^{n} a_{ij}\, C_{ij} = |A| \text{ for } i = 1, 2, \dots, n \text{ and } \sum_{j=1}^{n} a_{ij}\, C_{pj} = 0,\ i \ne p\right)$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = |A|\, I_n$$

similarly we can show that (adj A). $A = |A|\, I_n$

thus A (adj. A) $= |A|\, I_n =$ (adj. A) A

Note:

(i) $|A\,(\text{adj. } A)| = |A|^n$ $[\therefore |I_n| = 1]$

(ii) If $|A| \ne 0$, then $|\text{adj. } A| = |A|^{n-1}$

PROPERTIES OF adj(A)

(i) $A\,\text{adj}(A) = |A|\, I = \text{adj}(A) A$

(ii) $|\text{adj}(A)| = |A|^{n-1}$

(iii) $\text{adj}(A^T) = (\text{adj}(A))^T$

(iv) $\text{adj}(A^m) = (\text{adj}(A))^m$

(v) $\text{adj}(kA) = k^{n-1}\,\text{adj}(A)$ where k is scalar

(vi) $\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A)$

(vii) $\text{adj}(A_1A_2\dots A_{n-1}A_n) = \text{adj}(A_n)\text{adj}(A_{n-1})\dots\text{adj}(A_2)\text{adj}(A_1)$

(viii) $\text{adj}\,(I_n) = I_n$

(ix) $\text{adj}\, O = O$

(x) If A is symmetric matrix ⇒ adj A is also symmetric

(xi) If A is diagonal matrix ⇒ adj A is also diagonal

(xii) If A is triangular matrix ⇒ adj A is also triangular

(xiii) If A is singular matrix ⇒ $|\text{adj } A| = 0$

ILLUSTRATION 42

Find adjoint matrices of following matrices

(i) $P=\begin{bmatrix}-2 & 4\\ 1 & 7\end{bmatrix}$ **(ii)** $Q=\begin{bmatrix}-1 & 2 & -3\\ 4 & 0 & 3\\ 1 & 2 & -1\end{bmatrix}$ **(iii)** $R=\begin{bmatrix}-1 & 2 & 0\\ 1 & -1 & 2\\ -2 & 3 & 4\end{bmatrix}$

SOLUTION

(i) $P=\begin{bmatrix}-2 & 4\\ 1 & 7\end{bmatrix}$

$$a_{11}=-2 : a_{12}=4 : a_{21}=1 : a_{12}=7$$

$$C_{11}=(-1)^{1+1}\,7=7$$

$$C_{12}=(-1)^{1+2}\,1=-1$$

$$C_{21}=(-1)^{2+1}\,4=-4$$

$$C_{22}=(-1)^{2+2}\,(-2)=-2$$

$$\text{adj}(A)=C^T=\begin{bmatrix}+(7) & -(1)\\ -(4) & +(-2)\end{bmatrix}^T=\begin{bmatrix}7 & -4\\ -1 & -2\end{bmatrix}$$

(ii) $Q=\begin{bmatrix}-1 & 2 & -3\\ 4 & 0 & 3\\ 1 & 2 & -1\end{bmatrix}$

$$\text{adj}(Q)=C^T=\begin{bmatrix}+(-6) & -(-7) & +(8)\\ -(4) & +(4) & -(-4)\\ +(6) & -(-9) & +(-8)\end{bmatrix}^T$$

$$=\begin{bmatrix}-6 & -4 & 6\\ 7 & 4 & -9\\ 8 & 4 & -8\end{bmatrix}$$

(iii) $R=\begin{bmatrix}-1 & 2 & 0\\ 1 & -1 & 2\\ -2 & 3 & 4\end{bmatrix}$

$$\text{adj}(R)=C^T=\begin{bmatrix}+(-10) & -(8) & +(1)\\ -(8) & +(-4) & -(1)\\ +(4) & -(-2) & +(-1)\end{bmatrix}^T$$

$$=\begin{bmatrix}-10 & -8 & 4\\ -8 & -4 & +2\\ 1 & -1 & -1\end{bmatrix}$$

ILLUSTRATION 43

If $A=\begin{bmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 2 & 1 & 0\end{bmatrix}$ and $B=(\text{adj }A)$ and $C=5A$, Find the value of $\dfrac{|\text{adj }B|}{|C|}$.

SOLUTION

$$|A|=\begin{vmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 2 & 1 & 0\end{vmatrix}=1(0+3)+(0+6)+1(0-4)=5$$

$$\therefore\quad \frac{|\text{adj }B|}{|C|}=\frac{|\text{adj}(\text{adj }A)|}{|5A|}$$

$$=\frac{|\text{adj}(A)|^{3-1}}{5^3\,|A|}=\frac{\left(|A|^2\right)^2}{5^3\,|A|}=\frac{|A|^4}{5^3\,|A|}=\frac{5^4}{5^4}=1$$

ILLUSTRATION 44

$A_{3\times 3}$ is a matrix such that $|A|=a$, $B=(\text{adj }A)$ such that $|B|=b$, $\frac{1}{2}S=\frac{a}{b}+\frac{a^2}{b^3}+\frac{a^3}{b^5}+\ldots$ up to ∞ and $a=3$. Find the value of S.

SOLUTION

$$\frac{S}{2}=\frac{a}{b}+\frac{a^2}{b^3}+\frac{a^3}{b^5}+\ldots\infty$$

$$\left(\text{Infinite G.P. } A=\frac{a}{b} : r=\frac{a}{b^2}\right)$$

$$\Rightarrow\quad \frac{S}{2}=\frac{A}{1-r}=\frac{\frac{a}{b}}{1-\frac{a}{b^2}}=\frac{ab}{b^2-a}$$

$$\Rightarrow\quad S=\frac{2ab}{b^2-a}$$

$$B=\text{adj }A$$

$$|B|=|\text{adj }A|=|A|^{n-1}=|A|^2=a^2=b.$$

$$\Rightarrow\quad a=3;\ b=9$$

$$S=\frac{2ab}{b^2-a}=\left(\frac{2(3)(9)}{81-3}\right)=\frac{9}{13}$$

ILLUSTRATION 45

If matrix $A=\begin{bmatrix}3 & 2 & 1\\ 6x^2 & 2x^3 & x^4\\ 1 & a & a^2\end{bmatrix}$, then find the value of $\dfrac{d^2}{dx^2}$ $(|A(\text{adj }A)|^{1/3})$ at $x=a$.

SOLUTION

$$|A(\text{adj }A)|^{1/3}=(|A|\,|\text{ adj }A\ |)^{1/3}$$

$$=(|A|\,|\ A|^2)^{1/3}=|A|$$

$$\frac{d}{dx}\left(|A\text{ adj}(A)|^{1/3}\right)=\frac{d}{dx}\begin{vmatrix}3 & 2 & 1\\ 6x^2 & 2x^3 & x^4\\ 1 & a & a^2\end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & 1 \\ 12x & 6x^2 & 4x^3 \\ 1 & a & a^2 \end{vmatrix}$$

$$\frac{d^2}{dx^2}(|A(\text{adj}A)|^{1/3}) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12x & 12x^2 \\ 1 & a & a^2 \end{vmatrix}$$

at $x = a$; $\begin{vmatrix} 3 & 2 & 1 \\ 12 & 12a & 12a^2 \\ 1 & a & a^2 \end{vmatrix} = 0$ (As R_2 and R_3 are proportional)

NON-SINGULAR MATRIX

(i) A square matrix A is said to be a non-singular matrix if $|A| \neq 0$

(ii) A square matrix A is said to be a singular matrix if $|A| = 0$

ILLUSTRATION 46

$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & \lambda \\ 1 & 2 & -1 \end{bmatrix}$ find λ for which matrix is singular.

SOLUTION

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 3 & \lambda \\ 1 & 2 & -1 \end{vmatrix} = 1(-3-2\lambda) - 0 + (2-3) = 0$$

$\Rightarrow \quad 2\lambda = -4 \quad \Rightarrow \quad \lambda = -2$

INVERSE MATRICES (INVERTIBLE MATRICES)

Let A be a square matrices of order n. Then a matrices B (if such a matrix exists) is called the inverse of A if $AB = BA = I_n$.

Inverse of the square matrix A is denoted by A^{-1}.

The inverse of a square matrix A exists if and only if A is a non-singular matrix.

$$A\,(\text{adj. } A) = |A|\, I_n$$

$$A^{-1} A\,(\text{adj. } A) = |A|\, A^{-1} I_n$$

$$I(\text{adj. } A) = |A|\, A^{-1}$$

Inverse matrices of $A = A^{-1} = \dfrac{\text{adj. } A}{|A|}$

$$AA^{-1} = I_n$$

(i) If A and B be any two non-singular matrix, then AB is also a non-singular matrix and

$(AB)^{-1} = B^{-1} A^{-1}$

Proof: A, B are non-singular $\quad \therefore |A| \neq 0, |B| \neq 0$

$\therefore \quad |AB| = |A|\,|B| \neq 0.$

Hence AB is non-singular.

Using $A\,(\text{adj. } A) = |A|\, I_n$

We can also write it as $AB\,(\text{adj. } AB) = |AB|\, I_n$ (Replacing A by AB)

$$AB \frac{\text{adj.}(AB)}{|AB|} = I_n$$

or $AB(AB)^{-1} = I_n$

or $I_n B(AB)^{-1} = A^{-1}$

or $B(AB)^{-1} = A^{-1}$

or $B^{-1} B(AB)^{-1} = B^{-1} A^{-1}$

(Multiply by B^{-1} from left side)

or $I_n(AB)^{-1} = B^{-1} A^{-1}$

or $(AB)^{-1} = B^{-1} A^{-1}$

Hence $B^{-1} A^{-1}$ is the inverse of AB

(ii) If A is a non-singular matrix, then $(A^{-1})^{-1} = A$

(iii) $I_n^{-1} = I_n$ as $I_n^{-1} I_n = I_n$

ILLUSTRATION 47

Find inverse matrix of following square matrix

(i) $A = \begin{bmatrix} -1 & 3 \\ 2 & 7 \end{bmatrix}$ **(ii)** $B = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ -1 & 3 & 5 \end{bmatrix}$

SOLUTION

(i) $A = \begin{bmatrix} -1 & 3 \\ 2 & 7 \end{bmatrix}$

$$|A| = -7 - 6 = -13$$

$$\text{adj}(A) = C^T = \begin{bmatrix} 7 & -2 \\ -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{-1}{13}\begin{bmatrix} 7 & -3 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7/13 & 3/13 \\ 2/13 & 1/13 \end{bmatrix}$$

(ii) $B = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ -1 & 3 & 5 \end{bmatrix}$

$$|B| = \begin{vmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ -1 & 3 & 5 \end{vmatrix} = -1(5+6) - 0 + 2(3+1)$$

$$= -11 + 8 = -3$$

$$\text{adj}(B) = C^T = \begin{bmatrix} (11) & -(3) & (4) \\ -(-6) & (-3) & -(-3) \\ (-2) & -(0) & (-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & 6 & -2 \\ -3 & -3 & 0 \\ 4 & 3 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{adj(B)}{|B|} = \frac{1}{-3}\begin{bmatrix} 11 & 6 & -2 \\ -3 & -3 & 0 \\ 4 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -11/3 & -2 & 2/3 \\ 1 & 1 & 0 \\ -4/3 & -1 & 1/3 \end{bmatrix}$$

ILLUSTRATION 48

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$

Also compute A^{-1}.

SOLUTION

$$A^2 = A\,.\,A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Now, $A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus $A^2 - 4A - 5I = O$ [where O is the zero matrix]

∴ $A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O = O$

or $(A^{-1}A)A - 4(A^{-1}A) - 5A^{-1}I = O$;

or $IA - 4I - 5A^{-1} = O$

∴ $5A^{-1} = A - 4I$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

∴ $A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$

Alternate:

We can also use $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$

ILLUSTRATION 49

$\begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix} X = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$ find matrix X.

SOLUTION

Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$ we have $|A| = 8 + 2 = 10 \neq 0$

$$AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\Rightarrow \quad IX = A^{-1}B = \frac{\text{adj}(A)}{|A|}B$$

$$X = \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{10}\begin{bmatrix} -4+0 & 4-2 \\ -2+0 & 2+4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -4 & 2 \\ -2 & 6 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

Alternate

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now $\begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} 2a+c & 2b+d \\ -2a+4c & -2b+4d \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

Now we have $2a + c = -1, 2b + d = 1, -2a + 4c = 0, -2b + 4d = 2$

We have $a = -2/5, b = 1/5, c = -1/5, d = 3/5$

We get $X = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$

ILLUSTRATION 50

If $\begin{bmatrix}1 & -1\\ 2 & 3\\ -3 & 4\end{bmatrix} A = \begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & -5\\ 7 & -14 & -18\end{bmatrix}$, then find matrix A.

SOLUTION

Since the product matrix of 3×3 matrix and the premultiplier of A is a 3×2 matrix, then A is 2×3 matrix.

Let $A = \begin{bmatrix}\ell & m & n\\ x & y & z\end{bmatrix}$, then the given equation becomes

$$\begin{bmatrix}1 & -1\\ 2 & 3\\ -3 & 4\end{bmatrix}\begin{bmatrix}\ell & m & n\\ x & y & z\end{bmatrix} = \begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & -5\\ 7 & -14 & -18\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\ell - x & m - y & n - z\\ 2\ell + 3x & 2m + 3y & 2n + 3z\\ -3\ell + 4x & -3m + 4y & -3n + 4z\end{bmatrix} = \begin{bmatrix}-2 & 4 & 5\\ 1 & -2 & -5\\ 7 & -14 & -18\end{bmatrix}$$

$\Rightarrow$ $\ell - x = -2,\ m - y = 4,\ n - z = 5,\ 2\ell + 3x = 1,\ 2m + 3y = -2,$ $2n + 3z = -5, -3\ell + 4x = 7, -3m + 4y = -14, -3n + 4z = -18.$

On solving we get $\ell = -1, x = 1, m = 2, y = -2, n = 2, z = -3$

$$\Rightarrow \quad A = \begin{bmatrix}\ell & m & n\\ x & y & z\end{bmatrix} = \begin{bmatrix}-1 & 2 & 2\\ 1 & -2 & -3\end{bmatrix}$$

ILLUSTRATION 51

$A = \begin{bmatrix}-1 & 0 & 1\\ 2 & 1 & -2\\ 3 & 4 & -1\end{bmatrix}$ compute A^{-1} and $|A^{-1}|$.

SOLUTION

$$|A| = \begin{vmatrix}-1 & 0 & 1\\ 2 & 1 & -2\\ 3 & 4 & -1\end{vmatrix}$$

$$= -1(-1 + 8) - 0 + 1(8 - 3) = -7 + 5 = -2$$

$$\text{adj}(A) = C^T = \begin{bmatrix}+(7) & -(4) & +(5)\\ -(-4) & +(-2) & -(-4)\\ +(-1) & -(0) & +(-1)\end{bmatrix}^T$$

$$= \begin{bmatrix}+7 & 4 & -1\\ -4 & -2 & 0\\ 5 & 4 & -1\end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{-1}{2}\begin{bmatrix}7 & 4 & -1\\ -4 & -2 & 0\\ 5 & 4 & -1\end{bmatrix}$$

$$|A^{-1}| = 1/|A| = -1/2$$

ILLUSTRATION 52

Show that for square matrix of order $n \times n$

$\text{adj}(kA) = k^{n-1}\text{adj}(A)$ where k is scalar.

SOLUTION

Using the Identity $A\ \text{adj}(A) = |A|\ I$

Replace A by kA

$kA\ \text{adj}(kA) = |kA|\, I$

or $\quad kA\ \text{adj}(kA) = k^n |A|\, I$

or $\quad A\ \text{adj}(kA) = k^{n-1} |A|\, I$

or $\quad A^{-1}A\ \text{adj}(kA) = k^{n-1} |A|\ A^{-1}I$

(multiplying with A^{-1} from left side)

or $\quad I\ \text{adj}(kA) = k^{n-1} |A| \left(\dfrac{\text{adj}A}{|A|}\right)$

$\Rightarrow \quad \text{adj}(kA) = k^{n-1}\text{adj}(A)$

$\therefore \quad \text{adj}(3A) = 3^{n-1}\ \text{adj}(A)$

if order of matrix A is 3×3

$\therefore \quad \text{adj}(2A) = 2^2\ \text{adj}(A) = 4\text{adj}(A)$

ILLUSTRATION 53

Show that for non singular square matrix of order $n \times n$

$\text{adj}\ (AB) = \text{adj}(B)\ \text{adj}(A)$

SOLUTION

using the identity, $A\ \text{adj}\ (A) = |A|\ I$
replace A by AB

$AB\ \text{adj}\ (AB) = |AB|\, I$

(multiplying with A^{-1} from left side)

$A^{-1}AB\ \text{adj}\ (AB) = |A|\ |B|\ A^{-1}I$

$I\,B\ \text{adj}\ (AB) = |A|\ |B| \left(\dfrac{\text{adj}A}{|A|}\right) I$

$B\ \text{adj}\ (AB) = |B|\ \text{adj}(A)$

(multiplying with B^{-1} from left side)

$B^{-1}B\ \text{adj}\ (AB) = |B|\ B^{-1}\text{adj}(A)$

$I\ \text{adj}\ (AB) = |B| \left(\dfrac{\text{adj}B}{|B|}\right) \text{adj}(A)$

$\text{adj}\ (AB) = \text{adj}(B)\ \text{adj}(A)$

ILLUSTRATION 54

Show that for square matrix of order $n \times n$

$\text{adj}(\text{adj}(A)) = |A|^{n-2}A,$

also conclude that $|\text{adj}\ (\text{adj}\ (A))| = |A|^{(n-1)^2}$.

SOLUTION

Using the identity $A\ \text{adj}(A) = |A|\ I$
replace A by adj(A)

$$\text{adj}(A).\ \text{adj}\ (\text{adj}\ (A)) = |\text{adj}(A)|\ I$$

(multiplying with A from left side)

$$A\ \text{adj}(A).\ \text{adj}\ (\text{adj}\ (A)) = |\text{adj}\ (A)|\ AI$$

(using first identity and $|\text{adj}\ (A)| = |A|^{n-1}$)

$$|A|\ I\ \text{adj}\ (\text{adj}(A)) = |A|^{n-1}A$$

$$\because \quad \text{adj}(\text{adj}\ (A)) = |A|^{n-2}A$$

$$|\text{adj}\ (\text{adj}\ (A))| = |\,|A|^{n-2}A\,| \qquad \because\ (|kA| = k^n|A|)$$

$$|\text{adj}\ (\text{adj}\ (A))| = (|A|^{n-2})^n\ |A|$$

$$= |A|^{n^2-2n}\ |A| = |A|^{n^2-2n+1}$$

$$|\text{adj}\ (\text{adj}\ (A))| = |A|^{(n-1)^2}$$

ILLUSTRATION 55

If $M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$M(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$ then

prove that $[M(\alpha)\ M(\beta)]^{-1} = M(-\beta)\ M(-\alpha)$

SOLUTION

$$|M(\alpha)| = |M(\beta)| = 1$$

$$[M(\alpha)\ M(\beta)]^{-1} = (M(\beta))^{-1}\ (M(\alpha))^{-1}$$

Now $(M(\alpha))^{-1} = \dfrac{\text{adj}\,M(\alpha)}{|M(\alpha)|} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = M(-\alpha)$$

$$M(\beta)^{-1} = \frac{\text{adj}\,M(\beta)}{|M(\beta)|} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix} = M(-\beta)$$

$$\therefore \quad [M(\alpha)\ M(\beta)]^{-1} = (M(\beta))^{-1}\ (M(\alpha))^{-1} = M(-\beta)\ M(-\alpha)$$

ILLUSTRATION 56

If the non-singular matrix A is symmetric, then prove that A^{-1} is also symmetric.

SOLUTION

Since $|A| \neq 0$, therefore A^{-1} exists.

$$(A^{-1})^T = (AT)^{-1} = A^{-1} \qquad [\because A^T = A]$$

$\Rightarrow \quad A^{-1}$ is symmetric.

ILLUSTRATION 57

Let $A = \begin{bmatrix} 2\sin^2\theta & 0 \\ \sin^2\theta & \sin^2\theta \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ then find the most general values of θ where n is an integer.

SOLUTION

$$AA^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\sin^2\theta & 0 \\ \sin^2\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\sin^2\theta & 0 \\ 0 & 2\sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad 2\sin^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = \frac{1}{2} = \sin^2\frac{\pi}{4}$$

$$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}$$

ILLUSTRATION 58

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and $(10)\ B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$. If B is the inverse of matrix A, find α.

SOLUTION

Given, $10B = 10A^{-1} \Rightarrow 10BA = 10A^{-1}A$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} A = 10A^{-1}A$$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

Equating element a_{21}, we get $-5 + \alpha = 0 \quad \Rightarrow \quad \alpha = 5$

ILLUSTRATION 59

Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix}$.

If $AB = I$, where I is an identity matrix of order 3 then find $tr(B)$.

SOLUTION

$$AB = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -\alpha & 14\alpha & 7\alpha \\ 0 & 1 & 0 \\ \alpha & -4\alpha & -2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 5\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10\alpha - 2 & 5\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \alpha = \frac{1}{5}$$

$$tr(B) = 1 - 3\alpha = 1 - \frac{3}{5} = \frac{2}{5}$$

PROPERTIES OF A^{-1}

(i) $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$

(ii) $(A^{-1})^{-1} = A$

(iii) $|A^{-1}| = \dfrac{1}{|A|}$

(iv) $(kA)^{-1} = k^{-1}A^{-1} = \dfrac{1}{k}A^{-1}$ where k is scalar

(v) $(\text{diag.}(a_1\ a_2\ a_3 \ldots\ a_n))^{-1} = \text{diag}\left(\dfrac{1}{a_1}\ \dfrac{1}{a_2}\ \dfrac{1}{a_3} \ldots \dfrac{1}{a_n}\right)$

(vi) $\text{adj}(A^{-1}) = (\text{adj}(A))^{-1}$

(vii) $(AB)^{-1} = B^{-1}A^{-1}$

(viii) $(A^T)^{-1} = (A^{-1})^T$

(ix) $(A_1A_2 \ldots A_{n-1}\ A_n)^{-1} = A_n^{-1}\ A_{n-1}^{-1} \ldots A_2^{-1}\ A_1^{-1}$

(x) $(A^m)^{-1} = (A^{-1})^m$

(xi) If A is symmetric matrix $\Rightarrow A^{-1}$ is symmetric matrix.

(xii) If A is triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is a triangular matrix.

(xi) If A is scalar matrix $\Rightarrow A^{-1}$ is scalar matrix.

(xii) If A is diagonal matrix $\Rightarrow A^{-1}$ is diagonal matrix.

(xiii) $AB = AC \quad \Rightarrow \quad B = C$, iff $|A| \neq 0$.

(xiv) $|\text{adj}(\text{adj}(A))| = |A|^{(n-1)^2}$

ILLUSTRATION 60

If A is 3×3 order matrix such that $\det(A) = -2$ then find

(i) $|\text{adj}\ (A)|$ **(ii)** $|\text{adj}\ (A^{-1})|$

(iii) $|\text{adj}\ (3A)|$ **(iv)** $|A^{-1}|$

SOLUTION

(i) $|\text{adj}\ (A)| = |A|^{n-1} = |A|^2 = (-2)^2 = 4$

(ii) $|\text{adj}\ (A^{-1})| = |A^{-1}|^{n-1} = |A^{-1}|^2 = \left(\dfrac{1}{|A|}\right)^2 = \dfrac{1}{4}$

(iii) $|\text{adj}\ (3A)| = |3A|^{n-1} = |3A|^2 = (3^3|A|)^2 = 3^6(-2)^2 = 2916$

(iv) $|A^{-1}| = \dfrac{1}{|A|} = -\dfrac{1}{2}$

ILLUSTRATION 61

A is 3×3 order matrix such that $\det(A) = -2$ find

(i) $|4A^{-1}|$ **(ii)** $|(4A)^{-1}|$ **(iii)** $|\text{adj}\ (\text{adj}\ (A))|$

SOLUTION

(i) $|4A^{-1}| = 4^3|A^{-1}| = 64\left(\dfrac{1}{|A|}\right) = \dfrac{64}{-2} = -32$

(ii) $|(4A)^{-1}| = \dfrac{1}{|4A|} = \dfrac{1}{4^3|A|} = \dfrac{1}{64(-2)} = -\dfrac{1}{128}$

(iii) $|\text{adj}\ (\text{adj}\ (A))| = |A|^{(n-1)^2} = |A|^{2^2} = (-2)^4 = 16$

ILLUSTRATION 62

B is adjoint of matrix A of order $n \times n$ find $|AB + \lambda I|$ where λ is scalar.

SOLUTION

Given $B = \text{adj}(A)$

Using identity $A.\ \text{adj}(A) = (A)I$

$$\Rightarrow \quad AB = |A|\ I$$

$$|AB + \lambda I| = |\,|A|I + \lambda I| = |(|A| + \lambda)\ I| \qquad (\because\ |kA| = k^n|A|)$$

$$= (|A| + \lambda)^n\ |I|$$

$$= (|A| + \lambda)^n$$

ILLUSTRATION 63

Let A be matrix of order 3×3 such that $|A| = 1$.

Let $B = 2A^{-1}$ and $C = \dfrac{\text{adj}.A}{\sqrt[3]{2}}$ then find the value of $|AB^2C^3|$.

(where $|A|$ represent det. A)

SOLUTION

$$|B| = |2A^{-1}| = 2^3|A^{-1}| = \frac{8}{|A|} = 8$$

$$|C| = \left|\frac{\text{adj}.A}{\sqrt[3]{2}}\right| = \frac{|\text{adj}.A|}{\left(\sqrt[3]{2}\right)^3} = \frac{1}{2}|\text{adj}.A| = \frac{|A|^2}{2} = \frac{1}{2}$$

$$|AB^2C^3| = |A(2A^{-1})^2C^3| = |4A^{-1}C^3| = |4A^{-1}|\,|C|^3$$

$$= 4^3\frac{1}{|A|}\left(\frac{1}{2}\right)^3 = 8$$

ILLUSTRATION 64

A is a square matrix of order 5×5, $\det(A) = -\sqrt{3}$ then compute (i) $|\text{adj}(A^T)|$ (ii) $|\text{adj}(-A^{-1})|$

SOLUTION

(i) $|\text{adj}(A^T)| = |A^{\mathrm{T}}|^{n-1} = |A^T|^4 = |A|^4$ $\quad(\because n = 5)$

$$= (-\sqrt{3})^4 = 9$$

(ii) $|\text{adj}(-A^{-1})| = |(-1)^4\,\text{adj}(A^{-1})| = |\text{adj}(A^{-1})|$

$$= |A^{-1}|^4 = \frac{1}{|A|^4} = \frac{1}{\left(-\sqrt{3}\right)^4} = \frac{1}{9}$$

Using Elementary Row/Column Transformations to find Inverse Matrix

Any of the following operations is called an elementary transformation.

- **(i)** The interchange of any two rows (or columns)
- **(ii)** The multiplication of the elements of any row (or column) by any non-zero number.
- **(iii)** The addition to the elements of any row (or column) the corresponding elements of any other row (or column) multiplied by any number.

Note:

- **(i)** An elementary transformation is called a row transformation or column transformation according as it applies to rows or columns.
- **(ii)** Generally this approach is asked in Board examination

ILLUSTRATION 65

Obtain the inverse of the following matrix using elementary operations $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

SOLUTION

We know that $A = IA$,

i.e., $$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ (applying } R_1 \leftrightarrow R_2)$$

(applying $R_3 \to R_3 - 3R_1$)

Or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

(applying $R_1 \to R_1 - 2R_2$)

Or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

(applying $R_3 \to R_3 + 5R_2$)

Or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

(applying $R_3 \to \frac{1}{2}R_3$)

Or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

(applying $R_1 \to R_1 + R_3$)

Or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

(applying $R_2 \to R_2 - 2R_3$)

Or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

or $\quad I = A^{-1}A$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$

ILLUSTRATION 66

Using elementary row transformation, find the inverse of the matrix $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$.

SOLUTION

We have,

$$A = IA$$

or, $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

[Applying $R_1 \to R_1 - R_2$]

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[Applying $R_2 \to R_2 + (-2)R_1$ and $R_3 \to R_3 + (-3)R_1$]

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$$

[Applying $R_2 \to R_2(1/2)$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A$$

[Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 2R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

[Applying $R_3 \to 1/4\, R_3$]

[Applying $R_1 \to R_1 + (1/2)R_3$ and $R_2 \to R_2 - (1/2)\, R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

or $\quad I = A^{-1}A$

Hence, $A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$

Test Your Concepts-III

(Solutions on page H.313)

1. Find the adjoint of the matrix A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

2. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

3. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and verify that $A^{-1} A = I$.

4. If A is a square matrix of order 3 such that $|A| = 2$ then find $\left|\left(adj\ A^{-1}\right)^{-1}\right|$.

5. If A is a square matrix of order 3 such that $|A| = 5$ then find $|\text{Adj}\ (4A)|$.

6. If A and B are two non-singular matrices and both are symmetric and commute each other then show that both $A^{-1}B$ and $A^{-1}B^{-1}$ are symmetric.

7. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1} \cdots \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$, then find $a + b$.

8. Let $A = [a_{ij}]_{3 \times 3}$ be such that $a_{ij} = \begin{cases} 3, & when\, i = j \\ 0, & otherwise \end{cases}$ then find $\left\{ \dfrac{\det.(adj.(adj.A))}{5} \right\}$.

   [Note : $\{k\}$ denotes fractional part of k]

9. Let the matrix A and B be defined as $A = \begin{bmatrix} 3 & 2 \\ 2 & \alpha \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 7 & 3 \end{bmatrix}$. If $\det(2A^9 B^{-1}) = -2$, then find the number of distinct possible real values of α.

Answer keys

1. $\begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$ **2.** $\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$ **3.** $\begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$ **4.** 4

5. $5^2 \times 16^3$ **7.** -5050 **8.** $\frac{1}{5}$ **9.** 1

CALEY HAMILTON THEOREM (CHARACTERISTIC EQUATION OF SQUARE MATRIX)

For a given square matrix A of order n, and a column vector X, if there exists a scalar λ(lambda) such that $AX = \lambda X$, then λ is called an eigenvalue or characteristic of A. (i.e.,) $(A - \lambda I)X = 0$.

This equation has a solution only if $|A - \lambda I| = 0$. This equation is called the characteristic equation. The matrix A evidently satisfies the characteristic equation. The roots of the equation $|A - \lambda I| = 0$ are called the characteristic roots or eigenvalues of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}_{n\times n}$$

$\det(A - \lambda I) = 0$ where λ is scalar

$$\Rightarrow \quad a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_2 \lambda^2 + a_1\lambda + a_0 = 0$$

Characteristic equation of square matrix A.
Each matrix satisfies its own characteristic equation

$$a_n A^n + a_{n-1} A^{n-1} + \ldots + a_2 A^2 + a_1 A + a_0 I = \mathrm{O}$$

for 2×2 order matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (a - \lambda)(d - \lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a + d)\lambda + ad - bc = 0$$

Remark(s)

(i) If matrix A is of order 2×2 then $A^2 - (Tr(A))\,A + \det(A) = 0$

(ii) If matrix A is of order 3×3

then $A^3 - A^2(Tr(A)) + \frac{A}{2}\left((Tr(A))^2 - Tr\left(A^2\right)\right) - \det(A) = 0$

ILLUSTRATION 67

$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ Find relation in A^2 and A.

SOLUTION

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ -1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2 - \lambda)(1 - \lambda) + 3 = 0$$

$$\Rightarrow \quad 2 - 2\lambda - \lambda + \lambda^2 + 3 = 0$$

$\lambda^2 - 3\lambda + 5 = 0$ is characteristic equation of matrix A, hence A satisfies it

$$A^2 - 3A + 5I = O$$

Alternate

$$A^2 - 3A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6+3 \\ -2-1 & -3+1 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} 6 & 9 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= -5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -5I$$

$$\Rightarrow \quad A^2 - 3A + 5I = O$$

ILLUSTRATION 68

$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$ show that

(i) $5A^{-1} = 4I - A$

(ii) $A^3 - 4A^2 + 5A = O$

SOLUTION

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(3-\lambda) + 2 = 0$

$\Rightarrow \quad 3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$ it is a characteristic equation of matrix A

$A^2 - 4A + 5I = O$

(i) $A^2 - 4A + 5I = O$

(multiplying with A^{-1} from left side)

$A^{-1}A^2 - 4A^{-1}A + 5A^{-1}I = A^{-1}O$

$\Rightarrow A - 4I + 5A^{-1} = O$

$\Rightarrow 5A^{-1} = 4I - A$

(ii) $A^2 - 4A + 5I = O$

(multiplying with A from left side)

$A^3 - 4A^2 + 5A = AO = O$

Hence proved

ILLUSTRATION 69

Show that $A^{-1} = \dfrac{1}{4}(-A^2 + 2A + 3I)$

where matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

SOLUTION

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$|A| = 1(-2-1) + 1(1-2) = -3 - 1 = -4$

$\Rightarrow \quad (1-\lambda)[-2 - 2\lambda + \lambda + \lambda^2 - 1] + 1[1 + \lambda - 2] + 0 = 0$

$\Rightarrow \quad (1-\lambda)(\lambda^2 - \lambda - 3) + \lambda - 1 = 0$

$\Rightarrow \quad \lambda^2 - \lambda - 3 - \lambda^3 + \lambda^2 + 3\lambda + \lambda - 1 = 0$

$\Rightarrow \quad -\lambda^3 + 2\lambda^2 + 3\lambda - 4 = 0$

$\Rightarrow \quad \lambda^3 - 2\lambda^2 - 3\lambda + 4 = 0$

It is a characteristic equation of matrix A

$A^3 - 2A^2 - 3A + 4I = 0$

(multiplying with A^{-1} from left side)

$A^{-1}A^3 - 2A^{-1}A^2 - 3A^{-1}A + 4A^{-1}I = 0$

$\Rightarrow \quad A^2 - 2A - 3I + 4A^{-1} = 0$

$\Rightarrow \quad 4A^{-1} = -A^2 + 2A + 3I$

$\Rightarrow \quad A^{-1} = \dfrac{1}{4}(-A^2 + 2A + 3I)$

ILLUSTRATION 70

Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$ then find the value of $(x + y + z)$ where I is a unit matrix of order 3.

SOLUTION

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Using Sarrus Rule

$\Rightarrow \quad (2-\lambda)^2(1-\lambda) + 1 + 1 - (1-\lambda) - (2-\lambda) - (2-\lambda) = 0$

$\Rightarrow \quad (4 + \lambda^2 - 4\lambda) - 4\lambda - \lambda^3 + 4\lambda^2 - 3 + 3\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 5\lambda^2 - 5\lambda + 1 = 0$

$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 5\lambda - 1 = 0$

$\Rightarrow \quad A^3 - 5A^2 + 5A - I = 0$

$\Rightarrow \quad A^2 - 5A + 5I - A^{-1} = 0$

$\Rightarrow \quad A^{-1} = A^2 - 5A + 5I = xA^2 + yA + zI$

$x + y + z = 1 - 5 + 5 = 1$

ILLUSTRATION 71

If $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $|A - kI| = 0$ then find the sum of all the possible values of k.

SOLUTION

$$|A - kI| = 0 \quad \Rightarrow \quad \begin{vmatrix} -k & 1 & 0 \\ 2 & 5-k & 0 \\ 0 & 0 & 1-k \end{vmatrix} = 0$$

$(1-k)[-5k + k^2 - 2] = 0$

$\Rightarrow \quad k^3 - 6k^2 + 3k + 2 = 0$

Sum of the roots $= -\dfrac{\text{coefficient of } k^2}{\text{coefficient of } k^3} = 6$

Alternate:

$$|A-kI| = 0$$

$$\Rightarrow \quad k^3-(Tr(A))\,k^2+\frac{k}{2}\left[(Tr(A))^2-Tr(A^2)\right]-\det(A)=0$$

Sum of values of $k = \text{Tr}(A) = 6$

SPECIAL MATRIX

(i) Orthogonal Matrix

A matrix A is said to be orthogonal if $A'A = I$ where A' is the transpose of A

$$AA^T = I = A^TA$$

$$\because \quad AA^T = I$$

$$|AA^T| = |I|$$

$$\Rightarrow \quad |A|\;|A^T| = 1$$

$$\Rightarrow \quad |A|\;|A| = 1$$

$$\Rightarrow \quad |A| = \pm 1$$

hence we can conclude that orthogonal is non-singular matrix (Invertible matrix)

$$\because \quad AA^T = I$$

$$A^{-1}AA^T = A^{-1}I \qquad \text{(multiplying by } A^{-1} \text{ from left side)}$$

$$\Rightarrow \quad IA^T = A^{-1} \quad \Rightarrow \quad A^T = A^{-1}$$

for orthogonal matrix, transpose matrix is equal to inverse matrix

The determinant value of orthogonal matrix is either 1 or –1. Hence orthogonal matrix is always invertible

(ii) Idempotent Matrix

A matrix A is said to be Idempotent matrix if $A^2 = A$

(a) $A^n = A, n \geq 2, n \in N$

(b) Determinant value of idempotent matrix is either 0 or 1

(c) If idempotent matrix is invertible then its inverse will be identity matrix i.e. I.

(iii) Nilpotent Matrix

A matrix A is said to be Nilpotent matrix of order m if $A^m = O$ where $A^{m-1} \neq O$

A nilpotent matrix is not invertible.

(iv) Involutory Matrix

A matrix A is said to be Involutory Matrix if $A^2 = I$ or $A^{-1} = A$

(v) Periodic Matrix

A square matrix is which satisfies the relation

$A^{k+1} = A$, for some positive integer k, is a periodic matrix. The period of the matrix is the least value of k for which this holds true.

Note that period of an idempotent matrix is 1.

eq. If $A^3 = A, A^5 = A, A^7 = A, \ldots$ then it is a periodic matrix and $A^{2+1} = A$ so its period is $= 2$

DIFFERENTIATION OF A MATRIX

If $A = \begin{bmatrix} f(x) & g(x) \\ h(x) & \ell(x) \end{bmatrix}$ then $\dfrac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & \ell'(x) \end{bmatrix}$ is a differentiation of matrix A.

e.g. if $A = \begin{bmatrix} x^3 & \sin x \\ 2x & 6x^2 \end{bmatrix}$ then $\dfrac{dA}{dx} = \begin{bmatrix} 3x^2 & \cos x \\ 2 & 12x \end{bmatrix}$

ILLUSTRATION 72

If A is idempotent and $A + B = I$, then Show that

(i) B is idempotent **(ii)** $AB = O$ **(iii)** $BA = O$

where O is null matrix

SOLUTION

Here $A + B = I \Rightarrow B = I - A$

Now $B^2 = (I - A)(I - A)$

$$= I^2 - AI - IA + A^2$$

$$= I - A - A + A^2$$

$$= I - A - A + A \text{ since } A^2 = A$$

(A is idempotent)

$$= I - A = B$$

$\therefore$ B is idempotent is true

Again $AB = A(I - A) = AI - A^2 = A - A = 0$

Also $BA = (I - A)A = IA - A^2 = A - A = 0$

Hence all statements (i), (ii) and (iii) are true.

ILLUSTRATION 73

If $k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is an orthogonal matrix then find the value of k.

SOLUTION

Here let

$$A = k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Since A is orthogonal $\therefore AA^T = I$

$$\Rightarrow \quad k^2\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= k^2\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= k^2\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9k^2 I = I$$

$$\Rightarrow \quad 9k^2 = 1 \quad \Rightarrow \quad k^2 = \frac{1}{9} \quad \Rightarrow \quad k = \pm\frac{1}{3}$$

ILLUSTRATION 74

Identify the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ as Idempotent/ Nilpotent /Involutory/Orthogonal matrix.

SOLUTION

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Nilpotent matrix of index 2.

ILLUSTRATION 75

$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal find $2\alpha^2 + 6\beta^2 + 3\gamma^2$.

SOLUTION

$$AA^T = I$$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On equating we get

$$4\beta^2 + \gamma^2 = 1;\ \alpha^2 + \beta^2 + \gamma^2 = 1;\ 2\beta^2 - \gamma^2 = 0;\ \alpha^2 = \beta^2 + \gamma^2$$

$\Rightarrow \quad \gamma^2 = 2\beta^2$ from first relation we get $2\gamma^2 + \gamma^2 = 1$

$$\Rightarrow \quad \gamma^2 = \frac{1}{3} \text{ and } \beta^2 = \frac{1}{6}$$

$$\alpha^2 - \beta^2 - \gamma^2 = 0$$

$$\Rightarrow \quad \alpha^2 = \beta^2 + \gamma^2 = \frac{1}{2}$$

$$\because \quad 2\alpha^2 + 6\beta^2 + 3\gamma^2 = 1 + 1 + 1 = 3.$$

ILLUSTRATION 76

If P is orthogonal matrix and A periodic matrix with period with 4

$Q = PA\,P^T$, then find $P^T\,Q^{2005}\,P$.

SOLUTION

As given P is orthogonal matrix then $PP^T = I = P^TP$

$$P^T\,Q^{2005}\,P = P^T\,Q\,QQQ\ldots QQQQP$$

$$= P^T(PA\,P^T)\,(PA\,P^T)\,(PA\,P^T)\,(PA\,P^T)\ldots(PA\,P^T)P$$

$$= (P^TP)A(P^TP)A(P^TP)A(P^TP)\ldots A(P^TP)A(P^TP)$$

$$= IAIAIA\ldots AIAI = A^{2005}$$

As Matrix A is periodic with 4 then $A^5 = A$

$$A^{2005} = (A^5)^{401} = A^{401} = A(A^5)^{80} = AA^{80} = A(A^5)^{16}$$

$$= AA^{16} = A^2(A^5)^3 = A^2A^3 = A^5 = A$$

ILLUSTRATION 77

If $A^3 = O$, then show that $(I - A)^{-1} = I + A + A^2$.

SOLUTION

Given $A^3 = O$; $IA = AI = A$

$$\because \quad (I + A + A^2)(I - A) = I - A^3 = I$$

$$\therefore \quad I + A + A^2 = (I - A)^{-1}$$

SOLVING SYSTEM OF LINEAR EQUATION USING MATRIX

Let us consider the following system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots (1)$$

If $b_1 = b_2 = \ldots = b_n = 0$, then the system of equations (i) is called a system of homogeneous linear equations and if at least one of $b_1, b_2 \ldots, b_n$ is non zero, then it is called a system of non-homogeneous linear equations.

SOLUTION OF A SYSTEM OF EQUATIONS

A set of values $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $x_1, x_2, \ldots, x_n$ respectively which satisfy all the equations of the given system of equations is called a solution of the given system of equations. The system of equations is said to be consistent if its solution exist otherwise it is said to be inconsistent.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then the system of equations (1) can be written as $AX = B$

Here A is a $n \times n$ matrices and is called the coefficient matrices. Each of X and B is a $n \times 1$ matrices.

Let discuss three variable system of equation

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \quad \begin{bmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_2y + c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B = \frac{Adj\ A}{|A|}.B$$

(i) If $|A| \neq 0$, system is consistent having unique solution.

- (a) If $|A| \neq 0$ and (adj A). $B \neq$ Null matrix, system is consistent having unique non-trivial solution.
- (b) If $|A| \neq 0$ and (adj A). $B = 0$ (Null matrix), system is consistent having trivial solution.

(ii) If $|A| = 0$, system has no unique solution, system may or may not be consistent

- (a) If (adj A). $B = O$ (null matrix) then system has infinite solutions or no solution
- (b) If (adj A). $B \neq O$, then system is Inconsistent (no solution)

Working Rule

Generally system of linear equations are solved by using Matrix method if it is specifically asked like in Board examination

We can solve system of linear equations by using better method Cramer rule, which we have learnt in Determinant chapter

ILLUSTRATION 78

Solve the system of equations using matrix method

$$x + 2y + 3z = 1,\ 2x + 3y + 2z = 2,\ 3x + 3y + 4z = 1$$

SOLUTION

We have $x + 2y + 3z = 1, 2x + 3y + 2z = 2, 3x + 3y + 4z = 1$

The given system of equation in the matrix form are written as below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \quad \Rightarrow \quad X = A^{-1}B$$

$$\text{where } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$|A| = 1(12-6) - 2(8-6) + 3(6-9) = 6 - 4 - 9 = -7 \neq 0$$

$$\therefore \quad \text{Adj } A = C^T = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$\text{Since } X = A^{-1} \cdot B = \frac{\text{adj}(A).B}{|A|}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 3 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/7 \\ 8/7 \\ -2/7 \end{bmatrix}$$

$$\text{hence } x = -\frac{3}{7},\ y = \frac{8}{7},\ z = -\frac{2}{7}$$

ILLUSTRATION 79

Show that the homogeneous system of equations $x - 2y + z = 0$, $x + y - z = 0$, $3x + 6y - 5z = 0$ has non-trivial solution, Also find the solution.

SOLUTION

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = O$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{vmatrix}$

$$= 1\,(-5 + 6) + 2\,(-5 + 3) + 1\,(6 - 3) = 0$$

Thus, $|A| = 0$ So, the given system of equations has a non-trivial solution. To find these solutions, we put $z = k$ in the first two equations and write them as follows

$$x - 2y = -k \text{ and } x + y = k$$

$\Rightarrow \quad x = k/3,\ y = 2k/3$

These values of x, y and z also satisfy the third equation. Hence $x = k/3$, $y = 2k/3$ and $z = k$, where k is any real number satisfy the given system of equations.

ILLUSTRATION 80

By the method of matrix inversion, solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

SOLUTION

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

or $\quad AX = B \qquad \ldots(1)$

or $\quad X = A^{-1}B$

Where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix},\ X = \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$|A| = 1\,(-5 - 7) - 1\,(-2 - 14) + 1\,(2 - 10)$$

$$= -12 + 16 - 8 = -4 \neq 0$$

Let $\boldsymbol{C}$ be the matrix of cofactors of elements of $|\boldsymbol{A}|$.

$$\therefore \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = C' = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} = \frac{\text{Adj.}\,(A)}{|A|}$$

$$\text{Now } A^{-1}B = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

On equating the corresponding elements, we have

$$x = 1,\ u = -1,\ y = 3,\ v = 2, z = 5,\ w = 1$$

Test Your Concepts-IV

(Solutions on page H.314)

1. Find condition on a, b, c such that $PP' = I$, where $P = \begin{bmatrix} 2/3 & 3k & a \\ -1/3 & -4k & b \\ 2/3 & -5k & c \end{bmatrix}$

2. Solve the following equations by matrix method $5x + 3y + z = 16$, $2x + y + 3z = 19$, $x + 2y + 4z = 25$.

3. Suppose a Matrix A satisfies $A^2 - 5A + 7I = 0$ If $A^5 - aA + bI$, then find the value of $2a + b$.

4. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is an orthogonal matrix then find the value of $x + y$.

5. If A and B are two matrices such that $AB = B$ and $BA = A$, then show that $A - B$ is nilpotent.

6. If A is an orthogonal matrix of order n, then find the value of $|\text{adj.}(\text{adj } A)|$.

7. The system of homogeneous equations $tx+(t+1)y+(t-1)z=0$, $(t+1)x+ty+(t+2)z=0$, $(t-1)x+(t+2)y+tz=0$ has a non-trivial solution, find suitable t.

8. If the matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is invertible, then show that the planes $a_{11}x + a_{12}y + a_{13}z = 0$, $a_{21}x + a_{22}y + a_{23}z = 0$ and $a_{31}x + a_{32}y + a_{33}z = 0$ ($a_{ij} \in R, \forall i, j$) intersect in a point.

Answer keys

1. $a^2 + b^2 - c^2 = \frac{8}{9}$ **2.** $x = 1, y = 2, z = 5$ **3.** -87 **4.** -3 **6.** ± 1 **7.** $t = -1/2$

SOLVED PROBLEMS

PROBLEM 1

Let $f(x) = \begin{vmatrix} x & 1 & 1 \\ \sin \pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}$. $f(x)$ be an odd function and its odd value is equal to $g(x)$, then find $|f(1)g(1)|$.

SOLUTION

$$g(x) = f(-x) = -f(x) = -\begin{vmatrix} x & 1 & 1 \\ \sin \pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}$$

Now $f(x)\, g(x) = -\begin{vmatrix} x & 1 & 1 \\ \sin \pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}\begin{vmatrix} x & 1 & 1 \\ \sin \pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}$

$$f(1)g(1) = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix}^2$$

$$= -(2+0+1-2-3)^2$$

$$= -4$$

$\Rightarrow \quad |f(1)\, g(1)| = 4$

PROBLEM 2

If $x = by + cz + du$, $y = cz + ax + du$, $z = ax + by + du$, $u = ax + by + cz$. Then the find the value of the expression $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d}$.

SOLUTION

Let $E = ax + by + cz + du$ …(1)

$$E = ax + x$$

$$x = \frac{E}{1+a}$$

similarly $y = \frac{E}{1+b}$, $z = \frac{E}{1+c}$, $u = \frac{E}{1+d}$

put values of x, y, z and u in Eq.(1)

again $E\left[\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d}\right] = E$

$$\therefore \quad \frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1$$

PROBLEM 3

If $(l_1, m_1, n_1), (l_2, m_2, n_3) \& (l_3, m_3, n_3)$ are the direction cosines of three mutually perpendicular straight lines then show that the matrix A= $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal matrix.

SOLUTION

Recalling the concept from vector and 3 *D*

As $(l_1, m_1, n_1), (l_2, m_2, n_3) \& (l_3, m_3, n_3)$ are the direction cosines of straight lines

$$l_1{}^2 + m_1{}^2 + n_1{}^2 = 1, l_2{}^2 + m_2{}^2 + n_2{}^2 = 1, l_3{}^2 + m_3{}^2 + n_3{}^2 = 1$$

for perpendicular lines their dot product should be zero hence we have $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$; $l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$ and $l_3 l_1 + m_3 m_1 + n_3 n_1 = 0$

$$A\,A^T = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_1 l_3 & \Sigma l_2 l_3 & \Sigma l_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

$\Rightarrow$ *A* is orthogonal.

PROBLEM 4

For α, β, γ let $A = \begin{bmatrix} \alpha^2 & 6 & 8 \\ 3 & \beta^2 & 9 \\ 4 & 5 & \gamma^2 \end{bmatrix}$ and $B = \begin{bmatrix} 2\alpha & 3 & 5 \\ 2 & 2\beta & 6 \\ 1 & 4 & 2\gamma - 3 \end{bmatrix}$.

If $T_r(A) = T_r(B)$ then find the value of $\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$.

SOLUTION

$$T_r(A) = \alpha^2 + \beta^2 + \gamma^2$$

$$Tr(B) = 2\alpha + 2\beta + 2\gamma - 3$$

$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 = 2\alpha + 2\beta + 2\gamma - 3$

$\Rightarrow \quad (\alpha - 1) + (\beta - 1)^2 + (\gamma - 1)^2 = 0$

$\Rightarrow \quad \alpha = \beta = \gamma = 1$

Now $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 3$

PROBLEM 5

Let $A=\begin{bmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{bmatrix}$, where $0\le\theta<2\pi$, then find range of det(A).

SOLUTION

$$|A| = \begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}$$

$\therefore$ Expanding along R_1, we get

$$= 1\,(1+\sin^2\theta) - \sin\theta\,(-\sin\theta+\sin\theta) + 1\cdot(1+\sin^2\theta)$$

$\Rightarrow$ $|A| = 2(1+\sin^2\theta)$

As, $\sin^2\theta\in[0,1]$ $\therefore$ $|A|\in[2,4]$

We get $|A| = 2\,(\sin^2\theta+1)\in[2,4]$

PROBLEM 6

$A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$ then let us define a function

$f(x) = \text{det.}(A^TA^{-1})$ then show that $f^n(x) = \underbrace{f(f(f(f\ldots f(x))))}_{n\,\text{times}}$

is a constant function where $n\ge 2$.

SOLUTION

We have $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$

$\therefore$ det. A^T = det. $A = \begin{vmatrix}1 & \tan x\\ -\tan x & 1\end{vmatrix}$

$$= (1+\tan^2 x) = \sec^2 x$$

Now, $f(x) = \text{det.}\,(A^T A^{-1}) = \text{det.}(A^T)\,\det(A^{-1})$

$$= \frac{\det(A^T)}{\det(A)} = 1$$

Hence, $f(x) = 1$

$$f^n(x) = \underbrace{f(f(f(f\ldots\ldots f(x))))}_{n\,times} = 1$$

PROBLEM 7

Let $a = \lim\limits_{x\to1}\left(\dfrac{x}{\ell nx}-\dfrac{1}{x\ell nx}\right)$; $b = \lim\limits_{x\to0}\dfrac{x^3-16x}{4x+x^2}$;

$c = \lim\limits_{x\to0}\dfrac{\ell n(1+\sin x)}{x}$ and $d = \lim\limits_{x\to-1}\dfrac{(x+1)^3}{3(\sin(x+1)-(x+1))}$,

then comment on type of matrix $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

SOLUTION

We have

$$a = \lim_{x\to1}\left(\frac{x}{\ln x}-\frac{1}{x\ln x}\right)$$

$$= \lim_{x\to1}\left(\frac{x^2-1}{x\ln x}\right)\left(\frac{0}{0}\text{form, using L-H Rule}\right)$$

$$= \lim_{x\to1}\left(\frac{2x-0}{x\dfrac{1}{x}+\ln x.1}\right) = 2$$

$$b = \lim_{x\to0}\frac{x^3-16x}{4x+x^2} = \lim_{x\to0}\left(\frac{x^2-16}{4+x}\right) = \frac{-16}{4} = -4$$

$$c = \lim_{x\to0}\frac{\ell n(1+\sin x)}{x}$$

$$= \lim_{x\to0}\frac{\ell n(1+\sin x)}{\sin x}\left(\frac{\sin x}{x}\right) = (1)\,(1) = 1$$

$$d = \lim_{x\to-1}\frac{(x+1)^3}{3(\sin(x+1)-(x+1))}\quad\text{Put } x+1=h$$

$$= \lim_{x\to0}\frac{1}{3\left(\dfrac{\sinh-h}{h^3}\right)}$$

$$= \lim_{h\to0}\frac{1}{3\left(\dfrac{\left(h-\dfrac{h^3}{3!}+\dfrac{h^5}{5!}\ldots\right)-h}{h^3}\right)} = -2$$

Now, $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}$

As, $A^2 = AA = \begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}\begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow$ $A^2 = O$, but $A\ne O$

$\Rightarrow$ A is nilpotent matrix of order 2.

[Note that, any matrix of the form $A = \begin{bmatrix}a & -a^2\\ 1 & -a\end{bmatrix}$ is nilpotent]

PROBLEM 8

If $A=\begin{bmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{bmatrix}$, then show that $A^{-1}=A^3$.

SOLUTION

We have $A=\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}$

$\Rightarrow \quad |A| = 3(-3+4) + 3(2-0) + 4(-2-0) = 3 + 6 - 8 = 1,$

Hence A^{-1} exists

Now, $A^2 = AA$

$$= \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix}$$

$$A^4 = A^2\,A^2 = \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$\Rightarrow \quad A^4 = I$ (on multiplying by A^{-1})

$\therefore \quad A^3 = A^{-1}$

PROBLEM 9

If $A = \text{dia.}(d_1, d_2, d_3, d_4)$ where $d_i > 0\ \forall\ i = 1, 2, 3, 4$ is a diagonal matrix of order 4 such that $d_1 + 2d_2 + 4d_3 + 8d_4 = 16$ then find the maximum value of

$$f(x) = \log_{(\tan x + \cot x)}(\det.(A)) \text{ where } x \in \left(0, \frac{\pi}{2}\right).$$

SOLUTION

$$\text{Given, } A = \text{dia.}\begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}$$

$\therefore \quad \det.(A) = d_1 \cdot d_2 \cdot d_3 \cdot d_4$

Now, using A.M. ≥ G.M. for positive numbers

$$\frac{d_1 + 2d_2 + 4d_3 + 8d_4}{4} \geq (d_1 \cdot 2d_2 \cdot 4d_3 \cdot 8d_4)^{\frac{1}{4}}$$

$$\Rightarrow \quad \frac{16}{4} \geq 2^{3/2}\,(d_1 \cdot d_2 \cdot d_3 \cdot d_4)^{\frac{1}{4}}$$

$\Rightarrow \quad d_1 \cdot d_2 \cdot d_3 \cdot d_4 \leq 4$

$\Rightarrow \quad \det.(A) \in (0, 4]$

Now, $f(x) = \log_{(\tan x + \cot x)}(\det.(A)),\ x \in \left(0, \frac{\pi}{2}\right)$

$$= \frac{\log(\det.(A))}{\log(\tan x + \cot x)}$$

As, $(\tan x + \cot x) \geq 2\ \forall\ x \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad f(x)$ will be maximum, when Nr. Is maximum and Dr. is minimum.

Hence, $f(x)$ maximum $= \dfrac{\log 4}{\log 2} = \log_2(4) = 2$

PROBLEM 10

Show that the homogeneous system of equations

$$\begin{bmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = O \text{ has non trivial solutions for any } a, b, c, d;$$

where O is null matrix.

SOLUTION

For non-trivial solution of homogeneous system
The determinant of the coefficient matrix is zero

$$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 0 & a+b & c+d \\ 0 & ab & cd \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 \\ 1 & c+d & cd \\ 1 & a+b & ad \end{vmatrix} = 0 \text{ for all } a, b, c, d$$

Hence given homogeneous system of equations has non trivial solutions.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A square matrix P satisfies $P^2 = I - P$ where I is identity matrix. If $P^n = 5I - 8P$ then n is

(A) 4 (B) 5

(C) 6 (D) 7

2. Let A is a matrix of order 3 defined as, $A = [a_{ij}]_{3\times 3}$ where $a_{ij} = \lim\limits_{x\to 0}\dfrac{\sin(ix)}{\tan(jx)}$, $\forall\ 1 \le i, j \le 3$. Then A^2 is

(A) $4A$ (B) $3A$

(C) $2A$ (D) A

3. If A is a 3×3 matrix and det $A = 5$ then

det (Adj (Adj A)) = ____

(A) 5^2 (B) 5^3

(C) 5^4 (D) 5^5

4. Let A and B are two non-singular matrices of order 2 such that $(9A^2B - 6AB + B) = 9\left\{B - \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}\right\}$. If $B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ and $Tr(A) = \dfrac{10}{3}$, then the value of det. (A) can be

(A) 1 (B) 2

(C) 3 (D) 4

5. If A and B are square matrices of order 3 such that det.$(A) = -2$ and det. $(B) = 1$, then det. $(A^{-1}$ adj. B^{-1}. adj$(2A^{-1}))$ is equal to

(A) 8 (B) -8

(C) 1 (D) -1

6. If $M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $M^3 = (\alpha M - I)(\beta M - I)$, where α, β are integers and I is a 3×3 identity matrix, then the value of $(\alpha + \beta)$ is equal to

(A) 4 (B) 5

(C) 6 (D) 7

7. If A and B are two non-singular matrices of order 3 such that $AA^T = 2I$ and $A^{-1} = A^T - A\cdot \text{adj}(2B^{-1})$, then det.$(B)$ equals:

(A) 4 (B) $4\sqrt{2}$

(C) 16 (D) $16\sqrt{2}$

8. The equation $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a solution (x, y, z) besides $(0, 0, 0)$. The value of k equal to.

(A) 0 (B) 1

(C) 2 (D) 3

9. If M is a square matrix of order 2, then $(\text{tr.}(M))^2 - \text{tr.}(M^2)$ is equal to

(A) $\dfrac{\text{det.}(M)}{2}$ (B) 2 det.(M)

(C) 3 det.(M) (D) 4 det.(M)

10. For $\theta = \dfrac{3\pi}{5}$, let $B = [b_{ij}]$ be a square matrix of order 2 such that

$$b_{ij} = \begin{cases} \cos\theta, & i = j \\ \cos\left(\dfrac{j\pi}{2} + \theta\right), & i > j \\ \sin\left(\dfrac{j\pi}{2} - \theta\right), & i < j \end{cases}$$

Then find the trace of matrix B^5

(A) -1 (B) -2

(C) -3 (D) -4

11. Let M denote the matrix $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, where $i^2 = -1$, and let I denotes the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then the matrix $I + M + M^2 + M^3 + M^4 + \ldots M^{2010}$ is equal to

(A) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (B) $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ (D) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

12. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix}$ where $x, y, z \in R$.

If $B^T AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$ then the number of ordered triplets (x, y, z) is

(A) 2 (B) 6

(C) 8 (D) 9

13. Let $S = \left\{(a, b) \,\middle|\, A^3 = A, \text{ where } A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ a & b \end{pmatrix}\right\}$ then $n(S)$

(A) 1 (B) 2

(C) 3 (D) 0

14. Matrix A is given by $A = \begin{bmatrix} 3 & 11 \\ 2 & 8 \end{bmatrix}$ then the determinant of $A^{2011} - 5A^{2010}$ is

(A) -7.2^{2012} (B) 2^{2012}

(C) -7.2^{2010} (D) 7.2^{2010}

15. A is a 2×2 matrix such that $A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ and $A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$.

The sum of the elements of A, is

(A) -1 (B) 0

(C) 2 (D) 5

16. In a square matrix A of order 3 the elements, a_{ii} the sum of the roots, $a_{i,\,i+1}$ are product of roots of the equation $x^2-(a+b)x+ab=0$; $a_{i+1,\,i}$ are all unity and the rest of the elements are all zero. The value of the det. (A) is equal to

(A) 0 (B) $(a+b)^3$

(C) a^3-b^3 (D) $(a^2+b^2)(a+b)$

17. Let D_k is the $k\times k$ matrix with 0's in the main diagonal, unity as the element of 1st row and $(f(k))^{\text{th}}$ column and k for all other entries. If $f(x)=x-\{x\}$ where $\{x\}$ denotes the fractional part function then the value of det. (D_2) + det. (D_3) equals

(A) 32 (B) 34

(C) 36 (D) 44

18. For a matrix $A=\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$, the value of $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$ is equal to

(A) $\begin{bmatrix}1 & 100\\0 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & 4950\\0 & 1\end{bmatrix}$

(C) $\begin{bmatrix}1 & 5050\\0 & 1\end{bmatrix}$ (D) $\begin{bmatrix}1 & 2500\\0 & 1\end{bmatrix}$

19. A is an involutory matrix given by $A=\begin{bmatrix}0 & 1 & -1\\4 & -3 & 4\\3 & -3 & 4\end{bmatrix}$ then the inverse of $\frac{A}{2}$ will be

(A) 2A (B) $\frac{A^{-1}}{2}$

(C) $\frac{A}{2}$ (D) A^2

20. Consider a matrix $A=\begin{bmatrix}3 & 1\\-6 & -2\end{bmatrix}$, then $(1+A)^{99}$ equals (where I is a unit matrix of order 2)

(A) $I+2^{98}A$

(B) $I+2^{99}A$

(C) $I+(2^{99}+I)A$

(D) $I+(2^{99}-I)A$

21. Let $A=\begin{bmatrix}2 & 0 & 7\\0 & 1 & 0\\1 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-\alpha & 14\alpha & 7\alpha\\0 & 1 & 0\\\alpha & -4\alpha & -2\alpha\end{bmatrix}$.

If $AB=I$, where I is an identity matrix of order 3 then trace B has value equal to

(A) 0 (B) $\frac{2}{5}$

(C) $\frac{1}{5}$ (D) 5

22. If the product of n matrices $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\dots\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$ is equal to the matrix $\begin{bmatrix}1 & 378\\0 & 1\end{bmatrix}$ then the value of n is equal to

(A) 26 (B) 27

(C) 377 (D) 378

23. If $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ satisfies the equations $x^2-(a+d)x+k=0$, then

(A) $k=bc$ (B) $k=ad$

(C) $k=ad-bc$ (D) $k=a^2+b^2+c^2+d^2$

24. Let A, B be square matrix such that $AB=O$ and B is non singular then

(A) $|A|$ must be zero but A may non zero

(B) A must be zero matrix

(C) nothing can be said in general about A

(D) none of these

25. The number of solutions of the matrix equation $X^2=I$ other than I, is

(A) 0 (B) 1

(C) 2 (D) more than 2

26. Number of real values of x for which the matrix $A=\begin{bmatrix}3-x & 2 & 2\\2 & 4-x & 1\\-2 & -4 & -1-x\end{bmatrix}$ is singular, is

(A) 1 (B) 2

(C) 3 (D) indinite

27. If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and trace $(A)=12$, then

(A) $|A|=64$ (B) $|A|=16$

(C) $|A|=12$ (D) $|A|=0$

28. Let $A=\begin{bmatrix}\ell & m & n\\p & q & r\\1 & 1 & 1\end{bmatrix}$ and $B=A^2$. If $(\ell-m)^2+(p-q)^2=9$, $(m-n)^2+(q-r)^2=16$, $(n-\ell)^2+(r-p)^2=25$, then the value of det. B equals

(A) 36 (B) 100

(C) 144 (D) 169

29. If $A=\begin{bmatrix}-1 & \frac{3}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}$, then $I+A+A^2+\dots\infty=$

(A) $\begin{bmatrix}1 & -3\\1 & 4\end{bmatrix}$ (B) $\frac{2}{7}\begin{bmatrix}1 & -3\\-1 & 4\end{bmatrix}$

(C) $\frac{2}{7}\begin{bmatrix}1 & 3\\-1 & 4\end{bmatrix}$ (D) undefined

30. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $A^TA = AA^T = I$, then $xy =$

(A) -1
(B) 1
(C) 2
(D) -2

31. Number of 3×3 real matrices $A = [a_{ij}]$ such that $a_{ij} + a_{ji} = 0$ for all $i, j = 1, 2, 3$ and $|a_{ij}| \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is equal to

(A) 6856
(B) 6857
(C) 6858
(D) 6859

32. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$ and $A = \begin{bmatrix} \alpha+\beta & \alpha^2+\beta^2 & \alpha^3+\beta^3 \\ \alpha^2+\beta^2 & \alpha^3+\beta^3 & \alpha^4+\beta^4 \\ \alpha^3+\beta^3 & \alpha^4+\beta^4 & \alpha^5+\beta^5 \end{bmatrix}$, then

(A) A is involuntary matrix
(B) A is singular matrix
(C) $|A| = 26$
(D) None of these

33. Let $A = \begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$, then the value of $\dfrac{|A^{-1} + A^T|}{|adj.(adj.A)|}$ is equal to

(A) $-\dfrac{2^6\sqrt{3}}{3^8}$
(B) $\dfrac{2^3}{3^6\sqrt{3}}$
(C) $-\dfrac{2^9\sqrt{3}}{3^6}$
(D) $\dfrac{2^3}{3^5}$

34. Consider matrix $A = [a_{mn}]_{3\times 3}$. If $a_{mn} = \begin{cases} i, & m = n \\ \omega, & m > n \\ \omega^2, & m < n \end{cases}$ (where $i = \sqrt{-1}$ and ω is non real cube root of unity) and the value of det.(A) is $a + ib$; then $a + b$ is $(a, b \in R)$

(A) -1
(B) -3
(C) -4
(D) -5

35. Let $A = [a_{ij}]_{3\times 3}$ be a matrix.

If $A + A^T = \begin{bmatrix} 6 & 4 & 4 \\ a_{21}+a_{12} & 10 & a_{23}+a_{32} \\ a_{31}+a_{13} & 4 & 8 \end{bmatrix}$ where a_{12}, a_{23} and a_{31} are the positive root of the equation $x^3 - 6x^2 + px - 8 = 0$, $p \in R$, then the value of det. (A) is

(A) 28
(B) 24
(C) 1
(D) 0

36. If A and B are non singular square matrices of even order such that $A^T = A$, $B^T = -B$ and $AB + BA = O$ (where O is a null matrix), then

(A) B and A^{-1} commute each other on multiplication
(B) (AB) is skew symmetric matrix
(C) A^{-1} and B^{-1} commute each other on multiplication
(D) $(A^{-1}B)$ is symmetric matrix

37. If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then det (Adj (Adj A)) is

(A) $(14)^4$
(B) $(14)^6$
(C) $(14)^9$
(D) $(14)^2$

38. If matrix $A = [a_{ij}]_{3\times 3}$, matrix $B = [b_{ij}]_{3\times 3}$, where $a_{ij} + a_{ij} = 0$ and $b_{ij} - b_{ji} = 0 \ \forall\, i, j$, then $A^4.B^3$ is

(A) Non-Singular
(B) Zero matrix
(C) Symmetric
(D) Skew-Symmetric matrix

39. A and B are two square matrices such that $A^2B = BA$ and if $(AB)^{10} = A^kB^{10}$ then k is

(A) 1001
(B) 1023
(C) 1042
(D) none of these

40. If A and B are two non-singular matrices which commute then $(A(A+B)^{-1}B)^{-1}(AB) =$

(A) $A + B$
(B) $A^{-1} + B^{-1}$
(C) $A^{-1} + B$
(D) none of these

41. Let A be an idempotent matrix satisfying $(I - 0.4A)^{-1} = I - \lambda A$, where I is unit matrix of the same order as A. Then the value of λ is

(A) 2/5
(B) 2/3
(C) −2/3
(D) 1/2

42. If A and B are two square matrices and $AB = I$, then which of the following is not true?

(A) $BA = I$
(B) $A^{-1} = B$
(C) $B^{-1} = A$
(D) $A^2 = B$

43. The number of 2×2 matrices A, that are there with the elements as real numbers satisfying $\mathrm{A+A^T=I}$ and $\mathrm{AA^T=I}$ is

(A) zero
(B) one
(C) two
(D) infinite

44. If A, B, C, D be real matrices (not necessarily square) such that $A^T = BCD$, $B^T = CDA$, $C^T = DAB$, $D^T = ABC$ for the matrix $S = ABCD$ consider the statements

$I : S^3 = S$
$II : S^2 = S^4$

(A) II is true but not I
(B) I is true but not II
(C) both I and II are true
(D) both I and II are false

45. If $A = [a_{ij}]$ is a square matrix of even order such that $[a_{ij}] = i^2 - j^2$, then

(A) A is a skew-symmetric matrix
(B) A is a symmetric matrix and det A is a square
(C) A is a symmetric matrix and det A is zero
(D) none of these

46. Number of orthogonal matrices of order 2×2 whose entries are 0, 1 is

(A) 1
(B) 4
(C) 2
(D) 3

47. Let $A=\begin{bmatrix}0 & -\tan\theta/2\\ \tan\theta/2 & 0\end{bmatrix}(\theta\neq n\pi), B=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$,

$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ then the matrix $I+A$ is equal to

(A) $(I-A)B$ (B) $(I-A)^2B$

(C) $(I+A)^2B$ (D) $(I-A)^2$

48. If A and B are two square matrices such that $AB = B$ and $BA = A$ then for $n\in N$, $(A+B)^n=$

(A) $2^n(A+B)$ (B) $2^{n-1}(A+B)$

(C) $2^{n+1}(A+B)$ (D) $2^{\frac{n}{2}}(A+B)$

49. $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ then $A^4-5A^3-A^2-4A-2I=$

(A) 0 (B) A

(C) I (D) 2I

50. If A and P are different matrices of order n satisfying $A^3=P^3$ and $A^2P=P^2A$ (where $|A|\neq|P|$) then $\left|A^2+P^2\right|$ is equal to

(A) n (B) 0

(C) $|A||P|$ (D) $|A+P|$

51. If $A=\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}$ where ω is a complex cube root of unity then adj A equals

(A) $(\omega^2-\omega)\overline{A}$ (B) $(\omega-\omega^2)\overline{A}$

(C) $-\overline{A}$ (D) $\overline{A}$

52. If $A_1=[a_1]$, $A_2=\begin{bmatrix}a_2 & a_3\\ a_4 & a_5\end{bmatrix}$, $A_3=\begin{bmatrix}a_6 & a_7 & a_8\\ a_9 & a_{10} & a_{11}\\ a_{12} & a_{13} & a_{14}\end{bmatrix}\ldots A_n=[\ldots]$ and so on, where $a_r=[\log_2 r]$ (where [.] denotes greatest integer function). $T_r(A_{10})$ is equal to; (Tr stands for trace of the matrix)

(A) 800 (B) 80

(C) 792 (D) 160

53. Let A and B be square matrices of same order satisfying $AB = A$ and $BA = B$, then A^2B^2 equals (O being zero matrices of the same order as B)

(A) A (B) B

(C) I (D) O

54. The product of all the values of t, for which the system of the equations $(a-t)x+by+cz=0$, $bx+(c-t)y+az=0$, $cx+ay+(b-t)z=0$ has non-trivial solutions, is

(A) $\begin{vmatrix}a & -c & -b\\ -c & b & -a\\ -b & -a & c\end{vmatrix}$ (B) $\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}$

(C) $\begin{vmatrix}a & c & b\\ b & a & c\\ c & b & a\end{vmatrix}$ (D) none of these

55. A, B, C are three matrices of the same order such that any two are symmetric and the 3rd one is skew symmetric. If $X=ABC+CBA$ and $Y=ABC-CBA$, then $(XY)^T$ is

(A) symmetric (B) skew symmetric

(C) $I-XY$ (D) $-YX$

56. If a matrix A is such that $3A^3+2A^2+5A+I=0$ and $|A|\neq 0$ then $A^{-1}$is

(A) $3A^2+2A+5I$ (B) $-3A^2-2A-5I$

(C) $3A^2-2A+5$ (D) Can't be determined

57. Let A, B are square matrices of same order satisfying $AB = A$ and $BA = B$ then $(A^{2010}+B^{2010})^{2011}$ equals.

(A) $A+B$ (B) $2010\,(A+B)$

(C) $2011\,(A+B)$ (D) $2^{2011}\,(A+B)$

58. Let A be a 3×3 matrix given by $A=[a_{ij}]_{3\times 3}$. If for every column vector X satisfies $X'AX=0$ and $a_{12}=2008$, $a_{13}=2010$ and $a_{23}=-2012$. Then the value of $a_{21}+a_{31}+a_{32}=$ ____

(A) –6 (B) 2006

(C) –2006 (D) 0

59. A and B are two non-singular matrices so that $A^6=I$ and $AB^2=BA(B\neq I)$. A value of K so that $B^K=I$ is

(A) 31 (B) 32

(C) 63 (D) 64

60. For each real number x such that $-1<x<1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix}1 & -x\\ -x & 1\end{bmatrix}$ and $z=\dfrac{x+y}{1+xy}$ Then,

(A) $A(z)=A(x)+A(y)$ (B) $A(z)=A(x)\,[A(y)]^{-1}$

(C) $A(z)=A(x)\,A(y)$ (D) $A(z)=A(x)-A(y)$

61. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

(A) $A^{-n}B^nA^n$ (B) $A^nB^nA^{-n}$

(C) $A^{-1}B^nA$ (D) $n(A^{-1}BA)$

62. If A and B are any two different square matrices of order n with $A-B$ is non-singular $A^3=B^3$ and $A(AB)=B(BA)$, then

(A) $A^2+B^2=O$ (B) $A^2+B^2=I$

(C) $A^2+B^3=I$ (D) $A^3+B^3=O$

63. A square matrix A is said to be nilpotent of index m. If $A^m=0$, now, if for this A $(I-A)^n=I+A+A^2+\ldots+A^{m-1}$, then n is equal to

(A) 0 (B) m

(C) $-m$ (D) –1

64. If $A=\begin{bmatrix}i & -i\\ -i & i\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$, then A^8 equals $(i=\sqrt{-1})$

(A) $4B$ (B) $128\,B$

(C) $-128\,B$ (D) $-64\,B$

65. If A is involutory matrix, then which of the following is/are correct?

(A) $I+A$ is idempotent

(B) $I-A$ is idempotent

(C) $(I+A)(I-A)$ is singular

(D) $\dfrac{I+A}{3}$ is idempotent

66. If $A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$, $C=ABA^T$ then A^TC^nA equals to ($n\in \mathrm{I}^+$)

(A) $\begin{bmatrix}-n & 1\\ 1 & 0\end{bmatrix}$
(B) $\begin{bmatrix}1 & -n\\ 0 & 1\end{bmatrix}$
(C) $\begin{bmatrix}0 & 1\\ 1 & -n\end{bmatrix}$
(D) $\begin{bmatrix}1 & 0\\ -n & 1\end{bmatrix}$

67. If $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ then $A^{16}=$

(A) $\begin{bmatrix}0 & 256\\ 256 & 0\end{bmatrix}$
(B) $\begin{bmatrix}256 & 0\\ 0 & 256\end{bmatrix}$
(C) $\begin{bmatrix}-16 & 0\\ 0 & -16\end{bmatrix}$
(D) $\begin{bmatrix}0 & 16\\ 16 & 0\end{bmatrix}$

68. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$, $a, b\in N$, then number of matrix ‘B’ such that $AB=BA$ are

(A) 0
(B) 1
(C) finitely many
(D) infinite

69. Let A and B are two non-singular square matrices, A^T and B^T are the transpose matrices of A and B respectively, then which of the following is correct

(A) B^TAB is symmetric matrix if and only if A is symmetric
(B) B^TAB is symmetric matrix if and only if B is symmetric
(C) B^TAB is skew symmetric matrix for every matrix A
(D) B^TAB is skew symmetric matrix if B is skew symmetric

70. If A is a non-diagonal involutory matrix, then

(A) $A-I=O$
(B) $A+I=O$
(C) $A-I$ is non zero singular
(D) none of these

71. If $\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}A\begin{bmatrix}-3 & 2\\ 5 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, then $A=$

(A) $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$
(B) $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$
(C) $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$
(D) $-\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

72. Let $A=\begin{pmatrix}0 & \sin\alpha & \sin\alpha\cdot\sin\beta\\ -\sin\alpha & 0 & \cos\alpha\cdot\cos\beta\\ -\sin\alpha\cdot\sin\beta & -\cos\alpha\cdot\cos\beta & 0\end{pmatrix}$, then

(A) $|A|$ is independent of α and β
(B) A^{-1} depends only on α
(C) A^{-1} depends only on β
(D) none of these

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. $M=\{A \mid A$ is 3×3 matrix whose entries are -1 and $1\}$ then

(A) det A lies between -6 and 6
(B) det $A\in\{-4, 0, 4\}$
(C) Number of elements of $M=2^9$
(D) Number of elements of $M=3^9$

2. $A\in M_n(c)$ (where M is a matrix of order nxn) is non-singular matrix such that $3ABA^{-1}+A=2A^{-1}BA$ then

(A) A and B both are identity matrices
(B) $|A+B|=0$
(C) $\left|ABA^{-1}-AB^{-1}A\right|=0$
(D) $A+B$ is not a singular matrix

3. Consider a skew symmetric matrix $A=\begin{bmatrix}a & b\\ -b & c\end{bmatrix}$ such that a, b and c are selected from the set $S=\{0, 1, 2, 3, \ldots 9\}$. Then which of the following is/are correct?

(A) Number of possible matrix A is 10.
(B) Number of invertible matrix A is 9.
(C) If $|A|$ is divisible by 3, then number of possible matrix A is 4.
(D) If $b=1$, then $|A+I|=2$ where I is identity matrix

4. If ω is non-real cube root of unity and matrix $P=\begin{vmatrix}1 & \omega^3 & \omega^2\\ \omega^3 & 1 & \omega\\ \omega^2 & \omega & 1\end{vmatrix}$, then

(A) P is non-singular matrix
(B) P is orthogonal matrix
(C) P^{-1} is symmetric matrix
(D) $P^{-1}=\frac{1}{3}\begin{bmatrix}1-\omega^2 & 0 & \omega-\omega^2\\ 0 & 1-\omega & \omega^2-\omega\\ \omega-\omega^2 & \omega^2-\omega & 0\end{bmatrix}$

5. Let $A=[a_{ij}]_{3\times 3}$ be a matrix such that $a_{ij}=0\ \forall\ i\neq j$ and $a_{ij}>0\ \forall\ i=j$. If adj (A) satisfies the equation $x^3-9x^2+px-27=0$, $p\in R$ then

(A) tr. $(A)=3\sqrt{3}$
(B) det. (adj(adj A)) $=3^6$
(C) tr. $(A^{-1})=\frac{1}{3\sqrt{3}}$
(D) tr. $(A^{-1})=\sqrt{3}$

6. Let A_n is a $n\times n$ matrix in which diagonal elements are $1, 2, 3, \ldots, n$ (i.e., $a_{11}=1, a_{22}=2, a_{33}=3, \ldots, a_{ii}=i, \ldots a_{nn}=n$) and all other elements are equal to ’n’ then

(A) A_n is singular for all $'n'$
(B) A_n is non-singular for all $'n'$
(C) $\det.A_5 = 120$
(D) $\det.A_n = 0$

7. Let A and B are two square idempotent matrices such that $AB \pm BA$ is a null matrix, then the value of the det.$(A - B)$ may be
(A) –1 (B) 1
(C) 0 (D) 2

8. If the elements of a 2×2 matrix A are real positive and distinct such that $\det(A + A^T)^T = 0$ then
(A) $\det A > 0$ (B) $\det A \geq 0$
(C) $\det(A - A^T) > 0$ (D) $\det(AA^T) > 0$

9. If A is a square Matrix such that $A.(AdjA) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ then
(A) $|A| = 4$ (B) $|adjA| = 16$
(C) $\dfrac{|adj(adjA)|}{|adjA|} = 16$ (D) $|adj2A| = 128$

10. If $A = \begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$ and $B = \begin{bmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{bmatrix}$, then
(A) $|A| = |B|$
(B) $|A| = -|B|$
(C) $|A| = 2|B|$
(D) A is invertible if and only if B is invertible

11. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then
(A) adj(adj A) = A (B) | adj(adj A) | = 1
(C) | adj A | = 1 (D) None of these

12. Let A be a non singular, symmetric matrix of order three such that $A = \text{adj.}(A + A^T)$, then
(A) $A^{-1} = \dfrac{A}{16}$ (B) $A^{-1} = 16\text{A}$
(C) $\text{adj.}(A^{-1}) = 64\text{A}$ (D) $\text{adj.}(A^{-1}) = \dfrac{A}{1024}$

13. Let adj.$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 1 \\ -1 & 2 & 5 \end{bmatrix}$, then which of the following statement can be correct.
(A) $\text{tr}(A^{-1}) = 0.75$ (B) $|A| = -8$
(C) $|A| = 8$ (D) $\text{tr}(A^{-1}) = -0.75$

14. Let A, B, C be $n \times n$ order real matrices and product is pair-wise commutative, also $ABC = O_n$, if $\lambda = \det.\ (A^3 + B^3 + C^3)$. det. $(A + B + C)$ then λ can be -
(A) 0 (B) 2
(C) – 2 (D) 4

15. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, where $A^x = O$ (where O is null matrix and $x \in N$ less than 20), then which of the following is(are) false?
(A) Greatest value of x is 18
(B) Sum of values of x is 187
(C) Difference between largest and smallest value of x is 15
(D) Number of values of x is 7

16. If A and B are square matrices of order 2 such that A + adj.$(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and A^T – adj.$(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then
(A) B is symmetric matrix
(B) $A^n = A\ \forall\ n \in N$
(C) $|A + A^2 + A^3 + A^4 + A^5| = 0$
(D) $|B + B^2 + B^3 + B^4 + B^5| = 0$

17. Let $M = \begin{bmatrix} 1 & \lambda^3 & 0 \\ 0 & 1 & \mu^3 \\ \gamma^3 & 0 & 1 \end{bmatrix}$ where $\lambda, \mu, \gamma \in R$, then -
(A) Sum of roots of the equation $|M - xI| = 0$ is 3
(B) Sum of non real roots of the equation $|M - xI| = 0$ is $2 - \lambda\mu\gamma$.
(C) Product of roots of the equation $|M - xI| = 0$ is $1 + (\lambda\mu\gamma)^3$.
(D) Product of real roots of the equation $|M - xI| = 0$ is $1 + (\lambda\mu\gamma)$.

18. Let $A = \begin{bmatrix} 0 & \tan\theta & 2 \\ -\tan\theta & 0 & \sec\theta \\ -2 & -\sec\theta & 0 \end{bmatrix}$, where $\theta \neq (2n+1)\dfrac{\pi}{2}, (n \in I)$ and $B = \sum_{r=1}^{5} A^{(2r-1)}$ then which of the following statement is correct
(A) A is skew symmetric matrix
(B) B is skew symmetric matrix
(C) $(A + B)$ is skew symmetric matrix
(D) $(A + B)$ is symmetric matrix

19. Which of the following statements is (are) true
(A) Every square matrix can be expressed as the sum of a symmetric and skew- symmetric matrices in unique way
(B) For a square matrix 'A', $|\text{Adj}A| = |A|^{n-1}$ where 'n' is the order of 'A'
(C) If 'A' is a non singular square matrix then $|A^{-1}| = \dfrac{1}{|A|}$
(D) Two matrices 'A' and 'B' are said to be confirmable for multiplication in the same order iff the no. of rows of 'A; is equal to no. of columns of B

20. If for a square matrix $A = [a_{ij}]$, $a_{ij} = i^2 - j^2$ $i, j = 1,2 \ldots n$ is of even order then
(A) A is a skew-symmetric
(B) $|A|$ is a perfect square
(C) A is symmetric and $|A| = 0$
(D) A is neither symmetric nor skew-symmetric

21. Let $A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}$. Then
(A) $A^{3n}=I$ for all positive integers n
(B) $A^{-1}=A^2$
(C) $A^4=A$
(D) $|AdjA|=1$

22. If A and B are two matrices such that $B=-A^{-1}BA$ then
(A) $AB+BA=0$
(B) $(A+B)^2=A^2+B^2$
(C) $(A+B)^2=A^2+B^2+2AB$
(D) $(A+B)^2=A+B$

23. If A and B are two matrices such that $AB=BA$ then for every $n\in N$
(A) $(AB)^n=A^nB^n$
(B) $A^nB=BA^n$
(C) $A^{2n}-B^{2n}=(A^n-B^n)(A^n+B^n)$
(D) $(A+B)^n=A^n+{}^nC_1A^{n-1}B+{}^nC_2A^{n-2}B^2+.....+B^n$

24. If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$, then
(A) $A^3-A^2=A-1$
(B) $Det(A^{2010}-1)=0$
(C) $A^{50}=\begin{bmatrix}1 & 0 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}$
(D) $A^{50}=\begin{bmatrix}1 & 1 & 0\\ 25 & 1 & 0\\ 25 & 0 & 1\end{bmatrix}$

25. Let $A=\begin{bmatrix}-3 & -7 & -5\\ 2 & 4 & 3\\ 1 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}a\\ b\\ 1\end{bmatrix}$ if AB is a scalar multiple of B, then
(A) $4a+7b+5=0$
(B) $a+b+2=0$
(C) $b-a=4$
(D) $a+3b=0$

26. If 'A' is an odd order sequence matrix such that $A^2+A+2I=0$ then which of the following is/are true?
(A) A is non-singular
(B) A is symmetric
(C) A cannot be skew symmetric
(D) $A^{-1}=-\frac{1}{2}(A+I)$

27. If A and B are two invertible matrices of the same order, then adj(AB) is equal to
(A) $adj(B)\ adj(A)$
(B) $|B||A|B^{-1}A^{-1}$
(C) $|B||A|.A^{-1}B^{-1}$
(D) $|A||B|(AB)^{-1}$

28. If the sum of two idempotent matrices is idempotent then
(A) $AB+BA=0$
(B) $AB=BA=I$
(C) $AB\neq BA$
(D) $AB=BA=0$

29. Let $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, then
(A) $A^2-4A-5I_3=0$
(B) $A^{-1}=\frac{1}{5}(A-4I_3)$
(C) A^3 is not invertible
(D) A^2 is invertible

30. If $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$, then
(A) $A^3=9A$
(B) $A^3=27A$
(C) $A+A=A^2$
(D) A^{-1} does not exist

31. P is a non-singular matrix and A, B are two matrices such that $B=P^{-1}AP$ then the true statements among the following are
(A) A is invertible iff B is invertible
(B) $B^n=P^{-1}A^nP\ \forall\ n\in N$
(C) $\forall\ \lambda\in R,\ B-\lambda I=P^{-1}(A-\lambda I)P$ (I is the identity matrix)
(D) A, B are both singular matrices

32. If $x,y,z,w\in R$ satisfy the following system of equations
$x+y+z+w=1;\ x+2y+4z+8w=16;$
$x+3y+9z+27w=81$ and $x+4y+16z+64w=256$, then the pairs which has $H.C.F.$ as 2 is
(A) $(|w|,|z|)$
(B) $(|z|,|y|)$
(C) $(|y|,|x|)$
(D) $(|z|,|x|)$

33. Let $a_1, a_2, a_3\ldots$ be real numbers which are in arithmetic progression with common difference $d\neq 0$. Then
(A) $A=\begin{bmatrix}a_1 & a_2 & a_3\\ a_4 & a_5 & a_6\\ a_5 & a_6 & a_7\end{bmatrix}$ is singular
(B) $B=\begin{bmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_2 & a_4 & a_6\end{bmatrix}$ is non-singular
(C) The system of equations $a_1x+a_2y+a_3z=0$
$a_3x+a_1y+a_2z=0$
$a_4x+a_5y+a_6z=0$ has unique solution
(D) The system of equations $a_1x+a_2y+a_3z=0$
$a_4x+a_5y+a_6z=0$
$a_7x+a_8y+a_9z=0$ has infinitely many solutions

34. Let $A=\{1^2,3^2,5^2,........\}$. If 9 elements selected from set A to make a 3×3 matrix then det (A) will be divisible by
(A) 9
(B) 36
(C) 8
(D) 64

35. If $A=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$ and $A^{2012}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then which of the following is/are correct?
(A) $a=d$
(B) $a+b+c+d=4026$
(C) $a^2+b^2+d^2=2$
(D) $b=2012$

36. Which of the following statements are FALSE?

(A) If A and B are square matrices of the same order such that $ABAB = 0$, it follows that $BABA = 0$.

(B) Let A and B be different $n \times n$ matrices with real numbers. If $A^3 = B^3$ and $A^2B = B^2A$, then $A^2 + B^2$ is invertible.

(C) If A is a square, non-singular and symmetric matrix, then $\left(\left(A^{-1}\right)^{-1}\right)^{-1}$ is skew symmetric.

(D) The matrix of the product of two invertible square matrices of the same order is also invertible.

37. The value of the determinant

$$\begin{vmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) & \cos 2\alpha \\ \sin\theta & \cos\theta & \sin\alpha \\ -\cos\theta & \sin\theta & \lambda\cos\alpha \end{vmatrix}$$ is

(A) independent of θ for all $\lambda \in R$

(B) independent of θ and α when $\lambda = 1$

(C) independent of θ and α when $\lambda = -1$

(D) independent of λ for all θ

38. Suppose that a, b, c are real numbers such that $a + b + c = 1$.

If the matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ be an orthogonal matrix, then

(A) A is an involutory matrix

(B) $|A|$ is negative

(C) $a^3 + b^3 + c^3 - 3abc = 1$

(D) atleast one of a, b, c is negative

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Let A and B are non-singular matrices of order 3 such that det.$(A) = 5$ and $A^{-1}B^2 + AB = 0$.

1. The value of det.$(A^6 - 2A^4B + A^2B^2)$ is equal to

(A) 0 (B) 5^6

(C) $2^3.5^6$ (D) 10^6

2. A^2det.(A^2) – adj.(adj B) is equal to

(A) Null matrix (B) $25A^2 - 5B$

(C) $25A^2$ (D) $50A^2$

Comprehension 2

Let $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ and matrix B is such that $AB^TA = B$ and $BA^TB = I$ where I is unit matrix of order two.

On the basis of above information, answer the following question:

3. Matrix B^2 is -

(A) $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$ (D) $\begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix}$

4. If det.$(B) > 0$, then det.$(B^T - B)$ is

(A) 32 (B) –8

(C) –32 (D) 8

Comprehension 3

For $x > 0$, let $A = \begin{bmatrix} x+\frac{1}{x} & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 16 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{5x}{x^2+1} & 0 & 0 \\ 0 & \frac{3}{x} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ be two matrices.

Three other matrices X, Y and Z are defined as

$$X = (AB)^{-1} + (AB)^{-2} + (AB)^{-3} + \ldots (AB)^{-n},$$

$Y = \lim\limits_{n\to\infty} X$ and $Z = Y^{-1} - 2I$, where I is identity matrix of order 3.

5. The value of det.(adj.$(\sqrt{5}Y^{-1})$) is equal to

(A) $(5!)^2$ (B) $5^3(5!)^2$

(C) $5(5!)^2$ (D) $5^2(5!)^2$

6. Least positive integral value of tr(AY) is equal to

(A) 8 (B) 7

(C) 6 (D) 5

7. If tr$(Z + Z^2 + Z^3 + \ldots + Z^{10}) = 2^a + b$ where $a, b \in N$, then least value of $(a + b)$ is equal to

(A) 11 (B) 12

(C) 18 (D) 19

Comprehension 4

For a given square matrix A of order n, and a column vector X, if there exists a scalar λ such that $AX = \lambda X$, then λ is called an eigenvalue or characteristic of A. (i.e.,) $(A - \lambda I)X = 0$.

This equation has a solution only if $|A - \lambda I| = 0$. This equation is called the characteristic equation. The matrix A evidently satisfies the characteristic equation. The roots of the equation $|A - \lambda I| = 0$ are called the characteristic roots or eigenvalues of A.

8. The eigenvalues of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are

(A) (1, 2, 3) (B) (0, 3, 15)

(C) (3, 0, 1) (D) (4, 3, 2)

9. If A is a 3 × 3 real matrix with eigenvalues (0, 1, 1), then A satisfies

(A) $x^3 + 2x = 0$ (B) $-x^3 + 2x^2 - x = 0$

(C) $x^3 - 2x^2 = 0$ (D) $x^3 + 2x^2 + x = 0$

10. If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a matrix A, then the eigenvalues of A^{-1} are

(A) $\lambda_1, \lambda_2, \lambda_3$ (B) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

(C) $\lambda_1 - 1,\ \lambda_2 - 1,\ \lambda_3 - 1$ (D) $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2$

Comprehension 5

3×3 matrices are formed using the elements from the set $\{-3, -2, -1, 0, 1, 2, 3\}$ then answer the following.

11. Number of matrices that are skew symmetric matrices (or) scalar matrices is

(A) $7^7 - 7^4$ (B) $7^7 - 7^3$

(C) $7^3 + 6$ (D) $7^4\left(7^3 - 6\right)$

12. Number of matrices that are lower diagonal (or) upper diagonal is

(A) $7^6 - 7^3$ (B) $7^6 - 2\left(7^3\right)$

(C) $7^3\left(2 \times 7^3 - 1\right)$ (D) $7^9 - 7^6$

13. Number of matrices each of whose trace is atleast 7 is ___

(A) $7^9 - 7^7$ (B) $5\left(7^6\right)$

(C) $10\left(7^6\right)$ (D) 7^7

Comprehension 6

Let A be a square matrix of order 3 satisfies the matrix equation $A^3 - 6A^2 + 7A - 8I = 0$ and $B = A - 2I$. Also det. $A = 8$

14. The value of det. (adj.$(I - 2A^{-1})$) is equal to

(A) $\frac{25}{16}$ (B) $\frac{125}{64}$

(C) $\frac{64}{125}$ (D) $\frac{16}{25}$

15. If adj.$\left(\left(\frac{B}{2}\right)^{-1}\right) = \left(\frac{p}{q}\right) B$ where $p, q \in N$ then the least value of $(p + q)$ is equal to

(A) 7 (B) 9

(C) 29 (D) 41

Comprehension 7

$$A_0 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

If $B_n = \text{adj}(B_{n-1})$, $n \in N$ and I is an identity matrix of order 3, then answer the following question

16. Det. $\left(A_0 + A_0^2B_0^2 + A_0^3 + A_0^4B_0^4 + \ldots 10 \text{ terms}\right)$ is equal to

(A) 1000 (B) –800

(C) 0 (D) –8000

17. $B_1 + B_2 + \ldots + B_{49}$ is equal to

(A) B_0 (B) $7B_0$

(C) $49\,I$ (D) $49B_0$

Comprehension 8

Let A be a $m \times n$ matrix. If there exists a matrix L of type $n \times m$ such that $LA = I_n$, then L is called left inverse of A. Similarly, if there exists a matrix R of type $n \times m$ such that $AR = I_m$, then R is called right inverse of A.

For example to find right inverse of matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \text{ we take } R = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$$

and solve $AR = I_3$ i.e.

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow\ & x - u = 1 & y - v = 0 \qquad & z - w = 0 \\ & x + u = 0 & y + v = 1 \qquad & z + w = 0 \\ & 2x + 3u = 0 & 2y + 3v = 0 \qquad & 2z + 3w = 1 \end{aligned}$$

As this system of equations is inconsistent, we say there is no right inverse for matrix A.

18. Which of the following matrices is NOT left inverse of matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

(A) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & -7 & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 3 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

19. The number of right inverse for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

(A) 0 (B) 1

(C) 2 (D) infinite

Comprehension 9

In the foregoing passage, all matrices have each of their elements as either +1 or –1.

20. How many 3×3 matrices are symmetric ?

(A) 512 (B) 8

(C) 64 d0 128

21. How many 3×3 matrices are such that all the row products and all the column products are equal to –1 ?

(A) 64 (B) 16

(C) 128 (D) 512

22. How many 2×2 matrices have inverses ?

(A) 4 (B) 6

(C) 10 (D) 8

Comprehension 10

A Pythagorean triple is triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Define the matrices P, Q and R by

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, Q = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \text{ and } R = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

23. If we write Pythagorean triples (a, b, c) in matrix form as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then which of the following matrix product is not a Pythagorean triplet?

(A) $Q\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ (B) $P\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

(C) $R\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ (D) none of these

24. Which one of the following does not hold good ?

(A) $P^{-1} = \text{adj}.P$ (B) $(PQ)^{-1} = \text{adj}.(PQ)^{-1}$

(C) $(QR)^{-1} = \text{adj}.(QR)$ (D) $(PQR)^{-1} \neq \text{adj}.(PQR)$

25. Trace of matrix $(P + Q^T + 2R)$ equals

(A) 17 (B) 15

(C) 14 (D) 18

Comprehension 11

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, U_1, U_2, U_3 are column matrices satisfying $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and U is 3×3 matrix whose columns are U_1, U_2, U_3

26. The value of det U

(A) 3 (B) –3

(C) $\frac{3}{2}$ (D) 2

27. The sum of the elements of U^{-1} is

(A) –1 (B) 0

(C) 1 (D) 3

28. $\begin{bmatrix} 3 & 2 & 0 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} =$

(A) 5 (B) $\frac{5}{2}$

(C) 4 (D) $\frac{3}{2}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	ⓠ	ⓡ	●	●
B	ⓟ	●	●	ⓢ	ⓣ
C	●	●	ⓡ	ⓢ	ⓣ
D	ⓟ	ⓠ	ⓡ	●	●

1.

COLUMN-I	COLUMN-II
(A) M_r is defined as : $M_r = \begin{bmatrix} r-1 & \frac{1}{r} \\ 1 & \frac{1}{(r-1)^2} \end{bmatrix}$ and $\lvert M_r \rvert$ is the corresponding determinant value of M_r then $\underset{n\to\infty}{Lt} \left(\lvert M_2\rvert + \lvert M_3\rvert + \ldots \lvert M_n\rvert\right) =$	(p) 0

(Continued)

COLUMN-I	COLUMN-II
(B) If $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$	(q) 1

(Continued)

COLUMN-I	COLUMN-II
(C) If $A=\begin{bmatrix}3 & 1\\ -1 & 1\end{bmatrix}$ and a matrix C is defind as $C=\left(BAB^{-1}\right)\left(B^{-1}A^{T}B\right)$, where $\lvert B\rvert \neq 0$ then $\lvert C\rvert = K^2\,(K\in N)$ then $K=$	(r) 2
(D) If $A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ and $A^4=-\lambda I$ then λ is equal to	(s) 4

2. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ and $3B=A$ then match Column-I with Column-II

COLUMN-I	COLUMN-II
(A) $adjA$ is	(p) Orthogonal matrix
(B) B is	(q) Involutory matrix
(C) B^T is	(r) Cofactor matrix of A
(D) B^{-1}	(s) $-3\,A$

3. Consider a square matrix A of order 2 which has its elements as 0, 1, 2 and 4. Let N denote the number of such matrices, all elements of which are distinct.

COLUMN-I	COLUMN-II
(A) Possible non-negative value of det.(A) is	(p) 2
(B) Sum of values of determinants corresponding to N matrices is	(q) 4
(C) If absolute value of (det(A)) is least, then possible value of $\lvert$adj.(adj.(adj.A))$\rvert$	(r) -2
(D) If det(A) is algebraically least, then possible value of det($4A^{-1}$) is	(s) 0
	(t) 8

4.

COLUMN-I	COLUMN-II
(A) If $\lvert A\rvert = 2$ then $\left\lvert 2A^{-1}\right\rvert =$ (where A is a matrix of order 3)	(p) 1
(B) If $\lvert A\rvert = \frac{1}{8}$ then $\left\lvert adj\left(adj2A\right)\right\rvert =$ (where A is a matrix of order 3)	(q) 4
(C) If $\left(A+B\right)^2 = A^2+B^2$ and $\lvert A\rvert = 2$ then $\lvert B\rvert =$ (where A and B are matrices of odd order)	(r) Does not exist
(D) $\lvert A_{2\times2}\rvert = 2, \lvert B_{3\times3}\rvert = 3$ and $\lvert C_{4\times4}\rvert = 4$ and $\lvert ABC\rvert =$	(s) 0

5. Suppose matrix $A=\begin{bmatrix}7 & a & b & 1\\ c & \alpha & \beta & d\\ 3 & e & f & 10\end{bmatrix}$ All the unknown numbers are distinct integers from set $\{2,4,5,6,8,9\}$ such that sum of the entries in 1st row, 3rd row, 1st column and 4th column are equal to k then

COLUMN-I	COLUMN-II
(A) $k=$	(p) 20
(B) $a+b+c=$	(q) 17
(C) $2ef$	(r) 19
(D) $c+d$	(s) 16

6. Let $A=\begin{bmatrix}0 & \alpha\\ 0 & 0\end{bmatrix}$ and $\left(A+I\right)^{50}-50A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then match the following

COLUMN-I	COLUMN-II
(A) $\dfrac{a+b}{c+d}$	(p) 2
(B) $2a+b+c$	(q) 0
(C) $(a+d)^2-(b+c)^2$	(r) 4
(D) $(a-d)^2-(b+c)^2$	(s) 1

7.

COLUMN-I	COLUMN-II
(A) A is a matrix such that $A^2=A$. If $(I+A)^8 = I+\lambda A$, then $\lambda+1$ is equal to	(p) 64
(B) If A is a square matrix of order 3 such that $\lvert A\rvert = 2$, then $\lvert(\mathrm{adj}A^{-1})^{-1}\rvert$ is equal to	(q) 1
(C) Let $\lvert A\rvert = \lvert a_{ij}\rvert_{3\times3} \neq 0$. Each element a_{ij} is multiplied by λ^{i-j}. Let $\lvert B\rvert$ the resulting determinant, where $\lvert A\rvert = \lambda\lvert B\rvert$, then λ is equal to	(r) 256

(*Continued*)

COLUMN-I	COLUMN-II
(D) If A is a diagonal matrix of order 3×3 is commutative with every square matrix of order 3×3 under multiplication and trace $(A)=12$, then $\lvert A\rvert=$	(s) 4

8. Match the following:
A, B are two matrices of 3×3 order and O is a null matrix such that

COLUMN-I	COLUMN-II
(A) A is non singular and $AB=O$ then	(p) A = O
(B) B is non singular and $AB=O$ then	(q) $B=O$
(C) A and B both are non zero matrix and $AB=O$ then	(r) $\det A=0$
(d) $A^n=0$ for some $n\ge2$ then	(s) $\det B=0$

9. Match the following:

COLUMN–I	COLUMN–II
(A) A is a real skew symmetric matrix such that $A^2+I=0$. Then	(p) $BA-AB$
(B) A is a matrix such that $A^2=A$. If $(I+A)^n=I+\lambda A$, then λ equals $(n\in N)$	(q) A is of even order
(C) If for a matrix $A, A^2=A$, and $B=I-A$, then $AB+BA+I-(I-A)^2$ equals	(r) A
(D) A is a matrix with complex entries and A^* stands for transpose of complex conjugate of A. If $A^*=A$ & $B^*=B$, then $(AB-BA)^*$ equals	(s) 2^n-1
	(t) ${}^nC_1+{}^nC_2+\ldots+{}^nC_n$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Let $A=\begin{bmatrix}x^2 & 6 & 8\\ 3 & y^2 & 9\\ 4 & 5 & z^2\end{bmatrix}, B=\begin{bmatrix}2x & 3 & 5\\ 2 & 2y & 6\\ 1 & 4 & 2z-3\end{bmatrix}$. If trace A = trace B then the value of $x+y+z=$ _____

2. Let A be the set of all 3×3 symmetric matrices all of whose entries either 0 or 1. Five of these entries are 1 and four of them are 0. If n is the number of such matrices, then $n/2$ is

3. If $A^{-1}=\frac{1}{\lambda}\left(A^2-3A-9I\right)$ where $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ then $\lambda=$

4. If $A=\left[a_{ij}\right]_{4\times4}$ such that $a_{ij}=\begin{cases}2 & i=j\\ 0 & i\ne j\end{cases}$ if $\alpha=\det\left(adj\left(adjA\right)\right)$; then the remainder when α is divided by 7 is.

5. $(\alpha\beta)(\delta\beta)=\gamma\gamma\gamma$ and each of the letters represents uniquely a different non-zero digits then the trace of the matrix $A=\begin{bmatrix}\alpha & 1 & 2 & 0\\ 0 & \beta & 1 & 1\\ 0 & 0 & \gamma & 3\\ 1 & 1 & 0 & \delta\end{bmatrix}$ is equal to _________

6. If A and B are non-singular matrices of order three such that adj.$(AB)=\begin{bmatrix}1 & 1 & 1\\ 1 & p & 1\\ 1 & 1 & p\end{bmatrix}$ and $\lvert B^2\,\text{adj.}(A)\rvert=p^2-3$, then p is

7. Let $A=\begin{bmatrix}2 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. If B is a 3×3 matrix whose rows are B_1, B_2, B_3 satisfying $B_1A=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$, B_2A = [1 2 0] and B_3A = [2 1 1], then sum of elements of matrix B is

8. Let $Z=\begin{bmatrix}1 & 1 & 2\\ 2 & 3 & 5\\ 4 & 1 & 0\end{bmatrix}$ and $A=\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 0 & 3 & 1\end{bmatrix}$ if $Z=AB^{-1}$ where B is a square matrix of order 3 and t = trace of ((adj B).A), then $\frac{28}{\lvert t\rvert}$ is

9. The value of $\left|adj.\left(\frac{A}{3}\right)^{-1}\right|$, if $\lvert A\rvert_{3\times3}=9$ is

10. Let A be 2×3 matrix whereas B be a 3×2 matrix. If det.$(AB)=4$, then the value of det.(BA) is

11. Let $A=[a_{ij}]_{3\times3}$ is a square matrix of order 3 such that adj. $A=\begin{bmatrix}5 & -1 & -7\\ -1 & -7 & 5\\ -7 & 5 & -1\end{bmatrix}$ and det.$(A)<0$, then $\frac{1}{3}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}$ is equal to

12. Let B be a skew symmetric matrix of order 3×3 with real entries. Given $I-B$ and $I+B$ are non singular matrices. If $A=(I+B)(I-B)^{-1}$, where det.(A) > 0, then the value of (det.(2A) – det.(adj.A)) (where det.(P) means determinant of square matrix P) is

13. Let $x_1, x_2, x_3, x_4, x_5 > 0$ and $A = [a_{ij}]_{5\times 5}$ matrix such that

$$a_{ij} = \begin{cases} |x_i - x_j| & i \neq j \\ x_i, & i = j \end{cases} \text{ if } (x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0,$$

$$\left(x_2^2 - x_4x_1\right)\left(x_3^2 - x_4x_1\right) \leq 0, \left(x_3^2 - x_5x_2\right)\left(x_4^2 - x_5x_2\right) \leq 0$$

$$\left(x_4^2 - x_1x_3\right)\left(x_5^2 - x_1x_3\right) \leq 0, \left(x_5^2 - x_2x_4\right)\left(x_1^2 - x_2x_4\right) \leq 0,$$

then the sum of face value of the number det A is ______________ (given that $x_3 = 3$)

14. If $A = \begin{bmatrix} u & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, where $\left(u \neq -\frac{11}{3}\right)$ and det (adj(adj$A$)) = $(23)^4$, then the value of u is ____________

15. A is square matrix such that A.adj $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ then the value of $\dfrac{|adj(adjA)|}{|adjA|} =$ _______

16. Consider two matrices $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If matrix $A = \sum_{n=1}^{100} (B^n + C^n)$, then find the absolute value of determinant of matrix A.

17. If M be a non singular square matrix of order 3 such that $M^{-1} = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ then find the value of | 15 tr. (adj. M) |.

18. Let A be a square matrix of order 3 whose elements are real numbers and adj.(adj.(adj.A)) = $\begin{bmatrix} 16 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 3 & 4 \end{bmatrix}$ then find absolute value of Tr(A^{-1}).

19. Let M be a 3×3 matrix with real entries. If $M^TM = I_3$, where I_3 is a 3×3 identity matrix, then find the value of det. $(M^2 - I_3)$.

20. Consider $A = \begin{bmatrix} a & 2 & 1 \\ 0 & b & 0 \\ 0 & -3 & c \end{bmatrix}$ where, a, b, c are the roots of the equation $x^3 - 3x^2 + 2x - 1 = 0$. If matrix B is such that $AB = BA$, $|A + B - 2I| \neq 0$ and $A^2 - B^2 = 4I - 4B$, then find $\sqrt{\text{det.}(B)}$.

21. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = (\text{adj } A)$ and $C = 5A$, then find the value of $\dfrac{|adj\ B|}{|C|}$

22. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 4 & 7 & 3 \end{bmatrix}$, then the sum of squares of possible values of determinant of P is

23. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then det($A^{2005}$) equals to

24. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $A^2 = 8A + kI_2$, then find the value of $|k|$

25. If $\begin{bmatrix} 12 & \alpha^2 \\ -16 & -12 \end{bmatrix}$ is a nilpotent matrix of index 2, $\alpha > 0$, then $\alpha =$

26. A matrix of order 3×3 is formed with the elements –2, –1, 0, 1, 2 then the probability that it is skew symmetric is $\dfrac{1}{(p-1)^p}$ then p is ____

27. Let S be the set of all 2×2 matrices whose elements are 0 or 1. Then the number of non-singular matrices belonging to S is _____

28. If A is 3×3 non singular matrix and det $A = \frac{1}{2}$ then $\left|\frac{1}{81} adj(2adj3A)\right|$ equal to

29. If the number of idempotent diagonal matrices of order 4 by taking the elements 0, 1 is $8P$ then ‘P’ is

30. $\begin{bmatrix} 1 & 2k-1 \\ 0 & 1 \end{bmatrix}$, then the value of $\prod_{k=1}^{50} \begin{bmatrix} 1 & 2k-1 \\ 0 & 1 \end{bmatrix}$ is equal to $A = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ then r is equal to $500K$. Then K is ______________

31. Let ‘A’ is (4×4) matrix such that sum of elements in each row is 1. If sum of all the elements in A^{10} is n, then number of solution of the equation $\sin^{-1} x + \cos^{-1} x^2 = \dfrac{2\pi}{n}$ is/are ______________

32. A is 3×3 matrix such that $A'A = I$ and $|A| = 1$ then $1 + |A - I| =$

33. A is 3×3 matrix such that $|A| = 5$ then $\sqrt{\dfrac{|A.\ adj(A)|}{5}} =$

34. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$ and $f(2) = -5/4$, then the digit in the ten’s places of the vlaue of $\sum_{r=1}^{12} \left(3 - f(r)\right)$

35. A matrix A has x rows and $x + 5$ columns the matrix B has y rows and $11 - y$ columns both AB and BA exist. Then $y - x =$ ___

36. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then the trace of the matrix Adj(Adj A) is _____

37. If $\Delta = \begin{vmatrix} 1 & 3\cos\theta & 1 \\ \sin\theta & 1 & 3\cos\theta \\ 1 & \sin\theta & 1 \end{vmatrix}$, then the $\dfrac{|\text{maximum value of } \Delta - \text{Minimum value of } \Delta|^3}{250}$ is equal to

38. A be set of 3×3 matrices formed by entries 0, –1, and 1 only. Also each of 1, –1, 0 occurs exactly three times in each matrix. The number of symmetric matrices with trace $(A) = 0$ is k, then $\dfrac{k}{6} =$

39. Let A_n, $(n \in N)$ be a matrix of order $(2n - 1) \times (2n - 1)$, such that $a_{ij} = 0\ \forall i \neq j$ and $a_{ij} = n^2 + i + 1 - 2n\ \forall\ i = j$ where a_{ij} denotes the element of i^{th} row and j^{th} column of A_n. Let $T_n = (-1)^n \times$ (sum of all the elements of A_n). Find the value of $\left[\dfrac{\sum_{n=1}^{102} T_n}{520200}\right]$, where [.] represents the greatest integer function.

40. $\det P = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$, where '$P$' is an orthogonal matrix. Then the value of $|a + b + c|$ is

41. If $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\phi(x) = (1 + x)(1 - x)^{-1}$, such that $\phi(A) = -kA$ then find k.

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1. **[IIT JEE Mains 2020]**
The number of all 3×3 matrices A, with entries from the set $\{-1, 0, 1\}$ such that the sum of the diagonal elements of AA^T is 3, is ________.

2. **[IIT JEE Mains 2020]**
Let $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$. $x \in R$ and $A^4 = [a_{ij}]$. If $a_{11} = 109$, then a_{22} is equal to

3. **[IIT JEE Mains 2020]**
Let α be a root of the equation $x^2 + x + 1 = 0$ and the matrix $A = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$, then the matrix A^{31} is equal to:

(A) A　　(B) A^2
(C) I_3　　(D) A^3

4. **[IIT JEE Mains 2020]**
If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $10A^{-1}$ is equal to:

(A) $A - 6I$　　(B) $4I - A$
(C) $A - 4I$　　(D) $6I - A$

5. **[IIT JEE Mains 2020]**
Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3×3 real matrices such that $b_{ij} = (3)^{(I + j - 2)} a_{ji}$, where $i, j = 1, 2, 3$. If the determinant of B is 81, then the determinant of A is :

(A) 1/81　　(B) 1/3
(C) 3　　(D) 1/9

6. **[IIT JEE Mains 2020]**
If the matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$, $B = \text{adj } A$ and $C = 3A$, then $\dfrac{|\text{adj } B|}{|C|}$ is equal to:

(A) 8　　(B) 72
(C) 16　　(D) 2

7. **[IIT JEE Mains 2020]**
Let $a, b, c \in R$ be all non-zero satisfy $a^3 + b^3 + c^3 = 2$. If the matrix $A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

Satisfies $A^T A = I$, then a value of abc can be :

(A) $\dfrac{2}{3}$　　(B) $-\dfrac{1}{3}$
(C) 3　　(D) $\dfrac{1}{3}$

8. **[IIT JEE Mains 2020]**
Let $A = \{X = (x, y, z)^T : PX = 0 \text{ and } x^2 + y^2 + z^2 = 1\}$ where $P = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & -4 \\ 1 & 9 & -1 \end{bmatrix}$, then the set A:

(A) is a singleton
(B) contains exactly two elements
(C) contains more than two elements
(D) is an empty set

9. **[IIT JEE Mains 2020]**
Let A be a 2×2 real matrix with entries from $\{0, 1\}$ and $|A| \neq 0$. Consider the following two statements:

(P) If $A \neq I_2$, then $|A| = -1$
(Q) If $|A| = 1$, then $\text{tr}(A) = 2$.

where I_2 denotes 2×2 identity matrix and tr(A) denotes the sum of the diagonal entries of A. Then:

(A) (P) is true and (Q) is false
(B) Both (P) and (Q) are false
(C) Both (P) and (Q) are true
(D) (P) is false and (Q) is true

10. [IIT JEE Mains 2020]

Suppose the vectors x_1, x_2 and x_3 are the solutions of the system of linear equations, $Ax = b$ when the vector b on the right side is equal to b_1, b_2 and b_3 respectively. If $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $b_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ and $b_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, then the determinant of A is equal to

(A) $\frac{1}{2}$
(B) 4
(C) $\frac{3}{2}$
(D) 2

11. [IIT JEE Mains 2020]

Let A be a 3×3 matrix such that adj $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{bmatrix}$ and $B = \text{adj}\,(\text{adj}\,A)$.

If $|A| = \lambda$ and $|(B^{-1})^{\mathrm{T}}| = \mu$, then the ordered pair, $(|\lambda|, \mu)$ is equal to:

(A) $\left(9, \frac{1}{9}\right)$
(B) $\left(9, \frac{1}{81}\right)$
(C) $\left(3, \frac{1}{81}\right)$
(D) $(3, 81)$

12. [IIT JEE Mains 2020]

If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}, \left(\theta = \frac{\pi}{24}\right)$ and $A^5 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $i = \sqrt{-1}$, then which one of the following is not true?

(A) $0 \le a^2 + b^2 \le 1$
(B) $a^2 - d^2 = 0$
(C) $a^2 - b^2 = \frac{1}{2}$
(D) $a^2 - c^2 = 1$

13. [IIT JEE Mains 2020]

Let $\theta = \frac{\pi}{5}$ and $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. If $B = A + A^4$, then det(B):

(A) is one
(B) lies in (1, 2)
(C) is zero
(D) lies in (2, 3)

14. [IIT JEE Mains 2019]

If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to:

(A) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
(B) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
(C) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
(D) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

15. [IIT JEE Mains 2019]

If $A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$ then A is:

(A) invertible only if $t = \pi$
(B) invertible for all $t \in R$
(C) invertible only if $t = \frac{\pi}{2}$
(D) not invertible for any $t \in R$

16. [IIT JEE Mains 2019]

Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of det(A) is 8, then a value of d is

(A) –7
(B) $2(\sqrt{2}+2)$
(C) –5
(D) $2(\sqrt{2}+1)$

17. [IIT JEE Mains 2019]

Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$ where $b > 0$. Then the minimum value of $\frac{\det(A)}{b}$ is

(A) $\sqrt{3}$
(B) $-2\sqrt{3}$
(C) $-\sqrt{3}$
(D) $2\sqrt{3}$

18. [IIT JEE Mains 2019]

Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. If $AA^T = I_3$, then $|p|$ is

(A) $\frac{1}{\sqrt{2}}$
(B) $\frac{1}{\sqrt{5}}$
(C) $\frac{1}{\sqrt{6}}$
(D) $\frac{1}{\sqrt{3}}$

19. [IIT JEE Mains 2019]

Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det\,(BA^{-1}\,B^T)$ is equal to:

(A) $\frac{1}{4}$
(B) 16
(C) $\frac{1}{16}$
(D) 1

20. [IIT JEE Mains 2019]

Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$ and $Q = |q_{ij}|$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then $\frac{q_{21} + q_{31}}{q_{32}}$ is equal to:

(A) 15
(B) 9
(C) 135
(D) 10

21. **[IIT JEE Mains 2019]**

If $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$; then for all $\theta \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, det (A) lies in the interval:

(A) $\left(\frac{3}{2}, 3\right]$ (B) $\left(0, \frac{3}{2}\right]$

(C) $\left(\frac{5}{2}, 4\right]$ (D) $\left(1, \frac{5}{2}\right]$

22. **[IIT JEE Mains 2019]**

Let $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $(\alpha \in R)$ such that $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then a value of α is

(A) $\frac{\pi}{64}$ (B) 0

(C) $\frac{\pi}{32}$ (D) $\frac{\pi}{16}$

23. **[IIT JEE Mains 2019]**

Let the number 2, b, c be in an A.P. and $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$. If det $(A) \in [2, 16]$, then c lies in the interval

(A) $(2 + 2^{3/4}, 4)$ (B) $[4, 6]$

(C) $[3, 2 + 2^{3/4}]$ (D) $[2, 3)$

24. **[IIT JEE Mains 2019]**

If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$, then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is:

(A) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 13 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

25. **[IIT JEE Mains 2019]**

The total number of matrices $A = \begin{pmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{pmatrix}$, $(x, y \in R, x \neq y)$ for which $A^TA = 3I_3$ is:

(A) 2 (B) 4

(C) 3 (D) 6

26. **[IIT JEE Mains 2019]**

If A is a symmetric matrix and B is a skew-symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to:

(A) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$ (B) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} -4 & -2 \\ 1 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

27. **[IIT JEE Mains 2019]**

If $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which $\det(A) + 1 = 0$, is:

(A) 2 (B) –1

(C) 0 (D) 1

28. **[JEE Main 2017]**

If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, adj $(3A^2 + 12A)$ is equal to

(A) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (B) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

(C) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (D) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

29. **[JEE Main 2016]**

If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj $A = AA^T$, then $5a + b$ is equal to

(A) 13 (B) –1

(C) 5 (D) 4

30. **[JEE Main 2015]**

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to:

(A) (2, 1) (B) (–2, –1)

(C) (2, –1) (D) (–2, 1)

31. **[JEE Main 2014]**

If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}. A'$, the BB' equals:

(A) $I + B$ (B) I

(C) B^{-1} (D) $(B^{-1})'$

32. **[JEE Main 2013]**

If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to:

(A) 4 (B) 11

(C) 5 (D) 0

33. **[AIEEE 2012]**

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to:

(A) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

(C) $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

34. **[AIEEE 2011]**

Let A and B be two symmetric matrices of order 3.

Statement-1: $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement-2: AB is symmetric matrix if matrix multiplication of A and B is commutative.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is false.

(C) Statement-1 is false, Statement-2 is true.

(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

35. **[AIEEE 2010]**

The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is:

(A) less than 4 (B) 5

(C) 6 (D) at least 7

36. **[AIEEE 2009]**

Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define

Tr(A) = sum of diagonal elements of A and

$|A|$ = determinant of matrix A.

Statement-1: Tr$(A) = 0$

Statement-2: $|A| = 1$

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is false.

(C) Statement-1 is false, Statement-2 is true.

(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

37. **[AIEEE 2009]**

Let A be a 2×2 matrix

Statement-1: adj (adj A) = A

Statement-2: $|\text{adj } A| = |A|$

(A) Statement-1 is true, Statement-2 is false.

(B) Statement-1 is false, Statement-2 is true.

(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(D) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement -1.

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problem

1. **[IIT JEE 2019]**

Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$ where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and I is the 2×2 identity matrix. If α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$, Then the value of $\alpha^* + \beta^*$ is

(A) $-\dfrac{37}{16}$ (B) $-\dfrac{31}{16}$

(C) $-\dfrac{29}{16}$ (D) $-\dfrac{17}{16}$

2. **[IIT JEE 2017]**

How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of $M^T M$ is 5?

(A) 126 (B) 198

(C) 162 (D) 135

3. **[IIT JEE 2016]**

Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\dfrac{q_{31} + q_{32}}{q_{21}}$ equals

(A) 52 (B) 103

(C) 201 (D) 205

4. **[IIT-JEE 2012]**

Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is:

(A) 2^{10} (B) 2^{11}

(C) 2^{12} (D) 2^{13}

5. **[IIT-JEE 2012]**

If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that :

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$

(C) $PX = 2X$ (D) $PX = -X$

6. **[IIT-JEE 2011]**

Let M and N be two $2n \times 2n$ non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ is equal to

(A) M^2 (B) $-N^2$

(C) $-M^2$ (D) MN

7. **[IIT-JEE 2011]**
Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form : $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is :
(A) 2 (B) 6
(C) 4 (D) 8

Multiple Correct Choice Type Problem

1. **[IIT JEE 2020]**
Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix.
If $M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statements is/are ALWAYS TRUE?
(A) $M = I$ (B) $\det M = 1$
(C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

2. **[IIT JEE 2019]**
Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $adj\ M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$
where a and b are real numbers. Which of the following option is/are correct?
(A) $a + b = 3$
(B) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$
(C) $\det(\text{adj}M^2) = 81$
(D) If $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

3. **[IIT JEE 2019]**
Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$

$P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$

where P_k^T denotes the transpose of the matrix P_k. Then which of the following options is/are correct?
(A) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$
(B) X is a symmetric matrix
(C) The sum of diagonal entries of X is 18
(D) $X - 30I$ is an invertible matrix.

4. **[IIT JEE 2019]**
Let $x \in R$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.
Then which of the following option is/are correct?
(A) There exist a real number x such that $PQ = QP$
(B) $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in R$
(C) For $x = 0$, if $R \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then $a + b = 5$
(D) For $x = 1$, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which $R \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5. **[IIT JEE 2018]**
Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3, \in R$ and the system of equations (in real variables) $-x + 2y + 5z = b_1, 2x - 4y + 3z = b_2, x - 2y + 2z = b_3$ has at least one solution Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?
(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

6. **[IIT JEE 2017]**
Which of the following is (are) NOT the square of a 3×3 matrix with real entries ?
(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (D) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

7. **[JEE-Advanced 2016]**
Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$, Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$
(B) $4\alpha - k + 8 = 0$
(C) $\det(P\,\mathrm{adj}(Q)) = 2^9$
(D) $\det(Q\,\mathrm{adj}(P)) = 2^{13}$

8. **[JEE-Advanced 2015]**
Let X and Y be two arbitrary, 3×3 non-zero skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric?

(A) $Y^3Z^4 - Z^4Y^3$
(B) $X^{44} + Y^{44}$
(C) $X^4Z^3 - Z^3X^4$
(D) $X^{23} + Y^{23}$

9. **[JEE-Advanced 2014]**
Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if

(A) the first column of M is the transpose of the second row of M
(B) the second row of M is the transpose of the first column of M
(C) M is a diagonal matrix with nonzero entries in the main diagonal
(D) the product of entries in the main diagonal of M is not the square of an integer

10. **[JEE-Advanced 2014]**
Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) determinant of $(M^2 + MN^2)$ is 0
(B) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)$ U is zero matrix
(C) determinant of $(M^2 + MN^2) \geq 1$
(D) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix than U is the zero matrix

11. **[JEE-Advanced 2013]**
For 3×3 matrices M and N, which of the following statement(s) is(are) NOT correct?

(A) $N^TM\,N$ is symmetric or skew symmetric, according as M is symmetric or skew symmetric
(B) $MN - NM$ is skew symmetric for all symmetric matrices M and N
(C) MN is symmetric for all symmetric matrices M and N
(D) (adj M) (adj N) = adj $(M\,N)$ for all invertible matrices M and N

12. **[IIT-JEE 2012]**
If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is/are:

(A) –2
(B) –1
(C) 1
(D) 2

Linked Comprehension Type Questions

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a Paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options).

Comprehension 1

Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0] \qquad \ldots(E)$$

1. **[IIT-JEE 2011]**
If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is:

(A) 0
(B) 12
(C) 7
(D) 6

2. **[IIT-JEE 2011]**
Let ω be a solution of $x^3 - 1 = 0$ with $Im(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to:

(A) –2
(B) 2
(C) 3
(D) –3

3. **[IIT-JEE 2011]**
Let $b = 6$ with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is:

(A) 6
(B) 7
(C) $\frac{6}{7}$
(D) ∞

Comprehension 2

Let p be an odd prime number and T_p be the following set of 2×2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \{0, 1, 2, \ldots, p-1\} \right\}$$

4. **[IIT 2010]**
The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) divisible by p is

(A) $(p - 1)^2$
(B) $2(p - 1)$
(C) $(p - 1)^2 + 1$
(D) $2p - 1$

5. **[IIT 2010]**
The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[**Note:** The trace of a matrix is the sum of its diagonal entries.]

(A) $(p - 1)(p^2 - p + 1)$
(B) $p^3 - (p - 1)^2$
(C) $(p - 1)^2$
(D) $(p - 1)(p^2 - 2)$

6. **[IIT 2010]**
The number of A in T_p such that det (A) is not divisible by p is

(A) $2p^2$
(B) $p^3 - 5p$
(C) $p^3 - 3p$
(D) $p^3 - p^2$

Comprehension 3

Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

7. **[IIT 2009]**
The number of matrices in set A is
(A) 12 (B) 6
(C) 9 (D) 3

8. **[IIT 2009]**
The number of matrices A for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ has a unique solution, is}$$

(A) less than 4
(B) at least 4 but less than 7
(C) at least 7 but less than 10
(D) at least 10

9. The number of matrices A for which the system of linear equations

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is inconsistent, is}$$

(A) 0 (B) More than 2
(C) 2 (D) 1

Integer/Numerical Answer Type Question

1. **[IIT JEE 2020]**
The trace of a matrix is defined to the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is -18, then the value of the determinant of A is __________

2. **[IIT JEE 2018]**
Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ________.

3. **[IIT JEE 2017]**
For a real number α, if the system $\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

4. **[JEE 2016]**
Let $z = \dfrac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1,2,3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total numbered pairs (r, s) for which $P^2 = -I$ is

5. **[IIT-JEE 2011]**
Let M be a 3×3 matrix satisfying $M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$
Then the sum of the diagonal entries of M is.

6. **[IIT 2010]**
Let k be a positive real number and

Let $A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}$ If det (adj A) + det(adj B) = 10^6,

then $[k]$ is equal to
[Note: adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k]

ANSWER KEYS—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** B	**3.** C	**4.** B	**5.** B	**6.** C	**7.** D	**8.** C	**9.** B	**10.** B
11. B	**12.** C	**13.** C	**14.** A	**15.** D	**16.** D	**17.** B	**18.** D	**19.** A	**20.** D
21. B	**22.** B	**23.** C	**24.** B	**25.** D	**26.** B	**27.** A	**28.** C	**29.** C	**30.** C
31. D	**32.** B	**33.** A	**34.** D	**35.** A	**36.** D	**37.** A	**38.** C	**39.** B	**40.** A
41. C	**42.** D	**43.** C	**44.** C	**45.** A	**46.** C	**47.** A	**48.** B	**49.** B	**50.** B
51. B	**52.** B	**53.** A	**54.** B	**55.** D	**56.** B	**57.** D	**58.** C	**59.** C	**60.** C
61. C	**62.** A	**63.** D	**64.** B	**65.** C	**66.** D	**67.** B	**68.** D	**69.** A	**70.** C
71. A	**72.** A								

Multiple Correct Choice Type Questions

1. A, B, C	**2.** B, C	**3.** A, B, C, D	**4.** A, C, D	**5.** A, B, D
6. B, C	**7.** B, C	**8.** A, C, D	**9.** A, B, C	**10.** B, D
11. A, B, C	**12.** B, C	**13.** A, B, C, D	**14.** A, B, D	**15.** A, C, D
16. A, C	**17.** A, B, C, D	**18.** A, B, C	**19.** A, B, C	**20.** A, B
21. A, B, C, D	**22.** A, B	**23.** A, B, C, D	**24.** A, B, C	**25.** A, B, C, D
26. A, C, D	**27.** A,B,D	**28.** A, D	**29.** A, B, D	**30.** A, D
31. A,B,C	**32.** C	**33.** A, C, D	**34.** C,D	**35.** A,B,C
36. A, B, C	**37.** A, C	**38.** A, B, C, D		

Linked Comprehension Type Questions

1. D	**2.** A	**3.** B	**4.** C	**5.** C	**6.** A	**7.** D	**8.** B	**9.** B	**10.** B
11. C	**12.** C	**13.** C	**14.** A	**15.** A	**16.** C	**17.** D	**18.** C	**19.** D	**20.** C
21. B	**22.** D	**23.** B	**24.** D	**25.** C	**26.** A	**27.** B	**28.** A		

Matrix Match/Column Match Type Questions

1. A → (q)	B → (r)	C → (s)	D → (s)
2. A →(r, s)	B → (p, q)	C → (p, q)	D → (p, q)
3. A → (p, q, t)	B → (s)	C → (p, r)	D → (r)
4. A → (q)	B → (p)	C → (s)	D → (r)
5. A → (r)	B → (p)	C → (s)	D → (q)
6. A → (s)	B → (p)	C → (r)	D → (q)
7. A → (r)	B → (s)	C → (q)	D → (p)
8. A → (q, s)	B → (p, r)	C → (r, s)	D → (r)
9. A → (q)	B → (s, t)	C → (r)	D → (p)

Integer/Numerical Answer Type Questions

1. 3	**2.** 6	**3.** 5	**4.** 1	**5.** 21	**6.** 2	**7.** 6	**8.** 5	**9.** 9	**10.** 0
11. 6	**12.** 7	**13.** 9	**14.** 4	**15.** 4	**16.** 0	**17.** 5	**18.** 3	**19.** 0	**20.** 5
21. 1	**22.** 8	**23.** 1	**24.** 7	**25.** 3	**26.** 6	**27.** 6	**28.** 4	**29.** 2	**30.** 5
31. 2	**32.** 1	**33.** 5	**34.** 5	**35.** 5	**36.** 1	**37.** 4	**38.** 6	**39.** 2	**40.** 1
41. 1									

ARCHIVE: JEE MAIN

1. 672	**2.** 10	**3.** D	**4.** A	**5.** D	**6.** A	**7.** D	**8.** B	**9.** D	**10.** D
11. C	**12.** C	**13.** B	**14.** C	**15.** B	**16.** C	**17.** D	**18.** A	**19.** C	**20.** D
21. A	**22.** A	**23.** B	**24.** D	**25.** B	**26.** D	**27.** D	**28.** C	**29.** C	**30.** B
31. B	**32.** B	**33.** D	**34.** A	**35.** D	**36.** B	**37.** C			

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Single Correct Choice Type Questions

1. C	**2.** B	**3.** B	**4.** D	**5.** D	**6.** C	**7.** A

Multiple Correct Choice Type Questions

1. B, C, D	**2.** A, B, D	**3.** A, B, C	**4.** B, C	**5.** A, D	**6.** B, D	**7.** B, C	**8.** C, D
9. C, D	**10.** A, B	**11.** C, D	**12.** A, D				

Linked Comprehension Type Questions

1. D	**2.** A	**3.** B	**4.** D	**5.** C	**6.** D	**7.** A	**8.** B	**9.** B

Integer/Numerical Answer Type Questions

1. 5	**2.** 4	**3.** 1	**4.** 1	**5.** 9	**6.** 4

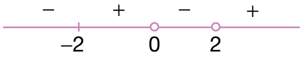
HINTS AND EXPLANATIONS

CHAPTER 1: FUNDAMENTAL OF MATHEMATICS

Test Your Concepts-I

1. $\Rightarrow \quad \dfrac{2}{x-2}-\dfrac{1}{x}\ge 0$

$\Rightarrow \quad \dfrac{2x-x+2}{(x-2)x}\ge 0$



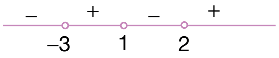
$\Rightarrow \quad \dfrac{x+2}{(x-2)x}\ge 0$

$x\in[-2,0)\cup(2,\infty)$

2. $\dfrac{1}{(x-1)(x-2)}-\dfrac{1}{(x-2)(x+3)}\ge 0$

$\dfrac{(x+3)-(x-1)}{(x-1)(x-2)(x+3)}\ge 0$

$\Rightarrow \quad \dfrac{4}{(x-1)(x-2)(x+3)}\ge 0$



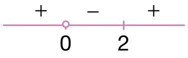
Poles $x\ne 1,2,-3$;

$x\in(-3,1)\cup(2,\infty)$

3. $\dfrac{x-1}{x}\le\dfrac{1}{2}$ (Do not Cross multiply as sign of x is not always positive)

$\Rightarrow \quad \dfrac{x-1}{x}-\dfrac{1}{2}\le 0$

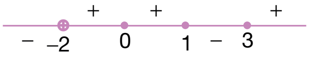
$\Rightarrow \quad \dfrac{2x-2-x}{2x}\le 0$



$\Rightarrow \quad \dfrac{x-2}{2x}\le 0$ we get $x\in(0,2]$

4. If $f(x)=\dfrac{x^2(x-1)}{(x+2)(x-3)}$, then its sign scheme will be

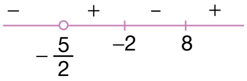
$\therefore$ for $f(x)\le 0$



$\Rightarrow \quad x\in(-\infty,-2)\cup[1,3)\cup\{0\}$

5. $\dfrac{x^2-1}{2x+5}-3<0 \quad\Rightarrow\quad \dfrac{x^2-1-6x-15}{2x+5}<0$

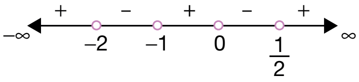
$\dfrac{x^2-6x-16}{2x+5} \quad\Rightarrow\quad \dfrac{(x-8)(x+2)}{2x+5}<0$



$\Rightarrow \quad x\in\left(-\infty,-\dfrac{5}{2}\right)\cup(-2,8)$

6. $2+\dfrac{3}{x+1}-\dfrac{2}{x}>0 \quad\Rightarrow\quad \dfrac{2x^2+2x+3x-2x-2}{x(x+1)}>0$

$\dfrac{2x^2+3x-2}{x(x+1)} \quad\Rightarrow\quad \dfrac{(2x-1)(x+2)}{x(x+1)}>0$



$\Rightarrow \quad x\in(-\infty,-2)\cup(-1,0)\cup\left(\dfrac{1}{2},\infty\right)$

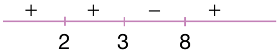
7. $x^2+x+1>0,\ \forall\, x\in R$

$\Rightarrow \quad x^2-1<x^2+x+1 \quad\Rightarrow\quad x>-2$

8. $(x-2)^{20}(x-3)^{20}(x-3)^{31}(x-8)^{31}\ge 0$

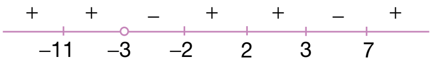
$(x-2)^{20}(x-3)^{51}(x-8)^{31}\ge 0$

$x\in(-\infty,3]\cup[8,\infty)$



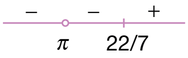
9. Pole $x\ne -3$

Zeroes $x=2,-2,3,-11,7$



$x\in(-3,-2]\cup[3,7]\cup\{-11,2\}$

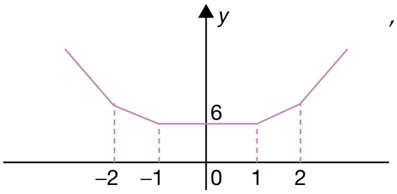
10. Pole $x\ne\pi$ and we know $\dfrac{22}{7}>\pi$



$x\in(-\infty,\pi)\cup\left(\pi,\dfrac{22}{7}\right]$ or $x\in\left(-\infty,\dfrac{22}{7}\right]-\{\pi\}$

Test Your Concepts-II

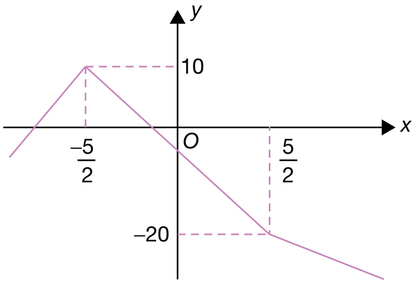
1. (i)



, Min. value = 6

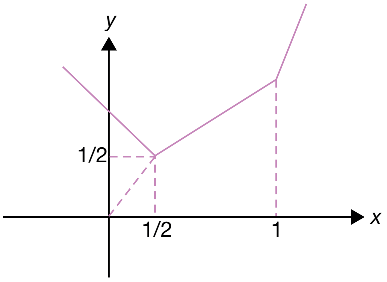
(ii) $y=\begin{cases}2x+15, & x<-\dfrac{5}{2}\\[2mm] -6x-5, & -\dfrac{5}{2}\le x<\dfrac{5}{2}\\[2mm] -2x-15, & x\ge\dfrac{5}{2}\end{cases}$

Max. value = 10



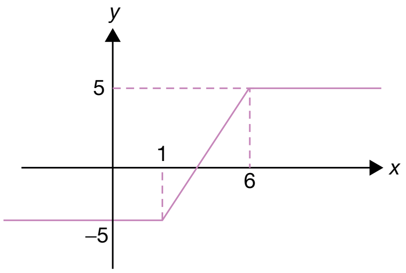
(iii) $y = \begin{cases} -3x+2, & x < -\dfrac{1}{2} \\ x, & \dfrac{1}{2} \le x < 1 \\ 3x-2, & x \ge 1 \end{cases}$

Min. value $= \dfrac{1}{2}$

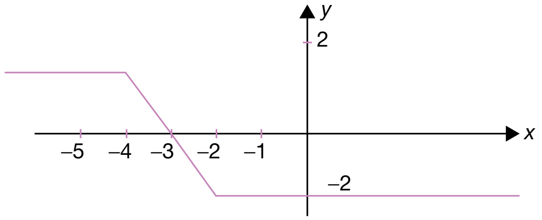


(iv) $y = \begin{cases} -5, & x < 1 \\ 2x-7, & 1 \le x < 6 \\ 5, & x \ge 6 \end{cases}$

Min. value $= -5$
Max. value $= 5$



2. Two modulus functions $|x+2|$ and $|x+4|$ are present so we have two critical points $x = -2, -4$



$|x+2| = 0 \Rightarrow x = -2$
$|x+4| = 0 \Rightarrow x = -4$
$f(-2) = 0 - |-2+4| = -2$
$f(-4) = |-2| - |0| = 2$
$f(-1) = |1| - |3| = -2$
$f(-5) = 3 - 1 = 2$

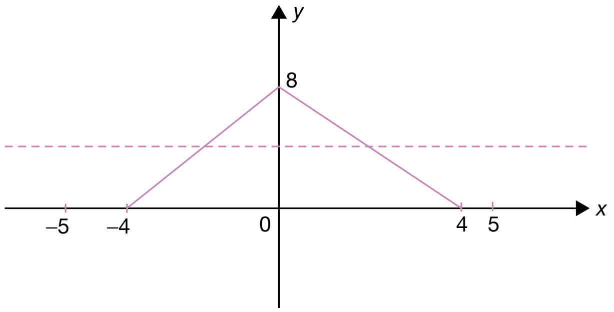
3. $|x-2| = \sqrt{36}$
or $|x-2| = 6$
or $x - 2 = \pm 6$
or $x - 2 = +6$ or $x - 2 = -6$
or $x = 8$ or $x = -4$

4. $||x+2| - 4| = 10$
or $|x+2| - 4 = 10$ or $|x+2| - 4 = -10$
or $|x+2| = 14$ or $|x+2| = -6$
or $x + 2 = \pm 14$ or $x \in \phi$
$\oplus\, x + 2 = 14 \Rightarrow x = 12$
$\Theta\, x + 2 = -14 \Rightarrow x = -16$
$x = 12$ or $x = -16$

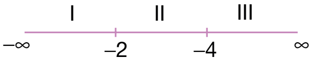
5. $|5 - |x-1|| = 2$
$5 - |x-1| = \pm 2$
$5 - |x-1| = 2$ or $5 - |4-1| = -2$
$|x-1| = 3$ or $|x-1| = 7$
$x - 1 = \pm 3$ or $x - 1 = \pm 7$
$\oplus\, x - 1 = 3 \Rightarrow x = 4$ or $\oplus\, x - 1 = 7 \Rightarrow x = 8$
$\Theta\, x - 1 = -3 \Rightarrow x = -2$ or $\Theta\, x - 1 = -7$
$\Rightarrow x = -6$
Final complete solution $x \in \{-6, -2, 4, 8\}$

6. $||x-4| - 3| = 100$
$|x-4| - 3 = \pm 100$
$|x-4| = 103$
or $|x-4| = -97$ (no solution)
$x - 4 = \pm 103$
$\oplus\, x = 107$
$\Theta\, x = -99$

7. $f(0) = 8; f(4) = 0; f(-4) = 0; f(5) = 0; f(-5) = 0$
(i) For No solution, $l \in (-\infty, 0) \cup (8, \infty)$
(ii) for 1 solution, we have $\lambda = 8$ only
(iii) For 2 solution, we have $0 < \lambda < 8$
(iv) for ∞ solution, we have $\lambda = 0$



8. $|x-4| = 0 \Rightarrow x = 4$
$|x+2| = 0 \Rightarrow x = -2$



we are solving it by forming three cases

CASE-1: $x < -2$
$-(x-4) - 2(-(x+2)) - 7x + 5 = 0$
$\Rightarrow -x + 4 + 2x + 4 - 7x + 5 = 0$
$\Rightarrow -6x + 13 = 0$
$x = 13/6$ but $x < -2$ so not acceptable

CASE-2: $-2 \le x \le 4$
$-(x-4) - 2(+(x+2)) - 7x + 5 = 0$
$\Rightarrow -x + 4 - 2x - 4 - 7x + 5 = 0$
$\Rightarrow -10x + 5 = 0$
$x = \dfrac{1}{2}$ acceptable solution

CASE-3: $x > 4$
$+(x-4) - 2(+(x+2)) - 7x + 5 = 0$
$x - 4 - 2x - 4 - 7x + 5 = 0$

$\Rightarrow \quad 8x = -3$

$x = -\dfrac{3}{8}$ but $x > 4$ so not acceptable

Final answer Case (I) ∪ (II) ∪ (III)

$x = \dfrac{1}{2}$ only

9. $||x-2|-2| = 8$

$|x-2| - 2 = \pm 8$

$|x-2| - 2 = +8$ or $|x-2| - 2 = -8$

$|x-2| = 10$ $\quad |x-2| = -6$ (No solution)

$x - 2 = \pm 10$

$x = 12$ or $x = -8$

10. $-6 \le |x-2| - 4 \le 6$

$-2 \le |x-2| \le 10$

$-2 \le |x-2|$ is always true for every real x ...(i)

and $|x-2| \le 10$

or $-10 \le x - 2 \le 10$

or $-8 \le x \le 12$ ...(ii)

common of both conditions we have

$x \in [-8, 12]$

11. $|x+2| + 4 \ge 6$ or $|x+2| + 4 \le -6$

$|x+2| \ge 2$ or $|x+2| \le -10$ (No solution)

Now solving $|x+2| \ge 2$

$x + 2 \ge 2$ or $x + 2 \le -2$

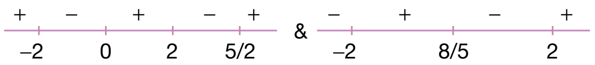
$x \ge 0$ or $x \le -4$

$x \in (-\infty, -4] \cup [0, \infty)$

12. $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1; x \ne \{-2, 2\} \Rightarrow -1 \le \dfrac{x^2-5x+4}{x^2-4} \le 1$

$\therefore \quad \dfrac{x^2-5x+4}{x^2-4} \ge -1$ and $\dfrac{x^2-5x+4}{x^2-4} \le 1$

$\Rightarrow \quad \dfrac{2x^2-5x}{x^2-4} \ge 0$ and $\dfrac{5x-8}{x^2-4} \ge 0$



$x \in (-\infty, -2) \cup [0, 2) \cup [5/2, \infty)$

and $x \in (-2, 8/5] \cup (2, \infty)$

Hence $x \in [0, 8/5] \cup [5/2, \infty)$

13. $\dfrac{x^2 - 7|x| + 10}{x^2 - 6x + 9} < 0 \Rightarrow \dfrac{x^2 - 7|x| + 10}{(x-3)^2} < 0$

$\Rightarrow \quad x^2 - 7|x| + 10 < 0$ but $x \ne 3$

CASE-1: $(x < 0)$

$x^2 + 7x + 10 < 0$

$(x+5)(x+2) < 0$

$\Rightarrow \quad x \in (-5, -2)$

CASE-2: $(x \ge 0)$

$x^2 - 7x + 10 < 0$

$\Rightarrow \quad (x-5)(x-2) < 0$

$\Rightarrow \quad x \in (2, 5)$ but $x \ne 3$

$\therefore$ Final solution $x \in (-5, -2) \cup (2, 3) \cup (3, 5)$

14. $|(x^2 + 2x + 2) + (3x + 7)| = |x^2 + 2x + 2| + |3x + 7|$

$\Rightarrow \quad x^2 + 5x + 9 < x^2 + 2x + 2 + |3x + 7|$

$\Rightarrow \quad |3x + 7| > 3x + 7$

CASE-1: $x \ge \dfrac{-7}{3} \Rightarrow 0 > 0$ not possible

CASE-2: $x < \dfrac{-7}{3} \Rightarrow -3x - 7 > 3x + 7 \Rightarrow 6x + 14 < 0$

$x < \dfrac{-7}{3}$

$\therefore$ Final solution $x \in \left(-\infty, \dfrac{-7}{3}\right)$

Test Your Concepts-III

1. $3^{2(x^2-x-6)} - 2.3^{(x^2-x-6)} + 1 = 0$

$\Rightarrow \quad 3^{(x^2-x-6)} = 1 \Rightarrow x^2 - x - 6 = 0$

$\Rightarrow \quad (x-3)(x+2) = 0 \Rightarrow x = -2, 3$

$\alpha^2 + \beta^2 = 4 + 9 = 13$

2. $3^{72}\left(\dfrac{1}{3}\right)^x\left(\dfrac{1}{3}\right)^{\sqrt{x}} > 1 \Rightarrow 3^{(72-x-\sqrt{x})} > 3^0$

$\Rightarrow \quad 72 - x - \sqrt{x} > 0$

$t^2 + t - 72 < 0$, Where $t = \sqrt{x} \Rightarrow (t+9)(t-8) < 0$

$\Rightarrow \quad -9 < t < 8 \Rightarrow \sqrt{x} < 8 \Rightarrow x \in [0, 64)$

3. $x^{\log_x 2} + x^2 = 3x$

$\Rightarrow \quad 2 + x^2 = 3x$

$\Rightarrow \quad x^2 - 3x + 2 = 0$, Domain $x > 0$ and $x \ne 1$

$\Rightarrow \quad (x-2)(x-1) = 0 \Rightarrow x = 2$ or $x = 1$

Hence only $x = 2$ is acceptable

$\therefore$ No. of real solutions = 1

4. $\log_{10}(x-1)^3 - 3\log_{10}(x-3) = \log_{10} 8$

$\log_{10}\left[\dfrac{(x-1)^3}{(x-3)^3}\right] = \log_{10} 8$

$\Rightarrow \quad 3\log_{10}\left(\dfrac{x-1}{x-3}\right) = 3\log_{10} 2$

$\Rightarrow \quad \log_{10}\left(\dfrac{x-1}{x-3}\right) = \log_{10} 2 \Rightarrow \dfrac{x-1}{x-3} = 2$

$\Rightarrow \quad x - 1 = 2x - 6 \Rightarrow x = 5$

so $\log_5 625 = 4$

5. $\log_4\left(\dfrac{4}{4}\right) - 2\log_4(4(-2)^4) = \log_4 1 - 2\log_4(4^3)$

$= 0 - 2 \times 3 = -6$

6. $x = \log_k b = \log_b c = \dfrac{1}{2}\log_c d$

we have $b = (k)^x$ and $c = (b)^x$ and $d = (c)^{2x}$

$\Rightarrow \quad d = (k)^{2x^3}$

$\log_k d = \log_k (k)^{2x^3} = 2x^3$

7. $N = \dfrac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \log_6^2 2$

$N = \dfrac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \dfrac{1}{(\log_2 6)^2} \quad \left(\because \dfrac{1}{(\log_2 6)} = \dfrac{1}{(1 + \log_2 3)^2}\right)$

$N = \dfrac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \dfrac{(\log_3 2)^2}{(1 + \log_3 2)^2}$

$N = \dfrac{(1 + \log_3 2)^2}{(1 + \log_3 2)^2} = 1$

8. $\sqrt{5-\log_2 x} = 3 - \log_2 x$

put $\log_2 x = y$ we have $\sqrt{5-y} = 3 - y$

$5 - y \geq 0 \Rightarrow y \leq 5$ and $3 - y \geq 0 \Rightarrow y \leq 3$

$5 - y = 3 - y$ squaring $5 - y = 9 + y^2 - 6y$

$\Rightarrow y^2 - 5y + 4 = 0 \Rightarrow (y-1)(y-4) = 0$

$\Rightarrow y = 1, 4$

$y = 1$ satisfy the equation $\log_2 x = 1 \Rightarrow x = 2$

9. $\log_{10}(3^x - 2^{4-x}) = \log_{10} 200 - \log_{10} 2^x$

$3^x\, 2^{4-x} = \frac{200}{2^x}$; $3^x . 2^x - 16 = 200$

$6^x = 216 \Rightarrow x = 3$;

put $x = 3$ in LHS of second equation

$$\log_3\left(3^{3^2-13(3)+28} + \frac{2}{9}\right) = \log_3\left(3^{-2} + \frac{2}{9}\right)$$

$$= \log_3 \frac{1}{3} = -1 = \log_5 0.2 = \text{RHS}$$

Hence $x = 3$ is the correct root.

10. $\log(-2x) = 2\log(x+1)$

$-2x = x^2 + 2x + 1 \Rightarrow x^2 + 4x + 1 = 0$

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} \Rightarrow -2 \pm \sqrt{3}$$

$x = -2 - \sqrt{3}$ and $x = -2 + \sqrt{3}$

Now $-x > 0$ and $x + 1 > 0$

so $x \in (-1, 0)$

so solution is $x = -2 + \sqrt{3}$

$\therefore$ Number of solutions = 1

11. $2^{-2\log(-x)} + 2 > 3.2^{-\log(-x)}$; Domain: $x < 0$

$2^{-\log(-x)} = t$

$t^2 + 2 > 3t$

$t^2 - 3t + 2 > 0$

$(t-2)(t-1) > 0$

i.e. $t \in (-\infty, 1) \cup (2, \infty)$

$2^{-\log(-x)} < 1$ or $2^{-\log(-x)} > 2$

$-\log(-x) < 0$ or $-\log(-x) > 1$

$\log(-x) > 0$ or $\log(-x) < -1$

$(-x) > 1$ or $(-x) < \frac{1}{10}$

$x < -1$ or $x > -\frac{1}{10}$ $\quad x \in (-\infty, -1) \cup \left(\frac{-1}{10}, \infty\right)$

intersection with domain $(-\infty, -1) \cup \left(\frac{-1}{10}, 0\right)$

12. $\log_2\left(\sqrt{4x+2} + 2\sqrt{x}\right) = 6^0 = 1$

$\Rightarrow \sqrt{4x+2} + 2\sqrt{x} = 2^1$

$\Rightarrow \sqrt{4x+2} = 2 - 2\sqrt{x}$

$\Rightarrow$ Squaring $4x + 2 = 4 + 4x - 8\sqrt{x}$

$\Rightarrow 8\sqrt{x} = 2 \Rightarrow x = \frac{1}{16}$

13. $5^{1+\log_4 x} + 5^{-1-\log_4 x} = 5^{1+\log_4 x} + \frac{1}{5^{1+\log_4 x}} = \frac{26}{5}$

$\therefore \quad t + \frac{1}{t} = \frac{26}{5}$ where $t = 5^{1+\log_4 x}$

$\Rightarrow t = 5, \frac{1}{5} \Rightarrow x = 1, \frac{1}{16}$

$\therefore \quad a = 1, b = \frac{1}{16} \Rightarrow \frac{a}{b} = 16$

14. $\log_2(4(4^x+1)) \cdot \log_2(4^x+1) = \log_{1/\sqrt{2}}\left(1/\sqrt{8}\right)$.

i.e., $\{\log_2 4 + \log_2(4^x+1)\}\log_2(4^x+1) = \log_{2^{-1/2}} 8^{-1/2}$

$\therefore \quad (2+y)\,y = \log_2 8 = 3 \quad$ [where $y = \log_2(4^x+1)$]

$\therefore \quad y^2 + 2y - 3 = 0$ or $y = -3, 1$

$\therefore \quad \log_2(4^x+1) = -3, 1$ or $4^x + 1 = 2^{-3}, 2$

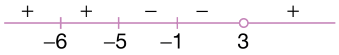
or $\quad 4^x + 1 = \frac{1}{8}, 2$

$4^x = -\frac{7}{8}$ is not possible

$\therefore \quad 4^x = 1 \quad \therefore \quad x = 0$

Single Correct Choice Type Questions

1. $\frac{(x+6)^2(x+5)(x+1)^2}{(x-3)} \leq 0$



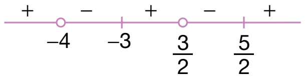
$x \in [-5, 3) \cup \{-6\}$

$\therefore$ 9 integral values of x.

Hence, the correct answer is (D).

2. Consider $\frac{8x^2+16x-51}{(2x-3)(x+4)} - 3 > 0$

$$\Rightarrow \frac{2x^2+x-15}{2x^2+5x-12} > 0 \Rightarrow \frac{(2x-5)(x+3)}{(2x-3)(x+4)} > 0$$



Hence both Nr and Dr are positive if $x < -4$ or $x > \frac{5}{2}$ and both negative if $-3 < x < \frac{3}{2}$

Hence all the statements are true.

Hence, the correct answer is (D).

3. $|x+1||x-1| = a^2 - 2a - 3 \Rightarrow |x^2 - 1| = a^2 - 2a - 3$

For solution to exist

$\therefore \quad a^2 - 2a - 3 \geq 0$

$\Rightarrow \quad (a+1)(a-3) \geq 0$

$\therefore \quad a \in (-\infty, -1] \cup [3, \infty)$

Hence, the correct answer is (A).

4. CASE-1: if $x \geq 0$

$\frac{x-1}{x-2} \geq 0$

$x \in (2, \infty) \cup [0, 1]$

CASE-2: if $x < 0$

$\frac{-x-1}{-x-2} \geq 0$ or $\frac{x+1}{x+2} \geq 0$

$x \in (-\infty, -2) \cup [-1, 0)$

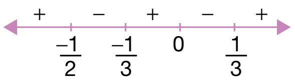
Final answer $x \in (-\infty, -2) \cup [-1, 0) \cup [0, 1] \cup (2, \infty)$

$x \in (-\infty, -2) \cup [-1, 1] \cup (2, \infty)$

Hence, the correct answer is (D).

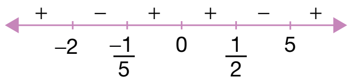
5. We have $\dfrac{3x+1}{x(6x^2+x-1)} > 0$

$\Rightarrow \quad \dfrac{3x+1}{x(3x-1)(2x+1)} > 0 \qquad \ldots(1)$



We have $x \in (-\infty, -1/2) \cup \left(\dfrac{-1}{3}, 0\right) \cup (1/3, \infty)$

Now, $\dfrac{x^2(x-5)(2x-1)}{(5x+1)(x+2)} < 0 \qquad \ldots(2)$



We have $x \in (-2, -1/5) \cup (1/2, 5)$

$\therefore$ Clearly, from (1) and (2),

$x \in (-2, -1/2) \cup (-1/3, -1/5) \cup (1/2, 5)$

Hence, the correct answer is (A).

6. $\dfrac{3x^2-7x+8}{x^2+1} > 1$ and $\dfrac{3x^2-7x+8}{x^2+1} \le 2$

$\Rightarrow \quad 2x^2 - 7x + 7 > 0$ and $x^2 - 7x + 6 \le 0$

$x \in R$ and $(x-1)(x-6) \le 0 \quad x \in [1, 6]$

Hence $x \in [1, 6]$

Hence, the correct answer is (D).

7. $\log_7 \dfrac{2x-6}{2x-2} > 0$ or $\dfrac{2x-6}{2x-2} > 1 \Rightarrow \dfrac{2}{x-1} < 0$

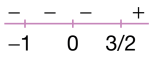
$\Rightarrow \quad x \in (-\infty, 1)$

Hence, the correct answer is (A).

8. $\dfrac{(e^x-1)(2x-3)(x^2+x+2)}{(2-\sin x)(x+1)^2 x} \ge 0$

$\because x^2 + x + 2 > 0, \forall x \in R$ and $2 - \sin x > 0, \forall x \in R$

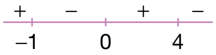
So, $\dfrac{(e^x-1)(2x-3)}{x(x+1)^2} \ge 0$



$\Rightarrow \quad x \in \left[\dfrac{3}{2}, \infty\right)$

Hence, the correct answer is (A).

9. $\dfrac{4^x\left(\left(\dfrac{3}{4}\right)^x - 1\right)}{(x-4)(x+1)} \ge 0$



$x \in (-\infty, -1) \cup [0, 4)$

Hence, the correct answer is (C).

10. $\log_{17}\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 0 \qquad \ldots(1)$

Equation (1) is defined if $x \ge 0$.

We can rewrite (1) as $\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 17^0 = 1$

$\Rightarrow \quad \sqrt{x+11} + \sqrt{x} = 11^1 = 11$

$\Rightarrow \quad \sqrt{x+11} = 11 - \sqrt{x}$

Squaring both sides we get $x + 11 = 121 - 22\sqrt{x} + x$

$\Rightarrow \quad 22\sqrt{x} = 110 \quad \Rightarrow \quad \sqrt{x} = 5$ or $x = 25$

This clearly satisfies (1). Thus, sum of all the values satisfying (1) is 25.

Hence, the correct answer is (A).

11. $6x + 10 - x^2 > 3$

$\Rightarrow \quad x^2 - 6x - 7 < 0$

$\Rightarrow \quad (x+1)(x-7) < 0$

Hence, the correct answer is (B).

12. $\log_{\frac{1}{2}}\left(\dfrac{x-1}{x-2}\right) \ge 0 \quad \Rightarrow \quad \dfrac{x-1}{x-2} > 0$ and $\dfrac{x-1}{x-2} \le 1$

$(-\infty, 1) \cup (2, \infty)$ and $(-\infty, 2)$

Hence, the correct answer is (C).

13. $\log_{x-3} \log_{2x^2-2x+3}(x^2+2x) = 0$

$\Rightarrow \quad \log_{2x^2-2x+3}(x^2+2x) = 1$

$\Rightarrow \quad (x^2+2x) = 2x^2 - 2x + 3$

$\Rightarrow \quad x^2 - 4x + 3 = 0$

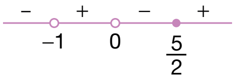
$\Rightarrow \quad (x-3)(x-1) = 0$

$\Rightarrow \quad x = 1$ or $x = 3$

Both do not belong to the domain. So, no solution

Hence, the correct answer is (D).

14. $\dfrac{(2x-5)}{x(x+1)} \ge 0; e^x > 0$ and $\cos x - 2 < 0$



Hence, the correct answer is (D).

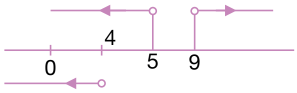
15. $(2^x - 1)(2^x - 7) = 7 \quad \Rightarrow \quad x = 3$ is the only solutions

Hence, the correct answer is (B).

16. $x^2 - 14x + 45 > 0$; $(x-9)(x-5) > 0$

$\Rightarrow \quad x > 9$ or $x < 5$ and $4 - x > 0$

$\Rightarrow \quad 4 > x \quad \Rightarrow \quad x < 4$



$\therefore \quad 0 < x < 4 \quad \Rightarrow \quad x \in \{1, 2, 3\}$ but $4 - x \ne 1$

$\Rightarrow \quad x \ne 3; x \in \{1, 2\}$

$\therefore$ There are two natural numbers.

Hence, the correct answer is (C).

17. We have $(x+1)^2 > 5x - 1$

$\Rightarrow \quad x^2 - 3x + 2 > 0$

$(x-1)(x-2) > 0 \quad \Rightarrow \quad x < 1$ or $x > 2 \qquad \ldots(1)$

And $(x+1)^2 < 7x - 3 \quad \Rightarrow \quad x^2 - 5x + 4 < 0$

$(x-1)(x-4) < 0 \quad \Rightarrow \quad 1 < x < 4 \qquad \ldots(2)$

$\therefore \quad (1) \cap (2) \quad \Rightarrow \quad x \in (2, 4)$

$\therefore$ Only one integral value of x which is 3.

Hence, the correct answer is (B).

18. $\sqrt{x^2+9} = 9 \Rightarrow x^2 + 9 = 81$

$\Rightarrow x^2 = 72 \Rightarrow x = \pm 6\sqrt{2}$

Hence, the correct answer is (A).

19. $x^2 - 2x - 3 \geq 0$

$\Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x = -3$

$x^2 - 2x - 3 < 0$

$\Rightarrow x^2 - 6x - 3 = 0$ no rational root

Hence, the correct answer is (B).

20. Let $\log_3 x = t \Rightarrow x = 3^t$

$$(3^t + 1)\, t^2 + 4t \cdot 3^t - 16 = 0$$

$$(t+4)\,(t \cdot 3^t + t - 4) = 0$$

$$\log_3 x = t = 1, -4 \Rightarrow x = 3, \frac{1}{81}$$

Alternate

$$x\log_3{}^2 x + \log_3{}^2 x + 4x\log_3 x - 16 = 0$$

$$x\log_3 x\,(\log_3 x + 4) + (\log_3 x + 4)\,(\log_3 x - 4) = 0$$

$$(\log_3 x + 4)\,(x\log_3 x + \log_3 x - 4) = 0$$

$$x = 3^{-4} \text{ or } (x+1)\log_3 x = 4 \Rightarrow x = 3$$

$$\Rightarrow x = \frac{1}{81}, 3$$

Hence, the correct answer is (C).

21. $|3x - 9| + 2 > 2$ or $|3x - 9| + 2 < -2$

$|3x - 9| > 0$ or $x \in \phi$

$x \in R - \{3\}$

Hence, the correct answer is (C).

22. $|x^2 + 4x + 4 + 4x + 3| = |x^2 + 4x + 4| + |4x + 3|$

$|a + b| = |a| + |b| \Rightarrow a \cdot b \geq 0$

$(x^2 + 4x + 4)\,(4x + 3) \geq 0$

$$x \geq -\frac{3}{4}$$

Hence, the correct answer is (C).

23. (A) $\log_{\frac{3}{2}}\log_4\log_{\sqrt{3}} 81 = \log_{\frac{3}{2}}\log_4 8 = \log_{\frac{3}{2}}\frac{3}{2} = 1$

(B) $\log_2 6 + \log_2\sqrt{\frac{2}{3}} = \log_2 2\sqrt{2}\sqrt{3}$

(C) $-\frac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\frac{64}{27}\right) = \frac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\frac{27}{64}\right) = \log_{\frac{\sqrt{3}}{2}}\left(\frac{\sqrt{3}}{2}\right) = 1$

Hence, the correct answer is (B).

24. $$\frac{2}{6\log_4 2000} + \frac{3}{6\log_5 2000} = \frac{1}{6}[2\log_{2000} 4 + 3\log_{2000} 5]$$

$$= \frac{1}{6}[\log_{2000} 4^2 \cdot 5^3] = \frac{1}{6}\log_{2000} 2000 = \frac{1}{6}$$

Hence, the correct answer is (B).

25. $\log_{10}{}^2 = \beta$

$$\log_{10}\left(\frac{1025}{1024} \times \frac{4}{4}\right) = \alpha$$

$\Rightarrow \log_{10} 4100 - \log_{10} 2^{12} = \alpha$

$\Rightarrow \log_{10} 4100 = \alpha + 12\beta$

Hence, the correct answer is (B).

26. $\log_3 M + 9\log_3 N = 3(1 + \log_{0.008} 5)$

$$\log_3 MN^9 = 3(\log_{0.008} 5 \times 0.008)$$

$$\log_3 (MN^9) = 3\log_{0.008} 0.04 = 3 \times \frac{2}{3} \quad \text{So, } MN^9 = 9$$

Hence, the correct answer is (B).

27. $(\ell nx)^2 + 3\ell nx - 4 > 0$

$(\ell nx + 4)(\ell nx - 1) > 0 \Rightarrow \ell nx < -4$ or $\ell nx > 1$

$x \in (0, e^{-4}) \cup (e, \infty)$

Hence, the correct answer is (D).

28. $$\frac{6a^{\ln b}\dfrac{\log b}{\log a^2}\dfrac{\log a}{\log b^2}}{a^{\ln b}} = \frac{3}{2} \text{ (independent of } a \text{ and } b)$$

Hence, the correct answer is (A).

29. $$N = \frac{81^{\frac{1}{\log_5 9}} + 3^{\frac{1}{\log_{\sqrt{6}} 3}}}{409} \cdot \left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}} - 125^{\log_{25} 6}\right)$$

$$N = \frac{81^{\log_9 5} + 3^{3\log_3 \sqrt{6}}}{409} \cdot (7^{\log_7 25} - 6^{\log_{25} 125})$$

$$= \frac{25 + 6\sqrt{6}}{409}\left(25 - 6\sqrt{6}\right) \Rightarrow N = 1 \Rightarrow \log_2 N = \log_2 1 = 0$$

Hence, the correct answer is (B).

30. $|x^4 \cdot 3^{|x-2|} \cdot 5^{x-1}| = -x^4 3^{|x-2|} \cdot 5^{x-1}$

$\Rightarrow x^4 \cdot 3^{|x-2|} \cdot 5^{x-1} = 0 \Rightarrow x^4 = 0$

so, $x = 0$

Hence, the correct answer is (C).

31. $\log_{3\sqrt{x}} x + \log_{3x}\sqrt{x} = 0;\ x > 0,\ x \neq \frac{1}{3}, \frac{1}{9}$

$\Rightarrow \log_{3\sqrt{x}} x = -\log_{3x}\sqrt{x}$

$$\Rightarrow \frac{1}{\log_x 3\sqrt{x}} = -\frac{1}{\log_{\sqrt{x}} 3x}$$

$$\Rightarrow \log_x\left(3\sqrt{x}\right) = -\log_{\sqrt{x}}(3x)$$

$$\Rightarrow \log_x\left(3\sqrt{x}\right) = \log_x(3x)^{-2}$$

$$\Rightarrow 3\sqrt{x} = \frac{1}{(3x)^2} \Rightarrow x^{5/2} = \frac{1}{27}$$

$$\Rightarrow x = \frac{1}{3^{6/5}} \Rightarrow \text{irrational solution.}$$

Hence, the correct answer is (C).

32. $7^{\log x} - 5^{\log x + 1} = 3 \cdot 5^{\log x - 1} - 13 \cdot 7^{\log x - 1}$

$$7^{\log x} + \frac{13 \cdot 7^{\log x}}{7} = \frac{3 \cdot 5^{\log x}}{5} + 5 \cdot 5^{\log x}$$

$$5 \cdot 7^{\log x - 1} = 7 \cdot 5^{\log x - 1}$$

$$7^{\log x - 2} = 5^{\log x - 2}$$

$\Rightarrow \log x - 2 = 0$

$\Rightarrow x = 100$

Hence, the correct answer is (C).

33. $12|x| \le 4x^2 + 9$

$(2x - 3)^2 \ge 0;\ x \in R$

Hence, the correct answer is (A).

34. Using $a^{\log_a N} = N$

We get $8x - 3 = 13;\ x = \left(\dfrac{16}{8}\right) = 2$

Hence, the correct answer is (A).

35. $N = 10^p;\ p = \log_{10} 8 - \log_{10} 9 + 2\log_{10} 6$

$$p = \log\left(\frac{8.36}{9}\right) = \log_{10} 32$$

$\therefore \quad N = 10^{\log_{10} 32} = 32$

Hence characteristic of $\log_3 32$ is 3.

Hence, the correct answer is (B).

36. $\dfrac{\log_2 16}{2\log_2 x} + \dfrac{\log_2 64}{\log_2 2x} = 3$

$\Rightarrow \quad \dfrac{2}{\log_2 x} + \dfrac{6}{1 + \log_2 x} = 3$

Let $\log_2 x = y$

$2 + 8y = 3y(1 + y) \quad \Rightarrow \quad 3y^2 - 5y - 2 = 0$

$3y^2 - 6y - y - 2 = 0 \quad \Rightarrow \quad 3y(y - 2) + 1(y - 2) = 0$

$\Rightarrow \quad \log_2 x = y = 2;\ \dfrac{-1}{3}$

$\Rightarrow \quad x = 4,\ x = 2^{-1/3}$ (which is irrational)

Hence, the correct answer is (C).

37. $2 \le \log_5 N < 3$

$\Rightarrow \quad 25 \le N < 125$

$\Rightarrow$ Number of integers $= 125 - 25 = 100$.

Hence, the correct answer is (C).

38. $4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$

$\Rightarrow \quad 4^{1/2} + 9^2 = 83^{\log_x 10}$

$\Rightarrow \quad 83 = 83^{\log_x 10}$

$\Rightarrow \quad 1 = \log_x 10$

$\Rightarrow \quad x = 10$

Hence, the correct answer is (C).

39. $(27) \cdot \dfrac{(9^x)}{3^x} = \dfrac{4^x}{8^x} \quad \Rightarrow \quad 27 \cdot 3^x = \dfrac{1}{2^x}$

$\Rightarrow \quad 27 = 6^{-x} \qquad \ldots(1)$

Now $(2)^{-(1+\log_2 3)x}$

$$= (2)^{-x} \cdot (2)^{-\log_2 (3)^x} = \frac{2^{-x}}{2^{\log_2 3^x}} = \frac{2^{-x}}{3^x} = 6^{-x} \qquad \ldots(2)$$

Hence $(2)^{-(1+\log_2 3)x} = 27$

Hence, the correct answer is (D).

40. We have, $\dfrac{\ell n\left(\sqrt{x^2 + 1 - x^2}\right)}{\ell n x} = x$

$\Rightarrow \quad \dfrac{\ell n 1}{\ell n x} = x \quad \Rightarrow \quad x = 0$ or $\ell n\, x = 0$

$\therefore \quad x = 0$ or $x = 1$ but both are unacceptable.

So no solution exists.

Hence, the correct answer is (D).

41. $(\ell n\, x)^2 - 1 + (\ell n\, x - 1)(\ell n^2 x - 2\ell n\, x - 7) = 0$

$(\ell n\, x - 1)[(\ell n\, x + 1) + (\ell n^2 x - 2\,\ell n\, x - 7)] = 0$

$\therefore \quad \ell n\, x = 1;\ \ell n\, x = -2;\ \ell n\, x = 3$

But $\ell n\, x = 1$ and $\ell n\, x = 3$ are acceptable only

$\therefore \quad x = e$ and $x = e^3$

Hence, the correct answer is (C).

42. $\log_x 144 = 2$

$144 = x^2$

$\therefore \quad x = 12$ or $x = -12$ (rejected)

Hence, the correct answer is (B).

43. $\log_6 16 = k \quad \Rightarrow \quad \dfrac{\log_2 16}{\log_2 6} = k$

$\dfrac{4}{1 + \log_2 3} = k \quad \Rightarrow \quad \log_2 3 = \dfrac{4 - k}{k}$

$$\log_{18} 24 = \frac{\log_2 24}{\log_2 18} = \frac{\log_2 8 + \log_2 3}{\log_2 9 + \log_2 2}$$

$$= \frac{3 + \left(\dfrac{4-k}{k}\right)}{2\left(\dfrac{4-k}{k}\right) + 1} = \frac{3k + 4 - k}{8 - 2k + k}$$

Hence, $\log_{18} 24 = \dfrac{2k + 4}{8 - k}$

Hence, the correct answer is (B).

44. $N = (0.18)^{20}$

$\log_{10} N = 20\log_{10}(0.18) = 20[\log_{10} 9 + \log_{10} 2 - 2]$

$\log_{10} N = -14.896$

Hence, the correct answer is (B).

45. Put $\log_3(x - 1) = t$

$\Rightarrow \quad t = 2 + \dfrac{1}{t^2} - 2t \quad \Rightarrow \quad 3t^3 - 2t^2 - 1 = 0$

$\therefore \quad (t - 1)(3t^2 + t + 1) = 0$

$\Rightarrow \quad t = 1 \quad \therefore \quad x = 4$

Hence, the correct answer is (B).

46. Using $a^{\log_a N} = N$ repeatedly, we get

$\left(\sqrt{5}\right)^{\log_5 x} = 2$

$(x)^{1/2} = 2 \quad \Rightarrow \quad x = 4$

Hence, the correct answer is (C).

47. $A = \dfrac{3}{\log_{27} 8}\left(\dfrac{9}{4}\right) = \dfrac{\dfrac{1}{1024}}{\dfrac{1}{2}} = \dfrac{1}{512}$

$\therefore \quad \log_{\sqrt{3}}\left(1024 \times \dfrac{1}{512} + 1\right) = 2$

Hence, the correct answer is (D).

48. $3^{\log_3 x} + 3^{2\log_3 x} + 3^{3\log_3 x} = 3$

$\Rightarrow \quad x + x^2 + x^3 = 3$

$\Rightarrow \quad (x-1) + (x^2-1) + (x^3-1) = 0$

$\Rightarrow \quad (x-1)\,(1 + x + 1 + x^2 + x + 1) = 0$

$\Rightarrow \quad x = 1 \quad (\because x^2 + 2x + 3 \neq 0)$

Hence, the correct answer is (A).

49. $5^{\frac{\log_3(\log_2 81)}{\log_3 5}} = 5^{\log_5(\log_2 81)} = 6 < \log_2 81 < 7$

Hence, the correct answer is (C).

50. Given $7^{3x^2} \cdot 5^x = 11$ taking log

$3x^2\,\ell n\,7 + x\,\ell n\,5 = \ell n\,11$

$\Rightarrow \quad (3\ell n\,7)x^2 + (\ell n\,5)\,x - \ell n\,11 = 0$

$\text{sum} = \dfrac{-\ell n 5}{3\ell n 7}$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

1. $\text{L.H.S.} = \log_2(x^2 - x) \times \log_2\left(\dfrac{x-1}{x}\right) + (\log_2 x)^2$

$= (\log_2(x-1) + \log_2 x)\,(\log_2(x-1) - \log_2 x) + (\log_2 x)^2$

$= (\log_2^2(x-1) - \log_2^2 x)\,(\log_2 x)^2 + \log_2^2(x-1)$

and hence $\log_2^2(x-1) = 4$

$\Rightarrow \quad x - 1 = \dfrac{1}{4}$ or 4

Hence, (A) and (B) are correct.

2. Let $N_1 = 3^{40}$

$\therefore \quad \log_{10}N_1 = 40\log_{10}3 = 40 \times 0.4771 = 19.084$

So, $L = 20$

Let $N_2 = 3^{-40}$

$\Rightarrow \quad \log_{10}N_2 = -40 \times \log_{10}3 = -40 \times 0.4771$

$= -19.084 = -19 - 1 + (-0.084) + 1$

$= 20 + 0.916 = \overline{20}.019$

So, $M = 19$, Now, verify alternatives.

Hence, (A) and (C) are correct.

3. $x^{\frac{3}{4}(\log_3 x)^2 + (\log_3 x) - \frac{5}{4}} = \sqrt{3}$

Let $y = \log_3 x$

$\therefore \quad \left(\dfrac{3y^2}{4} + y - \dfrac{5}{4}\right)y = \dfrac{1}{2} \quad \Rightarrow \quad (3y^2 + 4y - 5)y = 2$

$\Rightarrow \quad 3y^3 + 4y^2 - 5y - 2 = 0$

$\Rightarrow \quad 3y^2(y-1) + 7y(y-1) + 2(y-1) = 0$

$\Rightarrow \quad (y-1)\,(3y^2 + 7y + 2) = 0$

$\Rightarrow \quad (y-1)\,(3y+1)\,(y+2) = 0$

$y = 1, \dfrac{-1}{3}, -2$

$\therefore \quad \log_3 x = 1, \dfrac{-1}{3}, -2$

$\Rightarrow \quad x = 3, 3^{\frac{-1}{3}}, 3^{-2}.$

Hence, (A) and (B) are correct.

4. (A) $\log(1+2+3) = \log 6 = \log(1 \cdot 2 \cdot 3)$

$= \log 1 + \log 2 + \log 3$

(B) $\dfrac{\log_9 11}{\log_5 13} \times \dfrac{\log_{\sqrt{5}} 13}{\log_3 11} = \dfrac{\frac{\log_3 11}{2} \cdot 2\log_5 13}{\log_5 13 \cdot \log_3 11} = 1$

(C) $\log_2 7 = \dfrac{p}{q}, p \in I, q \in I, q \neq 0, \pm 1$

HCF $(p, q) = 1 \quad \Rightarrow \quad 7 = 2^{p/q}$

$\Rightarrow \quad 7^q = 2^p$ (not possible)

(D) $\log_2 x + \log_4(x+2) = 2$

$\Rightarrow \quad \log_2 x + \dfrac{1}{2}\log_2(x+2) = 2$

$\Rightarrow \quad 2\log_2 x + \log_2(x+2) = 4$

$\Rightarrow \quad \log_2 x^2\,(x+2) = 4$

$\Rightarrow \quad x^2(x+2) = 16$

$\Rightarrow \quad x^3 + 2x^2 - 16 = 0$

$\Rightarrow \quad x^2(x-2) + 4x(x-2) + 8(x-2) = 0$

$\Rightarrow \quad (x-2)\,(x^4 + 4x + 8) = 0$

$\Rightarrow \quad x = 2, -2 \pm 2i$

Hence, (A) and (B) are correct.

5. $N = \sqrt{(10)^2(10)^{\log_{10}4}} = \sqrt{400} = 20$ which form the sides of a right triangle with 99 and 101

Clearly N is coprime with 9.

Now verify other alternatives.

Hence, (A) and (D) are correct.

6. (A) As, $5^{\sqrt{\log_5 4}} = \left(5^{\log_5 4}\right)^{\frac{1}{\sqrt{\log_5 4}}} = 4^{\sqrt{\log_4 5}}$

So, $5^{\sqrt{\log_5 4}} - 4^{\sqrt{\log_4 5}} = 0$

(B) $\log_{\cot^2\frac{\pi}{6}}\left(\sqrt{27} - \sqrt{12}\right) = \log_3(3\sqrt{3} - 2\sqrt{3})$

$= \log_3\sqrt{3} = \dfrac{1}{2}$

(C) $\log_{2-\sqrt{3}}\left(\dfrac{2+\sqrt{3}}{2-\sqrt{3}}\right) = \log_{2-\sqrt{3}}\left(2+\sqrt{3}\right)^2$

$= 2\log_{2-\sqrt{3}}\left(2+\sqrt{3}\right)$, which is negative, as number and base of logarithm are on opposite side of unity.

(D) $\log_{\sec^2\frac{\pi}{3}}\left(\operatorname{cos}ec\dfrac{\pi}{6}\right) = \log_{2^2}(2)^1 = \dfrac{1}{2}$

Hence, (A) and (C) are correct.

7. $\dfrac{\frac{2}{3} - \frac{1}{3}\log_3 x}{\left(\log_3 x\right)^2} = 2$

Let $\log_3 x = t$; $\quad \dfrac{(2-t)}{3t^2} = 2$

$\Rightarrow \quad 6t^2 + t - 2 = 0$

$\Rightarrow \quad 6t^2 + 4t - 3t - 2 = 0$

$\Rightarrow \quad (2t - 1)(3t + 2) = 0$

$t = \frac{1}{2} \quad \text{or} \quad t = -\frac{2}{3};$

$\log_3 x = \frac{1}{2} \quad \text{or} \quad \log_3 x = \frac{-2}{3}$

$x = \sqrt{3} \quad \text{or} \quad x = (3)^{-2/3}$

Hence, (A) and (B) are correct.

8. Let $\log_{10} x + 2 = a$

and $\log_{10} x - 1 = b$

$\therefore \quad a + b = 2\log_{10} x + 1$

$\therefore$ given $a^3 + b^3 = (a + b)^3$

$\therefore \quad 3ab(a + b) = 0$

$(\log_{10} x + 2) \cdot (\log_{10} x - 1)\ (2 \log_{10} x + 1) = 0$

$\therefore \quad x = \frac{1}{100}$ or $x = 10$ or $x = \frac{1}{\sqrt{10}}$

Hence, (B), (C) and (D) are correct.

9. $N = (3\log_2 5 + 1)\ (2 + 2 \log_2 5) - (2 \log_2 5) \times (3 \log_2 5 + 4)$

Let $\log_2 5 = t$

$N = (3t + 1)\ (2 + 2t) - (2t)\ (3t + 4)$

$N = 6t^2 + 8t + 2 - 6t^2 - 8t = 2$

$\therefore \quad N = 2$

Hence, (A) and (D) are correct.

10. $\alpha = 2 \quad \text{or} \quad \alpha = 5$

and $M = 5 \quad M = 2$

$\therefore \quad N \in [25, 125)$ or $[32, 64)$

Hence $|b - 5a| = |125 - 125| = 0$

or $\quad |64 - 160| = 96$

Hence, (A) and (D) are correct.

11. $\log_2 x + \log_2 x + \ldots + \log_2 n$ (n times) $= 8$

$\log_2 x^n = 8 \quad \Rightarrow \quad x^n = 2^8$

Hence $x = 2$ or 4, 16, 256.

Hence, (A), (B), (C) and (D) are correct.

12. $a = (\log_3 \pi)\ (\log_2 3)\ (\log_\pi 2) = \frac{\log \pi}{\log 3} \cdot \frac{\log 3}{\log 2} \cdot \frac{\log 2}{\log \pi} = 1$

$b = \frac{\log 576}{3\log 2 + \log 3}$ the base of the log being 10

$b = \frac{2\log 24}{\log 24} = 2$

$c = (9)^{2x} - 9^x \cdot 3^{\log_3 12} + 27 = 0$

$\Rightarrow \quad t^2 - 12t + 27 = 0$

$\Rightarrow \quad t = 9 \quad \Rightarrow \quad 9^x = 9 \quad \Rightarrow \quad x = 1$

$\Rightarrow \quad t = 3 \quad \Rightarrow \quad 9^x = 3 \quad \Rightarrow \quad x = \frac{1}{2}$

$\Rightarrow \quad c = 2\left(1 + \frac{1}{2}\right) = 3.$

$d = 7^{(\log_7 2 + \log_7 3)} = 6$

$a + b + \frac{c}{d} = 1 + 2 + \frac{3}{6} = \frac{7}{2}.$

Hence, the correct answer is (A).

13. $\log_{18} 4 + \log_{18} 6 + 5 \cdot \log_{18} 3$

$= \log_{18}(4 \cdot 6 \cdot 3^5) = \log_{18}(2 \cdot 3^2 \cdot 2 \cdot 3^2 \cdot 18) = \log_{18}(18^3) = 3$

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. **The correct answer is (B).**
2. **The correct answer is (A).**
3. **The correct answer is (B).**

Combined Solution to 1–3

$|x + 1| - |2x + 1| + |2x| = 2$

CASE-1: $x < -1; -x - 1 + 2x + 1 - 2x = 2$

$x = -2$ (rejected)

CASE-2: $x < -1 \le x < \frac{-1}{2}; x + 1 + 2x + 1 - 2x = 2$

$x = 0$ (accepted)

CASE-3: $\frac{-1}{2} \le x < 0$

$x + 1 - (2x + 1) - 2x = 2; x + 1 - 2x - 1 - 2x = 2$

$-3x = 2$ (rejected)

CASE-4: $x \ge 0$

$(x + 1) - (2x + 1) + 2x = 2; x + 1 - 2x - 1 + 2x = 2$

$x = 2$ (accepted)

Hence, number of integral solutions are 2.

Now, $\alpha = 2$

Number of digit is 2^{50} are 16.

$\text{Log}_{10}(2)^{50} = 50 \log_{10} 2 = 15.05$

4. **The correct answer is (B).**
5. **The correct answer is (D).**
6. **The correct answer is (A).**

Combined Solution to 4–6

$\left(\sqrt{x}\right)^{-1 + \log_5 x} = 5$

$\Rightarrow \quad (\log_5 x - 1)\ \frac{1}{2} \log_5 x = 1$

$\Rightarrow \quad \log_5 x\ (\log_5 x - 1) = 2$

Put $\log_5 x = y$

$\Rightarrow \quad y^2 - y - 2 = 0$

$\Rightarrow \quad (y - 2)\ (y + 1) = 0$

$\Rightarrow \quad y = 2 \text{ or } -1$

$\Rightarrow \quad \log_5 x = 2 \text{ or } -1$

$x = 25 \quad \text{or} \quad x = \frac{1}{5}$

$\alpha = 25 \quad \text{and} \quad \beta = \frac{1}{5}$

4. Now, let $N = (25)^{10}$

$\log_{10} N = 10 \times 2\log_{10} 5 = 20\ [1 - 0.3010]$

$= 20\ [0.6990] = 13.980 = 13.98$

$\Rightarrow$ N is a 14 digit number

5. Let $x = \left(\frac{1}{5}\right)^{10}$; $\log x = 10(-\log 5) = -10(1 - \log 2)$

$= -10[0.699] = -6.99$

Hence, there are 6 zeroes after decimal before a significant figure.

6. $\left(\frac{1}{5}\right)^{\log_{25} 9} = \left(\frac{1}{5}\right)^{\log_5 3} = \frac{1}{5^{\log_5 3}} = \frac{1}{3}$

7. $\sqrt{f(x)} \log_{\sec^2(8.5)} \frac{25 - x^2}{16} > 0$

then $\frac{25 - x^2}{16} > 1 \Rightarrow 9 - x^2 > 0$

$\Rightarrow \quad x \in (-3, 3)$ ...(1)

$$f(x) \geq 0 \Rightarrow \log_{\frac{25-x^2}{16}}\left(\frac{24 - 2x - x^2}{14}\right) \geq 0$$

$$\frac{25 - x^2}{16} > 1 \Rightarrow x \in (-3, 3)$$

$$\frac{24 - 2x - x^2}{14} \geq 1 \Rightarrow x^2 + 2x - 10 \leq 0$$

$$x \in \left(-1 - \sqrt{11}, -1 + \sqrt{11}\right) \qquad \text{...(2)}$$

(1) ∩ (2)

$$x \in \left(-3, -1 + \sqrt{11}\right)$$

Hence, the correct answer is (C).

8. Given $0 < \frac{25 - x^2}{16} < 1$

$$\Rightarrow 0 < 25 - x^2 < 16$$

$$x \in (-5, 5) \cap \left((-\infty, -3) \cup (3, \infty)\right)$$

$$x \in (-5, -3) \cup (3, 5) \qquad \text{...(1)}$$

$$f(x) > 1 \Rightarrow \log_{\frac{25-x^2}{16}}\left(\frac{24 - 2x - x^2}{14}\right) > 1$$

$$\Rightarrow \frac{24 - 2x - x^2}{14} < \frac{25 - x^2}{16}$$

$$\Rightarrow x^2 + 16x - 17 > 0$$

$$x \in (-\infty, -17) \cup (1, \infty) \qquad \text{...(2)}$$

(1) ∩ (2)

$$x \in (3, 5)$$

Hence, the correct answer is (C).

9. $\frac{25 - x^2}{16} > 1 \Rightarrow x^2 - 9 < 0 \Rightarrow x \in (-3, 3)$ ...(1)

$$f(x) > 1 \Rightarrow \frac{24 - 2x - x^2}{14} > \frac{25 - x^2}{16}$$

$$\Rightarrow x^2 + 16x - 17 < 0$$

$$\Rightarrow x \in (-17, 1) \qquad \text{...(2)}$$

(1) ∩ (2)

$$x \in (-3, 1)$$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. A → (r); B → (p); C → (q); D → (s)

(A) $\log_4 (x + 1) + \log_4 (x + 8) = \frac{3}{2}$

$x > -1$ and $\log_4 (x + 1)(x + 8) = \frac{3}{2}$

or $(x + 1)(x + 8) = 8$

$x^2 + 9x = 0$ or $x = 0, -9$

but $x = -9$ is not possible $\therefore$ $x = 0$

(B) $\ln x = 0 \Rightarrow x = 1$

(C) $4\left(3\log_2 \frac{81}{80} + 5\log_2 \frac{25}{24} + 7\log_2 \frac{16}{15}\right)$

$$= 4 \cdot \log_2\left(\frac{3^{12}}{2^{12} . 5^3} \cdot \frac{5^{10}}{2^{15} . 3^5} \cdot \frac{2^{28}}{3^7 . 5^7}\right) = 4$$

(D) $\text{Rem} = P\left(-\frac{1}{2}\right) = -\frac{1}{16} + \frac{1}{8} + \frac{1}{4} + 1 = \frac{5}{4} + \frac{1}{16} = \frac{21}{16} = k$

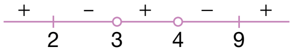
$$\therefore \frac{16k + 11}{16} = \frac{21 + 11}{16} = \frac{32}{16} = 2$$

2. A → (q); B → (t); C → (r); D → (p)

$$f(x) = \frac{x^2 - 11x + 18}{x^2 - 7x + 12} = \frac{(x - 9)(x - 2)}{(x - 3)(x - 4)}$$

(A) $0 < f(x) < 1$

$f(x) > 0 \Rightarrow x \in (-\infty, 2) \cup (3, 4) \cup (9, \infty)$ ...(1)



$f(x) < 1$

$$\Rightarrow \frac{x^2 - 11x + 18}{x^2 - 7x + 12} < 1$$

$$\Rightarrow \frac{-4x + 6}{(x - 3)(x - 4)} < 0$$

$$\Rightarrow \frac{2x - 3}{(x - 3)(x - 4)} > 0$$

$$\Rightarrow x \in \left(\frac{3}{2}, 3\right) \cup (4, \infty) \qquad \text{...(2)}$$

$\therefore$ From (1) and (2) $\Rightarrow \left(\frac{3}{2}, 2\right) \cup (9, \infty)$

(B) $f(x) < 1$ from (2)

$$\Rightarrow \left(\frac{3}{2}, 3\right) \cup (4, \infty)$$

(C) $f(x) < 0 \Rightarrow x \in (2, 3) \cup (4, 9)$

(D) $f(x) > 2$

$$\Rightarrow \frac{x^2-11x+18}{x^2-7x+12} > 2$$

$$\Rightarrow \frac{x^2-3x+6}{(x-3)(x-4)} < 0$$

$$\Rightarrow x \in (3, 4)$$

3. A → (q); B → (q, r, t); C → (s); D → (p)

(A) $\log_4(x-1) = \log_2(x-3) = \log_4(x-3)^2$

$\Rightarrow x - 1 = (x-3)^2$

$\Rightarrow x^2 - 7x + 10 = 0$

$\Rightarrow x = 2, 5$ But $x - 3 > 0$

$\therefore x = 5$

(B) $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4 = \log_\pi 12$

But $\pi^2 < 12 < \pi^3$

$\therefore 2 < \log_\pi 12 < 3 \Rightarrow 3$

(C) Taking $\log_3$ on both sides

$(2t + t^2 - 10)t = -2t, t = \log_3 x$

$t(t^2 + 2t - 8) = 0$

$\Rightarrow t = 0, 2, -4$

$\Rightarrow x = 1, 9, \dfrac{1}{81}$

(D) $\log_5(\sqrt{x+5} - \sqrt{x}) = 1$

$\Rightarrow \sqrt{x+5} - \sqrt{x} = 5$

$x + 5 = x + 25 + 10\sqrt{x}$

$\sqrt{x} = -2$(rejected)

No solutions.

4. A → (p); B → (p, r, s); C → (p, r); D → (p, q, r)

(A) Let $E = \log_2(\log_9 \sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}\,)$

Suppose $y = \sqrt{6+\sqrt{6+\sqrt{6+\ldots\infty}}}$

$\Rightarrow y = \sqrt{6+y}\ (y > 0)$

$\Rightarrow y^2 = 6 + y \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y-3)(y+2) = 0$

But $y = -2$(rejected), so $y = 3$

Hence $E = \log_2(\log_9 y) = \log_2(\log_9 3)$

$$= \log_2(\log_{3^2} 3) = \log_2\left(\frac{1}{2}\right) = -1$$

(B) We have

$N = 2^{(\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \ldots \cdot \log_{98} 99 \cdot \log_{99} 100)}$

$= 2^{\frac{\log 3}{\log 2}\cdot\frac{\log 4}{\log 3}\cdot\frac{\log 5}{\log 4}\cdots\frac{\log 99}{\log 98}\cdot\frac{\log 100}{\log 99}} = 2^{\log_2 100} = 100$

(C) Let $E = \dfrac{1}{\log_5 3} + \dfrac{1}{\log_6 3} - \dfrac{1}{\log_{10} 3}$

$= (\log_3 5 + \log_3 6) - \log_3 10 = \log_3\left(\dfrac{5\times 6}{10}\right)$

$= \log_3 3 = 1$

(D) We have $N = \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+\sqrt{14-6\sqrt{5}}}}$

$= \sqrt{2+\sqrt{5}-\sqrt{6-3\sqrt{5}+(3-\sqrt{5})}}$

(As, $14 - 6\sqrt{5} = (3-\sqrt{5})^2$) $= \sqrt{2+\sqrt{5}-\sqrt{9-4\sqrt{5}}}$

$= \sqrt{(2+\sqrt{5})-(\sqrt{5}-2)}$ (As, $9 - 4\sqrt{5} = (\sqrt{5}-2)^2$)

$= \sqrt{2+2} = \sqrt{4} = 2$

Integer/Numerical Answer Type Questions

1. $$\frac{1}{\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x} + \ldots\ldots.. + \dfrac{1}{\log_{2000} x}}$$

$$= \frac{1}{\log_x 2 + \log_x 3 + \ldots\ldots.. + \log_x 2000} = \frac{1}{\log_x(2, 3\ldots\ldots.2000)}$$

$$= \frac{1}{\log_x\left(\prod_{n=1}^{2000} n\right)} = \frac{1}{\log_x x} = 1$$

Hence, the correct answer is (1).

2. $a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}}$

$y = a^{\sqrt{\log_a b}}$

taking log both side on base b

$\log_b y = \log_b a^{\sqrt{\log_a b}}$

$\Rightarrow \log_b y = \sqrt{\log_a b}\ \log_b a;$

$\Rightarrow \log_b y = \sqrt{\dfrac{\log b}{\log a}}\ \dfrac{\log a}{\log b}$

$= \left(\log_b a = \dfrac{\log a}{\log b}\right)$ base changing property

$\Rightarrow \log_b y = \dfrac{\sqrt{\log a}}{\sqrt{\log b}};\quad \log_b y = \sqrt{\log_b a}$

$y = b^{\sqrt{\log_b a}};$

$\therefore a^{\sqrt{\log_a b}} - b^{\sqrt{\log_b a}} = 0$

Hence, the correct answer is (0).

3. $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$

$\Rightarrow (\log_2 x)^2 = \log_2 x - (\log_2 x)^2 + 1$

Let $\log_2 x = t$

$\Rightarrow 2t^2 - t - 1 = 0$

$\Rightarrow 2t^2 - 2t + t - 1 = 0$

$\Rightarrow 2t(t-1) + 1(t-1) = 0$

$\Rightarrow t = \dfrac{-1}{2}$ and $t = 1$

$\Rightarrow x = 2$

and $x = \dfrac{1}{\sqrt{2}}$(not possible).

Hence, the correct answer is (2).

4. $\frac{3}{4\log_3 x} - \frac{3\log_3 x}{3} = 2\log_3 x$

Let $\log_3 x = t$

$\frac{3}{4t} - t = 2t \Rightarrow \frac{3}{4t} = 3t$

$\Rightarrow t^2 = \frac{1}{4} \Rightarrow t = \frac{1}{2}$ or $\frac{-1}{2}$

$\log_3 x = \frac{1}{2} \Rightarrow x = \sqrt{3}$.

or $\log_3 x = \frac{-1}{2} \Rightarrow x = \frac{1}{\sqrt{3}}$

$\therefore$ Product of the two solutions is 1.

Hence, the correct answer is (1).

5. $\log_3\left(3+\sqrt{x}\right)\left(1+x^2\right) = 1 \Rightarrow \left(3+\sqrt{x}\right)\left(1+x^2\right) = 3$

$\Rightarrow 3x^2 + x^{5/2} + x^{1/2} = 0$ has no solution except $x = 0$, but $x = 0$ makes $\log_{(1+x^2)} 3$ into $\log_1 3$ which is undefined. Hence no solution

Hence, the correct answer is (0).

6. $|x-2|^2 + |x-2| - 2 = 0$

$\Rightarrow \left(|x-2|+2\right)\left(|x-2|-1\right) = 0 \Rightarrow |x-2| = -2, 1$

$\therefore |x-2| = 1$ or $x = 3, 1$

$\Rightarrow$ sum of the roots = 4

Hence, the correct answer is (4).

7. $|x^2 + 4x + 3| + 2x + 5 = 0$

CASE-1: $x^2 + 4x + 3 \geq 0 \Rightarrow x \in (-\infty, -3] \cup [-1, \infty)$

$x^2 + 4x + 3 + 2x + 5 = 0$

$\Rightarrow x^2 + 6x + 8 = 0 \Rightarrow x = -4, -2$ (Rejected)

CASE-2: $x^2 + 4x + 3 < 0 \Rightarrow x \in (-3, -1)$

$-x^2 - 4x - 3 + 2x + 5 = 0$

$\Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 + \sqrt{3}, -1 - \sqrt{3}$

$x = -1 + \sqrt{3}$ is rejected

Solutions are $x = -4, -1 - \sqrt{3}$

Hence, the correct answer is (2).

8. $|x - p| = x - p$ (Since $x \geq p$)

$|x - 15| = 15 - x$ (Since $x \leq 15$)

$|x - (p + 15)| = (p + 15) - x$ (as $15 + p > x$)

$\therefore$ expression reduces to

$E = x - p + 15 - x + p + 15 - x$

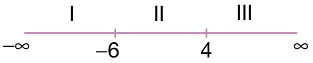
$E = 30 - x$

$\therefore E_{\min}$ occurs when $x = 15$

$\therefore \lambda = 15$

Hence, the correct answer is (3).

9. Two modulus are present hence we have critical points $x = -6$, $x = 4$,
we are solving by forming three cases



CASE-1: $x < -6$

$-(x-4) + 3(-(x+6)) - 2x = 5$

$-x + 4 - 3x - 18 - 2x = 5$

$\Rightarrow -6x = 19$

$x = -\frac{19}{6}$ not acceptable as $x < -6$

CASE-2: $-6 \leq x \leq 4$

$-(x-4) + 3(x+6) - 2x = 5$

$\Rightarrow -x + 4 + 3x + 18 - 2x = 5$

$22 \neq 5$

$x \in \phi$

CASE-3: $x > 4$

$+(x-4) + 3(+(x+6)) - 2x = 5$

$\Rightarrow x - 4 + 3x + 18 - 2x = 5$

$\Rightarrow 2x + 14 = 5$

$x = -\frac{9}{2}$ not acceptable as $x > 4$

Final values of x we have from case (I) ∪ (II) ∪ (III)

$x \in \phi$

Hence, the correct answer is (0).

10. $|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$

CASE-1: $x \geq 2$

$x + 1 - x + 3x - 3 - 2x + 4 = x + 2$, holds for all $x \geq 2$

CASE-2: $1 \leq x < 2$

$x + 1 - x + 3x - 3 + 2x - 4 = x + 2\ 4x = 8$

$x = 2$ (not possible)

CASE-3: $0 \leq x < 1$

$x + 1 - x - 3x + 3 + 2x - 4 = x + 2$

$2x = -2 \Rightarrow x = -1$ (not possible)

CASE-4: $-1 \leq x < 0$

$x + 1 + x - 3x + 3 + 2x - 4 = x + 2$

$x = x + 2$; not possible

CASE-5: $x < -1$

$x - 1 + x - 3x + 3 + 2x - 4 = x + 2$

$2x = -4 \Rightarrow x = -2$ (a solution)

Hence, the correct answer is (1).

11. Always true

$|a - b| \leq |a| + |b|$

Let $a = x - 6$, $b = x + 6$ and $a - b = -12$

$|a - b| \leq |a| + |b|$ Always true

Given inequality is of form $|a| + |b| > |a - b|$ it is not true hence $x \in \phi$

Hence, the correct answer is (0).

12. Let $a = x - 5$ and $b = x + 5$

$|a| + |b| = |a + b|$ is true for $ab \geq 0$

or $(x - 5)(x + 5) \geq 0$

$x \in (-\infty, -5] \cup [5, \infty)$

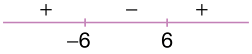
Hence, the correct answer is (5).

13. Let $a = x - 6$ and $b = x + 6$, $a - b = -12$

$||a| - |b|| = |a - b|$ is true then

$ab \geq 0$

$(x - 6)(x + 6) \geq 0$



$x \in (-\infty, -6] \cup [6, \infty)$

Hence, the correct answer is (4).

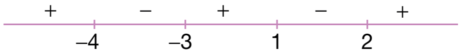
14. Let $a = x^2 - 3x + 2$, $b = x^2 + 7x + 12$

and $a - b == -10(x + 1)$

$|a| + |b| = |a - b|$ is true then $ab \leq 0$

$(x^2 - 3x + 2)(x^2 + 7x + 12) \leq 0$

$(x - 1)(x - 2)(x + 4)(x + 3) \leq 0$



$x \in [-4, -3] \cup [1, 2]$

Hence, the correct answer is (2).

15. $|x^2 - 1| + |x^2 - 4| \leq 3$

CASE-1: $x \geq 2 \quad \Rightarrow \quad x^2 - 1 + x^2 - 4 \leq 3$

$\Rightarrow \quad 2x^2 - 8 \leq 0$

$\Rightarrow \quad x^2 - 4 \leq 0$

$\Rightarrow \quad x \in [-2, 2] \quad x = 2$ …(1)

CASE-2: $1 < x < 2$

$\Rightarrow \quad x^2 - 1 - x^2 + 4 \leq 3$

$\Rightarrow \quad 3 \leq 3$ true $x \in (1,2)$ …(2)

CASE-3: $-1 \leq x \leq 1$

$\Rightarrow \quad -x^2 + 1 - x^2 + 4 \leq 3$

$\Rightarrow \quad 2x^2 - 2 \geq 0$

$\Rightarrow \quad x^2 - 1 \geq 0 \quad \Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$

$x = -1, 1$ …(3)

CASE-4: $-2 \leq x < -1$

$\Rightarrow \quad x^2 - 1 - x^2 + 4 \leq 3$

$3 \leq 3 \quad x \in [-2, -1)$ …(4)

CASE-5: $x < -2$

$x^2 - 1 + x^2 - 4 \leq 3$

$x \in [-2, 2]$

from (1) (2) (3) and (4)

$x \in [-2, -1] \cup [1,2]$

Alternate Solution

$\because \quad |x^2 - 1| + |x^2 - 4| \leq |(x^2 - 1) - (x^2 - 4)|$

$\Rightarrow \quad |a| + |b| \leq |a - b|$

So, $ab \leq 0$

$\Rightarrow \quad (x^2 - 1)(x^2 - 4) \leq 0$

$\Rightarrow \quad x \in [-2, -1] \cup [1, 2]$

Hence, the correct answer is (2).

16. $$-1 \leq \frac{x^2 - 5x + 4}{x^2 - 4} \leq 1$$

Solving left side inequality

$$\frac{x^2 - 5x + 4}{x^2 - 4} + 1 \geq 0 \quad \Rightarrow \quad \frac{2x^2 - 5x}{x^2 - 4} \geq 0$$

$$\Rightarrow \quad \frac{x(2x - 5)}{(x - 2)(x + 2)} \geq 0$$

$x \in (-\infty, -2) \cup [0, 2) \cup [5/2, \infty)$ …(1)

Solving right side inequality

$$\frac{x^2 - 5x + 4}{x^2 - 4} - 1 \leq 0 \Rightarrow \frac{x^2 - 5x + 4 - x^2 + 4}{x^2 - 4} \leq 0$$

$$\Rightarrow \quad \frac{8 - 5x}{x^2 - 4} \leq 0 \quad \Rightarrow \quad \frac{(5x - 8)}{(x - 2)(x + 2)} \geq 0$$

$x \in (-2, 8/5] \cup (2, \infty)$ …(2)

common of (1)and (2) we get $\left[0, \frac{8}{5}\right] \cup \left[\frac{5}{2}, +\infty\right)$

Hence, the correct answer is (0).

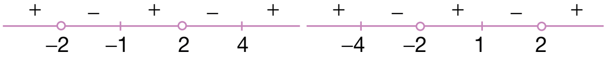
17. $$\left|\frac{3x}{x^2 - 4}\right| \geq 1$$

$$\frac{3x}{x^2 - 4} \geq 1 \quad \text{or} \quad \frac{3x}{x^2 - 4} \leq -1$$

$$\frac{3x}{x^2 - 4} - 1 \geq 0 \quad \text{or} \quad \frac{3x}{x^2 - 4} + 1 \leq 0$$

$$\frac{-x^2 + 3x + 4}{x^2 - 4} \geq 0 \quad \text{or} \quad \frac{x^2 + 3x - 4}{x^2 - 4} \leq 0$$

$$\frac{x^2 - 3x - 4}{x^2 - 4} = \frac{(x - 4)(x + 1)}{(x - 2)(x + 2)} \leq 0 \quad \text{or} \quad \frac{(x + 4)(x - 1)}{(x + 2)(x - 2)} \leq 0$$



$x \in (-2, -1] \cup (2, 4]$ …(1) or $x \in [-4, -2) \cup [1, 2)$ …(2)

Final answer (i) ∪ (ii)

$x \in (-2, -1] \cup (2, 4] \cup [-4, -2) \cup [1, 2)$

$x \in [-4, -2] \cup (-2, -1] \cup [1, 2) \cup (2, 4]$

Hence, the correct answer is (3).

18. $$\frac{(x - 2)^{100}(2 + x)^{101}(x - 1)^{10}}{x^{41}(x + 1)^{39}(4 - x)^{11}} \leq 0$$

Poles $x \neq 0, -1, 4$

Zeroes $x = 2, -2, 1$

$x \in (-\infty, -2] \cup (-1, 0) \cup (4, \infty) \cup \{1, 2\}$

Hence, the correct answer is (3).

19. $x^{(\log_2 7)^2} = 8^{\log_2 7} = 343$

$y^{(\log_3 5)^2} = 9^{\log_3 5} = 25;\ z^{(\log_5 216)^2} = 5^{\frac{1}{3}\log_5 216} = 6$

$\therefore$ Value is 374

Hence, the correct answer is (374).

20. $\log_4(x - 1) = \log_2(x - 3)$

$$\frac{\log_2(x - 1)}{2} = \log_2(x - 3)$$

$(x - 1) = (x - 3)^2$

$x - 1 = x^2 + 9 - 6x$

$x^2 - 7x + 10 = 0$

$(x - 5)(x - 2) = 0$

$\therefore \quad x = 5$ or 2

$x - 1 > 0$ and $x - 3 > 0$

$\therefore \quad x > 1 \quad \therefore \quad x > 3$

$\therefore \quad x = 5$ is the only solution

Hence, the correct answer is (5).

21. $$\frac{1}{\log_2(2x)}+\frac{\log_2(2x)}{2}=\frac{-3}{2}$$

Let $\log_2(2x)=y$

$$\frac{1}{y}+\frac{y}{2}=-\frac{3}{2}$$

⇒ $2+y^2=-3y$

⇒ $y^2+3y+2=0$ …(1)

⇒ $(y+2)\,(y+1)=0$

⇒ $y=-2$ or -1

If $y=-2$ then $\log_2(2x)=-2$ ⇒ $x=1/8$

If $y=-1$ then $\log_2(2x)=-1$ ⇒ $x=1/4$

∴ product $=\frac{1}{32}=p$

⇒ $\log_2\left(\frac{1}{p}\right)=\log_2(32)=5$

Hence, the correct answer is (5).

22. $3^{\log_2 x}-12\cdot x^{\log_{16} 9}+27=0$

or $x^{\log_2 3}-12\cdot x^{\log_4 3}+27=0$

or $x^{\log_2 3}-12\cdot x^{\frac{1}{2}\log_2 3}+27=0$

let $x^{\log_2 3}=y$

$y-12\sqrt{y}+27=0$

∴ $(\sqrt{y}-9)\,(\sqrt{y}-3)=0$

∴ $y=81$ or $y=9$

Hence $x^{\log_2 3}=3^4$ or $3^{\log_2 x}=3^4$

⇒ $x=16$ or $3^{\log_2 x}=3^2$ ⇒ $x=4$

So, $x_1^2+x_2^2=272$

Hence, the correct answer is (272).

23. $$\frac{\log_5 9\cdot\log_7 5\cdot\log_3 7}{\log_2\sqrt{6}}+\frac{1}{\log_9\sqrt{6}}$$

$$=\frac{\log_3 9}{\log_2\sqrt{6}}+\frac{1}{\log_9\sqrt{6}}=\frac{2}{\frac{1}{\log_{\sqrt{6}}2}}+\frac{1}{\frac{1}{\log_{\sqrt{6}}9}}$$

$=2\log_{\sqrt{6}}2+\log_{\sqrt{6}}9$

$=\log_{\sqrt{6}}4+\log_{\sqrt{6}}9=\log_{\sqrt{6}}36=4$

Hence, the correct answer is (4).

24. $$A=\frac{(\log_3 1-\log_3 4)(\log_3 9-\log_3 2)}{(\log_3 1-\log_3 9)(\log_3 8-\log_3 4)}$$

$$A=\frac{-(\log_3 4)\left(\log_3\frac{9}{2}\right)}{(-\log_3 9)(\log_3 2)}=\left(\log_3\frac{9}{2}\right)$$

⇒ $3^A=3^{\log_3\frac{9}{2}}=\frac{9}{2}$

Hence $2(3^A)=2\cdot\frac{9}{2}=9.$

Hence, the correct answer is (9).

25. $\left(x^2+3x+1\right)^{x^2+2x-8}=1$ is true if

(i) the exponent $=0$ and base is not zero

(ii) the base $=1$, or

(iii) the base $=-1$ and the exponent is even

(i) $x^2+2x-8=0$

⇒ $(x+4)\,(x-2)=0$

⇒ $x=-4,\,2$

(ii) $x^2+3x+1=1$

⇒ $x^2+3x=0$

⇒ $x(x+3)=0$

⇒ $x=-3,\,0$

(iii) $x^2+3x+1=-1$

⇒ $x^2+3x+2=0$

⇒ $(x+1)\,(x+2)=0$

⇒ $x=-2,\,-1.$

Now, check the exponent. If $x=-2$

⇒ x^2+2x-8 is even; if $x=-1$

⇒ x^2+2x-8 is odd; therefore case (iii) produces the solution -2 only. The sum of the solutions

$=-4+2+(-3)+0+(-2)=-7$

Hence modulus is 7.

Hence, the correct answer is (7).

26. Clearly $x>0$ and $x\neq\frac{1}{5}$

Now, $\log_{5x}\left(\frac{5}{x}\right)=\left(\frac{\log_5 5-\log_5 x}{\log_5 5+\log_5 x}\right)$

Putting $\log_5 x=t$, we get $t^2+\frac{(1-t)}{1+t}=1$

⇒ $t^3+t^2-2t=0$

⇒ $t(t-1)\,(t+2)=0$

⇒ $t=0,\,1,\,-2$

∴ $x=1,\,5$

So, $x_1=1$ and $x_2=5$

Hence Sum $=6.$

Hence, the correct answer is (6).

ARCHIVE: JEE MAIN

1. Let $3^x=t$; $t>0$

$t(t-1)+2=|t-1|+|t-2|$

⇒ $t^2-t+2=|t-1|+|t-2|$

CASE-1: $t<1$

$t^2-t+2=1-t+2-t$

⇒ $t^2+t-1=0$

$t=\frac{-1\pm\sqrt{5}}{2}$ but $t=\frac{\sqrt{5}-1}{2}$ is only acceptable

CASE-2: $1\le t\le 2$

$t^2-t+2=t-1+2-t$

$\Rightarrow \quad t^2 - t + 1 = 0$

$D < 0$ no real solution

CASE-3: $t \geq 2$

$t^2 - t + 2 = t - 1 + t - 2$

$t^2 - 3t + 5 = 0$

$\Rightarrow \quad D < 0$ no real solution

so we get only one solution

Hence, the correct answer is (D).

2. $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots \text{ to } \infty\right)} = \left(\frac{4}{25}\right)^{\log_{5/2}(1/2)}$

$= \left(\frac{1}{2}\right)^{\log_{\left(\frac{5}{2}\right)}\left(\frac{4}{25}\right)} = \left(\frac{1}{2}\right)^{-2} = 4$

Hence, the correct answer is (4).

3. $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

CASE-1: $x^2 - 5x + 5 = 1$

$\Rightarrow \quad x^2 - 5x + 4 = 0$

$\Rightarrow \quad x = 1, 4$

CASE-2: $x^2 - 5x + 5 = -1$

and $x^2 + 4x - 60$ is even

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$\Rightarrow \quad (x - 2)(x - 3) = 0$

$\Rightarrow \quad x = 2, 3$ but for $x = 2$, $x^2 + 4x - 60$ is even

CASE-3: base is non zero and power is zero

or $\quad x^2 + 4x - 60 = 0$

$\Rightarrow \quad x = 6, -10$

Sum of all solutions $= 1 + 4 + 2 + 6 - 10 = 3$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Problems

1. $(2x)^{\ell n\, 2} = (3y)^{\ell n\, 3}$ or $\ell n2(\ell n\, 2 + \ell nx) = \ell n3(\ell n3 + \ell ny)$ …(1)

$3^{\ell n\, x} = 2^{\ell n\, y}$ or $\ell nx\ \ell n\, 3 = \ell ny\ \ell n2$

Replace value of ln*y* in first equation

$$\ell n^2 2 + \ell n2\ \ell nx = \ell n^2 3 + \ell n3\left(\frac{\ell nx \cdot \ell n3}{\ell n2}\right)$$

$$\Rightarrow \quad \left(\frac{\ell n^2 3 - \ell n^2 2}{\ell n2}\right)\ell nx = \ell n^2 2 - \ell n^2 3$$

$$\Rightarrow \quad \ell nx = -\ \ell n2 = \ell n\,\frac{1}{2}$$

$x = \frac{1}{2}$ solution (x_0, y_0) we have $x_0 = \frac{1}{2}$

Hence, the correct answer is (C).

2. **CASE-1:** $x \geq -2$

$x^2 - x - 2 + x > 0 \quad \Rightarrow \quad x^2 - 2 > 0$

$\Rightarrow \quad (x - \sqrt{2})(x + \sqrt{2}) > 0$

$x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ but $x \geq -2$

$\therefore \quad x \in [-2, -\sqrt{2}) \cup [\sqrt{2}, \infty)$

CASE-2: $x < -2$

$x^2 + (x + 2) + x > 0$

or $\quad x^2 + 2x + 2 > 0$

or $\quad (x + 1)^2 + 1 > 0 \quad \therefore \quad x < -2$

Final answer $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Hence, the correct answer is (B).

3. $\log_{2^2}(x - 1) = \log_2(x - 3) \quad \Rightarrow \quad \log_2\sqrt{x - 1} = \log_2(x - 3)$

$\Rightarrow \quad \sqrt{x - 1} = x - 3 \qquad (\because x > 3)$

$\Rightarrow \quad x - 1 = x^2 + 9 - 6x$

$x^2 - 7x + 10 = 0 \quad \Rightarrow \quad x = 2, 5\ (\because x > 3)$

$x = 5$ is solution

Hence, the correct answer is (B).

4. Let $\log_2 7$ is rational number

$\log_2 7 = p\,/\,q$ where p and q are not integers and q is non zero

we have $7 = 2^{\,p\,/\,q}$

we have $7^q = 2^p$ which is not possible for any integral value of p and q

Hence assumption is wrong, $\log_2 7$ is an irrational number

Hence, the correct answer is (C).

Multiple Correct Choice Type Problems

1. $3^x = 4^{x-1}$

take log base 3 both sides

$x = (x - 1)\log_3 4$

$\Rightarrow \quad x = 2x\log_3 2 - 2\log_3 2$

$$\Rightarrow \quad x\left(2\log_3 2 - 1\right) = 2\log_3 2$$

$$\Rightarrow \quad x = \frac{2\log_3 2}{2\log_3 2 - 1} = \frac{2}{2 - \log_2 3}$$

$$= \frac{1}{1 - \frac{1}{2}\log_2 3} = \frac{1}{1 - \log_4 3}$$

Hence, (A), (B) and (C) are correct.

2. $x^{\frac{3}{4}(\log_x)^2 + \log x - \left(\frac{5}{4}\right)} = \sqrt{2}$

Take log with base 2.

$$\left\{\frac{3}{4}(\log x)^2 + \log x - \left(\frac{5}{4}\right)\right\}\log x = \frac{1}{2}$$

$$3(\log x)^3 + 4(\log x)^2 - 5\log x - 2 = 0$$

$\Rightarrow \quad (\log x - 1)(3\log x + 1)(\log x + 2) = 0$

$\Rightarrow \quad x = 2, 2^{-1/3}, \frac{1}{4}$

Hence, (A), (B), (C) and (D) are correct.

Matrix Match/Column Match Type Questions

1. A → (p, r, s); B → (q, s); C → (q, s); D → (p, r, s)

$$f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6} = \frac{(x-5)(x-1)}{(x-3)(x-2)} \qquad \ldots.(1)$$

$$\text{also } f(x) = \frac{x^2 - 5x + 6 - (x-1)}{x^2 - 5x + 6}$$

$$f(x) = 1 - \frac{(x+1)}{(x-3)(x-2)} \qquad \ldots(2)$$

(A) If $-1 < x < 1$

From (2)

$f(x) = 1 -$ (some positive quantity)

$f(x) < 1$ from (1) $f(x) > 0 \quad \Rightarrow \quad 0 < f(x) < 1$

(B) If $1 < x < 2$,

From (1) $f(x) < 0$ and $f(x) < 1$

(C) If $3 < x < 5$

From (1) $f(x) < 0$ and $f(x) < 1$

(D) If $x > 5$,

from (1) $f(x) > 0$

from (2) $f(x) < 1$

So, $0 < f(x) < 1$

Integer/Numerical Answer Type Question

1. $$\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$$

$$= (\log_2 9)^{2\times\log_{\log_2 9} 2} \times 7^{\frac{1}{2}\times\log_7 4}$$

$$= (\log_2 9)^{\log_{\log_2 9} 4} \times 7^{\log_7 2} = 4 \times 2 = 8$$

Hence, the correct answer is (8).

2. Let $$y = \left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}4 - \frac{1}{3\sqrt{2}}\ldots}}}\right)$$

$$= \frac{1}{3\sqrt{2}}\sqrt{4-y}$$

$$18y^2 = 4 - y$$

or $18y^2 + y - 4 = 0$

or $18y^2 + 9y - 8y - 4 = 0$

or $9y(2y+1) - 4(2y+1) = 0$

$$y = \frac{4}{9}, -\frac{1}{2} \qquad (\because y > 0)$$

required term $= 6 + \log_{3/2}\left(\frac{4}{9}\right) = 6 - 2 = 4$

Hence, the correct answer is (4).

CHAPTER 2: THEORY OF EQUATION

Test Your Concepts-I

1. By remainder theorem, we know that $p(x)$ when divided by $g(x) = \left(x - \frac{1}{2}\right)$ gives a remainder equal to $p\left(\frac{1}{2}\right)$.

Now, $p(x) = 4x^3 - 12x^2 + 14x - 3$

$$\Rightarrow \quad p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$$

$$\Rightarrow \quad p\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{12}{4} + \frac{14}{2} - 3$$

$$\Rightarrow \quad p\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 7 - 3$$

$$\Rightarrow \quad p\left(\frac{1}{2}\right) = \frac{3}{2}$$

Hence, required remainder $= p\left(\frac{1}{2}\right) = \frac{3}{2}$

2. Let $p(x) = 3x^2 + kx + 6$ be the given polynomial. Then, $(x + 3)$ is a factor of $p(x)$

$\Rightarrow \quad p(-3) = 0$
$\Rightarrow \quad 3(-3)^2 + k \times (-3) + 6 = 0$
$\Rightarrow \quad 27 - 3k + 6 = 0$
$\Rightarrow \quad 33 - 3k = 0 \quad \Rightarrow \quad k = 11$

Hence, $x + 3$ is a factor of $3x^2 + kx + 6$ if $k = 11$

3. Let $f(x) = px^2 + 5x + r$ be the given polynomial. Since $x - 2$ and $x - \frac{1}{2}$ are factors of $f(x)$. Therefore, $f(2) = 0$ and $f\left(\frac{1}{2}\right) = 0$

$$\left[\because x - 2 = 0 \Rightarrow x = 2 \text{ and } x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{2}\right]$$

$$\Rightarrow \quad p \times 2^2 + 5 \times 2 + r = 0 \text{ and } p\left(\frac{1}{2}\right)^2 + 5 \times \frac{1}{2} + r = 0$$

$$\Rightarrow \quad 4p + 10 + r = 0 \text{ and } \frac{p}{4} + \frac{5}{2} + r = 0$$

$$\Rightarrow \quad 4p + r = -10 \text{ and } \frac{p + 4r + 10}{4} = 0$$

$\Rightarrow \quad 4p + r = -10$ and $p + 4r + 10 = 0$
$\Rightarrow \quad 4p + r = -10$ and $p + 4r = -10$
$\Rightarrow \quad 4p + r = p + 4r$
[$\because$ RHS of the two equations are equal]
$\Rightarrow \quad 3p = 3r \quad \Rightarrow \quad p = r$

4. Let $p(x) = ax^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + a$ be the given polynomials. The remainders when $p(x)$ and $q(x)$ are divided by $(x - 3)$ are $p(3)$ and $q(3)$ respectively.

By the given condition, we have

$p(3) = q(3)$
$\Rightarrow \quad a \times 3^3 + 4 \times 3^2 + 3 \times 3 - 4 = 3^3 - 4 \times 3 + a$
$\Rightarrow \quad 27a + 36 + 9 - 4 = 27 - 12 + a$
$\Rightarrow \quad 26a + 26 = 0$
$\Rightarrow \quad 26a = -26$
$\Rightarrow \quad a = -1$

5. $(x - 1)g(x) = x^6 - 1$

$$\text{Now } g\left(x^{12}\right) = \left(\left(x^6\right)^{10} - 1\right) + \left(\left(x^6\right)^8 - 1\right) + \cdots + \left(\left(x^6\right)^2 - 1\right) + 6$$

$$\therefore g\left(x^{12}\right) = \left(x^6 - 1\right)P(x) + 6 = (x - 1)g(x)P(x) + 6.$$

Required remainder is 6

6. As α, β be roots of $x^2 + px + 1 = 0$ we have $\alpha + \beta = -p, \alpha\beta = 1$ and $\alpha^2 + p\alpha + 1 = 0$

As γ, δ are the roots of $x^2 + qx + 1 = 0$ we have $\gamma + \delta = -q, \gamma\delta = 1$ and $\gamma^2 + q\gamma + 1 = 0$

Now $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = \{\alpha\beta - \gamma(\alpha + \beta) + \gamma^2\}\{\alpha\beta + \delta(\alpha + \beta) + \delta^2\}$

$= \{1 + \gamma p + \gamma^2\}\{1 - p\delta + \delta^2\} = [(\gamma^2 + 1) + \gamma p][(\delta^2 + 1) - p\delta]$
$= (-q\gamma + \gamma p)(-q\delta - p\delta) = \gamma\delta(q^2 - p^2) = (q^2 - p^2)$

7. $\alpha^2 - 12\alpha + 4 = 0 \quad \Rightarrow (\alpha - 2)^2 = 8\alpha$

$$\frac{(\alpha - 2)^4 + (\beta - 2)^4}{17\alpha\beta} = \frac{64(\alpha^2 + \beta^2)}{17\alpha\beta} = 64\frac{((\alpha + \beta)^2 - 2\alpha\beta)}{17\alpha\beta}$$

$$= \frac{64}{17} \times \frac{136}{4} = 2^7$$

8. $x^2 + ax + b = 0$

$a + b = -a \quad \Rightarrow \quad 2a + b = 0$

and $ab = b \Rightarrow b = 0$ (rejected)

$\therefore \quad a = 1; b = -2 \quad \Rightarrow \quad (1, -2)$

9. $\alpha + \beta = -a$ and $\alpha\beta = b$

$\therefore \quad 3\alpha - 5 - \alpha^2 = \beta^2 - 3\beta$
$\Rightarrow \quad \beta^2 + \alpha^2 = 3(\alpha + \beta) - 5$
$\Rightarrow \quad (\alpha + \beta)^2 = 3(\alpha + \beta) + \alpha\beta - 5$
$a^2 = -3a + 2b - 5 \quad \Rightarrow \quad |a^2 + 3a - 2b| = 5$

10. $\alpha(\alpha\beta)^5 = 32 \quad \Rightarrow \alpha = 1$

The value of α will satisfy the given equation

$1 + a + 2 = 0 \quad \Rightarrow \quad a = -3$

11. From equations $\frac{\alpha}{2}$ and $\frac{\beta}{3}$ are roots of $x^2 - 5x + 3 = 0$

$\therefore \quad 3\alpha + 2\beta = 30$ and $\alpha\beta = 18$

Now $\frac{2\beta + 3\alpha}{\alpha\beta} = 5/3$

12. $x^2 + ax + b = 0$ has roots α, β

Replace x by $\frac{1}{x}$ then equation $bx^2 + ax + 1 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}$

similarly replacing x by $\frac{1}{x}$ in the equation $x^2 - ax + b - 2 = 0$, we get equation $(b - 2)x^2 - ax + 1 = 0$ has roots $\frac{1}{\gamma}, \frac{1}{\delta}$

Now, $\alpha\beta = b$, $\gamma\delta = b - 2$ and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{-a}{b}$, $\frac{1}{\gamma} + \frac{1}{\delta} = \frac{a}{b-2}$

Using given conditions we have

$\therefore \quad b(b-2) = 24$ and $\frac{-a}{b} + \frac{a}{b-2} = \frac{5}{12}$

$$\Rightarrow a\left[\frac{1}{b-2} - \frac{1}{b}\right] = \frac{5}{12} \quad \Rightarrow a\left[\frac{b-b+2}{24}\right] = \frac{5}{12} \quad \Rightarrow a = 5$$

13. $Px^2 + (1-P)x + 5 = 0$

$$\alpha + \beta = \frac{P-1}{P} \text{ and } \alpha\beta = \frac{5}{P}$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$$

$p^2 - 16p + 1 = 0$

$P_1 + P_2 = 16$

$P_1P_2 = 1$

$$\frac{P_1^4 + P_2^4}{(P_1P_2)^3} = \left(P_1^2 + P_2^2\right)^2 - 2(P_1P_2)^2$$

$((P_1 + P_2)^2 - 2P_1P_2)^2 - 2 = (256 - 2)^2 - 2 = 64514$

Test Your Concepts-II

1. Let α be the root of the given cubic where α can take values a, b, c

Hence $\alpha^3 + \alpha^2 - 333\alpha - 1002 = 0$

or $\quad \alpha^3 = 1002 + 333\alpha - \alpha^2$

$$\therefore \Sigma\alpha^3 = \Sigma 1002 + 333\Sigma\alpha - \Sigma\alpha^2$$

$$= 3006 + 333\Sigma\alpha - \left[(\Sigma\alpha)^2 - 2\Sigma\alpha_1\alpha_2\right]$$

But $\Sigma\alpha = -1; \Sigma\alpha_1\alpha_2 = -333$

$$\therefore a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666]$$

$$= 3006 - 333 - 667 = 3006 - 1000 = 2006 = P$$

$$\frac{P}{2006} = 1$$

2. Let $\alpha, \beta, \gamma, \delta$ be four roots

$\alpha\beta = -32$

$\alpha\beta\gamma\delta = -1984 \Rightarrow \gamma\delta = 62$

$x^4 - 18x^3 + kx^2 + 200x - 1984 = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$\equiv (x^2 - (\alpha + \beta)x - 32)(x^2 - (\gamma + \delta)x + 62)$

$\alpha + \beta = p \equiv (x^2 - px - 32)(x^2 - qx + 62)$

$\gamma + \delta = q$

equating co-eff. of x^3, x^2, x

$p + q = 18 \qquad \ldots(1)$

$-62p + 32q = 200 \qquad \ldots(2)$

$k = 62 + pq - 32 \qquad \ldots(3)$

from (1) and (2) $p = 4, q = 14$

from (3) $k = 86$.

3. $B^2 - 4AC = 0$

$(-2(1 + 3m))^2 - 4(1 + m)(1 + 8m) = 0$

$\Rightarrow \quad 4[(1 + 9m^2 + 6m) - (1 + 8m + m + 8m^2)] = 0$

$\Rightarrow \quad [m^2 - 3m] = 0 \Rightarrow m(m - 3) = 0$

$m = 0, 3$

4. If a quadratic equation can be written as perfect square of a linear term it means both roots are equal

$D = 0 \Rightarrow (-4)^2 - 4(2)K = 0$

$\Rightarrow \quad 16 - 8K = 0 \Rightarrow K = 2$

5. Let $f(x) = ax^2 + bx + c$

Since equation $ax^2 + bx + c = 0$ i.e. equation $f(x) = 0$ has no real root, therefore $f(x)$ will have same sign for all real values of x.

$\Rightarrow \quad f(0)$ and $f(1)$ will have same sign.

$\Rightarrow \quad f(1).f(0) > 0$

$\Rightarrow \quad (a + b + c)\, c > 0$

6. From given equation, we have product of roots $= \alpha\beta\gamma = -r$

New root $\alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha + \frac{\alpha}{r} = \alpha\left(1 + \frac{1}{r}\right)$

Thus the roots of the required equation are $\alpha\left(1 + \frac{1}{r}\right), \beta\left(1 + \frac{1}{r}\right)$ and $\gamma\left(1 + \frac{1}{r}\right)$. Hence the required equation is obtained from the given equation by replacing x by $x / \left(1 + \frac{1}{r}\right) = \frac{rx}{r+1}$.

Hence required equation is

$$\left(\frac{rx}{r+1}\right)^3 + p\left(\frac{rx}{r+1}\right)^2 + q\left(\frac{rx}{r+1}\right) + r = 0$$

$\Rightarrow \quad r^2x^3 + pr\,(r + 1)\,x^2 + q\,(r + 1)^2\,x + (r + 1)^3 = 0.$

7. $x^4 - 3x^3 + x^2 - 2x + 1 = 0$ has roots α, β, γ and δ

$\therefore \quad \Sigma\alpha = 3, \Sigma\alpha\beta = 1, \Sigma\alpha\beta\gamma = 2, \alpha\beta\gamma\delta = 1$

$$\Rightarrow \quad x^3 - 3x^2 + x - 2 + \frac{1}{x} = 0$$

$$\Rightarrow \quad x^3 = 3x^2 - x + 2 - \frac{1}{x}$$

$$\Rightarrow \quad \Sigma\alpha^3 = 3\Sigma\alpha^2 - \Sigma\alpha + \Sigma 2 - \Sigma\frac{1}{\alpha}$$

$$\Rightarrow \quad \Sigma\alpha^3 = 3((\Sigma\alpha)^2 - 2\Sigma\alpha\beta) - \Sigma\alpha + 8 - \frac{\Sigma\alpha\beta\gamma}{\alpha\beta\gamma\delta} = 3(9 - 2) - 3 + 8$$

$-2 = 24$

Test Your Concepts-III

1. (i) $y = -x^2 - 4x - 4$

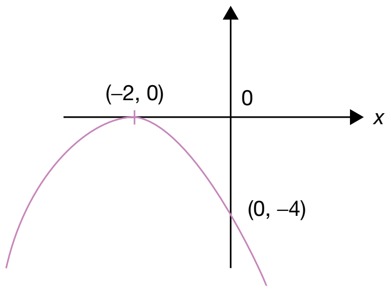
$A = -1 < 0$ downward parabola.

$B = -4, C = -4$

$D = B^2 - 4AC = 0$ real and equal roots

Vertex $\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = (-2, 0)$

It cuts y-axis at $(0, -4)$



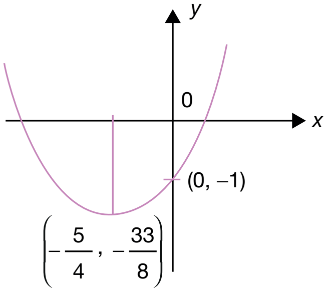
(ii) $y = 2x^2 + 5x - 1$

$A = 2 > 0$ upward parabola.

$D = B^2 - 4AC = 25 - 4(2)(-1) = 33 > 0$ real and distinct roots

Vertex $\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = \left(-\frac{5}{4}, -\frac{33}{8}\right)$ It lies in III^{rd} quadrant

It cuts y-axis at $(0, -1)$



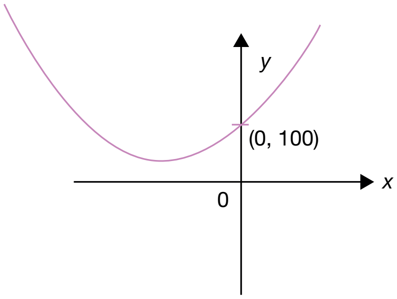
(iii) $y = x^2 + 7x + 100$

$A = 1 > 0$ upward parabola.

$D = B^2 - 4AC = 49 - 4 \times 100 = -351 < 0$ imaginary roots

Vertex $\left(\frac{-B}{2A}, \frac{-D}{4A}\right) = \left(-\frac{7}{2}, \frac{351}{4}\right)$ It lies in II^{nd} quadrant

It cuts y-axis at $(0, 100)$



2. $x^3 + x^2 - 4x - 4 \equiv (x + 1)(x - 2)(x + 2)$
$\equiv (x^2 - x - 2)(x + 2)$
$\Rightarrow \quad p = -1$

3. $x^2 + 3x + 4 = 0$ has imaginary roots so both roots are common

$\because \quad \frac{a}{1} = \frac{30}{3} = \frac{b}{4} \Rightarrow 4a - b = 0$

4. $D < 0 \Rightarrow (2m + 6)^2 - 4(4m + 12) < 0$
$4m^2 + 36 + 24m - 16m - 48 < 0$
$m^2 + 2m - 3 < 0$
$-3 < m < 1 \Rightarrow m = 0$

5. $D \le 0 \Rightarrow k^2 - 6k + 8 < 0 \Rightarrow 2 \le k \le 4$

$\Rightarrow k = 2, 3, 4 \Rightarrow 3$ values

6. $k - 2 < 0$ and $D \le 0$
$64 - 4(k - 2)(k + 4) \le 0$
$\Rightarrow \quad (k + 6)(k - 4) \ge 0$
$k \in (-\infty, -6]$

7. $D > 0; (3 - k)^2 - 4k > 0$
$\Rightarrow \quad 9 + k^2 - 6k - 4k > 0$
$\Rightarrow \quad k^2 - 10k + 9 > 0$
$\Rightarrow \quad (k - 9)(k - 1) > 0$
$k \in (-\infty, 1) \cup (9, \infty)$

8. Put $D > 0$, $(k + 2)^2 - k^2 - 12 > 0 \quad \Rightarrow \quad k > 2 \quad \Rightarrow \quad k = 3.$

9. (i) $x_0 = -\frac{b}{2a} = \frac{4}{4} = 1 \in [0, 4]$

$f(0) = 3$
$f(4) = 32 - 16 + 3 = 19$
$f(1) = 2 - 1 + 3 = 1$

Required range $y \in (1, 19)$

(ii) $x \in (2, 3]$

$x_0 = -\frac{b}{2a} = \frac{-(-4)}{2(2)} = 1 \notin (2, 3]$

$f(2) = 2(4) - 4(2) + 3 = 3$
$f(3) = 2(9) - 12 + 3 = 9$

Required range $y \in (3, 9]$

10. $y = \frac{x^2 - 2x + 2}{2x - 2}$

$\therefore \quad x^2 - 2x + 2 = 2xy - 2y \Rightarrow x^2 - 2x(y + 1) + 2(y + 1) = 0$

As $x \in R$, so $D \ge 0 \Rightarrow (y + 1)^2 - 2(y + 1) \ge 0$
$\Rightarrow \quad (y + 1)(y - 1) \ge 0$
$\Rightarrow \quad y \ge 1$ or $y \le -1$

11. $f(x) = \frac{x^2 - 5x + 6}{x^2 - 3x + 2} = \frac{(x - 2)(x - 3)}{(x - 1)(x - 2)}, x \ne 2, 1$

$f(x) = \frac{x - 3}{x - 1} = y \quad$ (let)

$x - 3 = yx - y$

$x = \frac{3 - y}{1 - y} = \frac{y - 3}{y - 1}$

Range is $R - \{1, -1\}$
$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

12. $y = \frac{2x^2 + 4x + 6 - 9}{x^2 + 2x + 3} = 2 - \frac{9}{(x + 1)^2 + 2}$

$y_{\max} = 2 - \frac{9}{\infty} \to 2$ and $y_{\min} = 2 - \frac{9}{2} = \frac{-5}{2}$

$\Rightarrow \quad y \in \left[-\frac{5}{2}, 2\right)$

13. $x \ne -4$, solve $x^2 + 6x - 7 < 0 \quad \Rightarrow \quad (x + 7)(x - 1) < 0$
$\Rightarrow \quad x \in (-7, -4) \cup (-4, 1); x \in \{-6, -5, -3, -2, -1, 0\}$

14. $y^2 - 5y < \frac{2}{(x + 1)^2 + 3}.$

Here, $(y^2 - 5y)$ should be less than minimum value of $\frac{2}{(x + 1)^2 + 3}$ $\forall\ k \in R$. But minimum value of $\frac{2}{(x + 1)^2 + 3}$ is just greater than 0, so $y^2 - 5y \le 0$
$\Rightarrow \quad y \in [0, 5]$

15. $2x^2-(a+3)x+2>0\ \forall\ x\in R$

$|a+3|<4 \Rightarrow -7<a<1$

16. $x^2+2ax+10-3a>0\ \forall\ x\in R$

$\Rightarrow\quad (x+a)^2-(a^2-10+3a)>0\ \forall\ x\in R$

$\Rightarrow\quad a^2+3a-10<0$

$\Rightarrow\quad (a+5)(a-2)<0$

$\Rightarrow\quad -5<a<2$

17. Let given equations $x^2+px+qr=0$ has roots α, β, $x^2+qx+rp=0$ has roots β,γ and $x^2+rx+pq=0$ has roots γ, α.

Now we have $\alpha\beta = qr$, $\beta\gamma = rp$ and $\gamma\alpha = pq$

$\Rightarrow\quad (\alpha\beta)(\beta\gamma)(\gamma\alpha)=(qr)(rp)pq)$

$\Rightarrow\quad (\alpha\beta\gamma)^2=(pqr)^2$

Test Your Concepts-IV

1. $D\geq 0 \Rightarrow 4a^2-4(a^2+a-3)\geq 0$

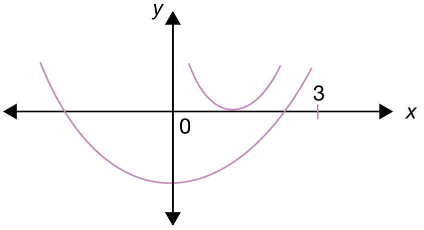
$\Rightarrow\quad 12-4a\geq 0 \Rightarrow a\leq 3 \quad \ldots(1)$

$f(3)>0 \Rightarrow 9-6a+a^2+a-3>0 \Rightarrow a^2-5a+6>0$

$\Rightarrow\quad (a-3)(a-2)>0 \Rightarrow a<2$ or $a>3 \quad \ldots(2)$

$\dfrac{-b}{2a}<3 \Rightarrow \dfrac{2a}{2}<3 \Rightarrow a<3. \quad \ldots(3)$

Hence, $a<2$



2. $\because$ Roots of quadratic $x^2-3ax+a^2=9$ has roots of opposite sign

$\therefore\quad a^2-9<0 \quad\Rightarrow\quad -3<a<3$

$\therefore$ Sum of all possible values of $a=0$

3. | $f(-1)<0$ | $f(2)<0$ |
|---|---|
| $(-1+k-1)(-1-2k-7)<0$ | $(2+k-1)(2-2k-7)<0$ |
| $\Rightarrow (k-2)(k+4)>0$ | $\Rightarrow (k+1)(2k+5)>0$ |
| $\Rightarrow k\in(-\infty,-4)\cup(2,\infty)$ | $\Rightarrow k\in\left(-\infty,-\dfrac{5}{2}\right)\cup(-1,\infty)$ |

$k\in(-\infty,-4)\cup(2,\infty)$

4. Let $f(x)=ax^2+bx+c$ and $f(1)=a+b+c>0$ and roots are imaginary.

It implies that graph (parabola) of $f(x)$ will be completely above x-axis.

So $f(-1)>0, f(2)>0$ and $f\left(\dfrac{-1}{2}\right)>0$

$\Rightarrow\quad a-b+c>0,\ 4a+2b+c>0,\ a-2b+4c>0$

5. Since leading co-efficient is +ve so x^2-mx+1 is negative in (1, 2)

If roots are real and (1, 2) lie between their roots.

$\Rightarrow\quad D\geq 0$ and $f(1)<0$ and $f(2)<0$

$m^2-4\geq 0 \qquad |m|\geq 2$

$1-m+2<0 \Rightarrow m>3$

$4-2m+1<0 \Rightarrow m>\dfrac{5}{2}$

Thus we have $m>\dfrac{5}{2}$

6. Given $|x|^2-4|x|+3-|k-1|=0 \quad \ldots(1)$

As above equation (1) have four distinct real roots so both roots of equation (1) must be positive and distinct.

Now,

(i) $D>0 \Rightarrow 16-12+4|k-1|>0$

$\Rightarrow\quad 1+|k-1|>0$, which is true $\forall\ k\in R. \quad \ldots(1)$

(ii) Sum of roots $>0 \Rightarrow 4>0$, which is true $\forall\ y\in R. \quad \ldots(2)$

(iii) Product of roots $>0 \Rightarrow 3-|k-1|>0. \quad \ldots(3)$

$\Rightarrow\quad |k-1|<3 \Rightarrow -2<k<4.$

must be satisfied simultaneously.

$\therefore\quad 1\cap 2\cap 3 \Rightarrow k\in(-2, 4)$

Clearly, possible integral values of k are $-1, 0, 1, 2, 3$.

7. Let $f(x)=x^2-6ax+2-2a+9a^2$, equation $f(x)=0$ will have its both roots greater then 3,

If (i) Discriminant $D\geq 0$

$D=36a^2-8+8a-36a^2\geq 0$

$\Rightarrow\quad -8+8a\geq 0 \Rightarrow a\geq 1 \quad \ldots(1)$

(ii) x coordinate of vertex $=-b/2a>3$

or sum of the roots $\alpha+\beta>6$

$\alpha+\beta>6 \Rightarrow 6a>6 \Rightarrow a>1 \quad \ldots(2)$

(iii) $f(3)>0 \Rightarrow 9-18a+2-2a+9a^2>0$

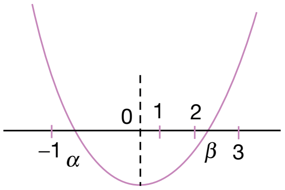
$\Rightarrow\quad 9a^2-20a+11>0 \Rightarrow 9a^2-11a-9a+11>0$

$\Rightarrow\quad (9a-11)(a-1)>0 \Rightarrow a<1$ or $a>\dfrac{11}{9} \quad \ldots(3)$

From (1), (2) and (3), we get common value as $a>11/9$.

8. $\because\quad f(0)<f(2)$

$\Rightarrow\quad f(1)<f(2)$ and $f(3)>f(-1)$



Single Correct Choice Type Questions

1. Vertex of $y=2x^2-4x+6$

$m=x=\dfrac{-b}{2a}=1$ and $n=y=\dfrac{-D}{4a}=4$

$\therefore$ Now check the alternatives for parabola having vertex $\left(2m,\dfrac{n}{2}\right)$ i.e., (2, 2)

Hence, the correct answer is (C).

2. The given equation may be written as

$x^2-px-(p+c)=0$

$\therefore\quad \alpha+\beta=p$ and $\alpha\beta+\alpha+\beta+1=-p-c+p+1$

$=1-c$

Hence, the correct answer is (A).

3. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(p-1)^2+1$

Hence, the correct answer is (B).

4. $x^3-2x^2-1=0$ or $a^3-1=2a^2$

$$\therefore \quad \frac{T_{11}-T_8}{T_{10}}=\frac{\alpha^{11}+\beta^{11}+\gamma^{11}-[\alpha^8+\beta^8+\gamma^8]}{\alpha^{10}+\beta^{10}+\gamma^{10}}$$

$$=\frac{\alpha^8(\alpha^3-1)+\beta^8(\beta^3-1)+\gamma^8(\gamma^3-1)}{\alpha^{10}+\beta^{10}+\gamma^{10}}$$

$$=\frac{2(\alpha^{10}+\beta^{10}+\gamma^{10})}{\alpha^{10}+\beta^{10}+\gamma^{10}}=2$$

Hence, the correct answer is (B).

5. $P(x)=4x^2+(k+8)x+9,\ x\in R$

$D<0 \quad \Rightarrow \quad (k+8)^2-144<0$

i.e., $(k+20)\ (k-4)<0$

$-20<k<4 \quad \Rightarrow \quad |k|_{\text{largest}}=19$

Hence, the correct answer is (B).

6. $3x^2+ax+1=0 \ \Rightarrow 6x^2+2ax+2=0$

$2x^2+bx+1=0 \quad \Rightarrow \quad 6x^2+3bx+3=0$

Take difference $(2a-3b)x-1=0$

$x=\dfrac{1}{2a-3b}$ sub in any equation

$2+b(2a-3b)+(2a-3b)^2=0$

$\Rightarrow \quad 2+2ab-3b^2+4a^2+9b^2-12ab=0$

$\Rightarrow \quad 4a^2+6b^2-10ab+2=0$

$\Rightarrow \quad \left|2a^2-5ab+3b^2\right|=|-1|=1$

Hence, the correct answer is (B).

7. $\alpha\beta=2,\ \beta\gamma=3,\ \gamma\alpha=6$

$(\alpha\beta\gamma)^2=36 \Rightarrow \alpha\beta\gamma=6$ we get $\alpha=2,\ \beta=1,\ \gamma=3$

$\alpha+\beta=2p,\ \beta+\gamma=2q,\ \gamma+\alpha=2r$

Adding all three equation

$\alpha+\beta+\gamma=p+q+r=6$

Hence, the correct answer is (D).

8. $(c_1a_2-c_2a_1)^2=(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)$

$(-1-b)^2=(1-b)(b^2+1)$

$\Rightarrow \quad b^3+3b=0 \Rightarrow b=0,\pm i\sqrt{3}$

Hence, the correct answer is (B).

9. α,β are roots of the equation $(x-a)(x-b)=c, c\neq 0$

$\therefore \quad (x-a)(x-b).-c=(x-\alpha)(x-\beta)$

$\Rightarrow \quad (x-\alpha)(x-\beta).+c=(x-a)(x-b)$

$\Rightarrow$ roots of $(x-\alpha)(x-\beta).+c=0$ are a and b.

Hence, the correct answer is (A).

10. $(a-1)x^2-(a+1)x+(a-1)\geq 0$

$a(x^2-x+1)-(x^2+x+1)\geq 0$

$$a\geq\frac{x^2+x+1}{x^2-x+1}=1+\frac{2x}{x^2-x+1}=1+\frac{2}{x+\frac{1}{x}-1}\Rightarrow a\in\left(1,\frac{7}{3}\right]$$

Hence, the correct answer is (C).

11. $(a-2)(x-[x])^2+2(x-[x])+a^2=0$ ...(1)

$f(y)=(a-2)y^2+2y+a^2=0$ ...(2)

since x cannot be an integer

when $2<x<3\Rightarrow[x]=2$

and $0<y<1$

$\therefore$ Equation (1) has exactly one solution in (2, 3)

$\therefore$ Equation (2) has exactly one solution in (0,1)

$\therefore \quad f(0).f(1)<0$

$a^2(a-2+2+a^2)<0$

$a^3(a+1)<0$

$a\in(-1,0)$

Hence, the correct answer is (D).

12. CASE-1: Suppose $x\geq 2$. Then the equation becomes

$(x-2)^2+(x-2)-2=0$

$x^2-3x=0$

Since $x\geq 2$, we get that $x=3$.

CASE-2: Suppose $x<2$. Then the equation becomes

$(2-x)^2+(2-x)-2=0$

$x^2-5x+4=0$

$(x-1)(x-4)=0$

But $x<2$. Therefore $x=1$. Thus the real roots of the equation are 1 and 3 and their sum is 4.

Hence, the correct answer is (D).

13. By hypothesis, $a\alpha^2+b\alpha+c=0$ and $a\beta^2-b\beta-c=0$

Let $f(x)=\dfrac{a}{2}x^2+bx+c$

Then $f(\alpha)=\dfrac{a}{2}\alpha^2+b\alpha+c=\dfrac{1}{2}(a\alpha^2+2b\alpha+2c)$

$=\dfrac{1}{2}(a\alpha^2-2a\alpha^2)=-\dfrac{a}{2}\alpha^2$

and $f(\beta)=\dfrac{a}{2}\beta^2+b\beta+c=\dfrac{1}{2}(a\beta^2+2b\beta+2c)$

$=\dfrac{1}{2}(a\beta^2+2a\beta^2)=\dfrac{3}{2}a\beta^2$

Therefore, $f(\alpha)f(\beta)=\dfrac{-3a^2\alpha^2\beta^2}{4}<0$

Hence $f(x)=0$ has a root in between α and β

Hence, the correct answer is (C).

14. Since a and b are the roots of $x^2-10cx-11d=0$ we have

(i) $a+b=10c$ and (ii) $ab=-11d$ ...(1)

Also since c and d are the roots of $x^2-10ax-11b=0$, we have

(i) $c+d=10a$ and (ii) $cd=-11b$ ...(2)

Adding part (i) of Eqs. (1) and (2) we get

$$a+b+c+d=10(a+c) \Rightarrow b+d=9(a+c) \qquad \ldots(3)$$

Multiplying (ii) of Eqs. (1) and (2) we get $abcd = 121\,bd \Rightarrow ac = 121$ $\qquad\ldots(4)$

Also, $a^2-10ca-11d=0=c^2-10ca-11b$

$\Rightarrow \quad a^2+c^2-20ca-11(b+d)=0$

From Eqs. (3) and (4) we have

$$a^2+c^2-20(121)-99(a+c)=0$$

$$(a+c)^2-2\times121-20\times121-99(a+c)=0$$

$$(a+c-121)(a+c+22)=0$$

$$a+c=121$$

$$a+c=-22$$

or

Since a, c are positive, $a+c\neq-22$. Therefore $a+c=121$ and

$$a+b+c+d=(a+c)+9\,(a+c)=1210$$

Hence, the correct answer is (D).

15. We have $16x^2+8(a+5)x-(7a+5)>0,\ \forall\ x\in R$

$\therefore$ Discriminant must be negative

Now, Discriminant $= D = 64\,((a+5)^2+7a+5)$

As $D<0$

$\Rightarrow \quad a^2+17a+30<0 \qquad \Rightarrow (a+15)\,(a+2)<0$

$\therefore \quad -15<a<-2$

Hence number of integral value(s) = 12

Hence, the correct answer is (A).

16. $\alpha+\beta=\dfrac{-b}{a}$

$\therefore \quad a\alpha+b=-a\beta$ and $a\beta+b=-a\alpha$

$$\therefore \quad S=\frac{-1}{a}\left(\frac{\alpha+\beta}{\alpha\beta}\right)=\frac{-1}{a}\left(\frac{-b/a}{c/a}\right)=\frac{b}{ac}$$

Hence, the correct answer is (A).

17. $x^4=3x-1$

$\therefore \quad x^3=3-\dfrac{1}{x}$ has roots α, β, γ and δ

$$\alpha^3=3-\frac{1}{\alpha};\ \beta^3=3-\frac{1}{\beta};\ \gamma^3=3-\frac{1}{\gamma};\ \delta^3=3-\frac{1}{\delta}$$

on adding $\sum\alpha^3 = 12-\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}+\dfrac{1}{\delta}\right)$

$$=12-\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}=12-3=9$$

Hence, the correct answer is (B).

18. $\tan^2\dfrac{3\pi}{8}=(\sqrt{2}+1)^2=3+2\sqrt{2}$

$\therefore\ x=3+2\sqrt{2}$ satisfy the equation

$$2x^2-3ax+4b=0 \qquad \ldots(1)$$

$$(x-3)^2=8 \quad \Rightarrow x^2-6x+1=0 \qquad \ldots(2)$$

Comparing (1) and (2), we get

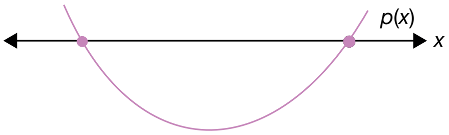
$\therefore \quad a=4$ and $b=\dfrac{1}{2}$

$\Rightarrow \quad a+4b=6$

Hence, the correct answer is (C).

19. Let $p(x)=x^2-(2-p)x+(p-2)$

We must have $D>0$



$\Rightarrow \quad (p-2)\,(p-6)>0$

$\therefore \quad p\in(-\infty,2)\cup(6,\infty)$

Hence, the correct answer is (A).

20. $111x^3-11x-1=0$

$\therefore \quad \alpha+\beta+\gamma=0$

$$\Sigma\alpha\beta=\frac{-11}{111}$$

$$\Rightarrow \quad \frac{1}{(\alpha\beta)^2}+\frac{1}{(\beta\gamma)^2}+\frac{1}{(\gamma\alpha)^2}$$

$$\alpha\beta\gamma=\frac{1}{111}$$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha\beta\gamma)^2}=\frac{(\alpha+\beta+\gamma)^2-2\sum\alpha\beta}{(\alpha\beta\gamma)^2}$$

$$=2442$$

Hence, the correct answer is (D).

21. $a(x-\ell)\,(x-m)+n=0$

If roots are α and β

then $a(x-\ell)\,(x-m)+n=a(x-\alpha)\,(x-\beta)$

$\Rightarrow \quad a(x-\ell)\,(x-m)=a(x-\alpha)\,(x-\beta)-n$

or we can say roots of $a(x-\alpha)\,(x-\beta)-n=0$ are ℓ, m.

Now by transformation for

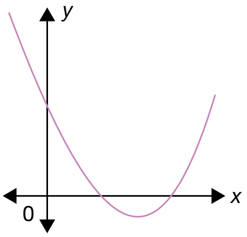
$$a\left(\frac{x}{x-k}-\alpha\right)\left(\frac{x}{x-k}-\beta\right)-n=0, \text{ we have}$$

$$\frac{x}{x-k}=\ell, m \quad \Rightarrow x=\frac{k\ell}{\ell-1}, \frac{km}{m-1}$$

Hence, the correct answer is (C).

22. $a>0;\ f(0)=c>0;\ \dfrac{-b}{a}>0$

$\therefore \quad -b>0 \Rightarrow b<0$



Hence, $abc<0$ and $D=b^2-4ac>0$.

Hence, the correct answer is (D).

23. $x^3+2x^2-4x+5\equiv(x-\alpha)\,(x-\beta)\,(x-\gamma)$

Put $x=2$, we get

$$13=(2-\alpha)\,(2-\beta)\,(2-\gamma) \qquad \ldots(1)$$

Also, $\dfrac{(\alpha^3+5)(\beta^3+5)(\gamma^3+5)}{13\alpha\beta\gamma}$

$$= \frac{2\alpha\cdot(2-\alpha)\cdot 2\beta\cdot(2-\beta)\cdot 2\gamma\cdot(2-\gamma)}{13\alpha\beta\gamma} = \frac{8}{13}(13) = 8$$

Hence, the correct answer is (B).

24. With the condition, $c^2 = 4d$, the equation $x^2 - cx + d = 0$ has equal roots. Each roots $= \frac{c}{2}$.

As $\frac{c}{2}$ is a root of $x^2 - ax + b = 0$

$\Rightarrow \quad d - \frac{ac}{2} + b = 0$

$\therefore \quad b + d = \frac{ac}{2} \Rightarrow 2(b + d) = ac.$

Hence, the correct answer is (B).

25. As the equation

$x^2 + 2x + 2 + e^{-2\alpha} - \cos\beta = 0$ has real roots, so discriminant ≥ 0

$\Rightarrow \quad 4 - 4(2 + e^{-2\alpha} - \cos\beta) \geq 0$

$\Rightarrow \quad 1 - 2 - e^{-2\alpha} + \cos\beta \geq 0$

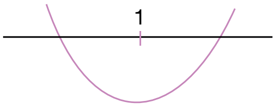
$\Rightarrow \quad \cos\beta \geq 1 + e^{-2\alpha},$

Which is not possible. So no real value of α, β is possible.

Hence, the correct answer is (D).

26. $\because\ a \neq 0$

$$f(x) = x^2 + x + \frac{a+1}{a} = 0$$



$\Rightarrow \quad f(1) < 0;\ 2 + \frac{a+1}{a} < 0$

$\Rightarrow \quad \frac{a+1/3}{a} < 0 \quad \Rightarrow \quad a \in \left(-\frac{1}{3}, 0\right)$

Hence, the correct answer is (C).

27. $x^2 + ax + b = 0$ has roots c and d.

$x^2 + cx + d = 0$ has roots a and b.

$c^2 + ac + b = 0$ ...(1)

$a^2 + ac + d = 0$ ...(2)

(1) – (2) $c^2 - a^2 = d - \text{b}$

Also $c + d = -\text{a}$

$a + b = -\text{c}$ ...(3)

$cd = b$ ...(4)

$ab = d$ ...(5)

(3) – (4) ...(6)

$c - a + d - b = -a + c$

$\Rightarrow d = b$

from (5) and (6)

$a = c = 1$

$b = -2 = d$

Hence, the correct answer is (D).

28. From diagram $\alpha - \beta = 3,\ \alpha > 0,\ \beta < 0,\ |\beta| > \alpha$

$\Rightarrow \quad b^2 - 4c = 9,\ c < 0$ and $b > 0$

$\Rightarrow \quad c = \frac{b^2 - 9}{4} < 0 \quad \Rightarrow \quad b \in (0, 3)$

$\therefore \quad b \in I \quad \Rightarrow \quad b = 1, 2$

but for $b = 2,\ c \notin I$

$\Rightarrow \quad b = 1 \quad \Rightarrow \quad c = -2$

$\Rightarrow$ Only one quadratic equation $x^2 + x - 2 = 0$

Hence, the correct answer is (A).

29. $\frac{\alpha^3\gamma + \beta^3\alpha + \gamma^3\beta}{\alpha\beta\gamma} = -8$

$\Rightarrow \quad \frac{(-a\alpha - a)\gamma + (-a\beta - a)\alpha + (-a\gamma - a)\beta}{\alpha\beta\gamma} = -8$

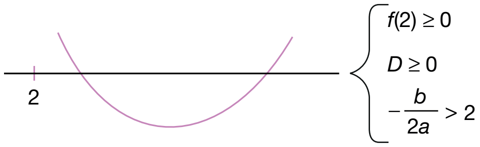
$\Rightarrow \quad \frac{-a\sum\alpha\beta - a\sum\alpha}{\alpha\beta\gamma} = -8 \quad \Rightarrow \quad \frac{-a.a}{-a} = -8$

$\Rightarrow \quad a = -8$

Hence, the correct answer is (A).

30. $\alpha + \frac{1}{\alpha} \geq 2,\ \beta + \frac{1}{\beta} \geq 2$

$\Rightarrow$ both the roots are real and greater than 2.

$\Rightarrow$ 

$$\Rightarrow \begin{cases} 4 - 10 - a \geq 0 \\ 25 + 4a \geq 0 \\ \frac{5}{2} > 2 \end{cases} \Rightarrow \quad a \in \left[\frac{-25}{4}, -6\right]$$

Hence, the correct answer is (C).

31. $\alpha^2 - \alpha + 11 = 0 \quad \Rightarrow \quad 3\alpha^3 - 3\alpha^2 + 33\alpha = 0$ ...(1)

$\beta^2 - \beta + 11 = 0 \quad \Rightarrow \quad 2\beta^3 - 2\beta^2 + 22\beta = 0$ ...(2)

(1) + (2)

$\Rightarrow \quad 3\alpha^3 - 3\alpha^2 + 2\beta^3 - 2\beta^2 + 33\alpha + 22\beta = 0$

$\Rightarrow \quad 3\alpha^3 - 3\alpha^2 + 2\beta^3 - 2\beta^2 + 11\alpha$

$= -22\,(\alpha + \beta) = -22$

Hence, the correct answer is (D).

32. Let $y = \frac{x}{x-1} \Rightarrow x = \frac{y}{y-1}$

$\therefore \quad \frac{y^2}{(y-1)^2} - \frac{2y}{y-1} - 5 = 0$

$\Rightarrow \quad y^2 - 2y(y-1) - 5(y-1)^2 = 0$

$\Rightarrow \quad 6y^2 - 12y + 5 = 0$

Hence, the correct answer is (A).

33. Given $f(f(1)) = 0$ and $f(f(2)) = 0$

i.e., equation $f(x) = 0$ has two roots $f(1)$ and $f(2)$

$\therefore \quad f(1) + f(2) = -\alpha$ and $f(1).f(2) = \beta$

$\Rightarrow \quad 5 + 3\alpha + 2\beta = -\alpha$ and $(1 + \alpha + \beta)\,(4 + 2\alpha + \beta) = \beta$

$\Rightarrow \quad 2\alpha + \beta = -\frac{5}{2}$ and $\left(1 - \frac{5}{4} - \frac{\beta}{2} + \beta\right)\cdot\left(4 - \frac{5}{2}\right) = \beta$

$\Rightarrow \quad \beta = -\frac{3}{2}$

$\therefore \quad f(0) = \beta = -\dfrac{3}{2}$

Hence, the correct answer is (A).

34. If $a > 0$, then $a + \dfrac{1}{a} > 0$

Now $D = 4 - 4\left(a + \dfrac{1}{a}\right) = 4\left(1 - \left(a + \dfrac{1}{a}\right)\right)$

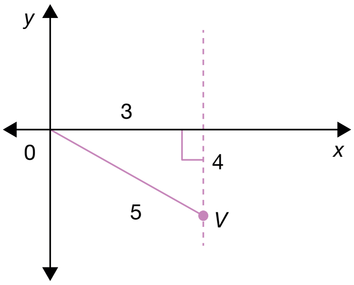
$a + \dfrac{1}{a} \geq 2$, when $a > 0$

then $D = -$ve

$\therefore$ graph never cuts x-axis.

Hence, the correct answer is (A).

35. Co-ordinates of vertex V are $(3, -4)$



$\Rightarrow \quad \dfrac{-b}{2} = 3$ and $\dfrac{-D}{4} = -4$

$\Rightarrow \quad b = -6,\ b^2 - 4c = 16$

$\Rightarrow \quad c = 5 \quad \Rightarrow \quad f(x) = x^2 - 6x + 5$

Hence, the correct answer is (D).

36. $\dfrac{-D}{4a} = 1 \quad \Rightarrow \quad D = -4a$

Given $D = b^2 - 4ac = -4 \quad \Rightarrow \quad a = 1$

Also $\dfrac{-b}{2a} = -4 \quad \Rightarrow \quad b = 8$

$\therefore \quad b^2 - 4ac = -4 \quad \Rightarrow \quad 64 - 4c = -4$ or $c = 17$

Hence, $a + b + c = 26$

Hence, the correct answer is (D).

37. $4x^3 + 8x^2 - x - 2 = 4(x - \alpha)(x - \beta)(x - \gamma)$

Put $x = -1$, we get $3 = 4(-1 - \alpha)(-1 - \beta)(-1 - \gamma)$

or $(\alpha + 1)(\beta + 1)(\gamma + 1) = \dfrac{-3}{4}$

Also $\alpha\beta\gamma = \dfrac{1}{2}$

$\therefore \quad \dfrac{4(\alpha+1)(\beta+1)(\gamma+1)}{\alpha\beta\gamma} = -6$

Hence, the correct answer is (D).

38. $\ell x^2 + mx + n = 0; \qquad px^2 + qx + r = 0$

Roots x_1, x_2; Roots x_3, x_4

As $|x_1 - x_2| = |x_3 - x_4|$

$\Rightarrow \quad (x_1 + x_2)^2 - 4x_1x_2 = (x_3 + x_4)^2 - 4x_3x_4$

$\Rightarrow \quad \dfrac{m^2}{\ell^2} - \dfrac{4n}{\ell} = \dfrac{q^2}{p^2} - \dfrac{4r}{p}$

$\Rightarrow \quad \dfrac{m^2 - 4rl}{q^2 - 4rp} = \left(\dfrac{\ell}{p}\right)^2$

$\Rightarrow \quad \dfrac{D_1}{D_2} = \dfrac{\ell^2}{p^2}$

Hence, the correct answer is (B).

39. The given equation can be written as $(x^2 + x + 1)(x^2 - ax + 1) = 0 \Rightarrow a^2 - 4 < 0 \Rightarrow a \in (-2, 2)$

Hence, the correct answer is (C).

40. $S_{n+1} = \alpha^{n+1} + \beta^{n+1} = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1}) = -\dfrac{b}{a} \cdot S_n - \dfrac{c}{a} \cdot S_{n-1}$

Hence, the correct answer is (A).

41. Let cost of clock $= x$

number of students $= n$

then $\dfrac{x}{n-2} = \dfrac{x}{n} + 1 \quad \Rightarrow \quad x = \dfrac{n^2 - 2n}{2}$

$\Rightarrow \quad 170 \leq \dfrac{n^2 - 2n}{2} \leq 195$

$341 \leq (n - 1)^2 \leq 391$

so $n = 20$ and $x = 180$

Hence, the correct answer is (C).

42. Let $ab + bc + ca = x$

$\Rightarrow \quad 2b^2 + 2(c - 2)b - 4c + c^2 + x = 0$

Since $b \in R$, $\therefore\ c^2 - 4c + 2x - 4 \geq 0$

Since $c \in R$, $\therefore\ x \leq 4$

Hence, the correct answer is (D).

43. If 'α' is the common root then $2\alpha^2 + k\alpha - 5 = 0$, $\alpha^2 - 3\alpha - 4 = 0$ solve the equations.

From $2\alpha^2 - 3\alpha - 4 = 0$, $\alpha = 4, -1$

Hence, the correct answer is (B).

44. $x = 1$, satisfy the equation

Product of the roots $= 1$

$\Rightarrow \quad r(P - Q) = P(Q - r) \Rightarrow \dfrac{2}{Q} = \dfrac{1}{P} + \dfrac{1}{r}$

Hence, the correct answer is (A).

45. $x^2 - 8x + (4y^2 + 12) = 0$ is a quadratic in 'x', 'x' is real then discriminate ≥ 0

$64 - 4(4y^2 + 12) \geq 0 \quad \Rightarrow \quad y^2 - 1 \leq 0$

Hence, the correct answer is (C).

46. $px^2 + qx + 1 = 0$ has real roots if $q^2 - 4p \geq 0$ or $q^2 \geq 4p$

Since $p, q \in \{1, 2, 3, 4\}$

The required points are(1,2), (1,3),(1, 4), (2,3), (2,4), (3,4), (4,4)

So the required number is 7

Hence, the correct answer is (A).

47. $\because \quad f(x + 1) - f(x) = 8x + 3$

$\Rightarrow \quad \{b(x + 1)^2 + c(x + 1) + d\} - \{bx^2 + cx + d\} = 8x + 3$

$\Rightarrow \quad b\{(x+1)^2 - x^2\} + c = 8x + 3$

$\Rightarrow \quad b(2x+1) + c = 8x + 3$ on comparing

$2b = 8$ and $b + c = 3$

Then, $b = 4$ and $c = -1$

Hence, the correct answer is (B).

48. $(x+\lambda)^2 + 1 = 0$ has clearly imaginary roots

So, both roots of the equations are common

$$\therefore \quad \frac{a}{1} = \frac{b}{2\lambda} = \frac{c}{\lambda^2+1} = k \text{ (say)}$$

Then $a = k$, $b = 2\lambda k$, $c = (\lambda^2+1)\,k$

As a, b, c are sides of triangle

$$a + b > c \Rightarrow 2\lambda + 1 > \lambda^2 + 1 \Rightarrow \lambda^2 - 2\lambda < 0$$

$\Rightarrow \quad \lambda \in (0, 2)$

The other conditions also imply same relation.

Hence, the correct answer is (A).

49. If x is real, $x^2 - 6|x| + 8 = 0 \Rightarrow |x|^2 - 6|x| + 8 = 0$

$\Rightarrow \quad |x| = 2, 4 \Rightarrow x = \pm 2, \pm 4$

If x is non – real, say $x = \alpha + i\beta$, then

$$(\alpha + i\beta)^2 - 6\sqrt{\alpha^2+\beta^2} + 8 = 0 \qquad \left(|\alpha + i\beta| = \sqrt{\alpha^2+\beta^2}\right)$$

$$\left(\alpha^2 - \beta^2 + 8 - 6\sqrt{\alpha^2+\beta^2}\right) + 2i\alpha\beta = 0$$

Comparing real and imaginary parts,

$\alpha\beta = 0 \quad \Rightarrow \quad \alpha = 0 \quad$ (if $\beta = 0$ then x is real.)

and $-\beta^2 + 8 - 6\sqrt{\beta^2} = 0$

$$\beta^2 \pm 6\beta - 8 = 0 \quad \Rightarrow \quad \beta = \frac{\mp 6 \pm \sqrt{68}}{2}$$

ie., $x = \alpha + i\beta = \pm(3 - \sqrt{17})i$

Hence $\pm(3 - \sqrt{17})i$ are non-real roots.

Hence, the correct answer is (A).

50. Put $2^x = y$. Equation becomes

$y^2 - 3(8y) + 128 = 0 \Rightarrow y^2 - 24y + 128 = 0$

$\Rightarrow \quad (y-8)(y-16) = 0 \Rightarrow y = 16, 8$

$\Rightarrow \quad 2^x = 16, 8 \Rightarrow x = 4, 3$

$\therefore \quad$ Sum of the roots is 7.

Hence, the correct answer is (C).

51. Note that we must have $3x^2 + x + 5 \geq 0$ and $x - 3 \geq 0$ or $x \geq 3$.

$$\sqrt{3x^2 + x + 5} = x - 3 \qquad \ldots(1)$$

Squaring both sides of (1), we get

$3x^2 + x + 5 = x^2 - 6x + 9$

$\Rightarrow \quad 2x^2 + 7x - 4 = 0 \quad \Rightarrow (2x-1)(x+4) = 0$

$\Rightarrow \quad x = 1/2, -4$

None of these satisfy the inequality $x \geq 3$. Thus, (1) has no solution.

Hence, the correct answer is (A).

52. $f(x) = (x+b)^2 + 2c^2 - b^2 \Rightarrow \min f(x) = 2c^2 - b^2$

Also $g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x+c)^2$

$\Rightarrow \quad \max g(x) = b^2 + c^2$

As $\min f(x) > \max g(x)$ we get $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}\,|b|$

Hence, the correct answer is (D).

53. Given equation can be written as $(x^2 - 3x + 3) - f = 0$ where $f = x - [x]$ and $0 \leq f < 1$

$\therefore \quad 0 \leq x^2 - 3x + 3 < 1$

Solving $x^2 - 3x + 3 = 0$; roots are Imaginary

$\therefore \quad x^2 - 3x + 3 \geq 0 \;\forall\; x \in R$

Solving $x^2 - 3x + 3 < 1 \quad \Rightarrow \quad 1 < x < 2$

if $1 < x < 2$; $[x] = 1$.

putting $[x] = 1$ in the given equation and solving we get $x = 2$. But $1 < x < 2$

$\therefore$ the given equation has no solution.

Hence, the correct answer is (A).

54. $6x^2 + 2ax + 2 = 0$ and $6x^2 + 3bx + 3 = 0$

Subtracting $x(2a - 3b) - 1 = 0$

$$\Rightarrow \quad x = \frac{1}{2a - 3b} \quad \text{(put in any equation)}$$

$$\therefore \quad 2\frac{1}{(2a-3b)^2} + \frac{b}{2a-3b} + 1 = 0$$

$2 + b(2a - 3b) + (2a - 3b)^2 = 0$

$4a^2 + 9b^2 - 12ab + 2ab - 3b^2 + 2 = 0$

$-10ab + 6b^2 + 4a^2 + 2 = 0$

$\Rightarrow \quad 5ab - 3b^2 - 2a^2 = 1$

Hence, the correct answer is (B).

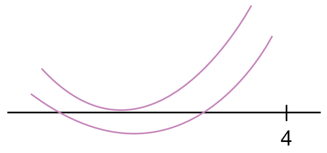
55. $f(x) = x^2 - 2kx + k^2 + 2k - 5$

(i) $f(4) > 0$

$16 - 8k + k^2 + 2k - 5 > 0$

$k^2 - 6k + 11 > 0$

$k \in R$



(ii) $D \geq 0; \quad 4k^2 - 4(k^2 + 2k - 5) \geq 0$

$$\Rightarrow \quad -2k + 5 \geq 0 \qquad \therefore \quad k \leq \frac{5}{2}$$

$$\text{(iii)} \quad \frac{-b}{2a} < 4; \quad \frac{2k}{2} < 4$$

$\therefore \quad k < 4$

Hence, from (i) $\cap$ (ii) $\cap$ (iii); $k \in \left(-\infty, \frac{5}{2}\right]$

Hence, the correct answer is (C).

56. $f(0) < 0$ not possible.

Hence, the correct answer is (A).

57. Roots of $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ are reciprocals of each other.

Hence, $\alpha < \frac{1}{2}$ and $\beta > -1$.

Hence, the correct answer is (A).

58. Given equation is $x^2 + 9y^2 - 4x + 3 = 0 \qquad \ldots(1)$

Or, $\quad x^2 - 4x + 9y^2 + 3 = 0$

Since x is real $\quad \therefore (-4)^2 - 4(9y^2 + 3) \geq 0$

Or, $\quad 16 - 4(9y^2 + 3) \geq 0 \quad$ or, $\quad 4 - 9y^2 - 3 \geq 0$

Or, $9y^2 - 1 \le 0$ or, $9y^2 \le 1$ or, $y^2 \le \dfrac{1}{9}$

Now $y^2 \le \dfrac{1}{9} \Leftrightarrow -\dfrac{1}{3} \le y \le \dfrac{1}{3}$ …(2)

Equation (1) can also be written as

$$9y^2 + 0y + x^2 - 4x + 3 = 0 \quad \ldots(3)$$

Since y is real $\therefore 0^2 - 4.9(x^2 - 4x + 3) \ge 0$

Or, $x^2 - 4x + 3 \le 0$

$\Rightarrow x \in [1, 3]$

Hence, the correct answer is (B).

59. $D = 0 \Rightarrow 16 \sin^2\theta = 4 \tan\theta$

$\Rightarrow \quad 4 \sin^2\theta = \tan\theta \quad \Rightarrow \quad 2 \sin 2\theta = 1$

$\text{Sin } 2\theta = \dfrac{1}{2} \quad \Rightarrow \quad 2\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

$\Rightarrow \quad \theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$

Hence, the correct answer is (C).

60. Clearly $\dfrac{-b}{2a} < 0$ and $\dfrac{c}{a} < 0$

Hence, the correct answer is (A).

61. $xf(x) - 1 = 0$ has roots 1, 2, 3 ____ 9

$xf(x) - 1 = A(x - 1)(x - 2)$ _____ $(x - 9)$

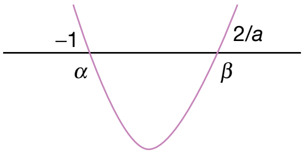
Put $x = 0 \quad \Rightarrow \quad A = \dfrac{1}{9!}$

Put $x = 10 \quad \Rightarrow \quad 10 f(10) - 1 = 1 \quad \Rightarrow \quad f(10) = \dfrac{1}{5}$

Hence, the correct answer is (D).

62. Let $f(x) = ax^2 + (a - 2)x - 2$

$f(x)$ is negative for two integral values of x, so graph should be vertically upward parabola i.e., $a > 0$



Let two roots of $f(x) = 0$ are α and β then $\alpha, \beta = \dfrac{-(a-2) \pm (a+2)}{2a}$

$\Rightarrow \quad \alpha = -1, \beta = \dfrac{2}{a} \Rightarrow 1 < \beta \le 2 \Rightarrow 1 < \dfrac{2}{a} \le 2 \Rightarrow a \in [1, 2)$

Hence, the correct answer is (B).

63. $f(x) = (x - a_1)(x - a_2)(x - a_3) + (a_1 - x) + (a_2 - x) + (a_3 - x)$

Now $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ are $x \to \infty$.

Again $f(a_1) = (a_2 - a_1) + (a_3 - a_1) > 0 \quad [\because a_1 < a_2 < a_3]$

$\Rightarrow$ One root belongs to $(-\infty, a_1)$

Also, $f(a_3) = (a_1 - a_3) + (a_2 - a_3) < 0$

$\Rightarrow$ One root belongs to (a_1, a_3) So $f(x) = 0$ has three distinct real roots.

Hence, the correct answer is (C).

64. Hence $x^2 + 10x - 36 = a(x - 3)^2 + b(x - 3)x + cx$

put $x = 0; \quad -36 = 9a \quad \Rightarrow \quad a = -4$

$x^2 + 10x - 36 = x^2(-4 + b) + x(24 - 3b + c) + (-36)$

comparing coefficients

also, $-4 + b = 1 \quad \Rightarrow \quad b = 5$

$24 - 15 + c = 10 \Rightarrow 9 + c = 10 \quad \Rightarrow \quad c = 1$

$a = -4; b = 5; c = 1$ i.e. $a + b + c = 2$

Hence, the correct answer is (A).

65. Let roots are α and 3α, then $4\alpha = 4a \Rightarrow \alpha = a$ and

$a + a^2 - 4a^2 + f(a) = 0 \quad \Rightarrow \quad f(a) = 3a^2 - a$

$f'(a) = 6a - 1, f''(a) = 6$, then minimum value of

$$f\left(\frac{1}{6}\right) = 3 \times \frac{1}{36} - \frac{1}{6} = \frac{1}{12} - \frac{1}{6} = -\frac{1}{12}$$

Hence, the correct answer is (D).

66. Let $x^2 + 19x + 89 = \lambda^2$

$\Rightarrow \quad x^2 + 19x + (89 - \lambda^2) = 0$ should have integral roots

$\therefore \quad D$ should be a perfect square.

$\Rightarrow \quad (19)^2 - 4(89 - \lambda^2) = \text{Perfect square } (m^2)$

$\Rightarrow \quad (m^2 - 4\lambda^2) = 5 \Rightarrow (m - 2\lambda)(m + 2\lambda) = 5$

$\therefore \quad (m - 2\lambda = 5, m + 2\lambda = 1)$

or $\quad (m - 2\lambda = -5, m + 2\lambda = -1)$

$\Rightarrow \quad m = 3, -3, \lambda = 1, -1$

For $\lambda = \pm 1$ equation becomes $x^2 + 19x + 88 = 0$

$\Rightarrow \quad (x + 11)(x + 8) = 0$

$\Rightarrow \quad x = -8, -11.$

Thus, required values of x are –8, –11.

Hence, the correct answer is (C).

67. Let $f(x) = ax^2 + bx + c$. Since $f(x)$ has no real zeroes, either $f(x) > 0$ or $f(x) < 0$ for all $x \in R$. since $f(0) = c < 0$, we get $f(x) < 0$ for all $x \in R$. Therefore, $a < 0$ as the parabola $y = f(x)$ must open downward. Obviously $f(1), f(-1)$ and $f(2) < 0$.

Hence, the correct answer is (A).

68. We have $\alpha + \beta = -\dfrac{b}{a}, \alpha\beta = -\dfrac{c}{a}$

Let γ, δ be the roots of $a^3x^2 + abcx + c^3 = 0$.

$$\text{Then } \gamma, \delta = \frac{-abc \pm \sqrt{(abc)^2 - 4a^3c^3}}{2a^3} = \frac{ac\left\{-b \pm \sqrt{b^2 - 4ac}\right\}}{2a^3} =$$

$$\frac{c}{2a}\left\{-\frac{b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4\frac{c}{a}}\right\}$$

$$= \frac{1}{2}(\alpha\beta)\left\{(\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\right\}$$

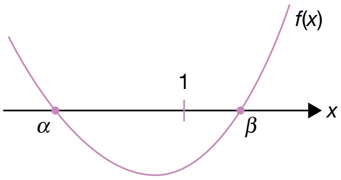
$$= \frac{1}{2}(\alpha\beta)\{(\alpha + \beta) \pm (\alpha - \beta)\} = \alpha^2\beta, \alpha\beta^2$$

Thus, roots of $a^3x^2 + abcx + c^3 = 0$ are $\alpha^2\beta$ and $\alpha\beta^2$

Hence, the correct answer is (D).

69. Let $f(x) = x^2 - 3x + \lambda$

Clearly $f(1) < 0$



$\Rightarrow \quad 1 - 3 + \lambda < 0 \quad \Rightarrow \lambda \in (-\infty, 2)$

Hence, the correct answer is (D).

70. For $5x^2 + 6x + 12 = 0$, $D < 0$ so imaginary roots
So both equation have both roots are common
$5x^2 + 6x + 12 = 0 \Rightarrow a = 5, b = 6, c = 12, a + b < c \Rightarrow$ triangle not exist.
Hence, the correct answer is (D).

71. $\cos(a + bx) = -1 - \dfrac{(x-1)^2}{4}$ exists only when $x = 1$
at $x = 1$; $a + b = \rightarrow \pi$
Hence, the correct answer is (A).

72. Let the roots are α and 2α

$$\Rightarrow \quad \alpha + 2\alpha = \frac{1-3a}{a^2-5a+3} \text{ and } \alpha.2\alpha = \frac{2}{a^2-5a+3}$$

$$\Rightarrow \quad 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right] = \frac{2}{a^2-5a+3}$$

$$\Rightarrow \quad 9a^2 - 6a + 1 = 9a^2 - 45a + 27$$

$$\Rightarrow \quad 39a = 26 \Rightarrow a = \frac{2}{3}$$

Hence, the correct answer is (A).

73. The polynomial $x^3 - 3x^2 - 9x + c$ can be written in the form of $(x - \alpha)^2 (x - \beta)$ if the equation
$x^3 - 3x^2 - 9x + c = 0$ has two equal roots. Let these be α, α, β.
We have $\alpha + \alpha + \beta = 3$ or $2\alpha + \beta = 3$ …(1)
$\alpha\alpha + \alpha\beta + \alpha\beta = -9$ or $2\alpha\beta + \alpha^2 = -9$ …(2)
Putting value of β in (2) we get
$2\alpha(3 - 2\alpha) + \alpha^2 = -9$
or $6\alpha - 3\alpha^2 = -9$
$\Rightarrow \alpha^2 - 2\alpha - 3 = 0$
$\Rightarrow (\alpha - 3)(\alpha + 1) \quad \Rightarrow \alpha = -1, 3$
We also have $\alpha^2\beta = -c$
When $\alpha = -1, \beta = 5, c = -5$ when $\alpha = 3, \beta = -3, c = 27$
Hence, the correct answer is (D).

74. First equation has irrational roots.
$\therefore$ both roots common as options have rational values of a, b, c

$$\frac{a}{2} = \frac{b}{-7} = \frac{2}{1}$$

Hence, the correct answer is (A).

75. Let $f(x) = px^2 + qx + r$
$\Rightarrow \quad f(0)\,f(1) > 0 \quad \Rightarrow \quad r(p + q + r) > 0$
Hence, the correct answer is (C).

76. $\alpha = \dfrac{x^2+9}{|x|} - 2 = |x| + \dfrac{9}{|x|} - 2$ (Using A.M. ≥ G.M.)
$\alpha \geq 4$.
Hence, the correct answer is (B).

77. $a\alpha^2 + 2b\alpha + c = 0$, $2b\alpha^2 + c\alpha + a = 0$; $c\alpha^2 + a\alpha + 2b = 0$
then $(a + 2b + c)(1 + \alpha + \alpha^2) = 0$
$\because a, b, c \in R^+$ then $\alpha + \alpha^2 = -1$
Hence, the correct answer is (A).

78. $x^2 - 7x + a$ is divisible by $x - 4$ and $x^2 + 5x + b$ is divisible by $x + 1$
$\therefore \quad a = 12; b = 4$
Hence, the correct answer is (C).

79. $f(x) = ax^2 + bx + c$: If $f(x) = x$ has no real solution then $f(f(x)) = x$ also has no real solution:
Hence, the correct answer is (D).

80. $D = 9a^2 - 16a(a + 1) \geq 0 \Rightarrow a(7a + 16) \leq 0$ …(1)
$x_1 > 1, x_2 > 1$
Where $x_1 + x_2 = \dfrac{3a}{a+1}$ and $x_1 + x_2 - 2 > 0 \Rightarrow \dfrac{a-2}{a+1} > 0$ …(2)
and $(x_1 - 1)(x_2 - 1) > 0 \Rightarrow x_1x_2 - (x_1 + x_2) + 1 > 0$
$\Rightarrow \quad \dfrac{2a+1}{a+1} > 0$ …(3)
Common of all we get $-\dfrac{16}{7} \leq a < -1$.
Hence, the correct answer is (C).

81. Let $f(x) = x^2 + 2bx + \log_3(b^2 - 4b + 4)$
For both roots to be of opposite sign
$f(0) < 0 \quad \Rightarrow \quad \log_3(b^2 - 4b + 4) < 0$
$\Rightarrow \quad b^2 - 4b + 4 < 1$
$\Rightarrow \quad b^2 - 4b + 3 < 0$
$\Rightarrow \quad (b - 1)(b - 3) < 0 \quad \Rightarrow \quad 1 < b < 3$
But $b \neq 2$
$\therefore \quad b \in (1, 2) \cup (2, 3)$.
Hence, the correct answer is (D).

82. $\alpha - \beta = \dfrac{\sqrt{\Delta_1}}{3}$

$$(\alpha + \delta) - (\beta + \delta) = \frac{\sqrt{\Delta_2}}{9}$$

$$\frac{\Delta_1}{9} = \frac{\Delta_2}{81}; \frac{\Delta_1}{\Delta_2} = \frac{1}{9}$$

Hence, the correct answer is (A).

83. Putting $3^x = y$, we have

$$(2a - 4)y^2 - (2a - 3)y + 1 = 0$$

This equation must have real solution
$\Rightarrow \quad (2a - 3)^2 - 4(2a - 4) \geq 0$
$\Rightarrow \quad 4a^2 - 20a + 25 \geq 0 \quad \Rightarrow \quad (2a - 5)^2 \geq 0$
This is true.
$y = 1$ satisfies the equation
Since 3^x is positive and $3^x \geq 3^0, y \geq 1$
Product of the roots $= 1 \times y > 1$

$$\Rightarrow \quad \frac{1}{2a-4} > 1 \quad \Rightarrow \quad 2a - 4 < 1 \quad \Rightarrow \quad a < \frac{5}{2}$$

Sum of the roots $= \dfrac{2a-3}{2a-4} > 1$

$$\Rightarrow \quad \frac{(2a-3)-(2a-4)}{2a-4} > 0$$

$$\Rightarrow \quad \frac{1}{2a-4} > 0 \quad \Rightarrow \quad a > 2 \quad \Rightarrow \quad 2 < a < \frac{5}{2}$$

Hence, the correct answer is (B).

84. $af(x)+bf(-x)=px^2+qx+r$ ...(1)

Replace x by $-x$

$af(-x)+bf(x)=px^2-qx+r$ ...(2)

$(a^2-b^2)f(x)=p(a-b)x^2+q(a+b)x+(a-b)r$

We get $f(x)=\dfrac{px^2(a-b)+q(a+b)x+(a-b)r}{a^2-b^2}=0$

now $D \geq 0$

$$q^2(a+b)^2-4p(a-b)^2r \geq 0$$

$$\Rightarrow \left(\frac{a+b}{a-b}\right)^2 \geq \frac{4pr}{q^2}$$

Hence, the correct answer is (D).

85. Let α be a common root

Then $\alpha^3+a\alpha+1=0$ ...(1)

And $\alpha^4+a\alpha^2+1=0$ ...(2)

$\alpha \times (1)-(2) \Rightarrow \alpha-1=0 \Rightarrow \alpha=1$

So, from $x^3+ax+1=0 \Rightarrow 1+a+1=0 \Rightarrow a=-2$

Hence, the correct answer is (B).

86. By hypothesis $\dfrac{\alpha}{\alpha-1}+\dfrac{\alpha+1}{\alpha}=-\dfrac{b}{a}$ and $\dfrac{\alpha}{\alpha-1}\cdot\dfrac{\alpha+1}{\alpha}=\dfrac{c}{a}$

$\Rightarrow \dfrac{2\alpha^2-1}{\alpha^2-\alpha}=-\dfrac{b}{a}$ and $\alpha=\dfrac{c+a}{c-a}$

$\Rightarrow (c+a)^2+2b(c+a)+b^2=b^2-4ac \Rightarrow (a+b+c)^2=b^2-4ac$

Hence, the correct answer is (C).

87. (i) $D>0$

$4a^2-4(a-5)(a-4)>0$

$9a-20>0 \Rightarrow a>\dfrac{20}{9} \Rightarrow a \in \left(\dfrac{20}{9},\infty\right)$ ...(1)

(ii) $(a-5)f(1)<0$

$\Rightarrow (a-5)(a-5-2a+a-4)<0$

$\Rightarrow a>5 \Rightarrow a \in (5,\infty)$ ...(2)

and $(a-5)f(2)<0 \Rightarrow (a-5)\{(a-5).4-4a+a-4\}<0$

$\Rightarrow (a-5)(a-24)<0 \Rightarrow 5<a<24$

$\Rightarrow a \in (5, 24)$ ...(3)

Using (1), (2) and (3)

The common condition is $a \in (5, 24)$

Hence, the correct answer is (A).

88. Roots of equation $ax^2-2bx+c=0$ are +ve then discriminant $\geq 0 \Rightarrow b^2 \geq ac$

Sum of roots $=\dfrac{b}{a}>0$, product of roots $=\dfrac{c}{a}>0$

Similarly for other two equations, we get $c^2 \geq ab$

$\Rightarrow \dfrac{c}{b}>0$ and $\dfrac{a}{b}>0$ and

$a^2 \geq bc \Rightarrow \dfrac{a}{c}>0$ and $\dfrac{b}{c}>0$

Using above conditions a, b, c are all +ve (or) all are –ve.

Multiplying we get $c^2a^2 \geq ab^2c$

$\Rightarrow ac(b^2-ac) \leq 0 \Rightarrow b^2-ac \leq 0$ ($\because ac>0$)

Also $a^2-bc \leq 0$ and $c^2-ab \leq 0$

And all, we get $a^2+b^2+c^2-ab-bc \leq 0$

$\Rightarrow \dfrac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$

$\Rightarrow a=b=c$

Hence, the correct answer is (C).

89. Let $f(x)=ax^2+2bx+2c$

Then, we have $f(\alpha)=a\alpha^2+2b\alpha+2c=-a\alpha^2+2(a\alpha^2+b\alpha+c)$

$=-a\alpha^2$ [$\because$ α is a root of $ax^2+bx+c=0$ $\backslash a\alpha^2+b\alpha+c=0$]

Also we have, $f(\beta)=a\beta^2+2b\beta+2c=3\alpha\beta^2+2(-\alpha\beta^2+b\beta+c)$

$=3a\beta^2$ [$\because$ β is a root of $-ax^2+bx+c=0$ $\therefore a^2\beta-b\beta-c=0$]

Now. $f(\alpha)\,f(\beta)=-3a^2\,\alpha^2\,\beta^2<0$ which implies that $f(\alpha)$, $f(\beta)$ are of opposite signs and hence, proves that the curve represented by $y=f(x)$ cuts the X-axis somewhere between α and β.

In other words $f(x)=0$ has a root lying between α and β.

Hence, the correct answer is (C).

90. $\dfrac{x^2+nx-2}{x^2-3x+4}-2 \leq 0$

$\Rightarrow x^2-(n+6)x+10 \geq 0$, true $\forall\, x \in R$ then

($\because x^2-3x+4>0\ \forall\, x \in R$)

$D \leq 0 \Rightarrow (n+6)^2-40 \leq 0 \Rightarrow -\sqrt{40}-6 \leq n \leq \sqrt{40}-6$ ...(1)

Similarly $\dfrac{x^2+nx-2}{x^2-3x+4}+1 \geq 0 \Rightarrow 2x^2+(n-3)x+2 \geq 0$

$\Rightarrow D \leq 0 \Rightarrow (n-3)^2-16 \leq 0 \Rightarrow -1 \leq n \leq 7$ ...(2)

Combined (1) and (2) we get $n \in \left[-1, \sqrt{40}-6\right]$

Hence, the correct answer is (A).

91. $x^2+3x+4>0\ \forall\, x \in R$.

None, $mx^2-3x+4 \leq 7x^2+21x+28$, $\forall\, x \in R$.

$\Rightarrow (m-7)x^2-24x-24 \leq 0$.

$\Rightarrow m-7<0$ and $24^2+24.4(m-7) \leq 0$

$\Rightarrow m \in (-\infty, 1]$.

Hence, the correct answer is (D).

92. $\alpha+\beta=-\dfrac{2b}{a}$, $\alpha\beta=-1$, $2\alpha-\dfrac{1}{\beta}+2\beta-\dfrac{1}{\alpha}=-\dfrac{6b}{a}$,

$$\left(2\alpha-\frac{1}{\beta}\right)\left(2\beta-\frac{1}{\alpha}\right)=-9$$

We get the quadratic equation as

$$ax^2+6bx-9a=0$$

so $p+q+r=6b-8a$

Hence, the correct answer is (C).

93. $D=9+4a_i=(2k+1)^2 \Rightarrow a_i=k(k+1)-2=(k+2)(k-1)$

Now for $a_i \in N$ and $a_i \leq 300 \Rightarrow k \in \{2, 3, ..., 15, 16\}$

$$\Rightarrow \sum a_i=\sum_{k=2}^{16}k^2+\sum_{k=2}^{16}k-\sum_{k=2}^{16}2=1600$$

Hence, the correct answer is (C).

94. Let α,α^2 be the roots of the given equation $ax^2+bx+c=0$

then $\alpha+\alpha^2=-\frac{b}{a},\alpha^3=\frac{c}{a}$

$$\left(\frac{c}{a}\right)^{1/3}+\left(\frac{c}{a}\right)^{2/3}=-\frac{b}{a}$$

$$\frac{c}{a}+\left(\frac{c}{a}\right)^2+\frac{3c}{a}\left(\left(\frac{c}{a}\right)^{1/3}+\left(\frac{c}{a}\right)^{2/3}\right)=-\frac{b^3}{a^3}$$

$$\frac{c}{a}+\frac{c^2}{a^2}+\frac{3c}{a}\left(-\frac{b}{a}\right)=-\frac{b^3}{a^3}$$

⇒ $b^3+a^2c+ac^2=3abc$

Hence, the correct answer is (C).

95. $b^2>(a+c)^2\Rightarrow(a-b+C)(a+b+C)<0\Rightarrow f(-1)f(1)<0$.

Hence, the correct answer is (A).

96. $$\frac{2a^2+x^2}{a^3-x^3}-\frac{2x}{ax+a^2+x^2}+\frac{1}{x-a}=0$$

⇒ $\frac{2x^2-3ax+a^2}{x^3-a^3}=0\ (x\neq a)$

The roots of $2x^2-3ax+a^2=0$ are $a,\frac{a}{2}$

∴ $\frac{a}{2}$ is the only root if $x\neq a$

∴ $a\in(-\infty,0)\cup(0,\infty)$

Hence, the correct answer is (D).

97. $f(x)=(x-\alpha)(x-\beta)(x-\gamma)$

α,β,γ are roots of $f(x)=0$

$P(x)=k(x-\alpha^2)(x-\beta^2)(x-\gamma^2)$

$x=0\Rightarrow P(0)=-1=-k\alpha^2\beta^2\gamma^2$

⇒ $k\alpha^2\beta^2\gamma^2=1$

We have $\alpha\beta\gamma=-1$

∴ $k=1$

$P(x^2)=(x^2-\alpha^2)(x^2-\beta^2)(x^2-g^2)$

$=(x-\alpha)(x-\beta)(x-\gamma)(x+\alpha)(x+\beta)(x+\gamma)=-f(x)f(-x)$

$P(4)=-f(2)f(-2)=-(11)(-9)=99$

Hence, the correct answer is (C).

98. $ax^2+bx+1=0$ has roots α_i,β_i

$x^2+bx+a=0$ has roots $\frac{1}{\alpha_i},\frac{1}{\beta_i}$. There are n^2 equations

$$G.E.=\sum\frac{1}{\alpha_i}+\sum\frac{1}{\beta_i}=\sum(-b)$$

$$=n(1+2+3+\ldots+n)=\frac{n.n(n+1)}{2}=\frac{n^2(n+1)}{2}$$

Hence, the correct answer is (B).

99. Let α,β be roots of $x^2-bx+c=0$,

Then $a+b=b$

⇒ one of the roots is '2' (Since a, b are primes and b is odd positive integer)

∴ $f(2)=0\Rightarrow 2b-c=4$ and $b+c=35$

∴ $b=13, c=22$

Minimum value $=f\left(\frac{13}{2}\right)=-\frac{81}{4}$

Hence, the correct answer is (C).

100. $f(x)=\ell x^2-mx+5$

$f(0)=5>0\Rightarrow f(x)\geq 0\,\forall x\in R$

$f(-5)\geq 0$

$25\ell+5m+5\geq 0$

$5\ell+m\geq -1$

Hence, the correct answer is (D).

101. $f(1)f(2)<0$

$(e^a-e^{2a}+e^a-1)(4e^a-2.e^{2a}+e^a-1)<0$

Let $e^a=y$

$(y^2-2y+1)(2y^2-5y+1)<0$

$(y-1)^2(2y^2-5y+1)<0$

$2y^2-5y+1<0$

$\frac{5-\sqrt{17}}{4}<y<\frac{5+\sqrt{17}}{4}$

$\log_e\frac{5-\sqrt{17}}{4}<a<\log_e\frac{5+\sqrt{17}}{4}$

Hence, the correct answer is (C).

102. $y=2x-1+\frac{6}{2x-5}$

for y to bc integer, $\frac{6}{2x-5}$ must be an integer

⇒ $2x-5=\pm1,\pm3,\pm6$, but $\pm3,\pm6$ are not possible

then solutions are (3, 11), (4, 9)

Hence, the correct answer is (C).

103. Let r_1,r_2 be the roots, from given condition

$r-2=\frac{1}{r}\Rightarrow r^2-2r-1=0, b=-2, c=-1, b^2+c^2=5$.

Hence, the correct answer is (D).

104. $ax^2-(4-2a)x-8=0$ for $x=-2,\frac{4}{a}$

By the condition $1<\frac{4}{a}\leq 2\Rightarrow a\in[2,4)$.

Hence, the correct answer is (A).

105. Either both quadratic expression should be perfect squares or both should have same roots. $D_1=0$ and $D_2=0$ do not hold for same k.

Both equation are identical if $\frac{2}{1}=\frac{-4(k-1)}{2k}=\frac{3}{k+1}$ for $k=\frac{1}{2}$.

Hence, the correct answer is (B).

106. $\alpha^2+\beta^3+\gamma^4=\alpha^2+(\beta^2+1)+(\gamma^3+\gamma)$

[since $\beta^3=\beta^2+1,\gamma^3=\gamma^2+1$]

$$= \alpha^2 + \left(\beta^2 + 1\right) + \left(\gamma^2 + 1 + \gamma\right)$$

$$= \left(\alpha^2 + \beta^2 + \gamma^2\right) + 2 + \gamma = 3 + \gamma$$

$\Rightarrow$ Now root $y = 3 + x \quad \Rightarrow \quad (y-3)^3 = (y-3)^2 + 1$

$$y^3 - 10y^2 + 33y - 37 = 0$$

Hence, the correct answer is (B).

107. Sum of roots $\dfrac{\alpha^2 + \beta^2}{\alpha\beta}$ and product $= 1$

Given, $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$

$$\Rightarrow \quad (\alpha + \beta)\left(\alpha^2 - \alpha\beta + \beta^2\right) = q$$

$$\therefore \quad \alpha^2 + \beta^2 - \alpha\beta = \frac{-q}{p} \qquad \ldots(1)$$

and $(\alpha + \beta)^2 = p^2$

$$\Rightarrow \quad \alpha^2 + \beta^2 + 2\alpha\beta = p^2 \qquad \ldots(2)$$

From Eqs. (1) and (2) we get

$$\alpha^2 + \beta^2 = \frac{p^3 - 2q}{3p} \text{ and } \alpha\beta = \frac{p^3 + q}{3p}$$

required equation is

$$x^3 - \left(\frac{p^3 - 2q}{p^3 + q}\right)x + 1 = 0$$

$$\Rightarrow \quad \left(p^3 + q\right)x^2 - \left(p^3 - 2q\right)x + \left(p^3 + q\right) = 0$$

Hence, the correct answer is (B).

108. $a\left(\dfrac{x}{1-x}\right)^2 + b\left(\dfrac{x}{1-x}\right) + c > 0$, Now α, β are roots of $ax^2 + bx + c = 0$

$$\alpha = \frac{x}{1-x}, \beta = \frac{x}{1-x} \text{ Hence } x = {}^{\alpha}\!/_{\alpha+1}, {}^{\beta}\!/_{\beta+1}$$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. $f(x) = x^2 - 4x + 6 = (x-2)^2 + 2$
$f(x) \geq 2 \ \forall \ x \in R$
$\therefore$ Option (C) is correct $f(x) > 0 \ \forall \ x \in R$
Option (B) is also correct $f(x) > 6 \ \forall \ x \in (-\infty, 0) \cup (4, \infty)$
Option (A) $f(x) \geq 2 \ \forall \ x < 4$ (incorrect)
Option (D) is also incorrect.
Hence, (B) and (C) are correct.

2. Let $a + b = x$, then $ab = \dfrac{x^2 - 7}{2}$

and $a^3 + b^3 = 10 \Rightarrow (a+b)^3 - 3ab(a+b) = 10$

$\Rightarrow \quad x^3 - 21x + 20 = 0 \Rightarrow x = 1, 4, -5$

Hence, (A), (B) and (D) are correct.

3. $ax^2 - x^2 - a^2x - 2x + a^2 + 2a = 0$

$$ax(x-a) - (x-a)(x+a) - 2(x-a) = 0$$

$$\Rightarrow \quad (x-a)\left(x - \frac{a+2}{a+1}\right) = 0$$

$$\Rightarrow \quad x = a \,;\, x = \frac{a+2}{a-1} \text{ and } x = b \,;\, x = \frac{b+2}{b-1}$$

But $a \neq b$ $(\because \text{distict})$

If $\dfrac{a+2}{a-1} = \dfrac{b+2}{a-1} \Rightarrow a = b$ (not possible)

The only possibility of common root

$$a = \frac{b+2}{b-1} \text{ (or) } b = \frac{a+2}{a-1}$$

$a = 1 + \dfrac{3}{b-1}$ $\quad \because a$ is +ve integer $\Rightarrow b - 1 = 1$ or $b - 1 = 3$

$$b = 2, 4$$

$$b = 2 \Rightarrow a = 4 \text{ or } b = 4 \Rightarrow a = 2$$

Hence, (B) and (C) are correct.

4. $f^2(x) - f(x) - 6 \geq 0$

$(f(x) - 3)(f(x) + 2) \geq 0$ hence solution is $(-\infty, 0] \cup [5, \infty)$

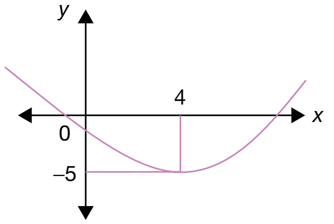
Hence, (C) and (D) are correct.

5. $E = \displaystyle\sum_{i=1}^{3} x_i^2 = 9m^2 - 2(-3) = 9m^2 + 6$

which is minimum when $m = 0$ and minimum value is 6,
$\therefore \quad k = 0$
For $m = 0$ $\quad f(x) = x^3 - 3x + 2$
$\therefore \quad f(x) = (x-1)^2 (x+2)$
$\therefore \quad$ Roots are all real but not all 3 distinct.
Hence, (B), (C) and (D) are correct.

6. From figure $a > 0$ and $\dfrac{-b}{2a} = 4$

$\Rightarrow \quad \dfrac{-b}{2a} > 0; \quad \therefore b < 0$



$f(0) = c < 0$

Also $\dfrac{-b}{2a} = 4 \quad \Rightarrow \quad 8a + b = 0$

Hence, (A), (B), (C) and (D) are correct.

7. $a < 0$
Sum of roots $= \dfrac{-b}{a} > 0 \quad \Rightarrow \quad b > 0$

Product of roots $= \dfrac{c}{a} < 0 \quad \Rightarrow \quad c > 0$

$bc \quad f(2) > 0$
$ab \quad f(-2) < 0$

Hence, (A), (B), (C) and (D) are correct.

8. $ax^2 + (2a - 1)x + (a - 2) = 0$
$D = (2a - 1)^2 - 4a(a - 2) = 4a + 1$ must be a perfect square.
Now, check from the options.
Hence, (A), (C) and (D) are correct.

9. $f(x) = x^2 - 6kx + k^2 + 6k,\ k \in [-5, 5]$

Here, $\frac{6k}{2} > 1 \Rightarrow k > \frac{1}{3}$ …(1)

$f(1) > 0 \Rightarrow 1 + k^2 > 0$ …(2)

$f(1) > 0 \Rightarrow k \le 0$ or $k \ge \frac{3}{4}$ …(3)

$\therefore$ (1) $\cap$ (2) $\cap$ (3)

$\Rightarrow k \ge \frac{3}{4}$

$\therefore$ k integers = 1, 2, 3, 4, 5

Also $f(1) = (1 + k^2)_{\text{min.}} = 1 + (1)^2 = 2$ (for $k = 1$)

Hence, (B) and (D) are correct.

10. We have $\log_5(5)(x^2 + 1) \ge \log_5(ax^2 + 4x + a)$

$\Rightarrow 5(x^2 + 1) \ge ax^2 + 4x + a\ \forall\ x \in R$

$\Rightarrow x^2(5 - a) - 4x + (5 - a) \ge 0\ \forall\ x \in R$

As $5 > a$ …(1)

and $D \le 0$

$\Rightarrow 16 - 4(5 - a)^2 \le 0 \Rightarrow (2 - (5 - a))\,(2 + 5 - a) \le 0$

$\Rightarrow (a - 3)\,(a - 7) \ge 0 \Rightarrow (-\infty, 3] \cap [7, \infty)$ …(2)

$\therefore$ (1) $\cap$ (2) $\Rightarrow a \in (-\infty, 3]$ …(3)

But for the domain of $\log_5(ax^2 + 4x + a)$

We must have $ax^2 + 4x + a > 0\quad \forall\ x \in R$

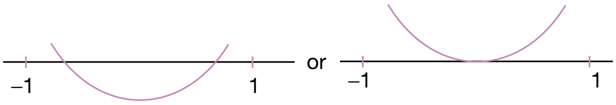
$\Rightarrow a > 0$ and $16 - 4a^2 < 0$

$\Rightarrow a \in (-\infty, -2) \cap (2, \infty)$ …(4)

$\Rightarrow$ (3) $\cap$ (4) $\Rightarrow a \in (2, 3]$

Hence, (A) and (B) are correct.

11. Let $f(x) = 4x^2 - 2x + k$



Here $D \ge 0 \Rightarrow k \le \frac{1}{4}$ …(1)

$-1 < \frac{-B}{2A} < 1 \Rightarrow -1 < \frac{1}{4} < 1$ …(2)

$f(-1) > 0 \Rightarrow k > -6$ …(3)

$f(1) > 0 \Rightarrow k > -2$ …(4)

So, (1) $\cap$ (2) $\cap$ (3) $\cap$ (4) $\Rightarrow k \in \left(-2, \frac{1}{4}\right]$

Hence, (A) and (B) are correct.

12. (A) $a < 0;\ \frac{-b}{2a} < 0 \Rightarrow b < 0$

$c < 0 \Rightarrow abc < 0$

(B) $a > 0, b > 0, c > 0 \Rightarrow abc > 0$
(C) $a > 0, b < 0, c > 0 \Rightarrow abc < 0$
(D) $a > 0, b < 0, c < 0 \Rightarrow abc > 0$
Hence, (B) and (D) are correct.

13. $f(x) = 2x^2 + kx + 50$

$|a - b| = 8 \Rightarrow \frac{\sqrt{D}}{|a|} = 8 \Rightarrow D = 256$

$\therefore$ Minimum value $= -\frac{D}{4a} = -\frac{256}{8} = -32$

$\because$ at $x = 0, f(x) = 50$
$\Rightarrow$ graph cuts positive y-axis.
Hence, (A) and (C) are correct.

14. Let $f(x) = ax^2 + c$
$f(1) = a + c, -4 \le a + c \le -1$
$f(2) = 4a + c, -1 \le 4a + c \le 5$

$\Rightarrow a = \frac{f(2) - f(1)}{3}, f(0) = c = \frac{4f(1) - f(2)}{3}$

$f(3) = 9a + c = 3f(2) - 3f(1) + \frac{4f(1)}{3} - \frac{f(2)}{3} = \frac{8f(2) - 5f(1)}{3}$

$\therefore f(3)_{\max} = \frac{8.5 - 5(-4)}{3} = 20$

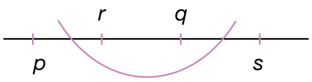
Hence, (A) and (C) are correct.

15. Let $x^3 + 4x^2 + px + q = 0$ has α, β, γ as roots
and $x^3 + 6x^2 + px + r = 0$ has α, β, δ as roots
$\therefore \alpha + \beta + \gamma = -4,$ …(1)
$\alpha\beta + \alpha\gamma + \beta\gamma = p$ …(2)
$\alpha\beta\gamma = -q$ …(3)
$\therefore \alpha + \beta + \delta = -6,$ …(4)
$\alpha\beta + \alpha\delta + \beta\delta = p$ …(5)
$\alpha\beta\delta = -r$ …(6)
$\Rightarrow$ (1) – (4) $\Rightarrow \gamma - \delta = 2$
and (2) – (5) $\Rightarrow (\alpha + \beta)(\gamma - \delta) = 0 \Rightarrow \alpha + \beta = 0$
$\Rightarrow \gamma = -4$ and $\delta = -6$
Equation having γ and δ as roots
$x^2 + 10x + 24 = 0$
$a = 5, c = 3 \Rightarrow a + c = 8$

Ratio of (3) and (6) $\Rightarrow \frac{\gamma}{\delta} = \frac{q}{r} \Rightarrow 3q - 2r = 0$

Hence, (A) and (B) are correct.

16. Let $f(x) = (x - p)\,(x - q) + 5(x - r)\,(x - s)$
$\because f(p) = +$ve
$f(r) = -$ve
$f(q) = -$ve and $f(s) = +$ve
$\therefore$ One root lies in (p, r) and other lies in (q, s)



Hence, (A) and (C) are correct.

17. $D = \{2(a + 1)\}^2 - 4(12a - 8) = 4(a - 1)\,(a - 9)$
$D \ge 0 \Rightarrow a \in (-\infty, 1] \cup [9, \infty) \Rightarrow$ roots are real

If $a < \frac{\pi}{6} \Rightarrow 12a - 8 < 0$

Product of roots < 0 hence roots are of opposite in sign.
If $a > \pi^2$
sum of roots $(-2(a + 1)) < 0$ and Product of roots $(12a - 8) > 0$
$\Rightarrow$ both roots are negative.
Hence, (A), (B) and (C) are correct.

18. $\beta > 1$ and $\alpha < -1 \Rightarrow \dfrac{c}{a} < -1$

since $a < 0 \Rightarrow c > 0$

also $\alpha + \beta > 0 \Rightarrow \dfrac{b}{a} > 0 \Rightarrow b > 0 \Rightarrow abc < 0$

if $x \in (0, \beta)$ graph lies above x-axis $\Rightarrow |\beta - \alpha| > 2$

Hence, (A), (B), (C) and (D) are correct.

19. $x = \sqrt{3+x} \Rightarrow x^2 = 3 + x$

$$x - \frac{3}{x} = 1$$

$$x^2 + \frac{9}{x^2} = 7$$

$$x^3 - x^2 - x - 1 = x(x^2 - x - 3) + 2x - 1$$
$$= 2x - 1$$

Now solving $x^2 - x - 3 = 0$

$$2x - 1 = \pm\sqrt{13} \Rightarrow 2x - 1 = \sqrt{13}$$

Hence, (A) and (B) are correct.

20. $D > 0 \Rightarrow (a-b)^2 + 4(a+b-1) > 0$

$\Rightarrow b^2 + (4-2a)b + (a^2 + 4a - 4) > 0, \forall b$

$\Rightarrow (4-2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow 32 - 32a < 0 \Rightarrow a > 1$

Hence, (A), (B) and (C) are correct.

21. $-3 < \dfrac{x^2 + kx + 1}{x^2 + x + 1} < 3$

$\Rightarrow -3(x^2 + x + 1) < x^2 + kx + 1 < 3(x^2 + x + 1)$

$\Rightarrow 4x^2 + (k+3)x + 4 > 0$

and $2x^2 + (3-k)x + 2 > 0 \ \forall x \in R$

$\Rightarrow (k+3)^2 - 64 < 0$ and $(k-3)^2 - 16 < 0$

$\Rightarrow k^2 + 6k - 55 < 0$ and $k^2 - 6k - 7 < 0$

$\Rightarrow k \in (-11, 5)$ and $k \in (-1, 7)$

$\Rightarrow k \in (-1, 5)$

The number of integral values of R is 5

If $f(x) = x^2 - (k^2 - 4k - 5)x - 1$

Then $f(0) \times f(1) = -1(1 - (k^2 - 4r - 5) - 1)$

$= k^2 - 4k - 5 = (k+1)(k-5) < 0$

$\therefore f(x) = 0$ has exactly one real root on (0, 1)

Hence, (A), (B) and (D) are correct.

22. Let $P = 3\alpha + 2$

$\Rightarrow \alpha = \dfrac{P-2}{3}$

Since $9\alpha^3 - 7\alpha + 6 = 0$

$\Rightarrow \dfrac{9(P-2)^3}{27} - \dfrac{7}{3}(P-2) + 6 = 0$

$\Rightarrow \dfrac{1}{3}(P^3 - 8 - 6P^2 + 12P) - \dfrac{7}{3}P + \dfrac{14}{3} + 6 = 0$

$\Rightarrow P^3 - 6P^2 + 12P - 8 - 7P + 14 + 18 = 0$

$\Rightarrow P^3 - 6P^2 + 5P + 24 = 0$

So, the equation $x^3 - 6x^2 + 5x + 24 = 0$ has roots $3\alpha + 2, 3\beta + 2, 3\gamma + 2$

Hence, (C) and (D) are correct.

23. a = sum of the roots = odd

Both roots can not be odd $\therefore$ one root must be 2.

$$f(2) = 0 \Rightarrow \left.\begin{aligned} 4 - 2a + b &= 0 \\ a + b &= 35 \end{aligned}\right\} a = 13, b = 22$$

Equation is $x^2 - 13x + 22 > 0$ and roots are 11, 2

Sum of roots = 13, Product of roots = 22, Difference of roots = 9

$$\Sigma f(i) = \sum_{i=1}^{10} (i^2 - 13i + 22) = -110$$

Hence, (B), (C) and (D) are correct.

24. Discriminant $\geq 0 \Rightarrow 4a^2 - 4(a^2 + a - 3) \Rightarrow a \leq 3$

x coordinate of vertex $= -\dfrac{(-2a)}{2} < 3 \Rightarrow a < 3$

$f(3) > 0 \Rightarrow 9 - 6a + a^2 + a - 3 > 0 \Rightarrow a \in (-\infty, 2) \cup (3, \infty)$

final $a \in (-\infty, 2)$

Hence, (C) and (D) are correct.

25. **CASE-1:** Suppose b is even, that is, $b = 2m$. Then

$$b^2 - 4ac = 4(m^2 - ac) = 4k$$

CASE-2: Supose b is odd, that is, $b = 2m - 1$. Then

$$b^2 - 4ac = (2m-1)^2 - 4ac$$
$$= 4m^2 + 4m + 1 - 4ac$$
$$= 4(m^2 + m - ac) + 1$$
$$= 4k + 1$$

Hence, (A) and (B) are correct.

26. $P(x) = x + a(x-1)(x-2)(x-3)$,

$\because P(4) = 16 \Rightarrow a = 2$

$\Rightarrow P(x) = 2(x-1)(x-2)(x-3) + x$

$\Rightarrow P(0) = -12, P(5) = 53, P(-1) = -49$

Hence, (A), (B) and (D) are correct.

27. $x^3 + 2x^2 - 3x + 1 = 0$ has roots α, β and γ

$x^3 - 3x^2 + 2x + 1 = 0$ has roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$

Given expression $\dfrac{1}{3 - \frac{1}{\alpha}} + \dfrac{1}{3 - \frac{1}{\beta}} + \dfrac{1}{3 - \frac{1}{\gamma}}$

$$= \frac{\sum\left(3 - \frac{1}{\alpha}\right)\left(3 - \frac{1}{\beta}\right)}{\left(3 - \frac{1}{\alpha}\right)\left(3 - \frac{1}{\beta}\right)\left(3 - \frac{1}{\gamma}\right)}$$

$$= \frac{27 - 6\left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right] + \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}}{\left(3 - \frac{1}{\alpha}\right)\left(3 - \frac{1}{\beta}\right)\left(3 - \frac{1}{\gamma}\right)} = \frac{11}{7}$$

Hence, (A), (B), (C) and (D) are correct.

28. $f(0) > 0$ and $f(3) < 0$ and $f(1) > 0$ and $f(2) = 0$

Hence, (A), (B), and (C) are correct.

29. $p_1x^2 + q_1x + r_1 = 0,\ \alpha + \beta = -\dfrac{q_1}{p_1}$

$$P_2x^2 + q_2x + r_2 = 0,\ \beta + \gamma = -\frac{q_2}{p_2}$$

$$P_3x^2 + q_3x + r_3 = 0,\ \alpha + \gamma = -\frac{q_3}{p_3}$$

$$\therefore \quad \alpha + \beta + \gamma = \frac{-1}{2}\left[\frac{q_1p_2p_3 + p_1q_2p_3 + p_1p_2q_3}{p_1p_2p_3}\right]$$

and $p_1p_2p_3(x - \alpha)^2 (x - \beta)^2 (x - \gamma)^2 = \prod_{i=1}^{3}(p_ix^2 + q_ix + r_i)$

Put $x = -1$

$$p_1p_2p_3[(1 + \alpha)(1 + \beta)(1 + \gamma)]^2 = \prod_{i=1}^{3}(p_i - q_i + r_i)$$

$$= \prod_{i=1}^{3}(p_i - q_i + r_i) = p_1p_2p_3(1 + \Sigma\alpha + \Sigma\alpha\beta + \alpha\beta\gamma)^2$$

Hence, (A) and (C) are correct.

30. $D > 0$

or $(a - 2)^2 - 4(a - 2) > 0$ or $(a - 2)(a - 6) > 0$

$\therefore \quad a\ (-\infty, 2) \cup (6, \infty)$

Hence, (A) and (D) are correct.

31. $x^2 + ax + 10 = 0 \qquad \ldots(1)$

$x^2 + bx + 8 = 0 \qquad \ldots(2)$

$x^2 + (2a + 3b)x + 60 = 0 \qquad \ldots(3)$

$2 \times (1) + 3 \times (2) - (3)$

$\Rightarrow \quad x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$

If $x = 2$; if $x = -2$

$a = -7$; if $a = 7$

$b = -6$; $b = 6$

Hence, (A) and (C) are correct.

32. $x = 1$ is a root and other root is $\dfrac{c + a - 2b}{a + b - 2c}$

$\because$ a, b, c are positive integers $\Rightarrow$ both roots are rational.

$\because$ second root in $(-1, 0)$ so product of roots is < 0

and $a + b - 2c > 0$ so $c + a - 2b < 0 \Rightarrow c + a < 2b$

$$\alpha + \beta = \frac{-2b}{a} < 0$$

$\alpha\beta = \dfrac{c}{a} > 0$ So, $ax^2 + 2bx + c = 0$ has both negative real roots.

Hence, (B), (C) and (D) are correct.

33. $D = 0 \Rightarrow a\left(a^3 + b^3 + c^3 - 3abc\right) = 0$

$\Rightarrow \quad a = 0\ (\text{or})\ a + b + c = 0$

But not $a^2 + b^2 + c^2 - ab - bc - ac = 0$ because a, b, c are distinct

Hence, (A), (B) and (C) are correct.

34. $a + b - 2c = (a - c) + (b - c) > 0 \Rightarrow$ mouth opens upwards

now $x = 1$ is a root Hence both roots are rational.

vertex $-\dfrac{(b + c - 2a)}{(a + b - 2c)} > 0 \Rightarrow \text{vertex} > 0$

Hence, (A), (B) and (C) are correct.

35. Since α and β are the roots of $ax^2 + bx + c = 0$ we have

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

Also since α' and β' are roots of the equation $a^3x^2 + (abc)x + c^3 = 0$

$$\alpha' + \beta' = \frac{-abc}{a^3} = \left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = (\alpha + \beta)\alpha\beta$$

$$\alpha'\beta' = \frac{c^3}{a^3} = (\alpha\beta)^3$$

Now $(\alpha' - \beta')^2 = (\alpha' + \beta')^2 - 4\alpha'\beta' = (\alpha + \beta)^2\alpha^2\beta^2 - 4\alpha^3\beta^3$

$= (\alpha\beta)^2[(\alpha + \beta)^2 - 4\alpha\beta] = (\alpha\beta)^2(\alpha - \beta)^2$

Also $|\alpha' - \beta'| = |\alpha\beta(\alpha - \beta)|$

Therefore $\alpha' - \beta' = \alpha\beta(\alpha - \beta) \Rightarrow \alpha' = \alpha^2\beta$ and $\beta' = \alpha\beta^2$

$\alpha' - \beta' = -\alpha\beta(\alpha - \beta) \Rightarrow \alpha' = \alpha\beta^2$ and $\beta' = \alpha^2\beta$

Hence, (C) and (D) are correct.

36. Let α be a root of equation $x^2 - 3x + c = 0$ then $-\alpha$ is a root of equation $x^2 + 3x - c = 0 \Rightarrow c = 0 \Rightarrow$ the solutions of $x^2 - 3x = 0$ are 0 and 3.

Hence, (C) and (D) are correct.

37. $(ay - bx)^2 + 4xy = 0$

$\Rightarrow \quad a^2y^2 + b^2x^2 + (4 - 2ab)xy = 0$

$\Rightarrow \quad \dfrac{a^2y}{x} + \dfrac{b^2x}{y} - (2ab - 4) = 0$

Put $\dfrac{y}{x} = t$

$\Rightarrow \quad a^2t^2 - (2ab - 4)t + b^2 = 0$

$D = (2ab - 4)^2 - 4a^2b^2$

$= 4(4 - 4ab) = 16(1 - ab)$ must be a perfect square.

Hence, (A) and (C) are correct.

38. Let $k = \alpha + i\beta$ and one real root γ, then $(\alpha + i\beta)\gamma^2 + \gamma + 1 = 0$

equating real and imaginary parts

$\beta\gamma^2 = 0$

$\gamma = 0(\text{since } \beta \neq 0)$

$\alpha\gamma^2 + \gamma + 1 = 0$ when $\gamma = 0 \Rightarrow 1 = 0$ which is impossible

so $k \in \phi$

Hence, (A), (B), (C) and (D) are correct.

39. a, b, c are roots of equation $\dfrac{1}{x} + \dfrac{1}{x - 1} + \dfrac{1}{x - 2} = 1$

$\Rightarrow (x - 1)(x - 2) + x(x - 2) + x(x - 1) = x(x - 1)(x - 2)$

$\Rightarrow x^3 - 6x^2 + 8x - 2 = 0$

Equation having roots $1 - a, 1 - b, 1 - c$ is given by

$$(1 - x)^3 - 6(1 - x)^2 + 8(1 - x) - 2 = 0$$

So product of roots $\pi(1 - a) = +(1 - 6 + 8 - 2) = 1$

Similarly equation having roots $a - 2, b - 2, c - 2$ is given by

$$(x + 2)^3 - 6(x + 2)^2 + 8(x + 2) - 2 = 0$$

$\Rightarrow \quad (a - 2)(b - 2)(c - 2)$

$= -(8 - 24 + 16 - 2)$

$= 2$

Hence, (A), (B), (C) and (D) are correct.

40. Assuming quadratic in a

$$a = x^2 + 2x,\ a = x^2 - 2x + 3$$

$\therefore$ given exp is $\left(a - x^2 - 2x\right)\left(a - x^2 + 2x - 3\right) = 0$

$\therefore \quad x = -1 \pm \sqrt{1 + a}$ and $x = 1 \pm \sqrt{a - 2}$

Hence, (A) and (B) are correct.

41. Suppose that both r and $r + 1$ are solutions to the equation $x^3 + 3x^2 - 34x = m$. Then $r^3 + 3r^2 - 34r = m$, and also $(r + 1)^3 + 3(r + 1)^2 - 34(r + 1) = m$. Subtracting the first of these equations from the second yields $(3r^2 + 3r + 1) + 3(2r + 1) - 34 = 0$, and simplification yields $3r^2 + 9r - 30 = 0$. Thus $0 = r^2 + 3r - 10 = (r + 5)(r - 2)$. We conclude that either $r = -5$ or $r = 2$. If $r = -5$, then $m = (-5)^3 + 3(-5)^2 - 34(-5) = -125 + 75 + 170 = 120$. The other possibility is $r = 2$, which yields $m = 8 + 12 - 68 = -48$. $m = 120$ and $m = -48$ are the two possibilities.

Hence, (A) and (C) are correct.

42. $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root $\Rightarrow$ $a^3 + b^3 + c^3 - abc = 0$

$\frac{1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2\,(c - a)^2] = 0 \Rightarrow a + b + c = 0$

$f(x) = bx^3 + cx^2 + ax - 5$

$f(0) = -5$

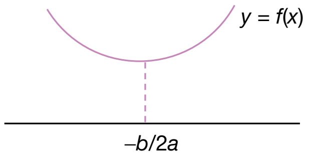
$f(1) = a + b + c - 5 = -5$

$\Rightarrow$ $f(x)$ will always pass through $(0, -5)$ and $(1, -5)$

Hence (a, b)

Hence, (A) and (B) are correct.

43. $f(x) = ax^2 + bx + 6 \quad \Rightarrow \quad f(0) = 6 > 0$



$\Rightarrow \quad a > 0$

for $x = 3 \quad \Rightarrow \quad a3^2 + 3b + 6 \ge 0 \quad \Rightarrow \ 3a + b \ge -2$

for $x = 6 \quad \Rightarrow \quad a6^2 + 6b + 6 \ge 0 \quad \Rightarrow \ 6a + b \ge -1$

Hence, (B) and (C) are correct.

44. $\left|x^2 + 2x - 8\right| + x - 2 = 0$

$x^2 + 3x - 10 = 0$ if $x \in (-\infty, -4] \cup [2, \infty)$

$(x + 5)(x - 2) = 0 \Rightarrow x = -5$ and 2 both are acceptable

$-x^2 - 2x + 8 + x - 2 = 0$ if $x \in (-4, 2)$

$x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

$x = -3$ or 2 then $x = -3$ is other root

No. of roots is 3

Sum of roots is –6

Product of roots is 30

Hence, (A), (B) and (C) are correct.

45. $f(0) = C$

As $f(x)$ is an integer for $x \in Z \quad \therefore C \in Z$

$f(1) = A + B + C$

$f(-1) = A - B + C$

$f(1) + f(-1) = 2(A + C)$

$\therefore \quad 2A$ is an integer

A + B is also an integer

Hence, (A), (B) and (D) are correct.

46. $\frac{p}{2x} = \frac{(a + b)x + c(b - a)}{x^2 - c^2}$

or $p(x^2 - c^2) = 2(a + b)x^2 - 2c(a - b)x$

or $(2a + 2b - p)x^2 - 2c(a - b)x + pc^2 = 0$

Now, $c^2(a - b)^2 - pc^2(2a + 2b - p) = 0 \qquad (\because$ equal roots)

$\Rightarrow \quad (a - b)^2 - 2p(a + b) + p^2 = 0 \ (\because c^2 \ne 0)$

$\Rightarrow \quad \left[p - (a + b)\right]^2 = (a + b)^2 - (a - b)^2$

$\Rightarrow \quad p = a + b \pm 2\sqrt{ab} = \left(\sqrt{a} \pm \sqrt{b}\right)^2$

Hence, (A) and (B) are correct.

Linked Comprehension Type Questions

1. Graph is symmetric about $x = -\frac{1}{2}$

$\Rightarrow \quad \frac{-b}{2a} = -\frac{1}{2} \ \Rightarrow \ \frac{b}{a} = 1 \quad \Rightarrow \quad a = b$

$\therefore \quad$ Sum of roots of $= \frac{-b}{a} = -1$

Hence, the correct answer is (D).

2. $4a + 2b + c < 0$

$\Rightarrow \quad a + c < -3a - 2b$

$a + c < -5a$

$\therefore \quad a + c$ is a negative number

Hence, the correct answer is (B).

3. **The correct answer is (A).**

4. **The correct answer is (C).**

Combined Solution to 3–4

$3x^3 - x^2 = kx - 9$ roots are α, β and γ

$\Rightarrow \quad 3x^3 - x^2 - kx + 9 = 0$

$\alpha + \beta + \gamma = \frac{1}{3} \quad \Rightarrow \quad \beta = \frac{1}{3} \qquad [\because \alpha + \gamma = 0]$

Put $\beta = \frac{1}{3}$ in cubic we get $k = 27$

Then $\alpha = 3,\ \gamma = -3$

$k = 27$

$\gamma = x^2 + 3x + \frac{1}{3}$

Minimum $= \frac{-D}{4a} = -\frac{(\alpha^2 - 4\beta)}{4} = \frac{-23}{12}$

5. **The correct answer is (A).**

6. **The correct answer is (B).**

Combined Solution to 5–6

5. Let $\dfrac{\alpha}{\alpha+1} = y \Rightarrow \alpha = \dfrac{y}{1-y}$

As, α is root of $x^3+2x^2+3x+3=0$

$$\Rightarrow \left(\frac{y}{1-y}\right)^3 + 2\left(\frac{y}{1-y}\right)^2 + \frac{3y}{1-y} + 3 = 0$$

[Using transformation]

$\Rightarrow y^3 - 5y^2 + 6y - 3 = 0$

Now, y_1, y_2, y_3 are roots of above equation,

$$\therefore \frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1} + \frac{\gamma}{\gamma+1} = 5$$

6. Also $\sum y_1 = 5, \sum y_1y_2 = 6, \sum y_1y_2y_3 = 3$

$$\therefore y_1^3 + y_2^3 + y_3^3 - 3y_1y_2y_3 = \sum y_1\left(\left(\sum y_1\right)^2 - 3\sum y_1y_2\right)$$

$\Rightarrow y_1^3 + y_2^3 + y_3^3 - 3(3) = 5[5^2 - 3(6)]$

$\therefore y_1^3 + y_2^3 + y_3^3 = 44$

Hence $\left(\dfrac{\alpha}{\alpha+1}\right)^3 + \left(\dfrac{\beta}{\beta+1}\right)^3 + \left(\dfrac{\gamma}{\gamma+1}\right)^3 = 44$

Combined Solution to 7–9

7. Let $x^4+ax^3+bx^2+cx+d = (x-x_1)(x-x_2)(x-x_3)(x-x_4)$

Let $(x-x_1)(x-x_2) = x^2+px+q$ and $(x-x_3)(x-x_4) = x^2+px+r$

$\therefore q = x_1x_2$ and $r = x_3x_4$

$\therefore x^4+ax^3+bx^2+cx+d$

$= x^4+2px^3+(p^2+q+r)x^2+p(q+r)x+qr$

$\therefore a = 2p, b = p^2+q+r, c = p(q+r), d = qr$

Clearly, $a^3 - 4ab + 8c = 0$ …(1)

If $a = 2 \Rightarrow b - c = 1$

Hence, the correct answer is (B).

8. Investigating the nature of the cubic equation of 'a'.

Let $f(a) = a^3 - 4ab + 8c$

$f'(a) = 3a^2 - 4b$

If $b < 0 \Rightarrow f'(a) > 0$

$\therefore$ The equation $a^3 + 4ab + 8c = 0$ hence only one real root.

Hence, the correct answer is (C).

9. Substituting $c = 1 - b$ in Eq. (1) we have

$(a+2)[(a-1)^2 + 3 - 4b] = 0 \Rightarrow 4b - 3 > 0$

$\Rightarrow b > \dfrac{3}{4} \Rightarrow c < \dfrac{1}{4}$

Hence, the correct answer is (A).

10. Let $x^4 + ax^3 + bx^2 + ax + 1 = (x^2+kx+1)^2$ comparing $2k = a$; $b = k^2 + 2 \Rightarrow a + b = (k+1)^2 + 1$ it can be 4

Hence, the correct answer is (D).

11. Let roots are α, β negative and γ, γ positive root

$f(\gamma) = 0 \Rightarrow \gamma^4 + a\gamma^3 + b\gamma^2 + a\gamma + 1 = 0$ …(1)

$f'(\gamma) = 0 \Rightarrow 4\gamma^3 + 3a\gamma^2 + 2b\gamma + a = 0$ …(2)

$(1) - \dfrac{\gamma}{2}(2)$ we get

$$(\gamma-1)\left((1+\gamma)(1+\gamma^2) + \frac{a\gamma}{2}\right) = 0$$

$\gamma = 1; \quad \alpha\beta\gamma^2 = 1; \quad \beta = \dfrac{1}{\alpha}$

The two equal +ve roots must be 1, 1, and let the –ve roots be $\alpha, \dfrac{1}{\alpha} \quad (\alpha \neq 1)$

Now $-a = 2 + \alpha + \dfrac{1}{\alpha} \Rightarrow a = -\alpha - \dfrac{1}{\alpha} - 2 > 0$

$\therefore$ The least integral value is '1'

Hence, the correct answer is (A).

12. **The correct answer is (A).**

13. **The correct answer is (D).**

Combined Solution to 12–13

Let $3^x = t \Rightarrow t^2 - ta - a + 3 \le 0 : t > 0$

Let $f(t) = t^2 - at + 3 - a$

Discriminant of $f(t) = 0$ is $a^2 - 4(3-a)a = a^2 + 4a - 12$

$D \ge 0 \Rightarrow a \le -6$ or $a \ge 2$

12. $f(t) \le 0$ has at least one negative solution.

If $x < 0$ then at least one t of $f(t) = 0$ lies in.

CASE-1: exactly one $t \in (0, 1)$ then $D \ge 0$ and $f(0)\,f(1) < 0$ then $a \in (2, 3)$

CASE-2: both roots lines in (0, 1) then $D \ge 0; f(0) > 0; f(1) > 0;$

$0 < \dfrac{a}{2} < 1$ Then $a \in \phi$

$\therefore a \in (2, 3)$

13. $f(t) \le 0$. has at least one positive solution

i.e., $x > 0 \Rightarrow t > 1$

CASE-1: exactly one root is greater than 1

(1) $D > 0$ (2) $f(1) < 0$ then $a > 2$

CASE-2: both roots greater than 1

(1) $D \ge 0$ (2) $f(1) > 0$ (3) $\dfrac{a}{2} > 1$

Then $a \in \phi$

$\therefore a \in (2, \infty)$

14. Similarly $x \in (1, 2)$ then $t \in (3, 9)$

Similar to the above the question

CASE-1: Exactly one root lies in (3, 9) $\Rightarrow f(3)\,f(9) < 0$

$\Rightarrow (9 - 3a + 3 - a)(81 - 9a + 3 - a) < 0 \Rightarrow a \in \left(3, \dfrac{42}{5}\right)$

CASE-2: Both roots lies in (3, 9) then $D \ge 0;$

$f(3) > 0; 2 < \dfrac{a}{2} < 3$ then $a \in \phi$

$\therefore a \in \left(3, \dfrac{42}{5}\right)$

Hence, the correct answer is (B).

15. $D = 4(1+2k)^2 - 4(1+k)(3+k) = 4(3k^2-2)$

$$3k^2 - 2 < 0 \Rightarrow -\sqrt{\frac{2}{3}} < k < \sqrt{\frac{2}{3}}$$

$\therefore$ Integral value of $k = 0$

Hence, the correct answer is (B).

16. (i) $D \geq 0 \quad \Rightarrow \quad k \leq -\sqrt{\frac{2}{3}}$ or $k \geq \sqrt{\frac{2}{3}}$

(ii) $a\,f(0) > 0 \quad \Rightarrow \quad (1+k)(3+k) > 0$

$\Rightarrow \quad k < -3$ or $k > -1$

(iii) $\frac{-b}{2a} > 0 \quad \Rightarrow \quad \frac{1+2k}{1+k} > 0$

$\Rightarrow \quad k < -1$ or $k > -\frac{1}{2}$

$\therefore$ Common values of (i), (ii) and (iii) are $k < -3$ or $k \geq \sqrt{\frac{2}{3}}$.

Hence, the correct answer is (B).

17. **The correct answer is (C).**

18. **The correct answer is (C).**

19. **The correct answer is (A).**

Combined Solution to 17–19

$(a-1)^2 + (b+3)^2 + (c-2)^2 = 0$

Hence, $a = 1$; $b = -3$; $c = 2$

$f(x) = x^2 - 2(k-1)x + 1$ and $g(x) = kx^2 - 3x + 2$

17. $f(0) + g(0) = 3$

18. $f(x) - g(x) = (1-k)x^2 - (2k-5)x - 1 = 0$

Now, $D = 0 \quad \Rightarrow 4k^2 - 24k + 29 = 0$

Sum of all real values of $k = 6$

19. Let D_1 be discriminant of $f(x) = 0$ and D_2 be discriminant of $g(x) = 0$

$D_1 = 4(k-1)^2 - 4$; $D_2 = (9 - 8k)$

Now,

CASE-1: $D_1 > 0$ and $D_2 = 0$

$(k-1)^2 - 1 > 0$ and $k = \frac{9}{8}$

Hence, no value of k from this case.

CASE-2: $D_2 > 0$ and $D_1 = 0$

$9 - 8k > 0$ and $(k-1)^2 - 1 = 0$

From this case $k = 0$ which is rejected.

Hence, no value of k from this case. Therefore no value of k is possible.

20. **The correct answer is (A).**

21. **The correct answer is (D).**

22. **The correct answer is (B).**

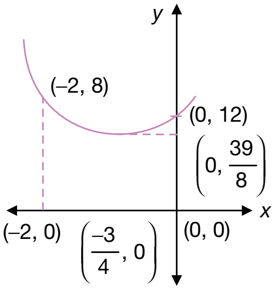
Combined Solution to 20–22

$\frac{-D}{4a} = 4 \quad \Rightarrow \quad \frac{8}{4a} = 4 \quad \Rightarrow \quad a = \frac{1}{2}$

$\frac{-b}{2a} = -4 \quad \Rightarrow \quad b = 8a \quad \Rightarrow \quad b = 4$

$b^2 - 4ac = -8 \quad \Rightarrow \quad 16 - 2c = -8 \quad \Rightarrow \quad c = 12$

$f(x) = \frac{x^2}{2} + 4x + 12$



y
(–2, 8)
(0, 12)
$\left(0, \frac{39}{8}\right)$
x
(–2, 0)
$\left(\frac{-3}{4}, 0\right)$
(0, 0)

20. $f(0) = 12$; $f(1) = \frac{33}{2}$

$f(0) + 2f(1) = 12 + 33 = 45$

21. $f(x) = 0$ has root α

$\therefore \quad \alpha^2 + 8\alpha + 24 = 0$

Hence, value of $(\alpha^3 + 10\alpha^2 + 40\alpha + 39)$

$= \alpha(\alpha^2 + 8\alpha + 24) + 2(\alpha^2 + 8\alpha + 24) - 9 = -9$

22. $h(x) = 2x^2 + 3x + 6$

$\because \quad -\frac{3}{4} \in [-2, 0]$

Range of $h(x)$ is $\left[\frac{39}{8}, 8\right]$

23. $y = \frac{x^2 - 3x - 4}{x^2 - 3x + 4}$

$\Rightarrow \quad (y-1)x^2 - 3(y-1)x + 4(y+1) = 0$

Since $x \in R$, so $D \geq 0$

$\Rightarrow \quad 9(y-1)^2 \geq 16(y+1)(y-1)$

$\Rightarrow \quad 7y^2 + 18y - 25 \leq 0$

$\Rightarrow \quad (y-1)(7y+25) \leq 0$

$\Rightarrow \quad \frac{-25}{7} \leq y \leq 1$

But $y \neq 1$, so range of $f(x) = \left[\frac{-25}{7}, 1\right)$

Clearly integers in the range of $f(x)$ are –3, –2, –1, 0.

$\therefore$ Sum of integers $= -6$.

Hence, the correct answer is (B).

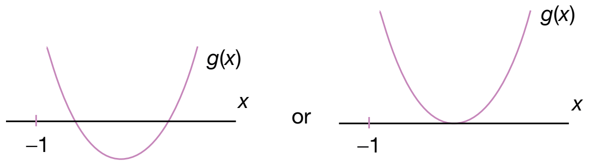
24. Different possibilities are as follows :

If both roots of $g(x) = 0$ are greater than –1, then 3 conditions should be satisfied simultaneously.

(1) $D \geq 0$ (2) $\frac{-B}{2A} > -1$ (3) $g(-1) > 0$

Now, (1) $\Rightarrow \quad (b+1) + (b-1) > 0$

$\Rightarrow \quad b^2 - 2b + 5 \geq 0 \;\forall\; b \in R$



(2) $\Rightarrow \quad \frac{b+1}{2} > -1 \quad \Rightarrow b > -3.$

and (3) $\Rightarrow 1 + (b+1) + (b-1) > 0 \quad \Rightarrow \quad b > \frac{-1}{2}$

Hence from (1) $\cap$ (2) $\cap$ (3), we get $b \in \left(\frac{-1}{2}, \infty\right)$

Hence, the correct answer is (D).

25. If $g(x) > -2\ \forall\ x \in R$, so

$$x^2 - (b+1)x + b - 1 > -2\ \forall\ x \in R$$

$$\Rightarrow \quad x^2 - (b+1)x + b + 1 > 0\ \forall\ x \in R,$$

So, discriminant should be negative.

$\therefore \quad D < 0$, gives

$$(b+1)^2 < 4(b+1) \quad \Rightarrow \quad (b-3)(b+1) < 0$$

$\therefore \quad -1 < b < 3$

Hence $b_{\text{largest natural number}} = 2$.

Hence, the correct answer is (B).

26. Co-ordinate of A is $\left(-2\sqrt{2}, 0\right)$ and Co-ordinate of C is $\left(2\sqrt{2}, 0\right)$

$AB = BC = 4$ units; $OB = 2\sqrt{2}$

$$y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$$

Hence, the correct answer is (D).

27. Minimum value of $y = f(x)$ is $-2\sqrt{2}$ at $x = 0$

Hence, the correct answer is (C).

28. $f(k) < 0 \quad \Rightarrow \quad \dfrac{k^2}{2\sqrt{2}} - 2\sqrt{2} < 0$

$\Rightarrow \quad k^2 - 8 < 0 \quad \Rightarrow \quad k \in \left(-2\sqrt{2}, 2\sqrt{2}\right)$

No. of integral values of 'K' which 'K' lies in $\left(-2\sqrt{2}, 2\sqrt{2}\right)$ is 5.

Hence, the correct answer is (B).

29. $D = (k-3)^2 - 4k = (k-1)(k-9) \geq 0 \Rightarrow k \in (-\infty, 1] \cup \quad [9, \infty)$

and $k - 3 > 0 \Rightarrow k > 3\ \&\ k > 0 \ \Rightarrow k \in [9, \infty)$

Hence, the correct answer is (A).

30. $D = 16 - 4k \geq 0 \Rightarrow k \leq 4$, sum $= -4 < 0$ and product $= k > 0$

$\Rightarrow k \in (0, 4]$

Hence, the correct answer is (D).

31. $D = 4\left(4\mu^2 - \left(\lambda^2 - 9\right)\right)$ and

Product of roots $= \dfrac{\lambda + 3}{\lambda - 3} < 0 \Rightarrow \lambda^2 - 9 < 0 \Rightarrow \lambda \in (-3, 3)$ and $\mu \in R$

Hence, the correct answer is (A).

32. **The correct answer is (D).**

33. **The correct answer is (D).**

34. **The correct answer is (C).**

Combined Solution to 32–34

Since, $x^4 + (\alpha-1)x^3 + (\alpha-1)x + 1 = 0$

$$\Rightarrow \quad x^2 + \frac{1}{x^2} + (\alpha-1)\left(x + \frac{1}{x}\right) = 0$$

$$\Rightarrow \quad t^2 - 2 + (\alpha-1)t = 0;\ \text{where } x + \frac{1}{x} = t$$

for real roots, we have

$(\alpha-1)^2 + 8 > 0$ which is always true

$\Rightarrow \quad t$ is always real

$\Rightarrow \quad x + \dfrac{1}{x}$ is always real

$\therefore$ for x to be real, $x + \dfrac{1}{x} \geq 2$ if $x > 0$ and $x + \dfrac{1}{x} < -2$ if $x < 0$

CASE-1: $x > 0$

Both the roots of $t^2 + (\alpha-1)t - 2 = 0$ are more than 2, such that

$af(2) > 0$ and $-\dfrac{b}{2a} > 2$

$\Rightarrow \quad 4 + (\alpha-1)2 - 2 > 0$ and $-\dfrac{(\alpha-1)}{2} > 2$

$\Rightarrow \quad \Rightarrow \alpha > 0$ and $\alpha < -3$

which contradicts and hence no solution.

CASE-2: $x < 0$

Both the roots of $t^2 + (\alpha-1)t - 2 = 0$ are less than 2, such that

$af(-2) > 0$ and $-\dfrac{b}{2a} < -2$

$\Rightarrow \quad 4 - (\alpha-1)2 - 2 > 0$ and $(\alpha-1) > 4 \Rightarrow \alpha < 2,\ \alpha > 5$.

35. Difference of the roots of $f(x)$ and $g(x)$ are equal $\Rightarrow$ $\dfrac{\Delta_1}{1} = \dfrac{\Delta_2}{1}$

The least value of $f(x) = \dfrac{-\Delta_1}{4} = \dfrac{-1}{4} \Rightarrow \Delta_1 = 1 \Rightarrow \Delta_2 = 1$

Hence, the correct answer is (A).

36. The least value of $g(x)$ occurs at $x = 7/2$ i.e

$$\frac{-b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7 \Rightarrow b_2^2 - 4c_2 = \Delta_2 \Rightarrow c_2 = 12$$

Hence, the correct answer is (D).

37. Roots of $x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4$

Hence, the correct answer is (C).

Matrix-Match Type

1. A → p, q, s, B → q, s, C → p, q, s, D → r

(A) Discriminant $= 1 + 4a$

$1 + 4a$ must be perfect square

$1 + 4a = (2\lambda + 1)^2; \lambda \in N$ (as $1 + 4a$ is odd)

$a = \lambda(\lambda + 1)$ which is an even number

(B) $f(x) = ax^2 + 2bx + 4c - 16 = 0$

$f(-2) = 4a - 4b + 4c - 16 = 4(a + c - b - 4)$

$f(-2) > 0$ as $(a + c > b + 4)$

$f(0) > 0 \ \Rightarrow\ 4c - 16 > 0 \ \Rightarrow\ c > 4$

(C) As both roots negative so $2b > 0$ and $9b - 14 > 0$ …(1)

$D \geq 0 \ \Rightarrow\ 4b^2 - 4(9b - 14) \geq 0$

$\Rightarrow \quad b^2 - 9b + 14 \geq 0$

$(b-2)(b-7) \geq 0 \ \Rightarrow\ b \in (-\infty, 2] \cup [7, \infty)$ …(2)

Intersection of (1) and (2) $b \in [7, \infty) \cup \left(\dfrac{14}{9}, 2\right]$

(D) $|x^2 - x - 6| = x + 2 \Rightarrow |(x-3)(x+2)| = x + 2$

$$\begin{cases} x = 4 & ; \quad x < -2 \\ x = -2, 2 & ; \quad x \in [-2, 3) \\ x = 4 & ; \quad x \geq 3 \end{cases}$$

$\therefore\ N = 3$

2. A → q; B → r, s; C → s; D → p

(A) (i) $f\left(\frac{1}{2}\right) > 0$, (ii) $D \geq 0$, (iii) $-\frac{b}{2a} > \frac{1}{2}$

(i) $4\left(\frac{1}{4}\right) - 4(\alpha - 2)\left(\frac{1}{2}\right) + (\alpha - 2) > 0$

$\Rightarrow 1 - 2(\alpha - 2) + \alpha - 2 > 0$

$\Rightarrow \alpha - 2 < 1 \Rightarrow \alpha < 3$

(ii) $(\alpha - 2)^2 - (\alpha - 2) \geq 0$

$\Rightarrow (\alpha - 2) - (\alpha - 3) \geq 0 \Rightarrow \alpha \in (-\infty, 2] \cup [3, \infty)$

(iii) $-\frac{-4(\alpha - 2)}{4} > \frac{1}{2} \Rightarrow \alpha > \frac{5}{2}$

From (i), (ii) and (iii) $\alpha \in \phi$

(B) (i) $f(0)\, f\left(\frac{1}{2}\right) < 0$,

$(\alpha - 2)(3 - \alpha) < 0 \Rightarrow (\alpha - 2)(\alpha - 3) > 0 \Rightarrow \alpha \in (-\infty, 2) \cup (3, \infty)$

(C) $\because$ leading coefficient > 0, $f(0) < 0$

$\Rightarrow \alpha - 2 < 0 \quad \Rightarrow \quad \alpha < 2$

(D) If both roots are zero then put $\alpha = 2$.
or coefficient of x = constant term = 0.

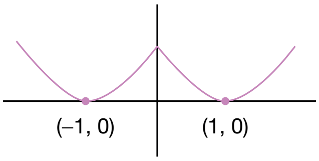
3. A → s, B → q, C → r, D → p

Let take $|x|^2 + a|x| + 1 = 0$

(A) If $a \geq 0$, vertex $= \left(\frac{-a}{2}, \frac{4 - a^2}{4}\right)$

expression is +ve $\therefore$ No solution

(B) If $a = -2$

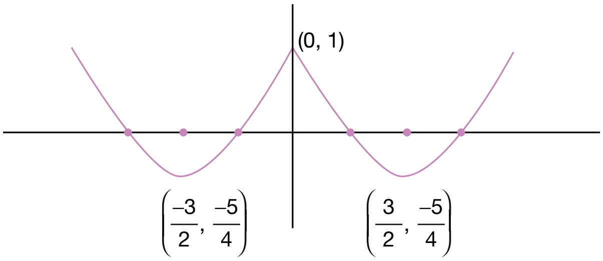


$\therefore$ Two real solution

(C) Three real solution not possible

(D) Four real solutions $\therefore a < -2$

For example : $a = -3$



(0, 1)
$\left(\frac{-3}{2}, \frac{-5}{4}\right)$
$\left(\frac{3}{2}, \frac{-5}{4}\right)$

Four real solutions

4. A → r; B → s; c → p; D → t

$(k - 1)(k + 1)x^2 + (2k + 7)(k - 1)x + (2k + 7)(k - 2) = 0$

(A) Exactly one root is '0' and other root is finite.
Product of roots. = 0 but sum of roots ≠ 0
Therefore $k = 2$.

(B) Both roots are zero means sum of roots = Product of roots = 0
Hence, $k = \frac{-7}{2}$ only

(C) Exactly one root infinity
i.e. coefficient of $x^2 = 0$ but coefficient of $x \neq 0$
Therefore $k = -1$ only.

(D) Both roots are infinity
i.e., coefficient of x^2 = coefficient of $x = 0$ but constant term $\neq 0$
Therefore $k = 1$ only.

5. A → p; B → q, r; C → p, q, r, s; D → s

$x^4 - 6x^3 + 8x^2 + 4ax - 4a^2 = 0$

$\Rightarrow \quad \left(x^2 - 2x - 2a\right)\left(x^2 - 4x + 2a\right) = 0$

Now $D_1 = 4(1 + 2a)$

$D_2 = 8(2 - a)$

(A) $D_1 > 0$ and $D_2 > 0$

$4(1 + 2a) > 0$ and $8(2 - a) > 0$

$a > -\frac{1}{2}$ and $a < 2 \quad \Rightarrow \quad a \in \left(-\frac{1}{2}, 2\right)$; $(A : P)$

(B) $D_1 > 0$ and $D_2 < 0$ or $D_1 < 0$ and $D_2 > 0$

$a > -\frac{1}{2}$ and $a > 2$ or $a < -\frac{1}{2}$ and $a < 2$

$a \in (2, \infty) \cup \left(-\infty, -\frac{1}{2}\right)$; $(B : q, r)$

(C) Product of roots is negative of any quadratic
So, $-2a < 0$ or $2a < 0$
$a > 0$ or $a < 0$
$\Rightarrow \quad a \in R - \{0\}$; $(C : p, q, r, s)$

(D) $D_1 = 0$ and $D_2 > 0$ or $D_1 > 0$ and $D_2 = 0$
$1 + 2a = 0$ and $8(2 - a) > 0$ or $1 + 2a > 0$ and $8(2 - a) = 0$

$a = -\frac{1}{2}$ and $a < 2$ or $a > -\frac{1}{2}$ and $a = 2$

$a = -\frac{1}{2}$ or $a = 2$

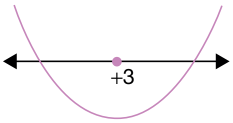
$a \in \left\{-\frac{1}{2}, 2\right\}$; $(D : S)$

6. (A) → q; (B) → s; C → r

(A) Maximum value of $P(x)$ in [1, 2] occurs at $x = 1$.
Hence, $P(1) = 1 \quad \Rightarrow \quad c - 10 = 1 \quad \Rightarrow \quad c - 11$

(B) Minimum value of $P(x)$ in [1, 5] occurs at $x = 3$,
Hence, $P(3) = -1 \quad \Rightarrow \quad c - 18 = -1 \quad \Rightarrow \quad c = 17$.

(C) Maximum value of $P(x)$ in [1, 4] occurs at $x = 1$
Hence, $P(1) = 2 \quad \Rightarrow \quad c - 10 = 2 \quad \Rightarrow \quad c = 12$.

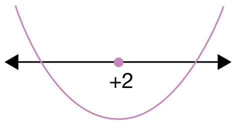


7. A → p; B → s; C → r

(A) $f(2) < 0; 3(2)^2 - 7k(2) + 5k - 3 < 0$

$12 - 14k + 5k - 3 < 0$

$9 - 9k < 0 \qquad \therefore k > 1$



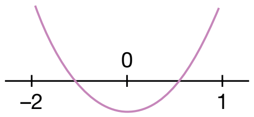
(B) $f(-2) > 0 \quad \Rightarrow \quad 3(4) + 14k + 5k - 3 > 0$

$\Rightarrow \quad 9 + 19k > 0 \quad \Rightarrow \quad k > \frac{-9}{19} \qquad \ldots(1)$

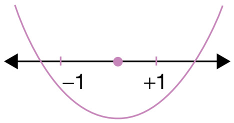
$f(0) < 0 \quad \Rightarrow 5k - 3 < 0 \quad \Rightarrow k < \frac{3}{5} \qquad \ldots(2)$

$f(1) < 0 \quad \Rightarrow 3 - 7k < 5k - 3 > 0 \quad \Rightarrow k < 0 \qquad \ldots(3)$

$(1) \cap (2) \cap (3), k \in \left(\frac{-9}{19}, 0\right)$



(C)



$f(-1) = 3 + 7k + 5k - 3 < 0 \quad \Rightarrow k < 0 \qquad \ldots(1)$

$f(1) = 3 - 7k + 5k - 3 < 0 \quad \Rightarrow k > 0 \qquad \ldots(2)$

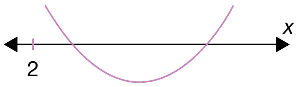
$(1) \cap (2)$ we get $k \in \phi$

8. A → s; B → q, r; C – p

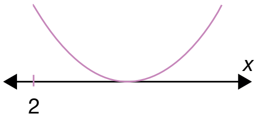
(A) We have $f(x) = x^2 + bx + 4$

$\therefore D \geq 0 \quad \Rightarrow b^2 - 16 \geq 0 \quad \Rightarrow b \in (-\infty, -4] \cap [4, \infty)$

and $f(2) > 0 \quad \Rightarrow 4 + 2b + 4 > 0 \quad \Rightarrow b > -4$



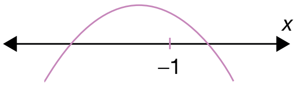
or



Also, $2 < \frac{-b}{2} \quad \Rightarrow 4 < -b \quad \Rightarrow b < -4.$

Hence $b \in \phi$.

(B)

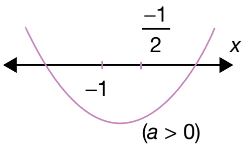


We have $f(x) = -x^2 + 5x + c$

Clearly $f(-1) > 0 \quad \Rightarrow -1 - 5 + c > 0$

Hence $c > 6$.

(C)



We have $f(x) = ax^2 + 6x + 1$

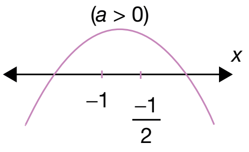
Now $D > 0 \quad \Rightarrow 36 - 4a > 0 \quad \Rightarrow a < 9$

CASE-1: If $a > 0$.

$f(-1) < 0 \quad \Rightarrow \quad a < 5$

and $f\left(\frac{-1}{2}\right) < 0 \quad \Rightarrow \quad a < 8$

Hence $a \in (0, 5)$



CASE-2: If $a < 0, f(-1) > 0$ and $f\left(\frac{-1}{2}\right) > 0$

$\Rightarrow a - 6 + 1 > 0$ and $\frac{a}{4} - 3 + 1 > 0 \Rightarrow a > 5$ and $\frac{a}{4} > 2$

$\therefore a > 8$ (Rejected)

Hence we conclude from CASE-1 and CASE-2 that $a \in (0, 5)$.

Integer/Numerical Answer Type Questions

1. $ab = d$ and $cd = b$. Also, $a + c + d = 0 = a + b + c$

Thus $b = d \ (\neq 0)$ and $a = c = 1$.

$\therefore \quad b = d = -2$

$a + b + c + d = -2$

Hence, the correct answer is (2).

2. Let $(x - x_1)(x - x_2) = x_2 + px + q$

$(x - x_3)(x - x_4) = x_2 + px + r$

$\Rightarrow \quad q = x_1x_2$ and $r = x_3x_4$

$x^4 + ax^3 + bx^2 + cx + d = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$

$= (x^2 + px + q)(x^2 + px + r)$

$= x^4 + 2px^3 + (q + r + p^2)x^2 + (pr + pq)x + qr$

$\Rightarrow \quad a = 2p, b = p^2 + q + r, c = p(q + r), d = qr$

$a^3 - 4ab + 8c = 0$ If $a = 2$, then $0 = 8 - 8b + 8c \Rightarrow b - c = 1$

Hence, the correct answer is (1).

3. If roots are opposite sign, then ‘0’ lies between the roots $\Rightarrow$ $a^2 - 9 < 0 \Rightarrow a \in (-3, 3)$

Hence, the correct answer is (6).

4. t is root of $x^{n+1} - 1 = 0 \Rightarrow t^{n+1} = 1$

Hence, the correct answer is (3).

5. Required roots are $1, 1, 1 + \alpha, 1 + \beta, 1 - \alpha, 1 - \beta$

Hence, the correct answer is (6).

6. $a^2 - b = b^2 - c \Rightarrow a + b = \frac{b - c}{a - b}$ similarly $b + c = \frac{c - a}{b - c}$ and $c + a = \frac{a - b}{c - a}$

$\therefore$ Product is equal to 1.

Hence, the correct answer is (1).

7. Here $P(x) = Q(x)(x^2 + 1) + x^2 - x + 1$

$$\therefore \sum P(\alpha) = \sum \alpha^2 - \sum \alpha + 4$$

$$=\left(\sum\alpha\right)^2-2\sum\alpha\beta-\sum\alpha+4=1+2-1+4=6$$

Hence, the correct answer is (6).

8. $D\ge 0\Rightarrow 4(a^2-a^2-a+3)\ge 0\Rightarrow a\le 3$ …(1)

$-\dfrac{B}{2A}<3 \quad\Rightarrow\quad a<3$ …(2)

$f(3)>0 \quad\Rightarrow\quad a\in(-\infty,2)\cup(3,\infty)$ …(3)

From (1), (2) and (3)

$$a\in(-\infty,2)$$

Hence, the correct answer is (1).

9. $[x]=-x^2+3x$

If $[x]=0\Rightarrow x\in[0,1)$ $x^2-3x=0\Rightarrow x=0,3$, solution is $x=0$

If $[x]=1\Rightarrow x\in[1,2)$ $x^2-3x+1=0\Rightarrow$ no solution

If $[x]=2\Rightarrow x\in[2,3)$ $x^2-3x+2=0\Rightarrow x=1,2$, solution $x=2$

If $[x]=-1\Rightarrow x\in[-1,0)$ $x^2-3x-1=0\Rightarrow$ one solution

$$n_1=3, n_2=0, n_3=2$$

Hence, the correct answer is (5).

10. $(a+c)^2-b^2<0$

$$(a-b+c)(a+b+c)<0$$

$$f(-1)f(1)<0$$

$f(x)=0$ has exactly one solution in $(-1, 1)$

Hence, the correct answer is (1).

11. $(x-1)(x-a)=1$

$x-1=1, x-a=1$ (or) $x-1=-1, x-a=-1$

$x=2, a=1$ (or) $x=0, a=1$

Hence, the correct answer is (1).

12. $2x^2-6x-5\sqrt{x^2-3x-6}=0$

$2x^2-6x=5\sqrt{x^2-3x-6}$

$2x^2-6x\in$ Rational Number

$\{\because x^2-3x-6$ is not a perfect square$\}$

$5\sqrt{x^2-3x-6}\in$ Irrational Number

So, no solution

Hence, the correct answer is (0).

13. Let $f(x)=x^2+2(m-1)x+m+5$

CASE-1: $f(x)>0\,\forall x$ if $\Delta<0 \Rightarrow m\in(-1,4)$ …(1)

CASE-2: $f(x)>0\,\forall\, x>1$ if both roots are smaller than 1

$$\Delta>0,\ f(1)>0,\ \frac{-b}{2a}<1$$

$\therefore \quad m\in(4,\infty)$ …(2)

Taking union of (1) and (2)

$$m\in(-1,\infty)$$

$\therefore \quad k=1$

Hence, the correct answer is (1).

14. $x^3+3x^2+3-a=0$

$$\alpha+\beta+\gamma=-3$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=0$$

$$\alpha^2+\beta^2+\gamma^2=9$$

$$\alpha,\beta,\gamma\in I$$

CASE-1: $\alpha^2=9, \beta^2=\gamma^2=0$

$$\alpha=-3$$

$$a=3+9(-3+3)=3$$

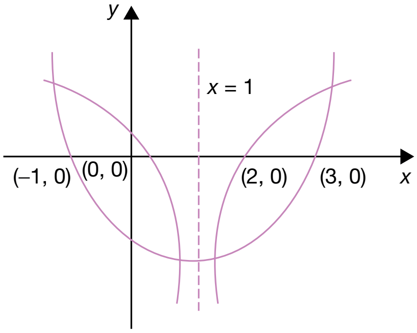
CASE-2: $\alpha^2=4, \beta^2=4, \gamma^2=1$

$$\alpha=-2, \beta=-2, \gamma=1$$

$$a=3+4(-2+3)=7.$$

Hence, the correct answer is (2).

15. $x^2-2x-\log_2|1-x|=3$



$$x^2-2x-3=\log_2|1-x|$$

so $y=x^2-2x-3$ …(1)

$y=\log_2|1-x|$ …(2)

(1) and (2) intersect at 4 points

Hence, the correct answer is (4).

16. $D\ge 0\Rightarrow a^2\ge\frac{1}{4}\Rightarrow a\le-\frac{1}{2}$ *or* $a\ge\frac{1}{2}$

CASE-1: If $a\le-\frac{1}{2}$, then a is negative and $x_1\ge a, x_2\ge 0$

$\Rightarrow$ atleast one root must be positive

$\Rightarrow$ either both roots are positive or exactly one root is positive.

Now $x^2-4ax+1=0$, can not have both roots positive. Since sum of the roots $=4a<0$.

Again exactly one root can not be positive since product >0

$\Rightarrow a\le-\frac{1}{2}$ is not possible

CASE-2: If $a\ge\frac{1}{2}$, then $x_1\ge+\frac{1}{2}, x_2\ge 0$

$\Rightarrow \quad x_1+x_2\ge 0\Rightarrow 4a\ge 0\Rightarrow a\ge\frac{1}{2}$

so $a\in\left[\frac{1}{2},\infty\right)\Rightarrow k=2.$

Hence, the correct answer is (2).

17. $|x^2-3x-1|<|3x^2+2x+1|+|2x^2+5x+2|, x^2-3x-1\ne 0$

$\Leftrightarrow |(3x^2+2x+1)-(2x^2+5x+2)|<|3x^2+2x+1|+|2x^2+5x+2|, x^2-3x-1\ne 0$

The inequality holds if and only if

$$(3x^2 + 2x + 1)\ (2x^2 + 5x + 2) > 0$$

i.e $2x^2 + 5x + 2 > 0$

i.e $(2x + 1)\ (x + 2) > 0$

i.e. $x \in (-\infty, -2) \cup (-1/2, \infty) \quad \Rightarrow a = 2$ and $b = \dfrac{1}{2}$

$\therefore \quad a + \log ab = 2$

Hence, the correct answer is (2).

18. $\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{b}\right) = 4$, for $a, b \in I$

$$a \in I \Rightarrow \frac{1}{a} \le 1$$

$$\Rightarrow \quad 1+\frac{1}{a} \le 2 \Rightarrow \left|1+\frac{1}{a}\right| \le 2 \text{ and } \left|1+\frac{1}{b}\right| \le 2$$

$$\Rightarrow \quad \left|1+\frac{1}{a}\right| . \left|1+\frac{1}{b}\right| \le 4$$

This is true only when $a = 1$ and $b = 1$

$\therefore$ Number of integral solutions is one.

Hence, the correct answer is (1).

19. $x = \dfrac{2z^2}{1+z^2} \Rightarrow z^2 = \dfrac{x}{2-x}$ and similarly others.

This implies that x, y, z all lie in [0, 2] and in view of symmetry all are equal.

$\therefore x^2 = \dfrac{x}{2-x} \Rightarrow x = 0$ or 1. Hence there are 2 solutions.

Hence, the correct answer is (2).

20. $9t - \dfrac{8}{t} + 1 = 0 \Rightarrow t = -1, \dfrac{8}{9}$; But $t = \sqrt{\dfrac{x}{2-x}}$ is +ve

$$\therefore \quad t = \frac{8}{9}$$

when $t = \dfrac{8}{9}, x = \dfrac{128}{145}$ which lies between 0 and 2.

Hence number of roots = 1

Hence, the correct answer is (1).

21. We have $a^2 - (2x^2 + 1)a + x^4 + x = 0$

$$\therefore \quad a = \frac{(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4(x^4+x)}}{2}$$

$$2a = (2x^2 + 1) \pm (2x - 1)$$

$\Rightarrow \quad x^2 + x - a = 0$ and $x^2 - x + 1 - a = 0$

Both have real roots hence $D = 1 + 4a \ge 0$ and $1 - 4(1 - a) \ge 0$

We get $a \ge \dfrac{3}{4}$

$\therefore \quad m + n = 7$

Hence, the correct answer is (7).

22. Given equation can be written as $(2x + 3)(x^{98} + x^{96} + \cdots + 1) = (2x + 3)\dfrac{(x^{100} - 1)}{x^2 - 1}$

So, the real roots are $x = \pm 1, \dfrac{-3}{2}$, out of which ±1 are not roots of given equation.

Hence, the correct answer is (1).

23. Let us form that equation having roots $y = g(x_i)$ i.e., $y = x^2 - 2$

$$x = \sqrt{y+2}$$

$$\Rightarrow \quad \left(\sqrt{y+2}\right)^5 + \left(\sqrt{y+2}\right)^2 + 1 = 0$$

$$\Rightarrow \quad y^5 + 20y^4 + 40y^3 + 79y^2 + 74y + 23 = 0$$

$$\therefore \quad g(x_1)\ldots g(x_2) = \text{ Product of roots} = -23$$

$$x_1x_2x_3x_4x_5 = -1 \Rightarrow g(x_1x_2\ldots x_5) = -1$$

$$\Rightarrow \quad g(x_1)g(x_2)\ldots g(x_5) - 30g(x_1x_2\ldots x_5) = 7$$

Hence, the correct answer is (7).

24. $|x_1 + x_2||x_1 - x_2| > \dfrac{1}{a} \Rightarrow \left|-\dfrac{1}{a}\right|\left|\sqrt{\dfrac{1}{a^2} - 4}\right| > \dfrac{1}{a}.(*)$

The inequation (*) has meaning if $\dfrac{1}{a^2} - 4 > 0$

If $a \in \left(-\dfrac{1}{2}, 0\right)$ then (*) is automatically satisfied

If $a \in \left(0, \dfrac{1}{2}\right)$ then (*) becomes equivalent to $\sqrt{\dfrac{1}{a^2} - 4} > 1$ (on canceling $\dfrac{1}{a} > 0$)

$$\Rightarrow \quad -\frac{1}{\sqrt{5}} < a < \frac{1}{\sqrt{5}}$$

but $a \in \left(0, \dfrac{1}{2}\right)$ was assumed $\Rightarrow a \in \left(0, \dfrac{1}{\sqrt{5}}\right)$

Thus all the values of a lie in the interval $\left(-\dfrac{1}{2}, 0\right) \cup \left(0, \dfrac{1}{\sqrt{5}}\right)$

$\Rightarrow k = 5$

Hence, the correct answer is (5).

25. Imposing the conditions; $\dfrac{b}{2a} > 2$, $b^2 \ge 48a$ and $f(2) = 4a - 2b + 12 > 0$ there is only one solution for $(a, b) \equiv (1, 7)$

Hence, the correct answer is (1).

26. $f(x) = x^2 - ax + 6a$

$$D > 0 \Rightarrow a^2 - 4.6a > 0 \Rightarrow a \in (-\infty, 0) \cup (24, \infty) \qquad \ldots(1)$$

$$f(1) < 0 \Rightarrow 1 - a + 6a < 0 \Rightarrow a < \frac{1}{5} \qquad \ldots(2)$$

$$f(-1) < 0 \Rightarrow 1 + a + 6a < 0 \Rightarrow a < -\frac{1}{7} \qquad \ldots(3)$$

From (1), (2), (3)

Value of a are $-4, -3, -2, -1$

$$(-4)^2 + (-3)^2 + (-2)^2 + (-1)^2 = 30$$

Hence, the correct answer is (5).

27. $\left(\dfrac{1}{3}\right)^{1/2\log_{1/3}(x^2 - \frac{10}{3}x+1)} \le 1$

$$0 < x^2 - \frac{10x}{3} + 1 \le 1$$

$$0 \le 3x^2 - 10x + 3 \le 3 \Rightarrow x \in [0, \frac{1}{3}) \cup (3, \frac{10}{3}]$$

Hence, the correct answer is (3).

28. $yx^2 - y = ax + 1 \Rightarrow yx^2 - ax - y - 1 = 0$

$\Rightarrow \quad a^2 + 4(y)(y+1) \geq 0$

$\Rightarrow \quad 4y^2 + 4y + a^2 \geq 0$ it is true for all real y

$D = 16 - 4 \times 4a^2 \leq 0 \Rightarrow 1 - a^2 \leq 0 \Rightarrow a^2 - 1 \geq 0$

$a \in (-\infty, -1] \cup [1, \infty)$ but $-1 \geq a \geq -2$

so $a = -2, -1$

Hence, the correct answer is (5).

29. Given $f(x) > g(x) \ \forall \ x \in R$

$\Rightarrow \quad px^2 + px - 2 > x^2 + (2p+1)x + p + 2 \ \forall \ x \in R$

$\Rightarrow \quad (p-1)x^2 - (p+1)x - p - 4 > 0 \ \forall \ x \in R$

So, $p - 1 > 0 \quad \Rightarrow p > 1$

and, $D < 0 \quad \Rightarrow (p+1)^2 + 4(p-1)\,(p+4) < 0$

$\Rightarrow \quad 5p^2 + 14p - 15 < 0$

$$\Rightarrow \quad \left(p - \frac{-14 - \sqrt{196 + 300}}{10}\right)\left(p - \frac{-14 + \sqrt{196 + 300}}{10}\right) < 0$$

$\Rightarrow \quad (p - (-3.6))\,(p - 0.8) < 0$

$p \in (-3.6, 0.8)$ but $p > 1$

Hence, there is no value of p for which $f(x) > g(x) \ \forall x \in R$

Hence, the correct answer is (0).

30. $A = \dfrac{3}{x^2 - x + 1} - 1 > 0$

$$\Rightarrow \quad \frac{3 - x^2 + x - 1}{x^2 - x + 1} > 0 \quad \Rightarrow \quad x^2 - x - 2 < 0 \ (x^2 - x + 1 > 0)$$

$\Rightarrow \quad (x-2)\,(x+1) < 0 \quad \Rightarrow \quad x \in (-1, 2)$

Number of integers $\{0, 1\} \quad \Rightarrow \quad A = 2$

B = coefficient of $x^2 = (k-3)\,(2k+1) = 0$

Coefficient of $x(k-3)(k-2) = 0$

absolute term $= (k-3)\,(k+3) = 0$

$B = 3$

$X = 2x^4 - 30x^2 + 8x + 10$

given $x = 2 + \sqrt{3} \quad \Rightarrow (x-2)^2 = 3 \quad \Rightarrow \quad x^2 - 4x + 1 = 0$

$$\underbrace{2x^2(x^2 - 4x + 1)}_{\text{zero}} + 8x^3 - 32x^2 + 8x + 10$$

$$\Rightarrow \quad \underbrace{8x^2(x^2 - 4x + 1)}_{\text{zero}} + 10 = 10 \quad \Rightarrow \quad X = 10$$

$(X \div A + B) = (10 \div 2 + 3) = 5 + 3 = 8$

Hence, the correct answer is (8).

31. Let $\alpha = n^2, \beta = (n+1)^2$

$\sqrt{\alpha\beta} = |\alpha - \beta| + 1$

Hence, $n = 2$

$\therefore \quad \alpha = 4, \beta = 9$

Now, $f(4) \cdot f(9) < 0$ and also checking boundary points.

We get $k \in \left[12, \dfrac{113}{9}\right)$.

Hence, the correct answer is (1).

32. $f(x) = x^2 + (2a-1)x + (a+3)$

$f(0) = (a+3)$

$f(2) = 5(a+1)$

$f(0) \cdot f(2) < 0 \quad \Rightarrow \quad a \in (-3, -1)$

Now, check when $f(0) = 0$ or $f(2) = 0$

$f(0) = 0 \quad \Rightarrow \quad a = -3$

when $a = -3$, the quadratic equation becomes

$x^2 - 7x = 0 \quad \Rightarrow \quad x = 0$ and $x = 7$

Hence, $a = -3$ is acceptable

when $f(2) = 0 \quad \Rightarrow \quad a = -1$

the quadratic equation becomes $x^2 - 3x + 2 = 0$

$\Rightarrow \quad x = 1$ or $x = 2$

Hence, $a = -1$ is not acceptable

$\therefore \quad a \in [-3, -1)$

Hence, the correct answer is (9).

33. Let $y = \log_{10}(4x^2 - ax + 3)$

Since y is non-negative for all $x \in [1, 3]$

$\therefore \quad y \geq 0$ for all $x \in [1, 3]$

$\Rightarrow \quad (4x^2 - ax + 3) \geq 1$ for all $x \in [1, 3]$

$4x^2 - ax + 2 \geq 0$ for all $x \in [1, 3]$

Let $f(x) = 4x^2 - ax + 2$

CASE-1: When $\dfrac{a}{8} \leq 1$

This case is not possible

CASE-2: $1 < \dfrac{a}{8} < 3 \quad \Rightarrow \quad a \in (8, 24)$

now, $f\left(\dfrac{a}{8}\right) \geq 0 \quad \Rightarrow \quad a \in \left[-4\sqrt{2}, 4\sqrt{2}\right]$

This is not possible.

CASE-3: $\dfrac{a}{8} \geq 3 \quad \Rightarrow \quad a \geq 24$

$f(3) \geq 0$

$a \leq \dfrac{38}{3}$, which is not possible

Hence, from the above cases $a \in \phi$.

Hence, the correct answer is (0).

34. $\left(x^2 + \dfrac{1}{x^2}\right) + 2k\left(x + \dfrac{1}{x}\right) + 1 = 0.$

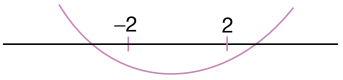
$$\left(x + \frac{1}{x}\right)^2 + 2k\left(x + \frac{1}{x}\right) - 1 = 0$$

Put $\left(x + \dfrac{1}{x}\right) = t$

$t^2 + 2kt - 1 = 0$

Now, one root of $t^2 + 2kt - 1 = 0$ will lie in $(-\infty, -2)$ and other root in $(2, \infty)$.

$\left.\begin{array}{l} f(-2) < 0 \\ f(2) < 0 \end{array}\right\} \cap$



Hence, $k \in \left(-\infty, \dfrac{-3}{4}\right) \cap \left(\dfrac{3}{4}, \infty\right)$

Hence, the correct answer is (8).

35. Let V_1 and V_2 be the vertices of the parabola $f(x)$ and $g(x)$ respectively.

[Hence max. value of $f(x) = a$ and min. value of $g(x) = 10 - 2a - a^2$]

We have $V_1 \equiv (1, a)$ and $V_2 \equiv (a, -a^2 + 10 - 2a)$

Now, $a < -a^2 + 10 - 2a$

$\Rightarrow \quad a^2 + 3a - 10 < 0$

$\therefore \quad (a + 5)(a - 2) < 0$

$\Rightarrow \quad -5 < a < 2$

$\therefore$ Largest natural number $a = 1$

Hence, the correct answer is (1).

36. We have

$$y = \log_2\left(\frac{(x^2+x+1)(x^2-x+1)}{x^4+x^3+2x^2+x+1}\right) = \log_2\left(\frac{x^2-x+1}{x^2+1}\right)$$

Assume $z = \dfrac{x^2-x+1}{x^2+1}$

$\Rightarrow \quad (z-1)x^2 + x + (z-1) = 0$. As $x \in R$, so

$D \geq 0 \quad \Rightarrow \quad 1 - 4(z-1)^2 \geq 0 \quad \Rightarrow \quad (z-1)^2 \leq \dfrac{1}{4}$

$\Rightarrow \quad \dfrac{1}{2} \leq z \leq \dfrac{3}{2}$

So, $y_{\min} = \log_2\left(\dfrac{1}{2}\right) = -1$

and $y_{\max} = \log_2\left(\dfrac{3}{2}\right) = (\log_2 3 - 1)$

Hence $y_{\min} + y_{\max} + \log_2 3 - 2 \equiv ((\log_2 m) - n)$

$\therefore \quad m = 3, n = 2$

Hence $(m + n) = 3 + 2 = 5$.

Hence, the correct answer is (5).

37. We have $\alpha + \beta = \dfrac{-6}{2} = -3$ and $\alpha\beta = \dfrac{b}{2}$

Now, $\left(\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}\right) = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$= \frac{-27 - 3\left(\frac{b}{2}\right)(-3)}{\frac{b}{2}} = \frac{-54+9b}{b} = 9 - \frac{54}{b}$$

Given $b < 0$, so $\left(\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}\right)$ will have least integral value 10,

which is possible when $b = -54$.

Hence, the correct answer is (10).

38. $\alpha^3 + 2\alpha^2 - \alpha + 3 = 0$

$\Rightarrow \quad \alpha^3 + \alpha^2 + 3 = \alpha - \alpha^2$

$= \alpha(1 - \alpha)$

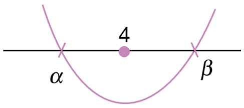
$$\therefore \quad \frac{(\alpha^3+\alpha^2+3)(\beta^3+\beta^2+3)(\gamma^3+\gamma^2+3)}{\alpha\beta\gamma}$$

$$= \frac{\alpha(1-\alpha)\beta(1-\beta)\gamma(1-\gamma)}{\alpha\beta\gamma} = (1-\alpha)(1-\beta)(1-\gamma)$$

$= 1 - (\alpha + \beta + \gamma) + (\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma = 5$

Hence, the correct answer is (5).

39.



4 lies between roots

$\Rightarrow \quad f(4) < 0$

$\Rightarrow \quad 16 - 4k - 16 + k^2 - 12 < 0$

$\Rightarrow \quad k^2 - 4k - 12 < 0$

$\Rightarrow \quad (k-6)(k+2) < 0$

$\Rightarrow \quad k \in (-2, 6) \quad \Rightarrow \quad$ 7 integral values

Hence, the correct answer is (7).

40. $(x^2 - 7x + 6)(9^x + 3) \leq 0$

or $\quad x^2 - 7x + 6 \leq 0 \qquad [\because\ 9^x + 3 > 0]$

or $\quad (x-1)(x-6) \leq 0$

or $\quad x \in [1, 6]$

Hence, the correct answer is (6).

41. $8x - 7 - x^2 > 0$

or $\quad x^2 - 8x + 7 < 0$

$x \in (1, 7)$

for $x \in (1, 7)$

$\log_3(x^2 + 4x - 3) > 0$ and $\log_5(x^2 + 6x - 5) > 0$

$x^2 + 4x - 3 > 1$

$\Rightarrow \quad x^2 + 4x - 4 > 0 \quad x \in (-\infty, -2-2\sqrt{2}) \cup (-2+2\sqrt{2}, \infty)$

$x^2 + 6x - 5 > 1$

$\Rightarrow \quad x^2 + 6x - 6 > 0$

$x \in (-\infty, -3-\sqrt{15}) \cup (-3+\sqrt{15}, \infty)$

So $x \in (1, 7)$

Hence, the correct answer is (5).

42. $\alpha + \beta + \gamma = c \quad \therefore\ \alpha\beta + \beta\gamma + \gamma\alpha = 4;\ \alpha\beta\gamma = 2$

$$\therefore \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{4}{2} = 2$$

$$\because \quad f\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = 2 \quad \Rightarrow \quad f(2) = 2$$

$\Rightarrow \quad 8 - 4c + 8 - 2 = 2 \quad \Rightarrow \quad c = 3$

Hence, the correct answer is (3).

43. $\min(f(x)) = \dfrac{-D}{4a} = \dfrac{-(4b^2 - 8c^2)}{4} = 2c^2 - b^2$

$\max(g(x)) = \dfrac{-D}{4a} = \dfrac{-(4c^2 + 4b^2)}{-4} = c^2 + b^2$

$\therefore$ For $\min(f(x)) < \max(g(x))$

$\Rightarrow \quad 2c^2 - b^2 < b^2 + c^2 \quad \Rightarrow \quad 2b^2 > c^2$

$\Rightarrow \quad \dfrac{b^2}{c^2} > \dfrac{1}{2} \Rightarrow$ minimum integral value of $\dfrac{b^2}{c^2} = 1$

Hence, the correct answer is (1).

44. Given, $6x^2 + 5x - 1 < 0 \Rightarrow (6x - 1)(x + 1) < 0$

$\Rightarrow \quad \underbrace{(k^2 - 3k + 2)x - 2}_{f(x)} > 0$ for some x in $\left(-1, \dfrac{1}{6}\right)$

Note : $f(-1) = -k^2 + 3k - 4 < 0,\ \forall\ k \in R$.

$\therefore \quad f\left(\dfrac{1}{6}\right) > 0$ for solution.

$\Rightarrow \quad k^2 - 3k - 10 > 0 \Rightarrow (k-5)(k+2) > 0$

$\therefore \quad k \in (-\infty, -2) \cup (5, \infty)$

$\Rightarrow \quad$ Minimum natural k is 6.

Hence, the correct answer is (6).

45. We have $3x^2 + x - 1 = 0$

Clearly, $|\alpha| < 1$, $|\beta| < 1$.

Also, $(\alpha + \beta) = \frac{-1}{3}$ and $\alpha\beta = \frac{-1}{3}$

$$\therefore \quad \sum_{r=1}^{\infty}(\alpha^r + \beta^r) = (\alpha + \alpha^2 + \alpha^3 + \ldots \infty) + (\beta + \beta^2 + \beta^3 + \ldots \infty)$$

$$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta}$$

$$= \frac{\alpha + \beta - 2\alpha\beta}{1-(\alpha+\beta)+\alpha\beta} = \frac{\frac{-1}{3}+\frac{2}{3}}{1+\frac{1}{3}-\frac{1}{3}} = \frac{1}{3}$$

Hence, the correct answer is (6).

46. $(a-1)^2 m^2 + m(a-1)(a+1) + (a-1)^2 - b = 0 \ \forall\ m \in R$.

$$\therefore \quad \left.\begin{array}{r}(a-1)^2 = 0\\ (a-1)(a+1) = 0\\ \text{and}\,(a-1)^2 - b = 0\end{array}\right\} \Rightarrow a = 1 \text{ and } b = 0$$

Hence $a^2 + b^2 = 1$

Hence, the correct answer is (1).

47. $3x + my = m$ ...(1)

$2x - 5y = 6$ ...(2)

Solving (1) and (2), we get

$$x = \frac{11m}{15+2m} \text{ and } y = \frac{2m-18}{2m+15}$$

Since $x > 0$ and $y < 0$

$$m \in \left(-\infty, -\frac{15}{2}\right) \cup (0, \infty) \text{ and } \left(-\frac{15}{2}, 9\right)$$

$\Rightarrow \quad m \in (0, 9)$

Number of integral value is 8.

Hence, the correct answer is (8).

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1. $\tan\alpha + \tan\beta = \frac{\lambda\sqrt{2}}{k+1}$; $\tan\alpha\tan\beta = \frac{k-1}{k+1}$

$$\tan(\alpha+\beta) = \frac{\frac{\lambda\sqrt{2}}{k+1}}{1-\frac{k-1}{k+1}} = \frac{\lambda\sqrt{2}}{2} = \frac{\lambda}{\sqrt{2}}$$

$\Rightarrow \quad \frac{\lambda^2}{2} = 50 \Rightarrow \lambda = 10$ and -10

Hence, the correct answer is (B).

2. $\alpha + \beta = 1$, $\alpha\beta = -1$ and $P_k = \alpha^k + \beta^k$

$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha^k - \alpha^{k-1} - \alpha^{k-2} = 0$ and $\beta^k - \beta^{k-1} - \beta^{k-2} = 0$

adding both we get $\alpha^k + \beta^k - (\alpha^{k-1} + \beta^{k-1}) - (\alpha^{k-2} + \beta^{k-2}) = 0$

$\Rightarrow P_k = P_{k-1} + P_{k-2}$

$P_1 = \alpha + \beta = 1$ and $P_2 = (\alpha+\beta)^2 - 2\alpha\beta = 1 + 2 = 3$

$P_3 = 4, P_4 = 7, P_5 = 11$

Hence, the correct answer is (A).

3. $e^{4x} + e^{3x} - 4e^x + e^x + 1 = 0$ Divide by e^{2x}

$$e^{2x} + e^x - 4 + \frac{1}{e^x} + \frac{1}{e^{2x}} = 0$$

$$\Rightarrow \left(e^{2x} + \frac{1}{e^{2x}}\right) + \left(e^x + \frac{1}{e^x}\right) - 4 = 0$$

$$\Rightarrow \left(e^x + \frac{1}{e^x}\right)^2 + \left(e^x + \frac{1}{e^x}\right) - 6 = 0$$

Let $e^x + \frac{1}{e^x} = 2 \quad \Rightarrow (e^x - 1)^2 = 0 \Rightarrow x = 0$.

$\therefore$ Number of real roots = 1

Hence, the correct answer is (C).

4. $ax^2 - 2bx + 5 = 0$ has root α, α then $\alpha = \frac{b}{a}$; $\alpha^2 = \frac{5}{a} \Rightarrow b^2 = 5a$

$x^2 - 2bx - 10 = 0$ has root α, $\beta \Rightarrow \alpha^2 - 2\beta\alpha - 10 = 0$

$\Rightarrow \quad \frac{5}{a} - 2\frac{b^2}{a} - 10 = 0 \Rightarrow a = \frac{1}{4}$ and $\alpha^2 = 20$; $\alpha\beta = -10$ and $\beta^2 = 5$

$\Rightarrow \quad \alpha^2 + \beta^2 = 25$

Hence, the correct answer is (B).

5. $f(x) = a(x-3)(x-\alpha)$

$f(2) = a(\alpha - 2)$ and $f(-1) = 4a(1+\alpha)$

$f(-1) + f(2) = 0 \Rightarrow a[(\alpha - 2) + 4 + 4a] = 0$

$a \neq 0$ we get $5\alpha = -2 \Rightarrow \alpha = -\frac{2}{5} = -0.4$

$\alpha \in (-1, 0)$

Hence, the correct answer is (C).

6. α and β are roots of $5x^2 + 6x - 2 = 0 \Rightarrow 5a^2 + 6\alpha - 2 = 0$

(By multiplying α^n)

$\Rightarrow \quad 5\alpha^{n+2} + 6\alpha^{n+1} - 2\alpha^n = 0$ ...(1)

Similarly $5\beta^{n+2} + 6\beta^{n+1} - 2\beta^n = 0$ ...(2)

Adding both

$$5(\alpha^{n+2} + \beta^{n+2}) + 6(\alpha^{n+1} + \beta^{n+1}) - 2(\alpha^n + \beta^n) = 0$$

$\Rightarrow \quad 5S_{n+2} + 6S_{n+1} - 2S_n = 0$

For $n = 4$ we get $5S_6 + 6S_5 = 2S_4$

Hence, the correct answer is (A).

7. $A : D \geq 0$

$\Rightarrow \quad (m+1)^2 - 4(m+4) \geq 0 \Rightarrow m^2 - 2m - 15 \geq 0$

$\Rightarrow \quad (m-5)(m+3) \geq 0 \Rightarrow m \in (-\infty, -3] \cup [5, \infty)$

$\therefore \quad A = (-\infty, -3] \cup [5, \infty)$ and $B = [-3, 5)$

$A - B = (-\infty, -3) \cup [5, \infty)$; $A \cap B = \{-3\}$

$B - A = (-3, 5)$; $A \cup B = R$

Hence, the correct answer is (A).

8. α, β are roots of $x^2 + px + 2 = 0$

$\Rightarrow \quad \alpha^2 + p\alpha + 2 = 0$ and $\beta^2 + p\beta + 2 = 0$

Also $\alpha + \beta = -p$ and $\alpha\beta = 2$

$\frac{1}{\alpha}, \frac{1}{\beta}$ are roots of $2x^2 + 2qx + 1 = 0$ then $\frac{1}{\alpha} + \frac{1}{\beta} = -q \Rightarrow p = 2q$

$$\left(\alpha - \frac{1}{\alpha}\right)\left(\beta - \frac{1}{\beta}\right)\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$$

$$= \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right)\left(\alpha\beta + 2 + \frac{1}{\alpha\beta}\right)$$

$$= \left(2 - \frac{(\alpha^2 + \beta^2)}{\alpha\beta} + \frac{1}{2}\right)\left(\frac{9}{2}\right) = \frac{9}{2}\left(\frac{5}{2} - \left(\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}\right)\right)$$

$$= \frac{9}{2}\left(\frac{9}{2} - \frac{p^2}{2}\right) = \frac{9}{4}(9 - p^2)$$

Hence, the correct answer is (C).

9. α and β are the roots of the equation $x^2 - x + 2\lambda = 0$ then $\alpha + \beta = 1$ and $a\beta = 2\lambda$

α and γ are the roots of the equation, $3x^2 - 10x + 27\lambda = 0$, then

$\alpha + \gamma = \frac{10}{3}$ and $a\gamma = \frac{27\lambda}{3} = 9\lambda$

$$\gamma - \beta = \frac{7}{3} \text{ and } \frac{\gamma}{\beta} = \frac{9}{2} \Rightarrow \gamma = \frac{9}{2}\beta$$

$$\gamma - \beta = \frac{9}{2}\beta - \beta = \frac{7}{3} \Rightarrow \frac{7}{2}\beta = \frac{7}{3} \Rightarrow \beta = \frac{2}{3} \text{ and } \gamma = 3$$

$$\alpha = 1 - \frac{2}{3} = \frac{1}{3} \text{ and } 2\lambda = \frac{2}{9} \Rightarrow \lambda = \frac{1}{9}$$

Now $\frac{\beta\gamma}{\lambda} = \frac{\frac{2}{3} \times 3}{\frac{1}{9}} = 18$

Hence, the correct answer is (D).

10. If exactly one root in (0, 1) then $f(0).f(1) < 0$

$\Rightarrow \quad 2(\lambda^2 - 4\lambda + 3) < 0 \Rightarrow 1 < \lambda < 3$

Now for $\lambda = 1$, $2x^2 - 4x + 2 = 0$

$(x - 1)^2 = 0$ we get $x = 1, 1$

So, both roots doesn't lie between (0, 1) $\therefore \lambda \neq 1$

Again for $\lambda = 3$

$$10x^2 - 12x + 2 = 0 \Rightarrow x = 1, \frac{1}{5}$$

so if one root is 1 then second root lie between (0, 1) so $\lambda = 3$ is correct.

$\therefore \quad \lambda \in (1, 3]$.

Hence, the correct answer is (B).

11. If $7x^2 - 3x - 2 = 0$ has roots α and β we have $\alpha + \beta = \frac{3}{7}$ and $\alpha\beta = \frac{-2}{7}$

$$\frac{\alpha}{1 - \alpha^2} + \frac{\beta}{1 - \beta^2} = \frac{\alpha + \beta - \alpha\beta(\alpha + \beta)}{1 - \alpha^2 - \beta^2 + \alpha^2\beta^2}$$

$$= \frac{\frac{3}{7} + \frac{2}{7}\left(\frac{3}{7}\right)}{1 - (\alpha + \beta)^2 + 2\alpha\beta + \alpha^2\beta^2} = \frac{27}{16}$$

Hence, the correct answer is (A).

12. $9x^2 - 18|x| + 5 = 0 \Rightarrow 9|x|^2 - 15|x| - 3|x| - 5 = 0$

$(\because x^2 = |x|^2)$

$\Rightarrow \quad 3|x|\,(3|x| - 5) - (3|x| - 5) = 0 \Rightarrow |x| = \frac{1}{3}, \frac{5}{3}$

$\Rightarrow \quad x = \pm\frac{1}{3}, \pm\frac{5}{3}$

Product of roots $= \frac{25}{81}$

Hence, the correct answer is (B).

13. α and β are the roots of the equation $4x^2 + 2x - 1 = 0$

$$4\alpha^2 + 2\alpha - 1 = 0 \quad \Rightarrow \quad \frac{1}{2} = 2a^2 + \alpha \qquad \ldots(1)$$

$$\beta = \frac{-1}{2} - \alpha$$

using equation (1)

$$\beta = -(2a^2 + \alpha) - \alpha = -2\alpha(\alpha + 1)$$

Hence, the correct answer is (C).

14. $x^2 - 64x + 256 = 0$ then $\alpha + \beta = 64$, $a\beta = 256$

$$\left(\frac{\alpha^3}{\beta^5}\right)^{1/8} + \left(\frac{\beta^3}{\alpha^5}\right)^{1/8} = \frac{\alpha^{3/8}}{\beta^{5/8}} + \frac{\beta^{3/8}}{\alpha^{5/8}} = \frac{\alpha + \beta}{(\alpha\beta)^{5/8}}$$

$$= \frac{64}{(256)^{5/8}} = 2$$

Hence, the correct answer is (D).

15. $D \geq 0 \Rightarrow (a - 10)^2 - 4.2.\left(\frac{33}{2} - 2a\right) \geq 0$

$\Rightarrow \quad a^2 - 4a - 32 > 0 \Rightarrow a \in (-\infty, -4] \cup [8, \infty)$

Hence, the correct answer is (8).

16. For rational D must be perfect square

$D = 121 - 24\alpha$

for $121 - 24\alpha$ to be perfect square a must be 3, 4, 5 So, $\alpha = 3$ is correct option

Hence, the correct answer is (D).

17. $x^2 - mx + 4 = 0$

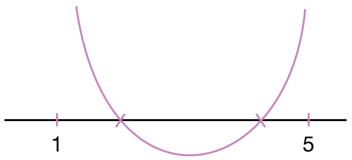
$$D > 0 \Rightarrow m^2 - 16 > 0 \Rightarrow m \in (-\infty, -4) \cup (4, \infty) \qquad \ldots(1)$$

$$1 < \frac{-b}{2a} < 5 \Rightarrow 1 < \frac{m}{2} < 5 \Rightarrow m \in (2, 10) \qquad \ldots(2)$$

$$f(1) > 0 \Rightarrow 1 - m + 4 > 0 \Rightarrow m < 5 \qquad \ldots(3)$$

$$f(5) > 0 \Rightarrow 25 - 5m + 4 > 0 \Rightarrow m < \frac{29}{5} \qquad \ldots(4)$$

we get final m as (4, 5)



Let one root is $x = 1$, $1 - m + 4 = 0 \Rightarrow m = 5$

Now equation $x^2 - 5x + 4 = 0 \Rightarrow (x - 1)(x - 4) = 0$

$x = 1$ i.e. $m = 5$ is also included hence $m \in (4, 5]$

Hence, the correct answer is (A).

18. $f(x) = (c - 5)x^2 - 2cx + (c - 4) = 0, c \neq 5$

$$f(0)\,f(2) < 0 \Rightarrow (c - 4)(c - 24) < 0 \Rightarrow c \in (4, 24) \qquad \ldots(1)$$

$$f(2)\,f(3) < 0 \Rightarrow (c-24)\,(4c-49) < 0 \Rightarrow \frac{49}{4} < c < 24 \qquad \ldots(2)$$

Eq. (1) $\cap$ (2)

$$\frac{49}{4} < c < 24$$

$c \in \{13, 14, \ldots, 23\}$ 11 elements

Hence, the correct answer is (D).

19. Sum of roots $\alpha + \beta = -(3-\lambda)$ and Product of roots $\alpha\beta = 2-\lambda$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (3-\lambda)^2 - 2(2-\lambda) = 9 + l^2 - 6\lambda - 4 + 2\lambda = l^2 - 4\lambda + 5$

$\alpha^2 + \beta^2$ is least then $\lambda = -\dfrac{b}{2a}$

$$\lambda = -\frac{-(-4)}{2} = 2$$

Hence, the correct answer is (B).

20. Let roots are α, α^3

$$\alpha + \alpha^3 = \frac{-k}{81} \text{ and } \alpha \cdot \alpha^3 = \frac{256}{81}$$

$$\alpha^4 = \frac{256}{81} = \frac{4^4}{3^4} \quad \Rightarrow \quad \alpha = \frac{4}{3}$$

$$\frac{4}{3} + \frac{64}{27} = \frac{-k}{81} \quad \Rightarrow \quad \frac{36+64}{27} = \frac{-k}{81} \quad \Rightarrow \quad k = -300$$

Hence, the correct answer is (B).

21. $3m^2x^2 + m(m-4)x + 2 = 0$

$$\lambda + \frac{1}{\lambda} = 1 \quad (\lambda \text{ is ratio of roots} = \alpha/\beta)$$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 1 \Rightarrow \alpha^2 + \beta^2 = \alpha\beta \Rightarrow (\alpha+\beta)^2 = 3\alpha\beta$$

$$\Rightarrow \quad \left(-\frac{m(m-4)}{3m^2}\right)^2 = \frac{3 \times 2}{3m^2}$$

$$\Rightarrow \quad \frac{(m-4)^2}{9m^2} = \frac{6}{3m} \Rightarrow (m-4)^2 = 18$$

$$\Rightarrow \quad m = 4 \pm \sqrt{18} = 4 \pm 3\sqrt{2}$$

Hence, the correct answer is (B).

22. $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$ is always positive, hence $2m + 1 > 0 \Rightarrow m > -1/2$

$D < 0 \Rightarrow 4(1+3m)^2 - 4(1+2m)\,4(1+m) < 0 \Rightarrow m^2 - 6m - 3 < 0$

$\Rightarrow 3 - \sqrt{12} < m < 3 + \sqrt{12}$ hence Integral value of $m = \{0, 1, 2, 3, 4, 5, 6\}$

Hence, the correct answer is (A).

23. $|\sqrt{x} - 2| + \sqrt{x}\,(\sqrt{x} - 4) + 2 = 0 \quad (x > 0)$

Let $\sqrt{x} = 1$

$|t-2| + t(t-4) + 2 = 0$

CASE-1: $t \geq 2$	CASE-2: $t < 2$
$t - 2 + t^2 - 4t + 2 = 0$	$2 - t + t^2 - 4t + 2 = 0$
$t^2 - 3t = 0$	$t^2 - 5t + 4 = 0$
$t = 0$ (Reject)	$t = 4$ (Reject)
$t = 3 \quad \sqrt{x} = 3$	$t = 1 \quad \sqrt{x} = 1$
$x = 9$	$x = 1$

Sum of all solution $= 9 + 1 = 10$

Hence, the correct answer is (B).

24. $4(1+3m)^2 - 4 \times (1+m^2)\,(1+8m) < 0$

$\Rightarrow \quad 1 + 9m^2 - 6m - (1 + 8m + m^2 + 8m^3) < 0$

$\Rightarrow \quad 1 + 9m^2 - 6m - 1 - 8m - m^2 - 8m^3 < 0$

$\Rightarrow \quad 8m^3 - 8m^2 + 2m < 0$

$\Rightarrow \quad m(4m^2 - 4m + 2) < 0$

$\Rightarrow \quad m[(2m-1)^2 + 1] < 0$ true for $m > 0$

Hence, the correct answer is (B).

25. If one root of equation $x^2 + px + q = 0$ is $2 - \sqrt{3}$ then other root will be $2 + \sqrt{3}$

$\therefore$ equation $x^2 - 4x + 1 = 0$

$\Rightarrow \quad p = -4$ and $q = 1$

$\Rightarrow \quad p^2 - 4q - 12 = 0$

Hence, the correct answer is (D).

26. $(m^2+1)x^2 - 3x + (m+1)^2 = 0$

$$\Rightarrow \quad \alpha + \beta = \frac{3}{m^2+1};\ \alpha\beta = \frac{(m+1)^2}{m^2+1}$$

$\because \ \alpha + \beta$ is maximum $\quad \therefore \ m^2 + 1$ is minimum $\quad \Rightarrow m = 0$

$\therefore \quad \alpha + \beta = 3$ and $\alpha\beta = 1$

$$|\alpha^3 - \beta^3| = |\alpha - b|\ (\alpha^2 + \alpha\beta + \beta^2)$$

$$= \left|\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\left\{(\alpha+\beta)^2 - \alpha\beta\right\}\right|$$

$$= \left|\sqrt{9-4}(9-1)\right| = 8\sqrt{5}$$

Hence, the correct answer is (D).

27. $\alpha + \beta = -\sin\theta;\ \alpha\beta = -2\sin\theta$

$$\text{Now, } \frac{\alpha^{12} + \beta^{12}}{\left(\dfrac{1}{\alpha^{12}} + \dfrac{1}{\beta^{12}}\right) \cdot (\alpha - \beta)^{24}} = \frac{(\alpha\beta)^{12}}{(\alpha-\beta)^{24}}$$

$$= \left(\frac{\alpha\beta}{(\alpha+\beta)^2 - 4\alpha\beta}\right)^{12}$$

$$= \left(\frac{-2\sin\theta}{\sin^2\theta + 8\sin\theta}\right)^{12} = \frac{2^{12}}{(\sin\theta + 8)^{12}}$$

Hence, the correct answer is (C).

28. CASE-1: $x \in [0, 9]$

$$2(3 - \sqrt{3}) + x - 6\sqrt{x} + 6 = 0$$

$$\Rightarrow \quad x - 8\sqrt{x} + 12 = 0 \Rightarrow \sqrt{x} = 4, 2 \Rightarrow x = 16, 4$$

Since $x \in |0, 9| \quad \therefore \ x = 4$

CASE-2: $x \in [9, \infty]$

$$2(\sqrt{x} - 3) + x - 6\sqrt{x} + 6 = 0$$

$$\Rightarrow \quad x - 4\sqrt{x} = 0 \Rightarrow x = 16, 0$$

Since $x \in [9, \infty] \quad \therefore \ x = 16$

Hence, $x = 4$ and 16

Hence, the correct answer is (D).

29. We have

$$\sum_{r=1}^{n} (x + r - 1)(x + r) = 10n$$

$$\Rightarrow \sum_{r=1}^{n}(x^2+(2r-1)x+r(r-1))=10n$$

$\Rightarrow$ $nx^2+\{1+3+5+\ldots,+(2n-1)\}x+\{1.2+2.3+\ldots+(n-1)n\}=10\,n$

$$\Rightarrow nx^2+n^2x+\frac{(n-1)n(n+1)}{3}=10n$$

$$\Rightarrow x^2+nx+\frac{n^2-31}{3}=0$$

Let a and $a+1$ be its two solutions

($\because$ it has two consecutive integral solutions)

$$\Rightarrow \alpha+(\alpha+1)=-n \Rightarrow \alpha=\frac{-n-1}{2} \quad \ldots(2)$$

$$\text{Also } \alpha(\alpha+1)=\frac{n^2-31}{3} \quad \ldots(2)$$

Putting value of (1) in (2), we get

$$-\left(\frac{n+1}{2}\right)\left(\frac{1-n}{2}\right)=\frac{n^2-31}{3} \Rightarrow n^2=121 \Rightarrow n=11$$

Hence, the correct answer is (A).

30. $(x^2-5x+5)^{x^2+4x-60}=1$

CASE-1: $x^2+4x-60=0 \Rightarrow (x-6)\,(x+100)=0$

$\Rightarrow$ $x=6,-10$

CASE-2: $x^2-5x+5=1 \Rightarrow x^2-5x+4=0$

$\Rightarrow$ $(x-1)\,(x-4)=0 \Rightarrow x=1,4$

CASE-3: $x^2-5x+5=-1$ and $x^2-4x-60$ is even number

$\therefore$ $x^2-5x+6=0 \Rightarrow (x-2)\,(x-3)=0 \Rightarrow x=2,3$

But for $x=3$, $x^2-4x-60$ is odd.

Hence $x=2$ only, thus solutions are $6,-10,1,4,2$

Sum of solutions is 3

Hence, the correct answer is (B).

31. $a_n=\alpha^n-\beta^n$ also $\alpha^2-6\alpha-2=0$

Multiply with a^8 on both sides

$$\Rightarrow \alpha^{10}-6\alpha^9-2\alpha^8=0 \quad \ldots(1)$$

$$\text{Similarly } \beta^{10}-6\beta^9-2\beta^8=0 \quad \ldots(2)$$

Subtracting (2) from (1) we have

$$\alpha^{10}-b^{10}-6(\alpha^9-b^9)=2(\alpha^8-\beta^8)$$

$$\Rightarrow a_{10}-6a_9=2a_8$$

$$\Rightarrow \frac{a_{10}-2a_8}{2a_9}=3$$

Hence, the correct answer is (A).

32. We have $\frac{1}{\alpha}+\frac{1}{\beta}=4$ and $2q=p+r$

$$\therefore \frac{2q}{p}=1+\frac{r}{p} \quad \Rightarrow \quad -2\,(\alpha+\beta)=1+\alpha\beta$$

$$\Rightarrow -2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\frac{1}{\alpha\beta}+1 \Rightarrow \frac{1}{\alpha\beta}=-9$$

$$\therefore \alpha\beta=-\frac{1}{9} \text{ and } \alpha+\beta=-\frac{4}{9}$$

Equation having roots α,β is $9x^2+4x-1=0$

$$\therefore |\alpha-\beta|=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{\frac{16}{81}+\frac{4}{9}}=\frac{2\sqrt{13}}{9}$$

Hence, the correct answer is (D).

33. For $x^2+2x+3=0$

$D=2^2-4.1.3<0$ i.e., both roots are complex.

Hence, both roots are common of $x^2+2x+3=0$ and $ax^2+bx+c=0$.

$$\therefore \frac{a}{1}=\frac{b}{2}=\frac{c}{3}$$

Hence, the correct answer is (A).

34. Let $e^{\sin x}=t$

$$e^{\sin x}-e^{-\sin x}-4=0 \Rightarrow t^2-4t-1=0 \Rightarrow t=\frac{4\pm\sqrt{16+4}}{2}$$

$$\Rightarrow t=e^{\sin x}=2\pm\sqrt{5}$$

we have $e^{\sin x}=2-\sqrt{5}<0$, which is not possible

or $\sin x=\text{in}\,(2+\sqrt{5})>1$, which is not possible

Hence no solution.

Hence, the correct answer is (C).

35. $P(x)=(a-a_1)x^2+(b-b_1)x+c-c_1$

$$X \text{ coordinate of vertex}=-\frac{b-b_1}{2(a-a_1)}=-1 \Rightarrow b-b_1$$

$$=2(a-a_1)$$

$$P(-1)=0 \Rightarrow (a-a_1)-(b-b_1)+c-c_1=0$$

$$\Rightarrow (a-a_1)-2(a-a_1)+c-c_1=0$$

$$\Rightarrow c-c_1=(a-a_1)$$

$$P(x)=(a-a_1)\,(x^2+2x+1)$$

$P(-2)=2 \Rightarrow (a-a_1)\,(4-4+1)=2$ we have $a-a_1=2$

$$P(x)=2(x^2+2x+1)$$

$$P(2)=18$$

Hence, the correct answer is (A).

36. Let $ax^2+bx+c=0$ original, equation.

Sachin : $a(x-4)\,(x-3)=0 \Rightarrow ax^2-7ax+12a=0$

So, $7a=-b$

Rahul : $a(x-3)\,(x-2)=0 \Rightarrow ax^2-5ax+6a=0$

So, $c=6a$

Now, $ax^2-7ax+6a=0 \Rightarrow x^2-7x+6=0$

So, Roots are $x=6,1$

Hence, the correct answer is (B).

37. $bx^2+cx+a=0$

Since the roots are imaginary,

$$c^2-4ab<0 \quad \Rightarrow \quad c^2<4ab \quad \Rightarrow \quad -c^2>-4ab$$

Now consider $3b^2x^2+6bcx+2c^2$

Since $3b^2>0$, the given expression has a minimum value

$$\text{So, minimum value}=\frac{4(3b^2)(2c^2)-36b^2c^2}{4(3b^2)}=-\frac{12b^2c^2}{12b^2}$$

$$=-c^2>-4ab$$

Hence, the correct answer is (A).

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Single Correct Choice Type Problems

1. $a + b = -20;\ ab = -2020$ and $c + d = 20;\ cd = 2020$

   Required expression $= ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

   $= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2$

   $= (a^2 + b^2)(c + d) - (a + b)(c^2 + d^2) = [(a + b)^2 - 2ab](c + d) - (a + b)[(c + d)^2 - 2cd]$

   $= [400 + 4040]\,20 + 20[400 - 4040] = 8000 + 8000 = 16000$

   Hence, the correct answer is (D).

2. Solving given equations and from given values of θ which lie in fourth quadrant

   For $x^2 - 2x\sec\theta + 1 = 0$ | For $x^2 + 2x\tan\theta - 1 = 0$

   $(x - \sec\theta)^2 = \tan^2\theta$ | $(x + \tan\theta)^2 = \sec^2\theta$

   $x = \sec\theta \pm \tan\theta$ | $\Rightarrow x = -\tan\theta \pm \sec\theta$

   $\alpha_1 = \sec\theta - \tan\theta$ and $\beta_1 = \sec\theta + \tan\theta$

   $\alpha_2 = \sec\theta - \tan\theta$ and $\beta_2 = -\sec\theta - \tan\theta$

   $\alpha_1 + \beta_2 = -2\tan\theta$

   Hence, the correct answer is (C).

3. Since $p(x) = 0$ has purely imaginary roots,

   $p(x) = ax^2 + c$, where a and c have same sign.

   Also, $p(p(x)) = 0$

   $\Rightarrow$ $p(x)$ is purely imaginary

   $\Rightarrow$ $ax^2 + c$ is purely imaginary

   Hence, x cannot be either purely real or purely imaginary.

   Hence, the correct answer is (D).

4. $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ ...(1)

   Subtracting, we get $(b - 1)x - 1 - b = 0 \Rightarrow x = \dfrac{b + 1}{b - 1}$

   This value of x satisfies equation (1)

   $\Rightarrow \dfrac{(b + 1)^2}{(b - 1)^2} + \dfrac{b + 1}{b - 1} + b = 0$

   $\Rightarrow b^2 + 1 + 2b + b^2 - 1 + b(b^2 - 2b + 1) = 0$

   $\Rightarrow b^2 + 1 + 2b + b^2 - 1 + b^3 + b - 2b^2 = 0$

   $\Rightarrow b^3 + 3b = 0$

   $b = \sqrt{3}i, -\sqrt{3}i, 0$

   Hence, the correct answer is (B).

5. $\alpha^3 + \beta^3 = q \Rightarrow (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = q$

   $\Rightarrow -p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \dfrac{q + p^3}{3p}$

   Required equation is

   $$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 0 \Rightarrow x^2 - \frac{(\alpha^2 + \beta^2)}{\alpha\beta}x + 1 = 0$$

   $$\Rightarrow x^2 - \left(\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}\right)x + 1 = 0$$

   $$\Rightarrow x^2 - \frac{p^2 - 2\left(\dfrac{p^3 + q}{3p}\right)}{\dfrac{p^3 + q}{3p}}x + 1 = 0$$

   $\Rightarrow (p^3 + q)x^2 - (3p^3 - 2p^3 - 2q)x + (p^3 + q) = 0$

   $\Rightarrow (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0.$

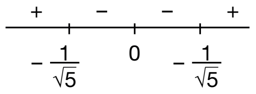
   Hence, the correct answer is (B).

Multiple Correct Choice Type Problems

1. $|x_1 - x_2| < 1 \quad \therefore (x_1 + x_2)^2 - 4x_1x_2 < 1$

   $\Rightarrow \dfrac{1}{\alpha^2} - 4 < 1 \quad \Rightarrow 5 - \dfrac{1}{\alpha^2} > 0$

   $\Rightarrow \dfrac{5\alpha^2 - 1}{\alpha^2} > 0$

   $\Rightarrow \alpha \in \left(-\infty, -\dfrac{1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \infty\right)$ ...(1)

   Also $D > 0 \quad \therefore 1 - 4\alpha^2 > 0 \quad \Rightarrow \alpha \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ ...(2)

   From (1) and (2) $\quad \alpha \in \left(-\dfrac{1}{2}, -\dfrac{1}{\sqrt{5}}\right) \cup \left(\dfrac{1}{\sqrt{5}}, \dfrac{1}{\sqrt{2}}\right)$

   Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. **The correct answer is (B).**
2. **The correct answer is (D).**

Combined Solution to 1–2

Since α, β be the roots of the equation, $x^2 - x - 1 = 0$ then $\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$ ...(1)

$a_n + a_{n-1} = p\alpha^n + q\beta^n + p\alpha^{n-1} + q\beta^{n-1}$

$= p\alpha^n + p\alpha^{n-1} + q\beta^n + q\beta^{n-1} = p\alpha^{n-1}(\alpha + 1) + q\beta^{n-1}(\beta + 1)$

$= p\alpha^{n-1}\cdot\alpha^2 + q\beta^{n-1}\cdot\beta^2$ [Using (1)]

$= p\alpha^{n+1} + q\beta^{n+1} = a_{n+1}$

So, $a_{12} = a_{11} + a_{10}$

$a_4 = a_3 + a_2 = a_2 + a_1 + a_1 + a_0 = a_1 + a_0 + 2a_1 + a_0$

$= 2a_0 + 3a_1 = 2(p + q) + 3(p\alpha + q\beta)$

$= 2(p + q) + 3(p\alpha + q(1 - \alpha)) = (2p + 5q) + \alpha(3p - 3q) = 28$ (given)

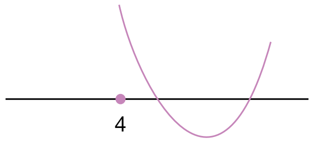
$\therefore 2p + 5q = 28$ and $3p - 3q = 0$ as $p, q \in Q$ and α is irrational.

$\therefore p = q = 4$

$\therefore p + 2q = 12$

Integer/Numerical Answer Type Question

1. $x^2 - 8kx + 16(k^2 - k + 1) = 0$



Let $f(x) = x^2 - 8kx + 16(k^2 - k + 1)$

(i) $D > 0 \Rightarrow 64k^2 - 64(k^2 - k + 1) > 0$

$\Rightarrow k > 1$

(ii) x coordinate of vertex $= -\dfrac{b}{2a} > 4 \Rightarrow -\left(\dfrac{-8k}{2}\right) > 4$

$\Rightarrow k > 1$

(iii) $f(4) \geq 0 \Rightarrow 16 - 32k + 16(k^2 - k + 1) \geq 0$

$\Rightarrow k^2 - k + 1 - 2k + 1 \geq 0$

$\Rightarrow k^2 - 3k + 2 \geq 0 \Rightarrow k \in (-\infty, 1] \cup [2, \infty)$

So, $k \in [2, \infty)$

So, least value of $k = 2$

Hence, the correct answer is (2).

CHAPTER 3: PROGRESSION AND SERIES

Test Your Concepts-I

1. $S_N = 50^2$;

$$S_D = \frac{2\cdot 50\cdot 51}{2} = 50\cdot 51$$

$$\Rightarrow \quad \frac{N}{D} = \frac{50}{51}$$

2. $$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_1+(n-1)d_2]} = \frac{5n+3}{3n+4} \Rightarrow \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_1+\left(\frac{n-1}{2}\right)d_2} = \frac{5n+3}{3n+4}$$

put $\frac{n-1}{2} = 16, n = 33$

We have $\frac{5n+3}{3n+4} = \frac{5\times 33+3}{3\times 33+4} = \frac{168}{103}$

3. Let $a,\ A_1, A_2 \ldots A_8, b \in A.P$

Where $a = 2, b = 20, n = 8$

$\therefore$ sum of the means $= \frac{n}{2}(a+b) = \frac{8}{2}(2+20) = 88$

4. Let m^{th} term of first A.P. be equal to the n^{th} term of the second A.P. then
2, 5, 8,…50 terms series 1
3, 5,7, …, 60 terms series 2
Common series 5, 11, 17, …, 119
40^{th} term of series 1 = 59^{th} term of series 2 = 119 = last term of common series

$\Rightarrow \quad a_n = 5+(n-1)d \Rightarrow 119+1 = 6n \Rightarrow n = 20.$

$\therefore$ Number of common terms is 20.

5. AP I = 12, 15, 18, … (common difference $d_1 = 3$)

AP II = 17, 21, 25, … (common difference $d_2 = 4$)
First term of the series of common numbers = 21

Here a = 21, common difference of the series of common numbers = L.C.M of d_1 and d_2 = 12
$\therefore$ Required sum of first hundred terms

$$= \frac{100}{2}\left[2\times 21+(100-1)12\right] = 100[21+594] = 61500$$

6. Since $A_1, A_2, A_3, \ldots, A_{11}$ be 11 A.M. s between 28 and 10.

$\therefore$ 28, $A_1, A_2, \ldots, A_{11}$, 10 are in A.P.

Let 'd' be the common difference of A.P.
Also the number of terms = 13.

$$10 = T_{13} = T_1 + 12d = 28 + 12d$$

$$\therefore \quad d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

$\therefore$ Number of integral A.M's is 5.

7. As log2, log $(2^x - 1)$ and log $(2^x + 3)$ are in A.P.,

$$2\log(2^x-1) = \log 2 + \log(2^x+3)$$

$$\Rightarrow \quad (2^x-1)^2 = 2(2^x+3)$$

$$\Rightarrow \quad 2^{2x} - 4\times 2^x - 5 = 0$$

$$\Rightarrow \quad (2^x-5)(2^x+1) = 0$$

As 2^x cannot be negative, we get

$$2^x - 5 = 0$$

$$\Rightarrow \quad 2^x = 5 \text{ or } x = \log_2 5$$

8. As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10.$

9. Sum of A.M. = 6 single A.M.

$$a+b+c+d+e+f = \frac{6(2+12)}{2} = 42$$

10. $a_{50} - a_{49} = \frac{3}{5} \quad \Rightarrow d = \frac{3}{5}$

Let T_{15} is 15^{th} term from end

$$T_{15} = a_n - 14d$$

$$= 100 - \frac{42}{5} = \frac{458}{5}$$

11. $\frac{11}{2}[2a+10d] = \frac{19}{2}[2a+18d]$

$$\Rightarrow \quad 11\times 2(a+5d) = 19\times 2(a+9d)$$

$$\Rightarrow \quad 11a + 55d = 19a + 171d$$

$$\Rightarrow \quad 8a + 116d = 0$$

$$\Rightarrow \quad 2a + 29d = 0$$

$$\Rightarrow \quad S_{30} = 0$$

12. $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \frac{1}{\sqrt{a_3}+\sqrt{a_4}} \ldots + \frac{1}{\sqrt{a_{49}}+\sqrt{a_{50}}}$$

$$= \frac{\left[\left(\sqrt{a_2}-\sqrt{a_1}\right)+\left(\sqrt{a_3}-\sqrt{a_2}\right)+\ldots+\left(\sqrt{a_{50}}-\sqrt{a_{49}}\right)\right]}{d}$$

where d = common difference of A.P.

$$= \frac{\sqrt{a_{50}}-\sqrt{a_1}}{d} = \frac{\sqrt{144}-\sqrt{4}}{\left(\frac{140}{49}\right)} = \frac{49}{14};\ d = \frac{140}{49}$$

13. Given $100(a+99d) = 50(a+49d)$

$$\Rightarrow \quad a + 149d = 0 \qquad \ldots(1)$$

and $T_{150} = a + 149d = 0$ (Using (1))

14. A.M. of 1, 2, 3, …, $n = \frac{n+1}{2}$

If k, $k+1$ are removed then

$$\text{A.M.} = \frac{\frac{n(n+1)}{2} - (2k+1)}{n-2}$$

$$\Rightarrow \quad 1 = \frac{n+1}{2} - \frac{n(n+1)-2(2k+1)}{2(n-2)}$$

$$2n-4 = (n+1)(n-2-n)+4k+2$$

$$\Rightarrow \quad n-k=1$$

15. $S_k = 153$

$$\Rightarrow \quad \frac{k}{2}\left[2a+(k-1)2\right] = 153$$

$$k^2 + k(a-1) = 153$$

$\Rightarrow \quad k^2 + k(a-1) - 153 = 0$ where a is the first term

Number of possible values of k = Number of factors of 153 excluding $1 = 5$

Test Your Concepts-II

1. $S = \dfrac{a\left(1-r^n\right)}{1-r}, P = a^n . r^{\frac{n(n-1)}{2}}$

$$R = \frac{1}{a}+\frac{1}{ar}+\frac{1}{ar^2}\ldots \text{ to } n \text{ terms} = \frac{1-r^n}{a(1-r)r^{n-1}}$$

$$S^n = R^n P^k \Rightarrow \left(\frac{S}{R}\right)^n = P^k \Rightarrow \left(a^2 r^{n-1}\right)^n = P^k$$

$$\Rightarrow \quad P^2 = p^k \Rightarrow k = 2$$

2. Let $a = x;\ b = xr;\ c = xr^2$

hence the number are $\dfrac{1}{b-a}, \dfrac{1}{2b}, \dfrac{1}{b-c}$

or $\dfrac{1}{x(r-1)}, \dfrac{1}{2xr}, \dfrac{1}{xr(1-r)}$

sine $\dfrac{1}{x(r-1)} + \dfrac{1}{xr(1-r)} = \dfrac{1}{2xr}$

hence numbers are in A.P.

3. $a^x = b^y = c^z = k,\ y^2 = xz,$

$$x = \log_a k = \frac{1}{\log_k a} \Rightarrow (\log b)^2 = \log a . \log c$$

Similarly $y = \dfrac{1}{\log_k b},\ z = \dfrac{1}{\log_k c}$

4. $a+(p-1)d = k$ ...(1)

$a+(q-1)d = kR$ ...(2)

$a+(r-1)d = kR^2$ ...(3)

Now $\dfrac{(3)-(2)}{(2)-(1)}$ we get $R = \dfrac{r-q}{q-p}$.

5. Let $S_n = 5 + 55 + 555 + \ldots n$ terms

$= 5\ \{1 + 11 + 111 + \ldots n \text{ terms}\}$

$= \dfrac{5}{9}\ \{9 + 99 + 999 + \ldots n \text{ terms}\}$

$= \dfrac{5}{9}\ \{(10-1) + (100-1) + (1000-1) + \ldots n \text{ terms}\}$

$$= \frac{5}{9}\left[\frac{10\left(10^n-1\right)}{10-1} - n\right]$$

Hence, $S_n = \dfrac{50}{81}\left(10^n - 1\right) - \dfrac{5}{9}n.$

6. Let the G.P. be $1, r, r^2, r^3, \ldots$

given condition $r = \dfrac{r^2}{1-r} \Rightarrow r = \dfrac{1}{2},$

Hence series is $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots \infty$

7. Three number in G.P. are $\dfrac{a}{r}, a, ar$

then $\dfrac{a}{r}, 2a,$ ar are in A.P. as given.

$\therefore \quad 2(2a) = a\left(r + \dfrac{1}{r}\right)$

or $\quad r^2 - 4r + 1 = 0$

or $\quad r = 2 \pm \sqrt{3}$

or $\quad r = 2 + \sqrt{3}$ as $r > 1$ for an increasing G.P.

8. $\dfrac{a}{1-r} = 7$ ...(1)

and $\dfrac{ar}{1-r^2} = 3$ ...(2)

Solving $r = 3/4$ and $a = 7/4 \quad \Rightarrow \quad a + r = 5/2$

9. $1 + \cos 60^\circ + \cos^2 60^\circ + \cos^3 60^\circ + \ldots$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$$

10. $a;\ a + 9;\ a + 18;\ \ldots$ Now, a_1, a_3 and a_8 in G.P. i.e., $a, a + 18$ and $a + 63$ in G.P.

$\therefore \quad (a + 18)^2 = a(a + 63) = a^2 + 63a$

$\Rightarrow \quad a = 12;\ d = 9$

So, $\quad a_7 = a + 6d = 12 + 54 = 66.$

11. The above series in an infinite G.P., whose first term $= \dfrac{1}{1+x}$ and common ratio $= \dfrac{-(1-x)}{(1+x)}$

$$\therefore \quad S_\infty = \frac{\frac{1}{1+x}}{1+\left(\frac{1-x}{1+x}\right)} = \frac{1}{2}$$

12. $\dfrac{ar^2}{a} = 9 \quad \Rightarrow \quad r = 3$

$$a + ar = \frac{4}{3} \quad \Rightarrow \quad a \cdot 4 = \frac{4}{3} \quad \Rightarrow \quad a = \frac{1}{3}$$

$$\therefore \quad a_4 = ar^3 = \frac{1}{3} \cdot 3^3 = 3^2 = 9.$$

13. $2x^3 + 9x^2 - 27x - 54 = 0$ ...(1)

$$\alpha = \frac{a}{r},\ \beta = a,\ \gamma = ar$$

$\therefore \quad a^3 = \dfrac{54}{2} = 27$

$\therefore \quad a = 3$ is root of (1)

Hence $2x^2(x-3) + 15x(x-3) + 18(x-3)$

$(x-3)\,(2x^2 + 15x + 18) = 0$

$(x-3)\,(x+6)\,(2x+3) = 0$

$\therefore \quad x = 3; -6; \dfrac{-3}{2}$

$\therefore \quad |\alpha| + |\beta| + |\gamma| = \dfrac{21}{2}$

Hence $\dfrac{2}{3}(|\alpha| + |\beta| + |\gamma|) = \dfrac{2}{3}\left(\dfrac{21}{2}\right) = 7$

Test Your Concepts-III

1. a_1, a_2, a_3, a_4, a_5 are in *H.P.*

$\Rightarrow \quad a_2 = \dfrac{2a_1a_3}{a_1 + a_3} \Rightarrow 2a_1a_3 = a_2a_1 + a_3a_2$

$a_4 = \dfrac{2a_3a_5}{a_3 + a_5} \Rightarrow 2a_3a_5 = a_3a_4 + a_5a_4$

$\Rightarrow \quad a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = 2a_1a_3 + 2a_3a_5 \quad \ldots(1)$

$a_3 = \dfrac{2(a_1a_5)}{a_1 + a_5} \Rightarrow a_1a_3 + a_5a_3 = 2a_1a_5 \quad \ldots(2)$

using (1) and (2)

$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = 2(2a_1a_5) = 4a_1a_5; k = 4$

2. Since $\log_{ax}x$, $\log_{bx}x$, $\log_{cx}x$ are in H.P.

$\therefore \quad \log_x ax, \log_x bx, \log_x cx$ are in A.P.

$\Rightarrow \quad 1 + \log_x a, 1 + \log_x b, 1 + \log_x c$ are in A.P.

$\Rightarrow \quad \dfrac{\log a}{\log x}, \dfrac{\log b}{\log x}, \dfrac{\log c}{\log x}$ are in A.P.

$\Rightarrow \quad \log a, \log b, \log c$ are in A.P.

$\Rightarrow \quad 2\log b = \log a + \log c = \log ac$

$\Rightarrow \quad \log b^2 = \log ac \Rightarrow b^2 = ac$

$\Rightarrow \quad a, b, c$ are in G.P.

3. $\dfrac{1}{c+a} - \dfrac{1}{b+c} = \dfrac{1}{a+b} - \dfrac{1}{c+a}$

$\dfrac{b+c-c-a}{(c+a)(b+c)} = \dfrac{a+c-a-b}{(a+b)(c+a)}$

On solving we get $2b^2 = a^2 + c^2$

4. $\because \quad a, b, c$ are in *H.P.*

$\because \quad b = \dfrac{2ac}{(a+c)}$

$\therefore \quad \dfrac{1}{b-a} + \dfrac{1}{b-c}$

$\Rightarrow \quad \dfrac{1}{\dfrac{2ac}{(a+c)} - a} + \dfrac{1}{\dfrac{2ac}{(a+c)} - c}$

$\Rightarrow \quad (a+c)\left\{\dfrac{1}{a(c-a)} + \dfrac{1}{c(a-c)}\right\} \Rightarrow \dfrac{(a+c)}{(a-c)}\left\{-\dfrac{1}{a} + \dfrac{1}{c}\right\}$

$\Rightarrow \quad \dfrac{(c+a)(a-c)}{ac(a-c)} \Rightarrow \dfrac{(a+c)}{ac} = \dfrac{2}{b}$

5. Let $S = \dfrac{1}{10} + \dfrac{2}{10^2} + \dfrac{3}{10^3} + \ldots \quad \ldots(1)$

So, $\dfrac{S}{10} = 0 + \dfrac{1}{10^2} + \dfrac{2}{10^3} + \ldots \quad \ldots(2)$

$\therefore \quad$ Subtracting (2) from (1), we get

$\dfrac{9S}{10} = \dfrac{1}{10} + \dfrac{2}{10^2} + \dfrac{3}{10^3} + \ldots = \dfrac{1/10}{1 - \dfrac{1}{10}} = \dfrac{1}{9}$

$\Rightarrow \quad S = \dfrac{1}{9}\left(\dfrac{10}{9}\right) = \dfrac{10}{81}$

6. Clearly, the given series is an arithmetic-geometric series whose corresponding G.P. is $1, -x, x^2, -x^3, \ldots$

The common ratio of this G.P. is $-x$. Let S_∞ denote the sum of the given infinite series. Then,

$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \ldots \infty \quad \ldots(1)$

$-xS_\infty = 0 - x + 3x^2 - 5x^3 + \ldots \infty \quad \ldots(2)$

Subtracting (2) from (1), we get

$(1+x)\,S_\infty = 1 + [-2x + 2x^2 - 2x^3 + \ldots \infty]$

$= 1 + \left[\dfrac{-2x}{1-(-x)}\right] = 1 - \dfrac{2x}{1+x} = \dfrac{1-x}{1+x}$

$\Rightarrow \quad S_\infty = \dfrac{1-x}{(1+x)^2}.$

7. Given a = A.M. between b and c

$\Rightarrow \quad a = \dfrac{b+c}{2} \quad \Rightarrow \quad 2a = b + c \quad \ldots(1)$

G_1, G_2 are two G.M.'s between b and c then b, G_1, G_2, c are in G.P.

If r is the common ratio of this G.P. Then $G_1 = br$, $G_2 = br^2$, $c = br^3$

Now : $G_1^3 + G_2^3 = (br)^3 + (br^2)^3 = b^3r^3 + b^3r^6$

$= b^3\left(\dfrac{c}{b}\right) + b^3\left(\dfrac{c}{b}\right)^2 \quad (\because c = br^3)$

$= b^2c + bc^2 = bc\,(b+c) = bc\,(2a) = 2abc \quad$ {from (1)}

Hence, $G_1^3 + G_2^3 = 2abc$

8. $b^2 = ac \Rightarrow 2\log_2 b = \log_2 a + \log_2 c$

$\log_2 a$, $\log_2 b$, $\log_2 c$ are in A.P.

$\dfrac{1}{\log_a 2}, \dfrac{1}{\log_b 2}, \dfrac{1}{\log_c 2}$ are in A.P.

$\therefore \quad \log_a 2$, $\log_b 2$, $\log_c 2$ are in *H.P.*

9. $\dfrac{16x + \dfrac{4}{x}}{2} \geq \left(16x.\dfrac{4}{x}\right)^{1/2} \quad \Rightarrow \quad 16x + \dfrac{4}{x} + 1 \geq 17$

$\therefore \quad$ Minimum value of $16x + \dfrac{4}{x} + 1$ is 17.

10. We have $x, A_1, A_2, A_3, \ldots A_{19}, A_{20}, A_{21}, \ldots A_{99}, 2y$ are in A.P.

$\therefore$ Common difference of A.P. $\Rightarrow d = \dfrac{2y - x}{100}$

$$\Rightarrow A_{20} = x + 20d = x + 20\left(\frac{2y - x}{100}\right) = x + \left(\frac{2y - x}{5}\right)$$

$$= \frac{4x + 2y}{5} \qquad \ldots(1)$$

Similarly $2x, A'_1, A'_2, A'_3, \ldots A'_{19}, A'_{20}, A'_{21}, \ldots A'_{99}, y$ are in A.P.

$\therefore$ Common difference of above A.P. $= d' = \left(\dfrac{y - 2x}{100}\right)$

$$A'_{20} = 2x + \left(\frac{y - 2x}{5}\right) = \frac{8x + y}{5} \qquad \ldots(2)$$

Now $A_{20} = A'_{20}$ (Given)

$\therefore$ From (1) and (2), we get

$$\frac{4x + 2y}{5} = \frac{8x + y}{5} \Rightarrow y = 4x; k = 4$$

11. Using A.M. $\geq$ G.M.

$$\Rightarrow \frac{a + b + c + d}{4} \geq \sqrt[4]{abcd}$$

$\Rightarrow (a + b + c + d)^4 \geq 256\, abcd$

But $(a + b + c + d)^4 \leq 256\, abcd$ (given)

$\Rightarrow$ A.M. = G.M. $\Rightarrow a = b = c = d$

$$\therefore \frac{a + b + c}{d} = 3$$

12. $\sqrt{3}, H, \sqrt{2}$ in H.P.

$$\Rightarrow H = \frac{2\sqrt{6}}{\sqrt{2} + \sqrt{3}}$$

$$\Rightarrow \frac{H}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2} + \sqrt{3}}, \frac{H}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{2} + \sqrt{3}}$$

Using componendo dividend

$$\frac{H + \sqrt{2}}{H - \sqrt{2}} + \frac{H + \sqrt{3}}{H - \sqrt{3}} = 2$$

13. Use A.M. $\geq$ G.M.

$$\frac{(x + 1) + (y + 2) + (z - 3)}{3} \geq ((x + 1)\,(y + 2)\,(z - 3))^{1/3}$$

$8 \geq ((x + 1)\,(y + 2)\,(z - 3))^{1/3}$

$(x + 1)\,(y + 2)\,(z - 3) \leq 512$

14. $4xy + 6yz + 8zx = 9$

A.M. $\geq$ G.M.

$$\frac{4xy + 6yz + 8zx}{3} \geq (4.6.8x^2y^2z^2)^{1/3}$$

$$3^3 \geq 4.6.8(xyz)^2 \Rightarrow (xyz)^2 \leq \frac{27}{4.6.8}$$

$$xyz \leq \frac{3}{8} \Rightarrow M = \frac{3}{8} \Rightarrow 2016M = 756$$

15. $pqr = 3^{-5/2}$;

$$\frac{p^2 + 9q^2 + 3pq + 3pq + r^2 + r^2}{6} \geq (3^4p^4q^4r^4)^{1/6}$$

$\Rightarrow p^2 + 9q^2 + 6pq + 2r^2 \geq 6(3^4.3^{-10})^{1/6} \geq 2$

16. $abc = 64$

$2b = a + c$

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc} = 4$$

$$\Rightarrow \frac{3b}{3} \geq 4 \Rightarrow b \geq 4$$

Test Your Concepts-IV

1. $T_r = (3r - 1)\,(3r + 2) = 9r^2 + 3r - 2$

$$S_n = \sum_{r=1}^{n} T_r = 9\sum_{r=1}^{n} T_r + 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 3\left(\frac{n(n+1)}{2}\right) - 2n$$

$$= 3n(n + 1)^2 - 2n$$

2. Given series is 5, 7, 13, 31, 85, …

After First difference, we get 2, 6, 18, 54 G.P. with common ratio 3.

$T_n = \lambda(3)^{n-1} + a$

$T_1 = \lambda + a = 5$

$T_2 = 3\lambda + a = 7 \Rightarrow \lambda = 1, a = 4$

$T_n = 3^{n-1} + 4$

$$S_n = \sum_{r=1}^{n} T_r = (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$

$$= \frac{3^n - 1}{2} + 4n = \frac{1}{2}[3^n + 8n - 1]$$

3. $$T_r = \frac{(r+3)}{r\cdot(r+1)(r+2)} = \frac{(r+2)+1}{r(r+1)(r+2)}$$

$$T_r = \frac{1}{r(r+1)} + \frac{1}{r(r+1)(r+2)}$$

$$T_r = \frac{(r+1) - r}{r(r+1)} + \frac{1}{2}\left[\frac{(r+2) - r}{r(r+1)(r+2)}\right]$$

$$T_r = \left[\frac{1}{r} - \frac{1}{r+1}\right] + \frac{1}{2}\left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$$

$$T_1 = \left[\frac{1}{1} - \frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\right]$$

$$T_2 = \left[\frac{1}{2} - \frac{1}{3}\right] + \frac{1}{2}\left[\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\right]$$

$$T_3 = \left[\frac{1}{3} - \frac{1}{4}\right] + \frac{1}{2}\left[\frac{1}{3\cdot 4} - \frac{1}{4\cdot 5}\right]$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$T_n = \left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$$

Adding all n terms,

we get $S_n = \left(1-\dfrac{1}{n+1}\right)+\dfrac{1}{2}\left(\dfrac{1}{2}-\dfrac{1}{(n+1)(n+2)}\right)$

$$=\left(\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}\right)$$

4. $$\sum_{r=2}^{\infty}\left(\frac{1}{r^2-1}\right)=\sum_{r=2}^{\infty}\left[\frac{1}{(r+1)(r-1)}\right]$$

$$=\frac{1}{2}\sum_{r=2}^{\infty}\left(\frac{(r+1)-(r-1)}{(r+1)(r-1)}\right)$$

$$=\frac{1}{2}\sum_{r=2}^{\infty}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$$

$$=\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots+\right.$$

$$\left.\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right]$$

$n\to\infty$

$$=\frac{1}{2}\left[1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right]=\frac{1}{2}\left(\frac{3}{2}-0-0\right)=\frac{3}{4}$$

5. $$\sum_{n=2}^{\infty}\frac{1}{t_n}=\sum_{n=2}^{\infty}\frac{2}{n(n-1)(n+1)}=\sum_{n=2}^{\infty}\left(\frac{(n+1)-(n-1)}{n(n-1)(n+1)}\right)$$

$$=\sum_{n=2}^{\infty}\left(\frac{1}{n(n-1)}-\frac{1}{n(n+1)}\right)$$

$$=\left(\frac{1}{2.1}-\frac{1}{2.3}+\frac{1}{3.2}-\frac{1}{3.4}+\frac{1}{3.4}-\frac{1}{4.5}+\ldots\right)=\frac{1}{2}$$

6. $$T_k=\frac{k}{(k^2+1)^2-k^2}=\frac{k}{(k^2+1+k)(k^2+1-k)}$$

$$=\frac{1}{2}\left(\frac{(k^2+1+k)-(k^2+1-k)}{(k^2+1+k)(k^2+1-k)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(k^2+1-k)}-\frac{1}{(k^2+1+k)}\right)$$

$$S=\frac{1}{2}\left(\sum_{k=1}^{n}\frac{1}{(k^2+1-k)}-\frac{1}{(k^2+1+k)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)$$

$$+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right)$$

$\vdots\ \vdots$

$$+\frac{1}{2}\left(\frac{1}{n^2+1-n}-\frac{1}{n^2+1+n}\right)$$

$$S_n=\frac{1}{2}\left(1-\frac{1}{n^2+1+n}\right)=\frac{1}{2}\left(\frac{n^2+1+n-1}{n^2+1+n}\right)$$

$$=\frac{n(n+1)}{2(n^2+n+1)}$$

put $n=100$ to get $\dfrac{5050}{10101}\quad\Rightarrow\quad N=5050$

7. $$T_n=\frac{1}{(2n-1)(2n+1)\ldots(2n+7)}$$

$$=\frac{1}{8}\left[\frac{1}{(2n-1)\ldots(2n+5)}-\frac{1}{(2n+1)\ldots(2n+7)}\right]$$

$$S_n=\Sigma T_n=\frac{1}{8}\left[\left(\frac{1}{1\cdot3\cdot5\cdot7}-\frac{1}{1\cdot5\cdot7\cdot9}\right)+\right.$$

$$\left.\left(\frac{1}{3\cdot5\cdot7\cdot9}-\frac{1}{5\cdot7\cdot9\cdot11}\right)+\ldots\right]$$

$$S_\infty=\frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7}\right)$$

$$\therefore\quad S_\infty=\frac{1}{120\times7}\quad\Rightarrow\quad 120\,S_\infty=\frac{1}{7}$$

8. $V_2=3V_1-2V_0=5$

$V_3=3V_2-2V_1=9$

$V_4=3V_3-2V_2=17$

$V_5=3V_4-2V_3=33$

$V_6=3V_5-2V_4=65$

$$\therefore\quad \frac{V_6}{13}=5$$

9. $$S_n=\sum n(n+1)=\frac{1}{3}n(n+1)(n+2)$$

$$\sigma_n=\frac{1}{n(n+1)(n+2)(n+3)}\left(\frac{(n+3)-n}{3}\right)$$

$$=\frac{1}{3}\left(\frac{1}{n(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$\sigma_n=\frac{1}{3}\left(\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$\sigma_{n-1}=\frac{1}{3}\left(\frac{1}{6}-\frac{1}{(n)(n+1)(n+2)}\right)=\frac{(n)(n+1)(n+2)-6}{18(n)(n+1)(n+2)}$$

$$18S_n\sigma_{n-1}=\frac{18}{3}.n(n+1)(n+2)\left(\frac{(n)(n+1)(n+2)-6}{18(n)(n+1)(n+2)}\right)$$

$$=\frac{(n)(n+1)(n+2)-6}{3}$$

$18S_n\sigma_{n-1}-S_n=-2$

$18S_n\sigma_{n-1}-S_n+2=0$

10. $$T_n=\frac{1}{n(n+2)(n+4)}$$

$$T_n=\frac{1}{4}\left(\frac{1}{n(n+2)}-\frac{1}{(n+2)(n+4)}\right)$$

$$T_1=\frac{1}{4}\left(\frac{1}{1.3}-\frac{1}{3.5}\right)$$

$$T_2=\frac{1}{4}\left(\frac{1}{2.4}-\frac{1}{4.6}\right)$$

$$T_3 = \frac{1}{4}\left(\frac{1}{3.5} - \frac{1}{5.7}\right)$$

$$S_\infty = \frac{1}{4}\left(\frac{1}{3} + \frac{1}{8}\right) = \frac{11}{96}$$

$$\frac{22}{S} = 192$$

11. $S = 2 + 5 + 12 + 31 + 86 + \ldots + t_n$

$S = 2 + 5 + 12 + 31 + \ldots + t_{n-1} + t_n$

$0 = 2 + 3 + 7 + 19 + 55 + \ldots - t_n$

$\Rightarrow \quad t_n = 2 + 3 + 7 + 19 + 55 + \ldots + t'_n$

$\Rightarrow \quad t_n = 0 + 2 + 3 + 7 + 19 + \ldots + t'_{n-1} + t'_n$

Subtract

$0 = 2 + 1 + 4 + 12 + 36 + \ldots$ n terms $-t'_n$

$\Rightarrow \quad t'_n = 3 + [4 + 12 + 36 + \ldots$ $(n - 2)$ terms$]$

$$\Rightarrow \quad t'_n = 3 + \frac{4\left(3^{n-2} - 1\right)}{3 - 1}$$

$\Rightarrow \quad t'_n = 2.3^{n-2} + 1, \left(n \geq 2\right)$

$$\text{Now } t_n = \sum_{n=1}^{n} t'_n = 2\sum_{n=2}^{n} 3^{n-2} + \sum_{n=2}^{n} 1$$

$t_n = 3^{n-1} + (n - 1)$

Now $S_n = \Sigma t_n$

$= \Sigma 3^{n-1} + \Sigma(n - 1)$

$$= \frac{3^n - 1 + n^2 - n}{2}$$

12. We have $t_n = S_n - S_{n\text{-}1} \; \forall \; n \geq 2$

$$\therefore \quad t_n = \frac{1}{6}\left[2\left(n^3 - \left(n - 1\right)^3\right) + 9\left(n^2 - \left(n - 1\right)^2\right) + 13\left(n - n + 1\right)\right]$$

$$= \frac{1}{6}\left[6n^2 - 6n + 2 + 9\left(2n - 1\right) + 13\right]$$

$$= \frac{1}{6}\left(6n^2 + 12n + 6\right) = \left(n + 1\right)^2$$

$$\therefore \sum_{r=1}^{n} \sqrt{t_r} = \sum_{r=1}^{n} \left(r + 1\right) = \frac{1}{2}\left(n + 1\right)\left(n + 2\right) - 1 = \frac{1}{2}n\left(n + 3\right)$$

Single Correct Choice Type Questions

1. As the number of terms are odd (7) let r, be thecommon ratio

So terms can be taken as $\frac{a}{r^3}, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, ar^3$

$\therefore$ Product of the term $= a^7 = 3^7$ as $(t_4 = a = 3)$

Hence, the correct answer is (B).

2. Let r is the common ratio

$$\frac{a + ar}{2} = 6 \textit{ and } \frac{ar^2 + ar^3}{2} = 54$$

$\Rightarrow \quad r^2 = 9, r = \pm 3 \Rightarrow r = 3\left(r \neq -3\right)$

when $r = 3$, $a = 3$, A.M. of a and $d = \frac{a + ar^3}{2} = 42$.

Hence, the correct answer is (D).

3. $$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_{4000} a_{4001}}$$

$$= \frac{1}{d}\left[\frac{a_2 - a_1}{a_1 a_2} + \frac{a_3 - a_2}{a_2 a_3} + \ldots + \frac{a_{4001} - a_{4000}}{a_{4000} a_{4001}}\right]$$

$$= \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \ldots + \left(\frac{1}{a_{4000}} - \frac{1}{a_{4001}}\right)\right]$$

$$= \frac{1}{d}\left[\left(\frac{1}{a_1} - \frac{1}{a_{4001}}\right)\right] = \frac{1}{d}\left[\frac{a_{4001} - a_1}{a_1 a_{4001}}\right]$$

$$\frac{4000}{a_1 a_{4001}} = 10 \Rightarrow a_1 a_{4001} = 400$$

also $a_1 + a_{4001} = 50$

$\Rightarrow \quad \left|a_1 - a_{4001}\right|^2 = 2500 - 1600$

$\Rightarrow \quad \left|a_1 - a_{4001}\right| = 30$

Hence, the correct answer is (B).

4. 7^4 ends with 01

$\Rightarrow \quad 7^{4k+r}$ ends with same two digit number as does 7^r

therefore the given number ends with the same two digits as 7^3.

Hence, the correct answer is (D).

5. $$\frac{\frac{p}{q} + 5^x}{\frac{p}{q} - 5^x} = \frac{\frac{q}{r} + 5^x}{\frac{q}{r} - 5^x} = \frac{\frac{r}{s} + 5^x}{\frac{r}{s} - 5^x}$$

$\Rightarrow \frac{p}{q} = \frac{q}{r} = \frac{r}{s} \Rightarrow p, q, r, s$ are in G.P.

Hence, the correct answer is (B).

6. $\left(a_{100} + b_{100}\right) - \left(a_1 + b_1\right) = 99d \Rightarrow d = 0$, $(a + b_1) + (a_2 + b_2) + \ldots +$ $a_{100} + b_{100} = 100(a_1 + b_1)$

$= 100 \times 100$

Hence, the correct answer is (C).

7. $$n^4 + 2n^3 + 3n^2 + 2n + 1 = n^2\left(n^2 + 2n + 3 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$= n^2\left(\left(n + \frac{1}{n}\right)^2 - 2 + 2\left(n + \frac{1}{n}\right) + 3\right)$$

$$= n^2\left(n + \frac{1}{n} + 1\right)^2$$

$$\sum_{n=1}^{2008} \frac{\sqrt{n^4 + 2n^3 + 3n^2 + 2n + 1}}{n\left(n + 1\right)}$$

$$= \sum_{n=1}^{2008}\left(1 + \frac{1}{n\left(n + 1\right)}\right) = \sum_{n=1}^{2008} 1 + \left(\frac{1}{n} - \frac{1}{n + 1}\right)$$

$$= 2008 + \frac{2008}{2009}$$

Hence, the correct answer is (A).

8. $\alpha + \beta = \frac{b}{a},\ \alpha\beta = \frac{c}{a}, \alpha, \beta \in (1,2)$

$\Rightarrow \quad \alpha - 1, \beta - 1, 2 - \alpha, 2 - \beta \in (0,1)$

Apply A.M. ≥ G.M.

$$\frac{(\alpha-1)+(2-\alpha)}{2} > \sqrt{(\alpha-1)(2-\alpha)}$$

and $\frac{(\beta-1)+(2-\beta)}{2} > \sqrt{(\beta-1)(2-\beta)}$

$\Rightarrow \quad (\alpha-1)(2-\alpha) < \frac{1}{4}$ and $(\beta-1)(2-\beta) < \frac{1}{4}$

$$(\alpha-1)(\beta-1)(2-\alpha)(2-\beta) < \frac{1}{16}$$

$\therefore \quad (\alpha-1).(2-\alpha).(\beta-1).(2-\beta) > 0$

$\Rightarrow \quad 0 < (\alpha-1)(2-\alpha)(\beta-1)(2-\beta) < \frac{1}{16}$

$$0 < \left(\frac{c}{a} - \frac{b}{a} + 1\right)\left(4 - \frac{2b}{a} + \frac{c}{a}\right) < \frac{1}{16} \qquad \ldots(1)$$

$(a-b+c)(4a-2b+c) \geq 1 \qquad \because a, b, c \in N$

From (1) $\frac{a^2}{16} > 1 \Rightarrow a \geq 5$

Hence, the correct answer is (B).

9. G.T is $T_n = \frac{n^2}{500 + 3n^3}$

for max $\frac{d}{dn}(T_n) = 0 \Rightarrow n = \left(\frac{1000}{3}\right)^{1/3}$

$$6 < \left(\frac{1000}{3}\right)^{1/3} < 7 \Rightarrow T_7 \text{ is greatest} \Rightarrow \frac{49}{500 + 3\left(7^3\right)}$$

Hence, the correct answer is (C).

10. $2n\,A_n = (2n-3)A_{n-1},\ n \geq 2$

$2K\,A_K = (2k-3)A_{k-1},\ k \geq 2$

$2K\,A_k - 2(K-1)A_{k-1} = -A_{k-1},\ k \geq 2$

$\therefore \quad 2(n+1)A_{n+1} - 2a_1 = -\sum_{k=1}^{n} A_K$

$$\sum_{k=1}^{n} A_K = 1 - 2(n+1)A_{n+1}$$

$$(n+1)A_{n+1} > 0$$

$\therefore \quad \sum_{k=1}^{n} A_k < 1, n \geq 1$

Hence, the correct answer is (C).

11. $T_n = \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$

$$S_n = \sum_{n=1}^{n} T_n = \frac{1}{2}\left[\frac{1}{1 \times 2} - \frac{1}{(n+1)(n+2)}\right]$$

$$S_\infty = \frac{1}{4}$$

$$8S_\infty = 2$$

Hence, the correct answer is (A).

12. $\frac{1}{x_{k+1}} = \frac{1}{x_k(x_k+1)} = \frac{1}{x_k} - \frac{1}{x_k+1} \Rightarrow \frac{1}{x_k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}}$

$\therefore \quad \frac{1}{x_1+1} + \frac{1}{x_2+1} + \ldots + \frac{1}{x_{100}+1} = \frac{1}{x_1} - \frac{1}{x_{101}}$

As $0 < \frac{1}{x_{101}} < 1$

$\therefore \quad \left[\frac{1}{x_1+1} + \frac{1}{x_2+1} + \ldots + \frac{1}{x_{100}+1}\right] = 1.$

Hence, the correct answer is (D).

13. $S = 1 + 4x + 7x^2 + 10x^3 + \ldots$

$$x.S = x + 4x^2 + 7x^3 + \ldots$$

Subtract

$$S(1-x) = 1 + 3x + 3x^2 + 3x^3 + \ldots$$

$$S(1-x) = 1 + 3x\left(\frac{1}{1-x}\right) \qquad |x| < 1$$

$$S = \frac{1+2x}{(1-x)^2}$$

Given $\frac{1+2x}{(1-x)^2} = \frac{35}{16}$

$\Rightarrow \quad 16 + 32x = 35 + 35x^2 - 70x$

$\Rightarrow \quad 35x^2 - 102x + 19 = 0$

$\Rightarrow \quad 35x^2 - 7x - 95x + 19 = 0$

$\Rightarrow \quad 7x(5x-1) - 19(5x-1) = 0$

$\Rightarrow \quad (5x-1)(7x-19) = 0 \quad \Rightarrow \quad x = \frac{1}{5}, \frac{19}{7}$

But $|x| < 1 \qquad \therefore \quad x = \frac{1}{5}$

Hence, the correct answer is (B).

14. $1 + a \geq 2\sqrt{a} \qquad \{AM \geq GM\}$

$$1 + b \geq 2\sqrt{b}$$

$$1 + c \geq 2\sqrt{c}$$

$$1 + d \geq 2\sqrt{d}$$

$\therefore \quad (1+a)(1+b)(1+c)(1+d) \geq 16\sqrt{abcd} = 16$

$\therefore \quad$ min. value = 16 (for $a = b = c = d = 1$)

Hence, the correct answer is (C).

15. $t_r = \dfrac{r(r+1)(2r+1)4}{6r^2(r+1)^2} = \dfrac{2(2r+1)}{3r(r+1)} = \dfrac{2}{3}\left[\dfrac{1}{r} + \dfrac{1}{r+1}\right]$

$$S_n = \frac{2}{3}\sum_{r=1}^{n}(-1)^r\left(\frac{1}{r} + \frac{1}{r+1}\right)$$

$$\Rightarrow \quad S_\infty = \frac{2}{3}\left[\left(-1+\frac{1}{2}-\frac{1}{3}+\ldots\right)+\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)\right]$$

$$= \frac{2}{3}[-1]$$

Hence, the correct answer is (D).

16. Let the numbers be 1, 2, 3, 4, ..., n and the erased number be x

$$\Rightarrow \quad \frac{\frac{n(n+1)}{2} - n}{n-1} \le 35\frac{7}{17} \le \frac{\frac{n(n+1)}{2} - 1}{n-1}$$

$$\Rightarrow \quad \frac{n+2}{2} \ge 35\frac{7}{17} \ge \frac{n}{2} \quad \Rightarrow \quad 68 + \frac{14}{17} \le n \le 70 + \frac{14}{17}$$

So when $n = 69$; $\dfrac{\frac{69\times 70}{2} - x}{68} = 35\dfrac{7}{17} \Rightarrow x = 7$ and when $n = 70$; $x \notin I$.

Hence, the correct answer is (B).

17. Let $\dfrac{n}{m}$ be the common ratio of G.P., where n and m are prime integers $n > m$.

Now $a_5 = a_1\dfrac{n_4}{m^4}$, so let $a_1 = km^4$, $k \in I^+$, so our G.P. becomes $km^4, km^3n, km^2n^2, kmn^3, kn^4$ and $kn^4 < 100$, if $n > 4$, $kn^4 \ge n^4 > 256 > 100$ so, $n \le 3$

Hence there are 3 cases

CASE-1: $n = 3$ and $m = 2$, $81k < 100$

$\Rightarrow \quad k = 1, 180^{\text{ln}}$ (16, 24, 36, 54, 81)

CASE-2: $n = 3$, $m = 1$, $81k < 100$ so $k = 1(1, 3, 9, 27, 81)$

CASE-3: $n = 2$, $m = 1$, $16k < 100$, so $k = 1, 2, \ldots 6$.

There are 6 solutions (1, 2, 4, 8, 16), (2, 4, 8, 16, 32), (3, 6, 12, 24, 48), (4, 8, 16, 32, 64), (6, 12, 24, 48, 96) and (5, 10, 20, 40, 80).

Hence, the correct answer is (C).

18. Here $t_r = \dfrac{r\cdot 2^r}{(r+2)!} = \dfrac{(r+2-2)2^r}{(r+2)!} = \dfrac{2^r}{(r+1)!} - \dfrac{2^{r+1}}{(r+2)!}$

$$\therefore \quad S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left[\frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}\right] = 1 - \frac{2^{n+1}}{(n+2)!}$$

$$\therefore \quad S_\infty = \lim_{n\to\infty}\left(1 - \frac{2^{n+1}}{(n+2)!}\right) = 1.$$

Hence, the correct answer is (D).

19. $n^2 + n(a-1) - 153 = 0$ where a is the first term.

Number of terms = Number of factors of 153 excluding 1.

$n(n + a - 1) = 153 = 3^2 \cdot 17$

Possible $n \to 3, 9, 17, 51, 153$

Hence, the correct answer is (B).

20. $\dfrac{\left(\frac{a}{b}\right)^n + \left(\frac{c}{b}\right)^n}{2} \ge \left(\dfrac{\left(\frac{a}{b} + \frac{c}{b}\right)}{2}\right)^n \ge 1 \Rightarrow n > 0$

as $\dfrac{a+c}{2} > \dfrac{2ac}{a+c} = b \Rightarrow \dfrac{a+c}{2b} > 1$

A.M. > H.M.

Hence, the correct answer is (D).

21. Let the sides be $a - d, a, a + d$

Where $a > d > 0$

We have

$$(a+d)^2 = (a-d)^2 + a^2$$

$$\Rightarrow \quad d = \frac{a}{4} \text{ we have } \sin\theta = \frac{a}{a+d}$$

$$\Rightarrow \quad \cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5}$$

Hence, the correct answer is (A).

22. $p = \dfrac{x^{\ln y - \ln z} + y^{\ln z - \ln x} + z^{\ln x - \ln y}}{3}$

$$\ge \sqrt{x^{\ln y - \ln z}.y^{\ln z - \ln x}.z^{\ln x - \ln y}}$$

$$\Rightarrow \quad x^{\ln y - \ln z} + y^{\ln z - \ln x} + z^{\ln x - \ln y} > 3$$

[as $x^{\ln y} = y^{\ln x}$ and $x \ne y \ne z$]

Hence $p \in (3, \infty)$

Hence, the correct answer is (C).

23. $f(n) = \sum(n^2 + 1) = \dfrac{n(n+1)(2n+1)}{6} + n$

$$g(n) = \sum(n^2 + n) = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6} + n}{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}$$

$$= \lim_{n\to\infty}\frac{1 + \frac{6}{(n+1)(2n+1)}}{1 + \frac{3}{(2n+1)}}$$

$$= 1$$

Hence, the correct answer is (B).

24. Since each a_1 is an odd number not divisible by a prime greater than 5, a_1 can be written as $a_1 = 3^r5^s$ where r, s are non-negative integers.

thus for all $n \in N$

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < \left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots +\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots +\right) = \frac{15}{8}$$

Hence, the correct answer is (A).

25. $a_0 = 17.23$, $a_1 = 33.23$ and $\alpha_{r+2} = \dfrac{\alpha_r + \alpha_{r+1}}{2}$

$$\forall r \ge 0 \Rightarrow \alpha_{r+2} - \alpha_{r+1} = -\frac{1}{2}(\alpha_{r+1} - \alpha_r)$$

$$\alpha_{10} - \alpha_9 = -\frac{1}{2}(\alpha_9 - \alpha_8) = \frac{1}{2^2}(\alpha_8 - \alpha_7) = \ldots$$

$$= -\frac{1}{2^9}(\alpha_1 - \alpha_0) = +\frac{-1}{2^9} \times 16 = \frac{-1}{2^5}$$

$$\therefore\ |\alpha_{10} - \alpha_9| = \frac{1}{32}$$

Hence, the correct answer is (C).

26. $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

$$= 2\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots\right) \Rightarrow \frac{1}{1.3} + \frac{1}{5.7} + \ldots = \frac{\pi}{8}$$

Hence, the correct answer is (A).

27. a, b, c, d are in G.P., let they are a, ar, ar^2, ar^3

$$\left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right)$$

$$= a^2 \times a^2\left[1 + r^2 + r^4\right]\left[r^2 + r^4 + r^6\right]$$

$$= a^4 r^2\left[1 + r^2 + r^4\right]^2$$

$$= \left[a^2 r\left[1 + r^2 + r^4\right]\right]^2$$

$$= (ab + bc + cd)^2$$

Hence, the correct answer is (B).

28. From A.M. ≥ G.M., we get

$$\frac{a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5}{5} \ge (a_1.2a_2.3a_3.4a_4.5a_5)^{\frac{1}{5}}$$

$$\Rightarrow\quad a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 \ge 5\ (5!\ a_1 a_2 a_3 a_4 a_5)^{1/5} = 5(6!)^{\frac{1}{5}}$$

Hence, the correct answer is (D).

29. Using the $AM \ge HM$ of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ we get,

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \ge \frac{3}{a + b + c} = \frac{3}{6} = \frac{1}{2} \text{ So, } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{3}{2}$$

Now, $$\frac{\left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2}{3}$$

$$\ge \left(\frac{a + \frac{1}{b} + b + \frac{1}{c} + c + \frac{1}{a}}{3}\right)^2 \ge \left(\frac{6 + \frac{3}{2}}{3}\right)^2$$

$$= \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

$$\therefore \left(a + \frac{1}{b}\right)^2 + \left(b + \frac{1}{c}\right)^2 + \left(c + \frac{1}{a}\right)^2 \ge \frac{75}{4}$$

Hence, the correct answer is (B).

30. Given expression equal to

$$\frac{(a + b)}{c} + \frac{(b + c)}{a} + \frac{(c + a)}{b} \text{ or } \frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b}$$

Using A.M. ≥ G.M. $$\frac{\frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b}}{6} \ge \sqrt{\frac{a}{c}\frac{b}{c}\frac{b}{a}\frac{c}{a}\frac{c}{b}\frac{a}{b}}$$

or $$\frac{a + b}{c} + \frac{b + c}{a} + \frac{c + a}{b} \ge 6$$

Hence, the correct answer is (C).

31. $$S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \ldots$$

$$\frac{S}{19} = 0 + \frac{4}{19^2} + \frac{44}{19^3} + \ldots$$

$$\Rightarrow\quad S - \frac{S}{19} = \frac{4}{19} + \frac{40}{19^2} + \frac{400}{19^3} + \ldots$$

$$\Rightarrow\quad \frac{18S}{19} = \frac{4}{19}\left(1 + \frac{10}{19} + \frac{100}{19^2} + \ldots\right)$$

$$\Rightarrow\quad \frac{18S}{19} = \frac{4}{19} \cdot \frac{1}{1 - \frac{10}{19}} = \frac{4}{9} \Rightarrow S = \frac{38}{81}.$$

Hence, the correct answer is (A).

32. $S_n - S_{n-1} = T_n = 2c(n - 1)$

Required Sum $= 4c^2\{0^2 + 1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2\}$

$$= \frac{2}{3}c^2 n(n - 1)(2n - 1)$$

Hence, the correct answer is (B).

33. $$\left(\frac{a + b + c + d}{4}\right)^2 \le \frac{a^2 + b^2 + c^2 + d^2}{4}$$

$$\Rightarrow \left(\frac{8 - e}{4}\right)^2 \le \frac{16 - e^2}{4} \Rightarrow 64 + e^2 - 16e \le 64 - 4e^2$$

$$0 \le e \le \frac{16}{5}$$

Hence, the correct answer is (A).

34. Let a, b be two numbers

$$A = \frac{a + b}{2},\ G = \sqrt{ab},\ H = \frac{2ab}{a + b}$$

Given $xA = yH = zH \Rightarrow \dfrac{a + b}{2\sqrt{ab}} = \dfrac{y}{x} = \dfrac{z}{y}$

$\therefore$ x, y, z are in G.P.

Hence, the correct answer is (B).

35. a, b, c are in A.P.

$\Rightarrow$ $2b = a + c$ …(1)

$(b - a), (c - b), a$ are in G.P.

$\Rightarrow$ $(c - b)^2 = a(b - a)$

$\Rightarrow$ $(b - a)^2 = a(b - a)$

$(b - a) = a$ …(2)

from (1) and (2)

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

Hence, the correct answer is (D).

36.

		No. of terms
Group (1)	1	1
Group (2)	2, 2, 2	3

(Continued)

		No. of terms
Group (3)	3, 3, ..., 3	6
Group (4)	4, 4, ..., 4	10
. . . . .		
. . . . .		
Group (r)	$r, r, ..., r$	$\frac{r^2+r}{2}$

Let 2008th term falls in rth group

$$\Rightarrow \quad 1+3+6+10+\ldots+\frac{(r-1)^2+(r-1)}{2}<2008\leq 1+3+6+\ldots+\frac{r^2+r}{2}$$

$$\Rightarrow \quad \frac{(r-1)r(r+1)}{6}<2008\leq\frac{r(r+1)(r+2)}{6}$$

$$\Rightarrow \quad r^3-\mathrm{r}<12048\leq(r+1)^3-(r+1) \qquad \ldots (1)$$

$\Rightarrow$ r will be nearer to cube root of 12048

Note: $22<\sqrt[3]{12048}<23$

for $r=22$ inequality (1) holds

for $r<22$ RHS of (1) is less than 12048

for $r\geq 23$ LHS of (1) is greater than 12048

$\Rightarrow$ $r=22$ is the required value $\Rightarrow$ 2008th term is 22

Hence, the correct answer is (C).

37. $$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$\left(\sum_{k=1}^{n}a_k\right)^2=\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}\right)^2=\frac{n^2}{(n+1)^2}$$

Hence, the correct answer is (B).

38. $$\sum_{k=1}^{\infty}\frac{1}{k!}\left(1+2+2^2+\ldots+2^{k-1}\right)$$

$$=\sum_{k=1}^{\infty}\frac{2^k-1}{k!}=e^2-e$$

Hence, the correct answer is (D).

39. $\frac{1}{2},\frac{1}{H_1},\frac{1}{H_2},\ldots,\frac{1}{H_{20}},\frac{1}{3}$ are in A.P.

Common difference $d=\dfrac{\frac{1}{3}-\frac{1}{2}}{20+1}=-\frac{1}{126}$;

$$\frac{1}{H_1}=\frac{1}{2}-\frac{1}{126}=\frac{31}{63};\ \frac{1}{H_{20}}=\frac{1}{3}+\frac{1}{126}=\frac{43}{126}$$

$$H_1=\frac{63}{31},\ H_{20}=\frac{126}{43}$$

Hence, the correct answer is (C).

40. $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots\right)=\frac{\pi^2}{6}$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)=\frac{\pi^2}{6}$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{6}-\frac{1}{2^2}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

Hence, the correct answer is (A).

41. Given 1, $\log_{9^2}\left(3^x+48\right),\log_9\left(3^x-8/3\right), \in$ A.P.

$$\Rightarrow \quad \log_9 9,\ \frac{1}{2}\log_9\left(3^x+48\right),\ \log_9\left(3^x-8/3\right)\in \text{A.P.}$$

$$\Rightarrow \quad 9,\left(3^x+48\right)^{1/2},3^x-8/3 \in\text{G.P. (By concept)}$$

$$\Rightarrow \quad \log a,\log b,\log c\in \text{A.P.}$$

$$\therefore \quad a,b,c\in \text{G.P.}$$

$$\therefore \quad 3^x+48=9\left(3^x-8/3\right)$$

$$\Rightarrow \quad 8.3^x=72$$

$$\Rightarrow \quad 3^x=9,\ 3^x=3^2,\ x=2.$$

Hence, the correct answer is (C).

42. Given $a,b,c\in$H.P.

So, $\frac{1}{a},\frac{1}{b},\frac{1}{c}\in$A.P.

$$\frac{a+b+c}{a},\frac{b+c+a}{b},\frac{c+a+b}{c}\in \text{A.P.}$$

$$\frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c} \text{ are in A.P.}$$

By using concept if $a,b,c\in$ A.P. Then their reciprocals are in H.P.

Hence, the correct answer is (B).

43. Let the first term is a, then first (2n + 1) terms are $a,a+2,a+4,\ldots a+2.2n$. Clearly the middle term of the sequence of $4n+1$ term is $\left(2n+1\right)^{th}$ term, i.e . $a+4n$ also the middle term of the A.P of $\left(2n+1\right)$term is $\left(n+1\right)^{th}$ term i.e., $a+2n$.Again for the last $\left(2n+1\right)$ terms the first term will be $\left(2n+1\right)^{th}$ term of the A.P i.e. $a+4n$ terms are in G.P whose common

$\therefore$ G.P. is $\left(a+4n\right),\left(a+4n\right)\left(0.5\right)^n$

Its middle term is $\left(a+4n\right)\left(0.5\right)^n$

According to the given condition,

$$a+2n=\left(a+4n\right)\left(0.5\right)^n$$

$$\therefore \quad a=\frac{2n-4n\left(0.5\right)^n}{\left(0.5\right)^n-1}$$

$\therefore$ Required middle term $= a+4n =$

$$\frac{2n-4n(0.5)^n}{(0.5)^n-1}+4n=\frac{2n}{1-\left(\frac{1}{2}\right)^n}=\frac{n.2^{n+1}}{2^n-1}$$

Hence, the correct answer is (D).

44. The general term of the series is $t_n=\dfrac{x^{2^{n-1}}}{1-x^{2^n}}$

$$=\frac{1+x^{2^{n-1}}-1}{\left(1+x^{2^{n-1}}\right)\left(1-x^{2^{n-1}}\right)}$$

$\therefore\quad t_n=\dfrac{1}{1-x^{2^{n-1}}}-\dfrac{1}{1-x^{2^n}}$

Now, $S_n=\sum\limits_{n=1}^{n}t_n=\left[\left\{\dfrac{1}{1-x}-\dfrac{1}{1-x^2}\right\}+\left\{\dfrac{1}{1-x^2}-\dfrac{1}{1-x^4}\right\}+\ldots+\right.$

$$\left.\left\{\frac{1}{1-x^{2^{n-1}}}-\frac{1}{1-x^{2^n}}\right\}\right]=\frac{1}{1-x}-\frac{1}{1-x^{2^n}}$$

$\therefore$ The sum to infinite terms

$$=\lim_{n\to\infty}S_n=\frac{1}{1-x}-1=\frac{x}{1-x}\qquad\left[\because \lim_{n\to\infty}x^{2^n}=0,\text{as }|x|<1\right]$$

Hence, the correct answer is (A).

45. Let $A_1,A_2\ldots A_n$ be airthmetric means between a and $2b$, then

$A_m=a+m\left(\dfrac{2b-a}{n+1}\right)$

Again, let $B_1,B_2\ldots B_n$ be arithmetic means

Between $2a$ and b then $B_m=2a+m\left(\dfrac{b-2a}{n+1}\right)$

Now, $A_m=B_m\Rightarrow a+m\left(\dfrac{2b-a}{n+1}\right)=2a+$

$$m\left(\frac{b-2a}{n+1}\right)\Rightarrow m\left(\frac{b+a}{n+1}\right)=a\Rightarrow\frac{a}{b}=\frac{m}{n-m+1}$$

Hence, the correct answer is (A).

46. Given $\dfrac{S_3}{S_6}=\dfrac{64}{91}=\dfrac{a\left(r^3-1\right)}{a\left(r^6-1\right)}$

$\Rightarrow\quad\dfrac{\left(r^3-1\right)}{\left(r^3+1\right)\left(r^3-1\right)}=\dfrac{64}{91}$

$\Rightarrow\quad r^3=\dfrac{27}{64}$

$\therefore\quad r=\dfrac{3}{4}$

Hence, the correct answer is (B).

47. $S_n=3+5+9+17+33+\ldots$

$=(2+1)+\left(2^2+1\right)\left(2^3+1\right)+\left(2^4+1\right)+\ldots$

$=\left(2+2^2+2^3+2^4+\ldots n\text{ terms}\right)+n$

$=2\left(2^n-1\right)+n=2^{n+1}+n-2$

Hence, the correct answer is (B).

48. Given $a+b=2A$

And $a,p,q,b\in$ G.P.

$\therefore\quad p^2=aq$ and $q^2=pb$

$\Rightarrow\quad p^3=apq$ and $q^3=bpq$

by adding we get

$p^3+q^3=apq+bpq=pq(a+b)=2Apq$

Hence, the correct answer is (B).

49. If n is a +ve number.

$n=P_1^{k_1}.P_2^{k_2}\ldots P_r^{k_r}$

(where $p_1,p_2,p_3,\ldots.p_r$ are prime number) then number of divisors of n are

$=(k_1+1)(k_2+1)\ldots(k_r+1)$

$\therefore$ Number of prime factor of 6912 are $=2^8.3^3$ so no. of divisors $=9\times4=36$

Prime factors of 52,480 are $=2^8.5.41$

$\therefore\quad$ No. of divisors $=9\times2\times2=36$

Prime factors of 32,000 are $=5^3\times2^8$

$\therefore\quad$ No. of divisors $=9\times4=36$

Now each number having same number of divisors i.e., 36, 36, 36

Each and every term is constant and constant sequence is always in A.P. and G.P. both as common difference is 0 and common ratio is 1.

Hence, the correct answer is (C).

50. We have $2b=a+c$ and a,p,b and b,q,c are in A.P

$\Rightarrow\quad p=\dfrac{a+b}{2},q=\dfrac{b+c}{2}$

Again, $p'=\sqrt{ab}$ and $q'=\sqrt{bc}$

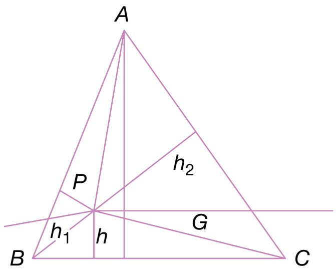
$\therefore\quad p^2-q^2=\dfrac{(a+b)^2-(b+c)^2}{4}$

$$=\frac{(a-c)(a+c+2b)}{4}=(a-c)b=p'^2-q'^2$$

Hence, the correct answer is (C).

51. $\Delta ABC=\Delta PBC+\Delta PAC+\Delta PAB$

$\dfrac{1}{2}a.3h=\dfrac{1}{2}a.h+\dfrac{1}{2}a\,h_1+\dfrac{1}{2}a.h_2$



$h_1+h_2=2h\Rightarrow h=\dfrac{h_1+h_2}{2}$

Hence, the correct answer is (C).

52. $\log_6(abc) = 6 \Rightarrow (abc) = 6^6$

Let $a = \frac{b}{r}$ and $c = br \Rightarrow b = 36$ and $a = \frac{36}{r}$

$\Rightarrow \quad r = 2, 3, 4, 6, 9, 12, 18$

Also $b - a = 36\left(1 - \frac{1}{r}\right)$ is a perfect cube. $\therefore \quad r = 4$

$\Rightarrow \quad a + b + c = 36 + 9 + 144 = 189$

Hence, the correct answer is (D).

53. $a_1 + a_3 + a_5 + \ldots + a_{199} = \beta$

$a_2 + a_4 + a_6 + \ldots + a_{200} = \alpha$

$a_2 - a_1 + a_4 - a_3 + a_6 - a_5 \ldots a_{200} - a_{199} = \alpha - \beta$

$d + d + d \ldots d = \alpha - \beta$

$d = \frac{\alpha - \beta}{100}$

Hence, the correct answer is (D).

54. Consider the function $f(x) = \frac{\sqrt{x}}{x + 2008}, x \geq 1$

$$f'(x) = \frac{(x + 2008) \times \frac{1}{2\sqrt{x}} - \sqrt{x}}{(x + 2008)^2}$$

$$= \frac{x + 2008 - 2x}{2\sqrt{x}\,(x + 2008)^2} = \frac{2008 - x}{2\sqrt{x}\,(x + 2008)^2}$$

$f'(x) = 0 \Rightarrow x = 2008$

$x \in (2008 - \delta, 2008), f'(x) > 0;$

$x \in (2008, 2008 + \delta), f'(x) < 0$

So $f(x)$ is maximum at $x = 2008$

Hence, the correct answer is (C).

55. $\frac{a + b}{2} = 2\sqrt{ab}$

$a + b - 4\sqrt{ab} = 0$

$\frac{a}{b} + 1 - 4\sqrt{\frac{a}{b}} = 0$ (Dividing by b)

or $\left(\sqrt{\frac{a}{b}}\right)^2 - 4\sqrt{\frac{a}{b}} + 1 = 0$

$\therefore \quad \sqrt{\frac{a}{b}} = \frac{4 \pm 2\sqrt{3}}{2} = \left(2 \pm \sqrt{3}\right)$

$\frac{a}{b} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$

Hence, the correct answer is (B).

56. $S_1 = \frac{T_1^2 + 3T_1 - 2}{3} = T_1 \Rightarrow T_1^2 = 2$

$$S_2 - S_1 = \frac{T_2^2 - T_1^2 + 3(T_2 - T_1)}{3} = T_2$$

$\Rightarrow \quad T_2^2 - T_1^2 + 3(T_2 - T_1) = 3T_2$

$\Rightarrow \quad T_2^2 = T_1^2 + 3T_1 = 2 + 3\sqrt{2}$

Hence, the correct answer is (C).

57. $s = ab + [ab + (a + b) + 1] + [ab + 2(a + b) + 2^2] + \ldots$

$[ab + (n - 1)(a + b) + (n - 1)^2]$

$$= nab + (a + b)\sum_{r=1}^{n-1} r + \sum_{r=1}^{n-1} r^2$$

$$= nab + (a + b)\frac{n(n - 1)}{2} + \frac{(n - 1))(n)(2n - 1)}{6}$$

$$= n\left[\frac{1}{6} + \frac{1}{6}(n - 1) + \frac{1}{6}(n - 1)(2n - 1)\right]$$

$$= \frac{n}{6}[1 + (n - 1)\{1 + 2n - 1\}]$$

$$= \frac{n}{6}[1 + 2n(n - 1)] = \frac{n}{6}(1 - 2n + 2n^2)$$

Hence, the correct answer is (C).

58. $\log(a + c) + \log(b + c) = 2\log(a + b)$

$(a + c)(b + c) = (a + b)^2$

$\Rightarrow \quad ab + c(a + b) + c^2 = (a + b)^2$

also, $c = \frac{2ab}{a + b} \Rightarrow 2ab = c(a + b)$ ...(1)

$\Rightarrow \quad 2ab + 2c(a + b) + 2c^2 = 2(a + b)^2$ ...(2)

From (1) and(2),

$c(a + b) + 2c(a + b) + 2c^2 = 2(a + b)^2$

$2(a + b)^2 - 3c(a + b) - 2c^2 = 0$

$\therefore \quad a + b = \frac{3c \pm \sqrt{9c^2 + 16c^2}}{4} = \frac{3c \pm 5c}{4} = 2c$ or $-\frac{c}{2}$

$\therefore \quad a + b = 2c \quad (\because a, b, c > 0)$

Hence, the correct answer is (A).

59. Let $S_n = 2 + 2.2^2 + 3.2^3 + \ldots + n.2^n$

$2.S_n = 0 + 2^2 + 2.2^3 + \ldots + n.2^{n+1}$

Take difference

$-S_n = 2 + 2^2 + 2^3 + \ldots + 2^n - n.2^{n+1}$

$S_n = n.2^{n+1} - 2(2^n - 1)$

$= 2^{n+1}(n - 1) + 2.$

Given that $2^{n+1}(n - 1) + 2 = 2^{n+10} + 2$

$\Rightarrow \quad (n - 1) = 2^9$

$\Rightarrow \quad n = 2^9 + 1 = 513$

Alternate

$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

$2^2 + 2^3 + \ldots 2^n = 2^{n+1} - 2^2$

$2^3 + \ldots + 2^n = 2^{n+1} - 2^3$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$+2^n = 2^{n+1} - 2^n$

$\Rightarrow \quad 2 + 2.2^2 + 3.2^3 + \ldots + n.2^n$

$= n\left(2^{n+1}\right) - \left(2^{n+1} - 2\right)$

$= 2^{n+1}(n - 1) + 2$

Given that $2^{n+1}(n-1)+2=2^{n+10}+2$

$\Rightarrow \quad (n-1)2^{n+1}=2^{n+10} \quad \Rightarrow \quad n-1=2^9$

$\Rightarrow \quad n=2^9+1=513$

Hence, the correct answer is (B).

60. $N=\frac{1}{9}\{9+99+999+\ldots+999999999\}$

$=1+11+111+\ldots+111111111$

$=123456789$

Hence, the correct answer is (A).

61. $E=9x\sin x+\frac{4}{x\sin x}$ [note that $x\sin x>0$ in $(0,\pi)$]

$$E=\left(3\sqrt{x\sin x}-\frac{2}{\sqrt{x\sin x}}\right)^2+12$$

$\therefore \quad E_{\min}=12$ which occurs when $3x\sin x=2$

$\Rightarrow \quad x\sin x=2/3]$

note that $x\sin x$ is continuous at $x=0$ and attains the value $\pi/2$ which is greater than $2/3$ at

$x=\pi/2$, hence it must take the $2/3$ in $(0,\pi/2)$]

Hence, the correct answer is (C).

62. sequence is $t_1+t_2+t_3+t_4+\ldots$

$t_3=t_1+t_2;\ t_7=1000$

$t_1=1$

but $t_7=t_1+t_2+t_3+t_4+t_5+t_6$

$1000=2(t_1+t_2+t_3+t_4+t_5)$

$=4(t_1+t_2+t_3+t_4)$

$=8(t_1+t_2+t_3)$

$1000=16(t_1+t_2)$

$t_1+t_2=\frac{125}{2}$

$\Rightarrow \quad t_2=\frac{125}{2}-1=\frac{123}{2}$

Hence, the correct answer is (B).

63. $T_n=[n(n+1)-(n-1)]\,n!=n.(n+1)!-(n-1).n!$

Now put $n=1,2,3,\ldots,n$ and add

$T_1+T_2+T_3+\ldots+T_n=n(n+1)!$

Hence, the correct answer is (B).

64. $\frac{1}{9}+\frac{1}{18}+\frac{1}{30}+\frac{1}{45}+\frac{1}{63}+\ldots$

$=\frac{1}{3}\left[\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots\right]$

$=\frac{2}{3}\left[\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\ldots\right]$

$T_n=\frac{2}{3}\frac{1}{(n+1)(n+2)}=\frac{2}{3}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

(using method of difference)

$\therefore \quad S_\infty=\frac{1}{3}$

Hence, the correct answer is (A).

65. $t_n=2+(7+11+15\ldots(n-1)\text{terms})$

$=2+\frac{19}{2}(2\times 7+18\times 4)=819$

Hence, the correct answer is (A).

66. $x>1, y>1, z>1$

$x, y, z \rightarrow$ G.P.

$\Rightarrow$ $\ln x$, $\ln y$, $\ln z$ are in A.P. $2\ln x$, $\ln xy$, $\ln xz$ are in A.P.

$(\ln x^2)^{-1}$, $(\ln xy)^{-1}$, $(\ln xz)^{-1}$ are in H.P.

Hence, the correct answer is (C).

67. $$\frac{(1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\left(1+\frac{1}{4}\right)+\ldots+\left(1+\frac{1}{n}\right)}{n}>$$

$$\left(2\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\ldots\times\frac{n+1}{n}\right)^{1/n}$$

$\Rightarrow \quad \frac{n+S_n}{n}>(n+1)^{1/n} \quad \Rightarrow \quad S_n>n(n+1)^{1/n}-n$

Hence, the correct answer is (A).

68. $2b=a+x \quad \Rightarrow \quad x=2b-a,$

$b^2=ay \quad \Rightarrow \quad y=\frac{b^2}{a}$

$b=\frac{2}{\frac{1}{a}+\frac{1}{z}} \quad \Rightarrow \quad z=\frac{ab}{2a-b} \quad \Rightarrow \quad xyz=\frac{b^3(2b-a)}{2a-b}$

Hence, the correct answer is (D).

69. $\left|\frac{2}{1-r}-2\right|<\frac{1}{10}$

$\left|\frac{r}{r-1}\right|<\frac{1}{20}$

$-\frac{1}{20}<1+\frac{1}{r-1}<\frac{1}{20}$

$-\frac{20}{21}>r-1>-\frac{20}{19}$

$-\frac{1}{19}<r<\frac{1}{21},\ r\neq 0$

$r\in\left(-\frac{1}{19},\frac{1}{21}\right)-\{0\}$

Hence, the correct answer is (D).

70. $S=\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ to ‘n’ terms

$S=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\ldots$ To ‘n’ terms

$=(1+1+1+\ldots)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots.\right)$

$$=n-\frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}=n-\left(1-\frac{1}{2^n}\right)$$

$=2^{-n}+n-1$

Hence, the correct answer is (D).

71. $t_{n+1} = \frac{t_n \times (2n+1)}{(n+1)} \times \frac{1}{2}$

$$(2n+2)t_{n+1} = (2n+1)t_n$$

$$(2n+3)t_{n+1} - (2n+1)t_n = t_{n+1}$$

Put $n = 1$

$$5t_2 - 3t_1 = t_2$$

$n = 2,\ 7t_3 - 5t_2 = t_3$

$n = n-1,\ (2n+1)t_n - (2n-1)t_{n-1} = t_n$

add $(2n+1)t_n - 2t_1 = S$

$$(2n+1)t_n - 3t_1 = t_2 + t_3 + \ldots + t_n$$

$$S = \frac{1.3.5\ldots(2n+1)}{\underline{|n} \times 2^n} - 1$$

Hence, the correct answer is (A).

72. We have $x_k + k = S - x_k$ where $x_1 + x_2 + \ldots + x_k = s$

$\Rightarrow\ 2x_k + k = S$

$\Rightarrow\ 2(S) + \frac{50.51}{2} = 50S$

$\Rightarrow\ 48(S) = 25.51$

$\Rightarrow\ x_{20} = \left(\frac{25.51}{48} - 20\right)\frac{1}{2} = \frac{315}{96}.$

Hence, the correct answer is (B).

73. Let the first term be a and $(2n-1)^{th}$ term be b then

$$p = a + (n-1)d = a + (n-1)\left(\frac{b-a}{2n-2}\right) = \frac{a+b}{2}$$

$$q = a.r^{n-1} = a\left(\frac{b}{a}\right)^{\frac{n-1}{2n-2}} = a\left(\frac{b}{a}\right)^{\frac{1}{2}} = \sqrt{ab}$$

$$\frac{1}{s} = \frac{1}{a} + (n-1)\left(\frac{\frac{1}{b} - \frac{1}{a}}{2n-2}\right) = \frac{\frac{1}{a} + \frac{1}{b}}{2}$$

p, q, r are the A.M, G.M, H.M of a, b.

$\therefore\ p \geq q \geq r$ and $ps = q^2$

Hence, the correct answer is (C).

74. $t_n = 2\,t_{n-1} - t_{n-2}$

$$t_n - t_{n-1} = t_{n-1} - t_{n-2}$$

$$a_n = t_n - t_{n-1}, n \geq 3$$

We have $a_n = a_{n-1}$

Thus $\{a_n\}$ is a constant sequence

$$a_5 = t_5 - t_4 = 1$$

Now $a_4 = t_4 - t_3 \Rightarrow 1 = 4 - t_3 \Rightarrow t_3 = 3$

Similarly $t_2 = 2,\ t_1 = 1$

Thus $\{t_n\}$ is an A.P with $r = 1$ and common difference 1

$$\sum_{n=1}^{2009} t_n = \frac{2009 \times 2010}{2} = 2003 \times 1005 = 2019045$$

Hence, the correct answer is (A).

75. a, b, c are in AP $\Rightarrow$ $a + c = 2b;$ …(1)

b, c, d are in GP $\Rightarrow$ $c^2 = bd$ …(2)

c, d, e are in HP $\Rightarrow$ $\frac{2ce}{c+e} = d$ …(3)

$$(1)\times(3) \Rightarrow \frac{(a+c)ce}{c+e} = bd = c^2$$

$\therefore\ (a+c)e = c(c+e)$

$ae = c^2 \Rightarrow a, c, e$ are in G.P

Hence, the correct answer is (B).

76. $a_1 = 1$

$\Rightarrow\ a_2 = 1 - \frac{1}{2}$

$\Rightarrow\ a_3 = 1 - \frac{1}{2} + \frac{1}{3}$

………

………

$$a_\infty = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln 2$$

Hence, the correct answer is (C).

77. $T_r = \frac{4r+1}{5^r r(r-1)}, r \geq 2$

$$\frac{5r - (r-1)}{5^r r(r-1)} = \frac{1}{5^{r-1}(r-1)} - \frac{1}{5^r r}$$

$$\sum_{r=2}^{\infty} T_r = \left(\left(\frac{1}{5^1.1} - \frac{1}{5^2.2}\right) + \left(\frac{1}{5^2.2} - \frac{1}{5^3.3}\right) + \left(\frac{1}{5^3.3} - \frac{1}{5^4.4}\right) + \ldots\infty\right) = \frac{1}{5}$$

Hence, the correct answer is (C).

78. $d = a^2 + b^2 + c^2$ $\quad\because a, b, c$ and d are in A.P.

$\Rightarrow\ a + 3t = (a+t)^2 + a^2 + (a+2t)^2$

$$5t^2 + 3(2a-1)t + 3a^2 - a = 0$$

$$D \geq 0 \ \Rightarrow\ 24a^2 + 16a - 9 \leq 0$$

$\Rightarrow\ -\frac{1}{3} - \frac{\sqrt{70}}{2} < a < -\frac{1}{3} + \frac{\sqrt{70}}{2}$

$\Rightarrow\ a = -1, 0$

$$a = 0, t = 0, \frac{3}{5}$$

$$a = -1, t = 1, \frac{4}{5}$$

$\Rightarrow\ t = 1$

$$a + b + c + d = 2$$

Hence, the correct answer is (C).

79. $\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

$$\log_e\left(1+\frac{1}{2}\right) = \log_e\frac{3}{2}$$

Hence, the correct answer is (C).

80. Given $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = 8\alpha\beta\gamma$

$$\text{But}\ \left.\begin{array}{l}\alpha+\beta \ge 2\sqrt{\alpha\beta}\\ \beta+\gamma \ge 2\sqrt{\beta\gamma}\\ \gamma+\alpha \ge 2\sqrt{\gamma\alpha}\end{array}\right]$$

$\therefore \quad (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) \ge 8(\alpha\beta\gamma)$

Hence $\alpha = \beta = \gamma$

$\therefore \quad \alpha\beta + \beta\gamma + \gamma\alpha = 12$

$\Rightarrow \quad 3\alpha^2 = 12 \qquad \Rightarrow \quad \alpha = 2$

Put $\alpha = 2$ in the given equation

$\therefore \quad 8 - 4m + 24 - 4n = 0 \qquad \Rightarrow \quad 32 = 4(m+n)$

$\Rightarrow \quad m + n = 8$

Hence, the correct answer is (D).

81. $\alpha + \beta = 10 \quad \Rightarrow \quad \gamma = 5$

$\therefore \quad$ Put value of γ in $x^2 - (3k+2)x + k^3 + 3 = 0$

$\Rightarrow \quad k^3 - 15k + 18 = 0$

$\Rightarrow \quad k = 3$

$\therefore \quad \alpha = 6, \beta = 4, \gamma = 5$

Hence, the correct answer is (B).

82. Let $\alpha - d, \alpha, \alpha + d$

$\therefore \quad (\alpha - d) + \alpha + (\alpha + d) = -a$

$\Rightarrow \quad 3\alpha = -a \qquad \ldots(1)$

$(\alpha - d)a + \alpha(a + d) + (\alpha - d)(\alpha + d) = b$

$\Rightarrow \quad 3a^2 - d^2 = b \qquad \ldots(2)$

$(\alpha - d)\alpha(\alpha + d) = \frac{1}{9}$

$\Rightarrow \quad \alpha(a^2 - d^2) = \frac{1}{9} \qquad \ldots(3)$

$\therefore \quad$ From (1) and (2), $d^2 = \frac{a^2}{3} - b$

$\therefore \quad$ From (3), $\frac{-a}{3}\left(\frac{-a}{9} + b - \frac{a^2}{3}\right) = \frac{1}{9}$

$\Rightarrow \quad 2a^3 - 9ab + 1 = 4$

Hence, the correct answer is (D).

83. Use A.M. $\ge$ G.M.

$$\Rightarrow \quad \frac{\left(\frac{4}{\alpha\beta} + \frac{3}{\beta\gamma} + \frac{2}{\gamma\alpha}\right)}{3} \ge \left(\frac{4}{\alpha\beta}\cdot\frac{3}{\beta\gamma}\cdot\frac{2}{\gamma\alpha}\right)^{\frac{1}{3}}\left(\alpha\beta\gamma = 16\sqrt{6}\right)$$

$$\therefore \quad \frac{4}{\alpha\beta} + \frac{3}{\beta\gamma} + \frac{2}{\gamma\alpha} \ge \frac{3}{4}.$$

Hence, the correct answer is (C).

84. Discriminant $D = 16b^2 - (4)(3a)(c) = 4(4b^2 - 3ac)$

Let three terms be $(r - d), (r), (r + d)$

$a = r - d \quad b = r \quad c = r + d$

$= 4[4r^2 - 3(r^2 - d)^2] = 4[r^2 + 3d^2]. \qquad$ So $D > 0$

As 'r' and 'd' cannot be simultaneously zero. So the equation have two distinct solutions.

Hence, the correct answer is (A).

85. $2b = a + c$ and $(\alpha+\beta)^2 = \left(\frac{\alpha+\beta}{\alpha\beta}\right)\left(\alpha^3 + \beta^3\right)$

$$\frac{b^2}{a^2} = \frac{-b}{c}\left(\frac{b^2}{a^2} - 2.\frac{c}{a}\right)$$

$$bc = -b^2 + 2ac$$

$$b(b + c) = 2ac$$

$$\left(\frac{a+c}{2}\right)\left(\frac{a+3c}{2}\right) = 2ac$$

$$a^2 + 3c^2 + 4ac = 8ac$$

$$\left(\frac{a}{c}\right)^2 - 4\left(\frac{a}{c}\right) + 3 = 0$$

$$\frac{a}{c} = 1; \frac{a}{c} = 3; a \ne b \ne c$$

Hence, the correct answer is (C).

86. Clearly $a = 2$ as sum of roots is odd.

Hence, the correct answer is (A).

87. Here p, q, r in H.P. $\quad \Rightarrow \quad q = \frac{2pr}{p+r} \qquad \ldots(1)$

Now $D = 4q^2 - 4pr$

$$= -4\left[pr - \left(\frac{2pr}{p+r}\right)^2\right] \qquad \text{using (1)}$$

$$= -4pr\left[\left(\frac{p-r}{p+r}\right)^2\right]$$

Since $pr > 0, p \ne r$ given,

$D \ne 0$ and $D < 0$.

Hence the roots are imaginary.

Hence, the correct answer is (C).

88. G.P. : $\frac{a}{r}, a, ar, \ldots$

$\frac{a}{r} + a + ar = 13 \Rightarrow \frac{a^2}{r^2}(1 + r + r^2) = 169 \qquad \ldots(1)$

$a^2\left(\frac{1}{r^2} + 1 + r^2\right) = 91 \qquad \ldots(2)$

From (1) and (2)

$a = 3$ and $r = \frac{1}{3}$ or 3 (rejected)

G.P. is 9, 3, 1, $\frac{1}{3}, \ldots \quad \Rightarrow \quad C$ is not correct

Hence, the correct answer is (A).

89. $S = 1 + \left(1 + \frac{1}{5}\right)\left(\frac{1}{2}\right) + \left(1 + \frac{1}{5} + \frac{1}{5^2}\right)\frac{1}{2^2} + \ldots \qquad \ldots(1)$

$\frac{S}{2} = 0 + \left(\frac{1}{2}\right) + \left(1 + \frac{1}{5}\right)\frac{1}{2^2} + \ldots \qquad \ldots(2)$

(Subtract)

$$\Rightarrow \quad \frac{S}{2} = 1 + \frac{1}{5}\cdot\frac{1}{2} + \frac{1}{5^2\cdot 2^2} + \ldots$$

$$\Rightarrow \quad \frac{S}{2} = \frac{1}{1-\frac{1}{10}} \quad \Rightarrow \quad \frac{S}{2} = \frac{10}{9} \quad \Rightarrow \quad S = \frac{20}{9}$$

Hence, the correct answer is (A).

90. $(2011)^{\log_{10} x} = 11^{\log_{10}(1+3+5+\ldots+4021)}$

Now $\underbrace{1+3+5+\ldots+4021}_{\text{sum of the first 2011 odd natural number}} = (2011)^2$

$\therefore \quad x^{\log_{10} 2011} = 11^{\log_{10}(2011)^2} = 11^{2\log_{10} 2011} = (121)^{\log_{10} 2011}$

Hence $x = 121$.

Hence, the correct answer is (D).

91. Let x and y be the 1st term and common ratio respectively of a G.P.

$\therefore \quad b = xy^{p-1}, a = xy^{p+q-1}, c = xy^{p-q-1}$

Thus, $ac = x^2y^{2p-2} = (xy^{p-1})^2 = b^2$

$\therefore \quad$ Discriminant $= 4(b^2 - ac) = 0$

Hence, $af(x) = 0$ at $x = \frac{b}{a}$ and $af(x) > 0$ at $x \neq \frac{b}{a}$

i.e., $af(x) \geq 0$

Hence, the correct answer is (A).

92. We have, $t_n = S_n - S_{n-1} = n(3n + 5) - (n - 1)(3n + 2)$

$= 6n + 2 = 32$ (Given)

$\Rightarrow \quad n = 5.$

Hence, the correct answer is (B).

93. $a_1, a_2, a_3 \ldots$ With $d = 2$

$$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a_1 + (3n-1)2]}{\frac{n}{2}[2a_1 + (n-1)2]} = \frac{3[(a_1-1)+3n]}{(a_1-1)+n}$$

This is independent of n if $a_1 = 1$

$\therefore \quad$ Arithmetic progression is $1 + 3 + 5 + 7 + \ldots$

$S_{10} = 10^2 = 100.$

Hence, the correct answer is (A).

94. $S = \frac{a}{1-r} = 20$ and $\frac{a^2}{1-r^2} = 80$

$\Rightarrow \quad r = \frac{2}{3}$

$\therefore \quad a = \frac{20}{3} = \frac{p}{q} \quad \Rightarrow \quad (p+q) = 23.$

Hence, the correct answer is (C).

95. For arithmetic progression, if $S_n = an^2 + bn$

$S_{n-1} = a(n-1)^2 + b(n-1)$

$T_n = S_n - S_{n-1}$

$\Rightarrow \quad T_n = a(2n-1) + b$

$\therefore d = 2a =$ common difference of arithmetic progression.

$S_n = 6n\sec^2\theta + n(n-1)(\sin^2\theta(4+\tan^2\theta) + \cos^2\theta(4+\cot^2\theta))$

Common difference, $d = 2 \times$ coefficient of n^2

$= 2 \times (\sin^2\theta(3 + \sec^2\theta) + \cos^2\theta(3 + \text{cosec}^2\theta))$

$$= 2\left(\underbrace{3 + \tan^2\theta + \cot^2\theta}_{\geq 2}\right)$$

$$d_{\min}\left(\theta, \frac{\pi}{4}\right) = 10.$$

Hence, the correct answer is (A).

96. $$\sum_{n=1}^{\infty}\frac{5^n - 2^n}{10^n} = \sum_{n=1}^{\infty}\left(\frac{5^n}{10^n} - \frac{2^n}{10^n}\right) = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{5}\right)^n\right)$$

the difference between two infinite geometric sequences.

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} - \frac{\frac{1}{5}}{1-\frac{1}{5}} = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, the correct answer is (D).

97. $72x^3 - 108x^2 + 46x - 5 = 0$

Let roots are $a - d, a, a + d$

$\therefore \quad 3a = \frac{108}{72} = \frac{3}{2} \qquad \Rightarrow \quad a = \frac{1}{2}$

Now $a(a^2 - d^2) = \frac{5}{72}$

$\frac{1}{4} - d^2 = \frac{5}{36} \qquad \Rightarrow \quad d^2 = \frac{1}{9}$

$\Rightarrow \quad d = \frac{1}{3} \qquad \Rightarrow \quad 2d = \frac{2}{3}$

Hence, difference between largest and smallest root $2d = \frac{2}{3}$ which lies in the interval $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

Hence, the correct answer is (B).

98. $$\frac{\frac{1}{x}}{1-\frac{1}{2x}} = \frac{1}{64} \quad \Rightarrow \quad \frac{\frac{1}{x}}{\frac{2x-1}{2x}} = \frac{1}{64}$$

$$\frac{2}{2x-1} = \frac{1}{64} \quad \Rightarrow \quad 128 = 2x - 1$$

$$\Rightarrow \quad x = \frac{129}{2}.$$

Hence, the correct answer is (C).

99. $S_n = \frac{n}{2}(a + \ell) \quad \Rightarrow \quad 400 = \frac{n}{2}(5 + 45) \quad \Rightarrow \quad n = 16$

Again, $45 = a + (n-1)d = 5 + 15d$

$\therefore \quad d = \frac{40}{15} = \frac{8}{3} \quad \Rightarrow \quad \left(\frac{n}{d}\right) = 3\left(\frac{16}{8}\right) = 6$

Hence, the correct answer is (D).

100. $\because$ first term $a_1 = 2$ {given}

$\therefore \quad a_n = \frac{a_{n-1}^2}{a_{n-2}} \quad \Rightarrow \quad \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}}$

ratio of consecutive terms is constant

$\therefore \quad$ sequence is G.P.

Let $a_2 = x$

then common ratio $= \dfrac{x}{2}$

$\therefore \quad a_5 = 2\left(\dfrac{x}{2}\right)^4 \quad \Rightarrow \quad a_5 = \dfrac{x^4}{8}$

$\because \quad a_5 \to$ +ve integer

$\therefore \quad x$ must be an even and integer.

$\because \quad$ Given $\Rightarrow a_5 \le 512$

$$\frac{x^4}{8} \le 512 \quad \Rightarrow \quad x^4 \le 4096 \quad \Rightarrow \quad x \le 8 \qquad \{\because x \in \text{even}\}$$

$\therefore \quad x = 2, 4, 6, 8$

So, $a_5 = 2, 32, 162, 512$

Hence, the correct answer is (D).

101. $T_r = r(r+1)\,[(r+1)^2 - r^2]$

$$S = \sum_{r=1}^{50} T_r = \sum_{r=1}^{50} r(2r+1)(r+1)$$

$$= \sum_{r=1}^{50} (2r^3 + 3r^2 + r)$$

$$= 2.\left(\frac{50.51}{2}\right)^2 + 3.\frac{50.51.101}{6} + \frac{50.51}{2}$$

$$= 25.50.51^2 + 25.51.101 + 25.51$$

$$= 50.51^2.26$$

Hence, the correct answer is (B).

102. $g_2 g_4 g_6 \cdots g_{200} = \alpha \qquad \ldots(1)$

$g_1 g_3 g_5 \cdots g_{199} = \beta \qquad \ldots(2)$

Divide (1) and (2)

$$\left(\frac{g_2}{g_1}\right)\left(\frac{g_4}{g_3}\right)\cdots\left(\frac{g_{200}}{g_{199}}\right) = \frac{\alpha}{\beta}$$

$$\Rightarrow \quad r^{100} = \frac{\alpha}{\beta} \quad \Rightarrow \quad r = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{100}}$$

Hence, the correct answer is (B).

103. Degree $n = 1 + 3 + 5 + \ldots + 21 = 121$

Hence, the correct answer is (A).

104. $$S_n = \sum_{K=1}^{2008} \frac{K+2}{K! + (K+1)! + (K+2)!}$$

$$= \sum_{K=1}^{2008} \frac{K+2}{K!\,(K+2)^2}$$

$$= \sum_{K=1}^{2008} \frac{1}{K!(K+2)} = \sum_{K=1}^{2008} \frac{K+1}{(K+2)} = \sum_{K=1}^{2008} \frac{K+2-1}{(K+2)!}$$

$$= \sum_{1}^{2008} \left(\frac{1}{(K+1)!} - \frac{1}{(K+2)!}\right)$$

$$= \frac{1}{2} - \frac{1}{2008!}$$

Hence, the correct answer is (B).

105. Let $b_n = \dfrac{1}{a_n} \Rightarrow b_{n+1} + \dfrac{1}{3} = 2\left(b_n + \dfrac{1}{3}\right)$ i.e. the sequence $< b_n + \dfrac{1}{3} >$ is a G.P with common ratio 2.

So $b_n + \dfrac{1}{3} = 2^n\left(b_n + \dfrac{1}{3}\right) \Rightarrow b_n = \dfrac{1}{3}\left(2^{n+1} - 1\right)$.

Required $\displaystyle\sum_{i=1}^{n} \frac{1}{a_i} = b_0 + b_1 + b_2 + \ldots. + b_n$

$$= \frac{1}{3}(2-1) + \frac{1}{3}\left(2^2 - 1\right) + \ldots + \frac{1}{3}\left(2^{n+1} - 1\right)$$

$$= \frac{1}{3}\left[2\left(\frac{2^{n+1}-1}{2-1}\right) - (n+1)\right] = \frac{1}{3}\left(2^{n+2} - n - 3\right).$$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. Lengths of sides of $S_1, S_2, S_3 \ldots$ are

$10, \dfrac{10}{\sqrt{2}}, \dfrac{10}{\left(\sqrt{2}\right)^2}, \ldots$ respectively

$\therefore \quad S_n$ has side $= 10\left(\dfrac{1}{\sqrt{2}}\right)^{n-1}$

Area of $S_n = 100\left(\dfrac{1}{2}\right)^{n-1}$

$$\frac{200}{2^n} < 1 \Rightarrow 2^n > 200$$

Hence, (B), (C) and (D) are correct.

2. $f_{n-1} + f_n = 2n \qquad \ldots(1)$

$f_{n-2} + f_{n-1} = 2n - 2 \qquad \ldots(2)$

$f_n - f_{n-2} = 2$

$$\Rightarrow \quad \begin{cases} f_3 - f_1 = 2 \Rightarrow f_3 = 102 \text{ and} \\ f_2 + f_3 = 6 \Rightarrow f_2 = 6 - 102 = -96 \end{cases}$$

sequence is 100, –96, 102 …

$$f_n = \begin{cases} n - 98, \; n \text{ even} \\ n + 99, \; n \text{ odd} \end{cases}$$

Hence, (A), (B) and (C) are correct.

3. $x_1 + x_2 + x_3 + \ldots + x_{100} = \dfrac{100}{2}\,(x_1 + x_{100}) = -1$

$x_2 + x_4 + \ldots + x_{100} = \dfrac{50}{2}\,(x_1 + d + x_{100}) = 1$

from these $d = \dfrac{3}{50}$

$50(x_1 + x_1 + 99d) = -1 \quad \Rightarrow \quad x_1 = \dfrac{-149}{50}$

$$x_{100} = x_1 + 99d$$

$$= -\frac{149}{50} + \frac{99 \times 3}{50} = \frac{148}{50} = \frac{74}{25}$$

$$x_1^2 + x_2^2 + \ldots + x_{100}^2 = \sum_{r=1}^{100} (x_1 + (r-1)d)^2$$

$$= \sum_{r=1}^{100} (x_1^2 + (r-1)^2 d^2 + 2(r-1)x_1 d)$$

$$= \frac{14999}{50}$$

Hence, (A), (B), (C) and (D) are correct.

4. Let the roots of the given equation be α/r, α, αr where $\alpha > 0$ and $r > 1$. Then

$$\frac{\alpha}{r} + \alpha + \alpha r = -b \qquad ....(1)$$

$$\frac{\alpha}{r} \cdot \alpha + \alpha \cdot \alpha r + \frac{\alpha}{r} \cdot \alpha r = c \qquad ....(2)$$

and $\left(\frac{\alpha}{r}\right)(\alpha)\,(\alpha r) = 1$ ....(3)

From (3) we get $\alpha^3 = 1$ or $\alpha = 1$

From (1) we get $\frac{1}{r} + 1 + r = -b$ ....(4)

$$\Rightarrow \quad \left(\frac{1}{\sqrt{r}} - \sqrt{r}\right)^2 + 3 = -b$$

$\Rightarrow \quad -b - 3 > 0$ or $b < -3 \Rightarrow b \in (-\infty, -3)$

Also, from (2), $1/r + r + 1 = c$ ...(5)

From (4) and (5) $-b = c$ or $b + c = 0$

As $r > 1$, $\alpha/\,r = 1/r < 1$ and $\alpha r = r > 1$

Hence, (A), (B), (C) and (D) are correct.

5. $x + y + z = 3\left(\frac{a+b}{2}\right)$

$$\Rightarrow \quad 15 = 3\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \quad a + b = 10 \qquad ...(1)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3\left(\frac{1}{a} + \frac{1}{b}\right)}{2}$$

$$\Rightarrow \quad \frac{5}{3} = \frac{3(a+b)}{2ab} = \frac{3 \times 10}{2ab}$$

$$\Rightarrow \quad ab = 9 \qquad ...(2)$$

From (1) and (2), $a = 9, b = 1$ or $a = 1$ and $b = 9$.

Hence GM $= \sqrt{ab} = 3$, $a + 2b = 11$ or 19.

Hence, (A), (B), (C) and (D) are correct.

6. $a_1 + a_4 + a_7 + \ldots + a_{16} = 147$

$\Rightarrow \quad 3(a_1 + a_{16}) = 147 \quad \Rightarrow \quad a_1 + a_{16} = 49$

Again $a_1 + a_4 + a_7 + a_{10} + \ldots + a_{16}$

$$= a_1 + a_1 + 3d + a_1 + 6d + \ldots + a_1 + 15d$$

$$= 6a_1 + 45d = 147$$

$$\Rightarrow \quad 2a_1 + 15d = 49$$

$$a_1 + a_6 + a_{11} + a_{16} = a_1 + a_1 + 5d + a_1 + 10d + a_1 + 15d$$

$$= 4a_1 + 30d$$

$$= 2(2a_1 + 15d)$$

$$= 2(49) = 98$$

Now using A.M. $\geq$ G.M.

$$\frac{a_1 + a_2 + \ldots + a_{16}}{16} \geq (a_1 a_2 a_3 \ldots a_{16})^{\frac{1}{16}}$$

$$\frac{8(a_1 + a_{16})}{16} \geq (a_1 a_2 a_3 \ldots a_{16})^{\frac{1}{16}}$$

$$\left(\frac{49}{2}\right)^{16} \geq a_1 a_2 a_3 \ldots a_{16}$$

Hence, (A), (B), (C) and (D) are correct.

7. If a, b, c, d are in H.P.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{c}, \frac{1}{b}, \frac{1}{d}$ are in A.P.

(A) Let $\sqrt{\frac{a+d}{b+c}} = p$, $\sqrt{\frac{ad}{bc}} = q$

$$p + \frac{1}{p} = q + \frac{1}{q} \quad \Rightarrow \quad (p - q) = \frac{1}{q} - \frac{1}{p}$$

$$p - q = \frac{p-q}{pq}, \; p - q = 0 \quad \text{or} \quad pq = 1$$

$$\Rightarrow \quad p = q \quad \text{or} \quad pq = 1$$

$$\Rightarrow \quad \sqrt{\frac{a+d}{b+c}} = \sqrt{\frac{ad}{bc}} \quad \text{or} \quad \left(\sqrt{\frac{a+d}{b+c}}\right)\left(\sqrt{\frac{ad}{bc}}\right) = 1$$

$$\Rightarrow \quad \frac{a+d}{b+c} = \frac{ad}{bc} \quad \text{or} \quad \frac{a+d}{b+c} = \frac{bc}{ad}$$

$$\Rightarrow \quad \left(\frac{1}{d} + \frac{1}{a} = \frac{1}{b} + \frac{1}{c}\right)$$

(B) $abc + bcd = abd + acd \Rightarrow \left(\frac{1}{d} + \frac{1}{a}\right) = \left(\frac{1}{c} + \frac{1}{b}\right)$

(C) $\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$ wrong

(D) If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P. $\Rightarrow \frac{1}{a} \times \frac{1}{d} < \frac{1}{b} \times \frac{1}{c}$

$\frac{1}{a} + 1, \frac{1}{b} + 1, \frac{1}{c} + 1,$ $\frac{1}{d} + 1$ are also in A.P.

$$\Rightarrow \quad \left(\frac{1}{a} + 1\right)\left(\frac{1}{d} + 1\right) < \left(\frac{1}{b} + 1\right)\left(\frac{1}{c} + 1\right)$$

Hence, (A), (B) and (D) are correct.

8. $(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots (10^n - 1) = a$

$$\Rightarrow \quad \frac{10(10^n - 1)}{9} - n = a$$

Now (A) option

$$\frac{7}{9}[9 + 99 + 999 + \ldots \text{ upto } n \text{ terms}] = \frac{7}{9}a$$

Now (C) option

$$\frac{4}{81}[9^2 + 99^2 + 999^2 + \ldots \text{ upto } n \text{ terms}]$$

$$\Rightarrow \quad \frac{4}{81}(10^2 + 10^4 + 10^6 + \ldots 10^{2n} + n - 2(10 + 10^2 + 10^3 + \ldots 10^n))$$

$$= \frac{4}{81}\left[\frac{100(10^{2n} - 1)}{99} + n - 2(10)\frac{(10^n - 1)}{9}\right]$$

$$= \frac{4}{81}\left[\frac{100(10^{2n}-1)}{99} + n - 2(a+n)\right]$$

Hence, (A) and (C) are correct.

9. $$\frac{1}{a_{n-1}a_{n+1}} = \frac{a_n}{a_{n-1}a_na_{n+1}} = \frac{a_{n+1}-a_{n-1}}{a_{n-1}a_na_{n+1}} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_na_{n+1}}$$

$$\sum_{n=2}^{\infty}\frac{1}{a_{n-1}a_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{a_1a_2} - \frac{1}{a_{n-1}a_{n+1}}\right) = 1$$

$$\sum_{n=2}^{\infty}\frac{a_n}{a_{n-1}a_{n+1}} = \lim_{n\to\infty}\left(\frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_{n-1}} - \frac{1}{a_n}\right) = 2.$$

$a_1 = a_2 = 1$, so$\{a_n\}$ is not an increasing sequence.

Hence, (A), (B) and (C) are correct.

10. b is H.M. of a and c

H.M. < A.M. $\Rightarrow b < \frac{a+c}{2}$

$$b - a < c - b$$

$$a - b > b - c$$

$\frac{1}{a-b} - \frac{1}{b-c} < 0$ (for both cases $a > b > c$ or $a < b < c$)

also $b^2 < \left(\frac{a+c}{2}\right)^2$ or $b^2 < ac$

also $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\frac{1}{a} - 1, \frac{1}{b} - 1, \frac{1}{c} - 1$ are in A.P.

so $bc(1-a)$, $ac\,(1-b)$, $ab\,(1-c)$ are in A.P.

Hence, (B), (C) and (D) are correct.

11. $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

$$\frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\frac{a+b+c}{3\sqrt[3]{abc}} \geq 1$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a+b+c}{3\sqrt[3]{abc}} \geq 4$$

$$\left(\frac{a+b}{\sqrt{ab}}\right)\left(\frac{b+c}{\sqrt{bc}}\right)\left(\frac{c+a}{\sqrt{ca}}\right) \geq 8$$

$$a^2 + b^2 \geq 2ab, b^2 + c^2 \geq 2bc, c^2 + a^2 \geq 2ac$$

$$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ca$$

$$\frac{a^3}{bc} + b + c \geq 3a$$

$$\frac{b^3}{ca} + c + a \geq 3b$$

$$\frac{c^3}{ab} + a + b \geq 3c$$

$$\therefore \quad \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

Hence, (A), (B) and (D) are correct.

12. $$\frac{a + b + \frac{c}{2} + \frac{c}{2}}{4} \geq \sqrt[4]{a.b.\frac{c}{2}.\frac{c}{2}}$$

i.e. $$\frac{a+b+c}{4} \geq \sqrt[4]{\frac{abc^2}{4}}$$

$$\therefore \quad \frac{(a+b+c)^4}{4^4} \geq \frac{abc^2}{4} \Rightarrow abc^2 \leq \frac{1}{64}(a+b+c)^4$$

$\therefore$ the greatest value of abc^2, we have $a = b = \frac{c}{2}$

$\therefore \quad a + b + c = 1.$

Hence, (B) and (D) are correct.

13. $$\left(a.2^x + b.3^y + 1\right)^2 \geq \left(4^x + 9^y + 1\right)\left(a^2 + b^2 + 1\right)$$

$$\Rightarrow \quad \left(a.3^x - b.2^x\right)^2 + \left(2^x - a\right)^2 + \left(3^y - b\right)^2 \leq 0$$

$$\Rightarrow \quad a = 2^x, b = 3^y.$$

Hence, (A) and (D) are correct.

14. The 1st elements of $S_1, S_2, S_3, S_4, \ldots$ are $\frac{1^2-1}{1}, \frac{2^2-1}{2}, \frac{3^2-1}{3}, \ldots$

$\Rightarrow \quad S_i$ must be a set starting at $\frac{i^2-1}{i}$

$$S_{20} = \left\{\frac{399}{20}, \frac{399+20}{20}, \frac{399+40}{20}, \text{....20 term}\right\}$$

$\Rightarrow \quad$ third term is $\frac{439}{20}$

$$S_{20} = \frac{20}{2}\left[\frac{2\times 399}{20} + 19\times 1\right] = 589.$$

Hence, (A) and (C) are correct.

15. If a, b, c are p^{th}, q^{th} and r^{th} terms of an A.P. then $\frac{b-c}{a-b}$ is always a rational number

Hence, (C) and (D) are correct.

16. $$1^2 + 2^2 + 3^2 + \ldots + n^2 - \{a_1 + a_2 + a_3 + \ldots + a_n\}$$

$$= \frac{n\left(n^2-1\right)}{3}$$

$$\therefore \quad S_n = \frac{n(n+1)(2n+1)}{6} - \frac{n\left(n^2-1\right)}{3}$$

$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

$$a_n = S_n - S_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n.$$

Hence, (A), (B) and (D) are correct.

17. Let the first term be a and $(2n-1)^{\text{th}}$ term be b then

$$p = a + (n-1)d = a + (n-1)\left(\frac{b-a}{2n-2}\right) = \frac{a+b}{2}$$

$$q = ar^{n-1} = a\left(\frac{b}{a}\right)^{\frac{n-1}{2n-2}} = a\left(\frac{b}{a}\right)^{\frac{1}{2}} = \sqrt{ab}$$

$$\frac{1}{s} = \frac{1}{a} + (n-1)\left(\frac{\frac{1}{b}-\frac{1}{a}}{2n-2}\right) = \frac{\frac{1}{a}+\frac{1}{b}}{2}$$

p, q, s are the A.M, G.M, H.M of a, b.

$\therefore \quad p \geq q \geq s$ and $ps = q^2$

Hence, (A) and (C) are correct.

18. $a_{n+1}^2 - 10\, a_n a_{n+1} + a_n^2 - 1 = 0$ and

$$a_n^2 - 10\, a_n a_{n-1} + a_{n-1}^2 - 1 = 0$$

Thus for $n \geq 2, a_{n+1}, a_{n-1}$ are two distinct roots of $t^2 - 10t\, a_n + a_n^2 - 1 = 0$

$\therefore \quad a_{n+1} + a_{n-1} = 10a_n \Rightarrow a_1 = 0, a_2 = 1, a_3 = 10$ etc...

Hence, (A), (B), (C) and (D) are correct.

19. $t_n = \dfrac{n^2+n-1}{\underline{|n+2}} = \dfrac{n^2+2n-(n+1)}{\underline{|n+2}}$

$$= \left(\frac{1}{\underline{|n}} - \frac{1}{\underline{|n+1}}\right) - \left(\frac{1}{\underline{|n+1}} - \frac{1}{\underline{|n+2}}\right)$$

$$t_{100} = \frac{100^2+100-1}{\underline{|102}} = \frac{10099}{\underline{|102}}$$

putting $n = 1, 2, 3, \ldots$ successively and adding

$$S_n = \left[1 - \frac{1}{\underline{|n+1}}\right] - \left[\frac{1}{2} - \frac{1}{\underline{|n+2}}\right] = \frac{1}{2} - \frac{1}{\underline{|n+1}} + \frac{1}{\underline{|n+2}}$$

Hence, (A), (C) and (D) are correct.

20. Since r^{th} row contains r elements.

The number of elements in the first r rows $= 1 + 2 + 3 + \ldots + r =$

$$\frac{1}{2}r(r+1) = K(say)$$

$$T_r = 1^3 + 2^3 + \ldots + K^3 = \frac{1}{4}K^2(K+1)^2$$

$$= \frac{1}{64}r^2(r+1)^2 + (r^2+r+2)^2$$

Now that $S_n = T_n - T_{n-1} = \dfrac{1}{8}n^3(n^2+1)(n^2+3)$

$8S_n$ is dibisible by n^2, n^3, n^2+1, n^2+3

Hence, (A), (B), (C) and (D) are correct.

21. Given $a_1 = 2;\ \dfrac{a_n}{a_{n-1}} = \dfrac{a_{n-1}}{a_{n-2}} \quad \Rightarrow \quad a_1, a_2, a_3, a_4, a_5, \ldots$ In G.P.

Let $a_2 = x$ then for $n = 3$ we have

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} \Rightarrow a_1^2 = a_1 a_3 \quad \Rightarrow \quad a_3 = \frac{x^2}{2}$$

i.e. $2, x, \dfrac{x^2}{2}, \dfrac{x^3}{4}, \dfrac{x^4}{8}, \ldots$ With common ratio $r = \dfrac{x}{2}$

given $\dfrac{x^4}{8} \leq 162 \quad \Rightarrow \quad x^4 \leq 1296 \quad \Rightarrow \quad x \leq 6$

Also x and $\dfrac{x^4}{8}$ are integers $\Rightarrow x$ must be even then only $\dfrac{x^4}{8}$ will be an integer.

hence possible values of x is 4 and 6. ($x \neq 2$ as terms are distinct)

hence possible value of $a_5 = \dfrac{x^4}{8}$ is $\dfrac{4^4}{8}, \dfrac{6^4}{8}$

$32, 162 \quad \Rightarrow \quad$ B, D

Hence, (B) and (D) are correct.

22. a, b, c are in G.P.; $\qquad b = ar, c = ar^2$

$$12 = \frac{2ab}{a+b} \quad \Rightarrow \quad \frac{ab}{a+b} = 6$$

$$\Rightarrow \quad \frac{a^2r}{a(1+r)} = 6 \quad \Rightarrow \quad \frac{ar}{1+r} = 6 \qquad \ldots(1)$$

again $36 = \dfrac{2bc}{b+c} \quad \Rightarrow \quad \dfrac{bc}{b+c} = 18$

$$\frac{a^2r^3}{ar(1+r)} = 18 \Rightarrow \frac{ar^2}{1+r} = 18 \qquad \ldots(2)$$

from (1) and (2), $a = 8, r = 3$

now verify all the alternaties.]

Hence, (A), (C) and (D) are correct.

23. Let S_n and s_n are sum to 'n' terms of 2AP's.

Let $\; S_n = (2n-3)nk$

and $\; s_n = (4n+1)nk$

$\Rightarrow \quad T_n = S_n - S_{n-1}$

$= k[n(2n-3) - (n-1)(2(n-1)-3)]$

$= (k)\,[n(2n-3) - (n-1)(2n-5)] = k(4n-5)$

Now $t_n = s_n - s_{n-1}$

$= k[n(4n+1) - (n-1)(4(n-1)+1)]$

$= k[n(4n+1) - (n-1)(4n-3)] = k(8n-3)$

$$\Rightarrow \quad \frac{T_{10}}{t_{11}} = \frac{k(35)}{k(85)} = \frac{7}{17} \text{ and } \frac{t_6}{T_6} = \frac{k(45)}{k(19)} = \frac{45}{19}$$

Hence, (A) and (C) are correct.

24. $b_k = \dfrac{n}{2}\left[a_k + a_{n+k-1}\right]$

$$= \frac{n}{2}\left[a_1 + (k-1)d + a_1 + (n+k-2)d\right]$$

$$= \frac{n}{2}\left[a_n + a_1 + 2(k-1)d\right]$$

$$\sum_{k=1}^{n} b_k = \frac{n}{2}\left[na_n + na_1 + 2d\sum_{k=1}^{n}(k-1)\right] = n^2 a_n$$

Hence, (A) and (C) are correct.

25. Let $b = a + p$, $c = a + 2p$, $d = a + 3p$

$$\frac{\frac{1}{a}+\frac{1}{d}}{\frac{1}{b}+\frac{1}{c}} = \frac{\frac{1}{a}+\frac{1}{a+3p}}{\frac{1}{a+p}+\frac{1}{a+2p}} = \frac{(a+p)(a+2p)}{a(a+3p)}$$

$$= \frac{a^2+3ap+2p^2}{a^2+3ap} > 1$$

$$\therefore \quad \frac{1}{a}+\frac{1}{d} > \frac{1}{b}+\frac{1}{c}$$

$$\left(\frac{1}{b}+\frac{1}{c}\right)(a+d)\left(\frac{1}{a+p}+\frac{1}{a+2p}\right)(a+a+3p)$$

$$= \frac{(2a+3p)^2}{a^2+3ap+2p^2} = 4 + \frac{p^2}{a^2+3ap+2p^2} > 4$$

Hence, (C) and (D) are correct.

26. Let A.P. be $A, A+d, A+2d, \ldots.$ Then

$t_{2n-1} = A + (2n-2)d = x$(say), Then

$$(n-1)d = \frac{x-A}{2}$$

$$\therefore \quad a = t_n = A + (n-1)d = A + \frac{x-A}{2} = \frac{A+x}{2}$$

Let G.P be $A, AR, A\,R^2 \ldots.$ Then

$$t_{2n-1} = A\,R^{2n-2} = x \quad \Rightarrow \quad R^{n-1} = \left(\frac{x}{A}\right)^{\frac{1}{2}}$$

$$\therefore \quad b = t_n = A\,R^{n-1} = A\left(\frac{x}{A}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \sqrt{Ax}$$

Let H.P be $\frac{1}{A}, \frac{1}{\frac{1}{A}+D}, \frac{1}{\frac{1}{A}+2D} \ldots$ then

$$t_{2n-1} = \frac{1}{\frac{1}{A}+(2n-2)} = x \text{ then}$$

$$(n-1)\mathrm{D} = \frac{1}{2}\left(\frac{1}{x}-\frac{1}{A}\right)$$

$$\therefore \quad c = t_n = \frac{1}{\frac{1}{A}+(n-1)D} = \frac{1}{\frac{1}{A}+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{A}\right)}$$

$$= \frac{1}{\frac{1}{2}\left(\frac{1}{x}+\frac{1}{A}\right)}$$

Clearly a,b, and c are A.M., G.M and H.M between the numbers, x and A respectively

Hence $a \geq b \geq c$ also $b^2 = ac$

Hence, (B) and (C) are correct.

27. Let a, ar, ar^2, ar^3 are the first four terms of the G.P

$$\therefore \quad a^4r^6 = 4 \text{ and } a^2r^4 = 1 \Rightarrow r^2 = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2} \text{ and } a = \pm 4$$

$$\therefore \quad \text{Sum of infinite } G.P = \frac{a}{1-r} = 8, -8, \frac{8}{3}, -\frac{8}{3}$$

Hence, (A), (B), (C) and (D) are correct.

28. $\because \quad 2b = a + c$

$$\frac{a+c}{2} \geq \sqrt{ac}$$

$$2b \geq 2\sqrt{ac}$$

$$b \geq \sqrt{ac} \quad \Rightarrow \quad b^3 \geq abc$$

$$\Rightarrow \quad b^3 \geq 4 \quad \text{or} \quad b \geq (4)^{1/3} \quad \Rightarrow \quad [b] \geq 1$$

Hence, (A), (B), (C) and (D) are correct.

29. $a, ar, ar^2 \to$ G.P.

H.M. of a and b is 12 $\Rightarrow \dfrac{2.a.ar}{a+ar} = 12 \Rightarrow \dfrac{ar}{1+r} = 6$ …(1)

A.M. of b and c is 3 $\Rightarrow \dfrac{ar+ar^2}{2} = 3 \Rightarrow ar(1+r) = 6$ …(2)

$r = -2$ and $a = 3$

Hence, (A) and (B) are correct.

30. $$S(n) = 1 + \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) + \ldots + \left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}\right)$$

$$\leq 1 + \left(\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) + \ldots + \left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\ldots+\frac{1}{2^{n-1}}\right)$$

$$\leq 1 + 1 + 1 + \ldots + 1 \ (n \text{ terms}) \leq n$$

Also $$S(n) \geq 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{n-2}+1}+\frac{1}{2^{n-2}+2}+\ldots+\frac{1}{2^{n-1}}\right)$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = 1 + \left(\frac{n-1}{2}\right) = \frac{n+1}{2}$$

$$\therefore \quad S(2n) > \frac{2n+1}{2} = n + \frac{1}{2} > n$$

Hence, (A) and (D) are correct.

31. $$T_r = \frac{r\left(\sqrt{r+1}\right)-(r+1)\sqrt{r}}{r^2(r+1)-(r+1)^2r} = \frac{r\sqrt{r+1}-(r+1)\sqrt{r}}{-r^2-r}$$

$$= \frac{(r+1)\sqrt{r}}{r(r+1)} - \frac{r\sqrt{r+1}}{r(r+1)} = \frac{1}{\sqrt{r}} - \frac{1}{\sqrt{r+1}}$$

$$\Rightarrow \quad \sum_{r=1}^{99} T_r = \frac{1}{1} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \ldots - \frac{1}{\sqrt{100}} = 1 - \frac{1}{\sqrt{100}} = \frac{9}{10}$$

Hence, (A), (C) and (D) are correct.

32. $S_{(1)} + S_{(2)} + \ldots + S_{(n-1)}$

$S_{(1)} : 1$

$S_{(2)} : 1 + \frac{1}{2}$

$S_{(3)} : 1 + \frac{1}{2} + \frac{1}{3}$

…………

…………

$S_{(n-1)} : 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n-1}$

Adding vertically:

$$= (n-1) + \frac{(n-2)}{2} + \frac{(n-3)}{3} + \dots \left(\frac{n-(n-1)}{(n-1)}\right)$$

$$= n\left[1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{n-1}\right] - [1+1+1+\dots 1]$$

$$= nS_{(n-1)} - (n-1) = nS_n - n$$

Hence, (A) and (D) are correct.

33. We have

$$\frac{1}{2}(2+a_1) \ge \sqrt{2a_1}$$

$$\frac{1}{2}(2+a_2) \ge \sqrt{2a_2}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{1}{2}(2+a_n) \ge \sqrt{2a_n}$$

Multiplying above inequalities, we get

$$\frac{1}{2^n}(2+a_1)(2+a_2)\dots(2+a_n) \ge 2^{n/2}\sqrt{a_1a_2\dots a_n} = 2^{n/2}$$

$$\Rightarrow \quad (2+a_1)(2+a_2)\dots(2+a_n) \ge 2^{3n/2}$$

As all $a_i \ne 2$, thus we have strict inequality in the above inequality.

Hence, (A) and (B) are correct.

34. T_p of A.P. $= \dfrac{1}{q(p+q)} = A + (p-1)D$ ...(1)

T_q of A.P. $= \dfrac{1}{P(p+q)} = A + (q-1)D$ ...(2)

$$\frac{1}{T_{p+q}} = A + (p+q-1)D$$

and $\dfrac{1}{T_{pq}} = A + (pq-1)D.$

Now, solving Eqs. (1) and (2), we get

$$A = D = \frac{1}{pq(p+q)}$$

$$\therefore \quad \frac{1}{T_{p+q}} = A + (p+q-1)D = (p+q)D = \frac{1}{pq}$$

And $\dfrac{1}{T_{pq}} = A + (pq-1)D = pqD = \dfrac{1}{p+q}$

$\Rightarrow \quad T_{p+q} = pq$ and $T_{pq} = p+q$

Also, $\because pq > p+q$

i.e, $\quad T_{p+q} > T_{pq}$

Hence, (A), (B) and (C) are correct.

35. $\dfrac{a+b}{2} = 2\sqrt{ab} \quad \Rightarrow \quad \sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}} = 4 \quad \sqrt{\dfrac{a}{b}} = 2\pm\sqrt{3}$

$$\frac{\sqrt{\frac{a}{b}}}{\sqrt{\frac{b}{a}}} = \frac{2+\sqrt{3}}{2-\sqrt{3}} \quad \Rightarrow \quad \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$\frac{a}{b} = 7+4\sqrt{3} \quad \Rightarrow \quad a:b = \left(7+4\sqrt{3}\right):1$$

$$\sqrt{\frac{a}{b}} = 2-\sqrt{3} \quad \Rightarrow \quad a:b = 1:7-4\sqrt{3}$$

Hence, (A), (B) and (C) are correct.

36. $b_2 = b_1r,\ b_3 = b_1r^2$

$$b_1r^2 > 4b_1r - 3b_1$$

$$\Rightarrow \quad r^2 > 4r - 3$$

$$\Rightarrow \quad r^2 - 4r + 3 > 0$$

$$\Rightarrow \quad (r-1)(r-3) > 0$$

$$r > 3 \quad \text{or} \quad r < 1$$

Hence, (A) and (C) are correct.

37. a, b, c are in H.P.

$\Rightarrow \quad \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

$\Rightarrow \quad \dfrac{a+b+c}{a}, \dfrac{a+b+c}{b}, \dfrac{a+b+c}{c}$ are in A.P.

$\Rightarrow \quad \dfrac{b+c}{a}, \dfrac{c+a}{b}, \dfrac{a+b}{c}$ are in A.P. [subtracting 1 from each term]

Thus $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P.

$\dfrac{b+c}{a} - 1, \dfrac{c+a}{b} - 1, \dfrac{a+b}{c} - 1$ are in A.P.

And $\dfrac{a}{b+c-a}, \dfrac{b}{c+a-b}, \dfrac{c}{a+b-c}$ are also in H.P.

Also $b = 2ac/(a+c)$, so $\dfrac{1}{b-a} + \dfrac{1}{b-c} = \dfrac{2b-(a+c)}{(b-a)(b-c)}$

$$= \frac{2b - 2ac/b}{b^2 - b(a+c) + ac}$$

$$= \frac{2}{b}\cdot\frac{b^2-ac}{b^2-ac} = \frac{2}{b}$$

Lastly, $\left(a - \dfrac{b}{2}\right)\left(c - \dfrac{b}{2}\right) = ac - \dfrac{b}{2}(a+c) + \dfrac{b^2}{4}$

$$= ac - \frac{b}{2}\cdot\frac{2ac}{b} + \frac{b^2}{4} = \frac{b^2}{4}$$

$a - \dfrac{b}{2}, \dfrac{b}{2}, c - \dfrac{b}{2}$ are in G.P.

Hence, (A), (B), (C) and (D) are correct.

38. $a + c = 2b$

$$a + b > c$$

$$\Rightarrow \quad a + b + c > 2c$$

$$\Rightarrow \quad 3b > 2c$$

$$b + c > a$$

$$\Rightarrow \quad b + 2c > a + c$$

$$2c > b$$

$$\Rightarrow \quad \frac{2}{3} < \frac{b}{c} < 2$$

Similarly for $\dfrac{b}{a}$

Hence, (A) and (C) are correct.

39. $S_r = r + r\left(\frac{1}{r+1}\right) + r\left(\frac{1}{r+1}\right)^2 + \ldots \infty = \frac{r}{1-\frac{1}{r+1}} = r+1$ verify a, b, c an correct and d is false.

(A) $\sum S_r = S_1 + S_2 + \ldots + S_n = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{3}$

(B) $S_1 \cdot S_2 \cdot S_3 \ldots S_n = 2 \cdot 3 \ldots (n+1) = (n+1)!$

(C) $\frac{1}{S_1S_2} + \frac{1}{S_2S_3} + \ldots + \frac{1}{S_{n-1}S_n} = \sum_{r=1}^{n=1} \frac{1}{S_rS_{r+1}}$

$$= \sum_{r=1}^{n=1}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}$$

Hence, (A), (B) and (C) are correct.

40. Let removed numbers are $a - d, a, a + d$
sum of removed numbers $= 3a$

$\Rightarrow \quad 6 \le 3a \le 3n - 3$

$\Rightarrow \quad 2 \le a \le n - 1 \qquad \ldots(1)$

also $3A = \frac{n(n+1)}{2} - \frac{43}{4}(n-3)$

$$A = \frac{2n^2 - 41n + 129}{12} \qquad \ldots(2)$$

from (1) and (2)

$17.5 \le n \le 23.5 \; n \in N$

$n = 18, 19, 20, 21, 22, 23$

for $a \in N$, n must be odd

n may be 19, 21, 23

when $n = 19$, $a = 6$, d can be 2 or 5

when $n = 21$ $A \notin N$ not possible

when $n = 23$ $A \notin N$ not possible.

Hence, (A) and (B) are correct.

41. $$S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \ldots$$

r^{th} term $T_r = \frac{1}{r^2}(1+2+\ldots+r)^2 = \frac{1}{r^2}\left\{\frac{r(r+1)}{2}\right\}^2 = \frac{r^2+2r+1}{4}$

$\therefore \quad T_7 = 16$

and $S_{10} = \sum_{r=1}^{10} T_r$

$$= \frac{1}{4}\left\{\frac{(10)(10+1)(20+1)}{6} + (10)(10+1) + 10\right\}$$

$$= \frac{505}{4}$$

Hence, (A) and (C) are correct.

42. $$\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{1}{6}n.\,(n+1)[2n+1+3] = \frac{n(n+1)(n+2)}{3}$$

$a = 0, b = 1, c = 2$

Hence, (A), (B) and (C) are correct.

43. Let a, r be the first term and common ratio respectively, then $t_3 = ar^2 = 64$

$\therefore \quad t_1t_2t_3t_4t_5 = a.ar.ar^2.ar^3.ar^4 = a^5r^{10}$

$$= \left(ar^2\right)^5 = 64^5 = 4^{15} = 2^{30} = \frac{2^{33}}{8}$$

Hence, (A) and (B) are correct.

44. As a_{912}, a_{951} and a_{480} are divisible by 3, none of them is prime, and for a_{91}, we have

$$a_{91} = \frac{10^{91}-1}{10-1} = \frac{10^{91}-1}{10^7-1} \times \frac{10^7-1}{10-1}$$

$$= \left(1 + 10^7 + \ldots + 10^{84}\right)\left(1 + 10 + \ldots + 10^6\right)$$

$\Rightarrow \quad a_{91}$ is not prime

Hence, (A), (B), (C) and (D) are correct.

45. Let $b = ar, ar, c = ar^2, d = ar^3$

$2b - 14 = a + c - 11, \qquad 2c - 18 = b + d - 12$

$2ar - 14 = a + ar^2 - 11, \qquad 2ar^2 - 18 = ar + ar^3 - 12$

$a(r-1)^2 = -3, \quad \ldots(1) \qquad ar(r-1)^2 = -6 \quad \ldots(2)$

$a = -3, r = 2$

$a + b + c + d = -45$

Hence, (B) and (C) are correct.

46. $\because \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \to$ A.P.

$\Rightarrow \quad \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \to A.P.$

$\Rightarrow \quad \frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c} \to A.P.$

$\Rightarrow \quad \frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b} \to H.P. \quad \therefore$ (a) is true.

$\Rightarrow \quad \frac{b+c}{a} - 1, \frac{a+c}{b} - 1, \frac{a+b}{c} - 1 \to$ A.P.

$\Rightarrow \quad \frac{b+c-a}{a}, \frac{a+c-b}{b}, \frac{a+b-c}{c} \to A.P.$

$\Rightarrow \quad \frac{a}{b+c-a}, \frac{b}{a+c-b}, \frac{c}{a+b-c} \to H.P.$

$\therefore$ (b) is true

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \to$ A.P.

$\Rightarrow \quad -bc, -ac, -ab \to A.P.$ (dividing by $-abc$)

$\Rightarrow \quad (ab+bc+ca) - bc, (ab+bc+ac) - ac, (ab+bc+ca) - ab \to A.P.$

$\Rightarrow \quad a(b+c), b(a+c), c(a+b)$ are in A.P.

Hence, (A), (B), (C) and (D) are correct.

47. $$T_n = \frac{n^2(n+1)}{n(n+1)} + \frac{2}{n(n+1)} = n + \frac{2}{n(n+1)}$$

$$S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n} n + \sum_{n=1}^{n} \frac{2}{n(n+1)}$$

$$S_n = \frac{n(n+1)}{2} + 2\left(1 - \frac{1}{n+1}\right) = \frac{n(n+1)}{2} + \frac{2n}{(n+1)}$$

$$S_9 = \frac{9\times 10}{2} + \frac{2\times 9}{10} = 45 + \frac{9}{5}$$

$\Rightarrow \quad 5S_9 = 234$

$$S_5 = \frac{5\times 6}{2} + \frac{10}{6} = 15 + \frac{5}{3}$$

$\Rightarrow \quad 3S_5 = 50$

Hence, (A) and (C) are correct.

48. $\frac{45}{2} = 5 + 5d \quad \Rightarrow \quad \frac{35}{2} = 5d \quad \Rightarrow \quad d = \frac{7}{2}$

$$A_1 = 5 + \frac{7}{2} = \frac{17}{2}$$

$$A_2 = \frac{24}{2}, A_3 = \frac{31}{2}, A_4 = \frac{38}{2}$$

(A) $\frac{12}{2}, \frac{24}{2}, \frac{31}{2} \quad \Rightarrow \quad$ G.P.

(B) $\frac{24}{2}, \frac{36}{2}, \frac{54}{2} \quad \Rightarrow \quad$ G.P.

(C) 18, 12, 8 $\quad \Rightarrow \quad$ G.P.

(D) 7, 14, 28 $\quad \Rightarrow \quad$ G.P.

Hence, (B) and (C) are correct.

49. $a_n = S_n - S_{n-1}$

$$(n+1)\left[\frac{(n+2)(2n+3)-6}{6}\right] - n\left[\frac{(n+1)(2n+1)-6}{6}\right]$$

$$= (n+1)^2 - 1 = n^2 + 2n$$

a_r, a_{r+1}, a_{r+2} in A.P.

$$2a_{r+1} = a_r + a_{r+2}$$

$\Rightarrow \quad 2(r+1)^2 + 2(r+1) = r^2 + 2r + (r+2)^2 + 2(r+2)$

$\Rightarrow \quad r = -2$ reject.

$$V_n = \sum_{i=1}^{n}\left(\frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]\right)$$

$$V_8 = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{6} - \frac{1}{8} + \frac{1}{7} - \frac{1}{9} + \frac{1}{8} - \frac{1}{10}\right] = \frac{29}{45}$$

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. **The correct answer is (B).**
2. **The correct answer is (C).**

Combined Solution to 1–2

$x = 1$ is a root of Eq Let the root is α, then by solving we get $\alpha = 1$

$$\alpha = \frac{c(a-b)}{a(b-c)} = \frac{c\left(a - \frac{2ac}{a+c}\right)}{a\left(\frac{2ac}{a+c} - c\right)}$$

$$= \frac{c\left(a^2 - ac\right)}{a\left(ac - c^2\right)} = \frac{a-c}{a-c} = 1$$

The roots of $x^2 - Px + Q = 0$ are also 1, 1, $p = 2$, $Q = 1$

$a > b$ then, $\frac{a+b}{2} + \sqrt{ab} = a - b$ by solving $a = 9b$

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{2}{1} \quad \Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}} = \frac{3}{1}$$

3. **The correct answer is (B).**
4. **The correct answer is (C).**
5. **The correct answer is (D).**

Combined Solution to 3–5

$$ax^2 + 2bx + b = 5x^2 - 3bx - a$$

$\Rightarrow \quad (a-5)x^2 + 5bx + (b+a) = 0$

If $a \neq 5$, then since $x \in R$

$$D = 25b^2 - 4(b+a)(a-5) \geq 0 \quad \forall\, b \in R$$

$\Rightarrow \quad 25b^2 - 4(b+a)\,(a-5) \geq 0 \quad \forall\, b \in R$

$\therefore \quad 16(a-5)^2 + 16(25)\,a(a-5) \leq 0$

$\Rightarrow \quad 16(a-5)\,(a-5+25a) \leq 0$

$\Rightarrow \quad (a-5)\,(26a-5) \leq 0$

$\therefore \quad a \in \left[\frac{5}{26}, 5\right)$

If $a = 5$, $5bx + (b+5) = 0$

not satisfied for $b = 0$

$\therefore \quad a_m \in \{1, 2, 3, 4\}$

$$t_r = (r-1)\,(r-2)\,(r-3)\,(r-4)$$

$$S_n = \frac{1}{5}\sum_{r=1}^{n}(r-4)(r-3)(r-2)(r-1)(r-(r-5))$$

$$= \frac{1}{5}\sum_{r=1}^{n}((r-4)(r-3)(r-2)(r-1)(r-(r-5))$$

$$= \frac{1}{5}\sum_{r=1}^{n}[(r-4)\,(r-3)\,(r-2)\,(r-1)r - (r-5)\,(r-4)\,(r-3)\,(r-2)\,(r-1)]$$

$$= \frac{1}{5}n(n-1)(n-2)(n-3)(n-4)$$

$S_n = 0 \quad \Rightarrow \quad n = 1, 2, 3, 4 \quad n = 0$ (rejected)

$$\sum\frac{1}{t_r} = \frac{1}{3}\lim_{n\to\infty}\sum_{r=5}^{n}\frac{(r-1)-(r-4)}{(r-4)(r-3)(r-2)(r-1)}$$

$$= \frac{1}{3}\lim_{n\to\infty}\sum_{r=5}^{n}\left(\frac{1}{(r-4)(r-3)(r-2)} - \frac{1}{(r-3)(r-2)(r-1)}\right)$$

$$\lim_{n\to\infty}\frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n-3)(n-2)(n-1)}\right] = \frac{1}{18}$$

6. **The correct answer is (A).**
7. **The correct answer is (B).**
8. **The correct answer is (D).**

Combined Solution to 6–8

$a_i = 0$ or $1\ \forall\ 1 \leq i \leq n-1$, as we have 2 options for every a_i

$$T_n = 2^{n-1}$$

Frequency of $a_i = 1$ is same as $a_i = 0$ for $1 \leq i \leq n-1$ and $a_n = 1$

$$\therefore \quad S_n = \frac{1}{2}\times 2^{n-1}\left\{\frac{1}{10}+\frac{1}{10^2}+\ldots+\frac{1}{10^{n-1}}\right\}+2^{n-1}\times\frac{1}{10^n}$$

$$= \frac{2^{n-1}}{18}\left(1-\frac{10}{10^n}+\frac{18}{10^n}\right)=\frac{2^{n-1}}{18}\left(1+\frac{8}{10^{n-1}}\right).$$

$$\lim_{n\to\infty}\frac{S_n}{T_n}=\lim_{n\to\infty}\frac{2^{n-1}}{18}\left(1+\frac{8}{10^{n-1}}\right)\frac{1}{2^{n-1}}$$

$$=\frac{1}{18}(1+0)=\frac{1}{18}$$

9. **The correct answer is (D).**
10. **The correct answer is (C).**

Combined Solution to 9–10

$$a_r=\frac{5^{r+1}}{r+1}-\frac{5^r}{r}$$

$$S_n=f(n)-f(0)=\frac{5^{n+1}}{n+1}-5$$

$$S_\infty\to\infty$$

$$a_n=(n+2)((n+2)!)$$

$$=(n+3)!-(n+2)!$$

$$S_n=(n+3)!-6$$

$$S_{50}=53!-6$$

So $a-b=53-6=47$.

11. **The correct answer is (B).**
12. **The correct answer is (B).**
13. **The correct answer is (A).**

Combined Solution to 11–13

$a_1+a_2+a_3=126$

a_1, a_2, a_3 are in A.P $a_2=\frac{126}{3}=42$

The numbers in A.P. are $42-d$, 42, $42+d$

The numbers in G.P. are $43+d$, 34,42 $-d$

Applying the condition, we get $d=-26$, 25

The two possible sets of numbers are

(1) A.P. 68, 42,16 G.P. 17,34,68

(2) A.P. 17, 42, 67 G.P. 68,34,17

14. **The correct answer is (B).**
15. **The correct answer is (A).**
16. **The correct answer is (D).**

Combined Solution to 14–16

Given $P(x)=3x^2-x+\frac{25}{12}$

$$=3\left[x^2-\frac{1}{3}x+\frac{25}{36}\right]=3\left(\left(x-\frac{1}{6}\right)^2+\frac{2}{3}\right)$$

$\Rightarrow \quad P(x)|_{\min}=2$, occurs at $x=\frac{1}{6}$.

Given, $\frac{a}{1-r}=2$ …(1)

and $a=r^2$ …(2)

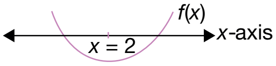
∴ solving (1) and (2), we get

$$\frac{r^2}{1-r}=2 \Rightarrow r^2=2-2r \Rightarrow r^2+2r=2$$

$\therefore \quad r=(\sqrt{3}-1)$ [As $r\in(0,1)$]

$\Rightarrow \quad a=r^2=4-2\sqrt{3}$

Let $f(x)=x^2+(k+1)x+a+2\sqrt{3}$



Now, $f(x)=0 \Rightarrow x^2+(k+1)x+4=0.$

Since 2 lies between the roots, hence $f(2)<0$

$\Rightarrow \quad 4+2k+2+4<0$

$\Rightarrow \quad 2k+10<0$

$\Rightarrow \quad k<-5$

So, $k_{\text{max.}}=-6$

$$\text{Let } y=\frac{P(x)}{P'(x)}=\frac{3x^2-x+\frac{25}{12}}{6x-1}$$

$$\Rightarrow \quad 3x^2-x+\frac{25}{12}=6xy-y$$

$$\Rightarrow \quad 3x^2-x(1+6y)+\frac{25}{12}+y=0$$

As, $x\in R$ so $D\geq 0$

$$\Rightarrow \quad (6y+1)^2-12\left(\frac{25+12y}{12}\right)\geq 0$$

$-\sqrt{2/3}$
$\sqrt{2/3}$

$\Rightarrow \quad 36y^2+12y+1-25-12y\geq 0$

$\Rightarrow \quad 36y^2-24\geq 0$

$\Rightarrow \quad 3y^2-2\geq 0$

$$\text{Range of } y=\left(-\infty,-\sqrt{\frac{2}{3}}\right]\cup\left[\sqrt{\frac{2}{3}},\infty\right)$$

$\infty(-\infty,p]\cup[q,\infty)$

So, $p=-\sqrt{\frac{2}{3}}$ and $q=\sqrt{\frac{2}{3}}$

Hence, $(p+q)=0$

17. **The correct answer is (C).**
18. **The correct answer is (A).**
19. **The correct answer is (B).**

Combined Solution to 17–19

$\alpha, \beta, \gamma$are in H.P., hence

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

$$\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{2}{\beta} \Rightarrow \beta=\frac{2\alpha\gamma}{\alpha+\gamma};\ \frac{1}{\alpha}+\frac{1}{\gamma}+\frac{1}{\beta}=\frac{3}{\beta}$$

$$=\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}=\frac{q}{q}=1$$

$\beta = 3$ which is a root.

$\Rightarrow \quad 27 + 9p + 3q - q = 0$

$\Rightarrow \quad 9p + 2q + 27 = 0$

$\Rightarrow \quad 9p + 2q = -27$

(ii) $2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) \geq 2\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)$

Using ($a^2 + b^2 + c^2 \geq ab + bc + ca$ with $a = \frac{1}{\alpha}$, $b = \frac{1}{\beta}$ and $c = \frac{1}{\gamma^2}$)

$$\Rightarrow \quad 3\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right) \geq \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 = 1$$

(add $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ both sides)

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} \geq \frac{1}{3}$$

(iii) for equality, $\alpha = \beta = \gamma = 3$

$\Rightarrow \quad \Sigma\alpha\beta = 27$ and $\alpha\beta\gamma = 27$

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} \leq \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

$(ab + bc + ca \leq a^2 + b^2 + c^2)$

$$3\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right) \leq \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)^2 = 1$$

$$\left(\text{adding } 2\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)\text{both sides}\right)$$

$$\frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} \leq \frac{1}{3}$$

$$\Rightarrow \quad -\frac{p}{q} \leq \frac{1}{3} \quad \Rightarrow \quad \frac{p}{q} \geq -\frac{1}{3}.$$

20. **The correct answer is (B).**

21. **The correct answer is (A).**

22. **The correct answer is (A).**

Combined Solution to 20–22

$$G_1G_2\ldots G_n = \left(\sqrt{1 \times 1024}\right)^n = 2^{5n}$$

$\therefore \quad 2^{5n} = 2^{45}$

$\therefore \quad n = 9$

$A_1 + A_2 + A_3 + \ldots + A_{m-1} + A_m = 1025 \times 171$

$$\therefore \quad m\left(\frac{-2 + 1027}{2}\right) = 1025 \times 171$$

$\therefore \quad m = 342$

$m + n = 351$

Since $n = 9$,

$$\therefore \quad = (1024)^{\frac{1}{9+1}} = 2$$

$\therefore \quad G_1 = 2, r = 2$

$$G_1 + G_2 + \ldots + G_n = \frac{2.\left(2^9 - 1\right)}{2 - 1} = 1024 - 2 = 1022$$

Common difference of sequence $A_1, A_2, \ldots, A_m$ is $\frac{1027 + 2}{342 + 1} = 3$

$\therefore \quad$ common difference of sequence $A_1, A_3, A_5, \ldots, A_{m-1}$ is 6

we have $A_{171} + A_{172} = -2 + 1027 = 1025$

$$\therefore \quad \frac{2A_{171} + 2A_{172}}{2} = 1025$$

Also $G_5 = 1 \times 2^5 = 32$

$\therefore \quad G_5^2 = 1024 \qquad$ i.e. $G_5^2 + 1 = 1025$

$\therefore \quad 2A_{171}, G_5^2 + 1, 2A_{172}$ are in A.P.

23. **The correct answer is (C).**

24. **The correct answer is (B).**

25. **The correct answer is (B).**

Combined Solution to 23–25

We have first term $= a = r - 1$, common difference $= d = 2r - 3$.

Now, $S_r = \frac{r}{2}[2(r-1) + (r-1)(2r-3)] = \frac{r}{2}(2r^2 - 3r + 1)$

Hence $S_r = \left(r^3 - \frac{3}{2}r^2 + \frac{r}{2}\right)$

(i) We have $\sum_{r=1}^{n} S_r = S_1 + S_2 + \ldots + S_n$

$$= \sum_{r=1}^{n}\left(r^3 - \frac{3}{2}r^2 + \frac{r}{2}\right) = \sum_{r=1}^{n} r^3 - \frac{3}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n} r$$

$$= \frac{n^2(n+1)^2}{4} - \frac{3}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{2}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{4}[n(n+1) - (2n+1) + 1] = \frac{n(n+1)}{4}(n^2 - n)$$

$$\therefore \quad \sum_{r=1}^{n} S_r = \frac{n^2(n^2 - 1)}{4}$$

$$\text{Hence } \sum_{r=1}^{13} S_r = \frac{(12)^2((12)^2 - 1)}{4} = \frac{144 \times 143}{4}$$

$= 36 \times 143 = 5148 \quad \Rightarrow \quad$ option (C)

(ii) We have $3T_r = S_{r+1} - S_r$

$$\Rightarrow \quad 3T_r = \left((r+1)^3 - \frac{3}{2}(r+1)^2 + \frac{(r+1)}{2}\right) - \left(r^3 - \frac{3}{2}r^2 + \frac{r}{2}\right)$$

$$= ((r+1)^3 - r^3) - \frac{3}{2}((r+1)^2 - r^2) + \frac{(r+1) - r}{2}$$

$$= ((r+1)^2 + r^2 + r(r+1)) - \frac{3}{2}((r+1) + \frac{1}{2}$$

$$= (3r^2 + 3r + 1) - + \left(3r + \frac{3}{2}\right) + \frac{1}{2} = 3\mathrm{r}^2$$

Hence $T_r = r^2$

$$\therefore \quad \sum_{r=1}^{n} T_r = T_1 + T_2 + \ldots + T_n =$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(iii) We have $V_r = T_{r+2} - T_r + 1 = (r+2)^2 - r^2 + 1$

$\Rightarrow \quad V_r = (4r+4) + 1 = 4r + 5$

Hence $V_1, V_2, V_3, \ldots$ are in A.P. with common difference 4.

26. Given $c_n = a_1 + a_2 + a_3 + \ldots + a_n$

where $a_1, a_2, \ldots, a_n$ are in A.P. with $d = 2$

and $d_n = b_1 + b_2 + b_3 + \ldots + b_n$

where $b_1, b_2, b_3, \ldots, b_n$ are in A.P. with $d = 2$

Also (a_n, c_n) lies on $y = px^2 + qx + r)$

Now $c_n = pa_n^2 + qa_n = r$ $\quad \ldots(1)$

$$c_{n-1} = pa_{n-1}^2 + qa_{n-1} + r \quad \ldots(2)$$

$\therefore$ From (1) and (2), we get

$$c_n - c_{n-1} = p\left(a_n^2 - a_{n-1}^2\right) + q(a_n - a_{n-1})$$

$$\Rightarrow \quad a_n = p(a_n + a_{n-1})\,(a_n - a_{n-1}) + q(a_n - a_{n-1})$$

$$a_n = (a_n + a_{n-1})\,[p(a_n + a_{n-1}) + q] \quad \ldots(3)$$

$$(a_n - a_{n-1} = d)$$

On putting $n = 2$ and 3 in equation (3), we get

$$a_2 = d[p(a_2 + a_1) + q] \quad \ldots(4)$$

$$a_3 = d[p(a_3 + a_2) + q] \quad \ldots(5)$$

Now (5) – (4), we get

$$a_3 - a_2 = d[p(a_3 - a_1)]$$

$$\Rightarrow \quad d = d\,p(4)$$

$$\Rightarrow \quad 4p = 1 \quad \Rightarrow \quad p = \frac{1}{4}$$

Hence, the correct answer is (A).

27. To find q:

$$c_n = pa_n^2 + qa_n + r$$

On putting $n = 1, 2$ in above equation, we get

$$c_1 = a_1 = pa_1^2 + qa_1 + r \quad \ldots(1)$$

$$c_1 = a_1 + a_2 = pa_2^2 + qa_2 + r \quad \ldots(2)$$

but $a_2 = a_1 + 2$

$$\therefore \quad 2a_1 + 2 = p(a_1 + 2)^2 + q(a_1 + 2) + r$$

$$= (pa_1^2 + qa_1 + r) + 4a_1 p + 4p + 2q(4p = 1)$$

$$2a_1 + 2 = c_1 + a_1 + 1 + 2q \quad (\because c_1 = a_1)$$

$$2a_1 + 2 = 2a_1 + 1 + 2q \quad \Rightarrow \quad q = \frac{1}{2}$$

Hence, the correct answer is (C).

28. If $r = 0$, then

$$c_1 = pa_1^2 + qa_1$$

$$a_1 = \frac{1}{4}a_1^2 + \frac{1}{2}a_1 \quad (\because c_1 = a_1)$$

$$a_1^2 - 2a_1 = 0 \quad \Rightarrow \quad a_1 = 0 \quad \text{or} \quad a_1 = 2$$

Also $d_1 = \frac{1}{4}b_1^2 + qb_1$ $\quad \left(\therefore q = \frac{1}{2} \text{and } d_1 = b_1\right)$

$$b_1 = \frac{1}{4}b_1^2 + \frac{1}{2}b_1$$

$$b_1^2 - 2b_1 = 0 \quad \Rightarrow \quad b_1 = 0 \text{ or } b_1 = 0$$

But $a_1 < b_1$ $\quad \Rightarrow \quad a_1 = 0$ or $b_1 = 2$

Hence, the correct answer is (C).

29. **The correct answer is (C).**

30. **The correct answer is (B).**

Combined Solution to 29–30

29. $b = a + (n+1)d; \quad d = \dfrac{(b-a)}{(n+1)}$

$$A_2 = a + 2d = \frac{2b + a(n-1)}{(n+1)}$$

30. $A_2 = \dfrac{2b + a(n-1)}{(n+1)}$

Also $\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in A.P.

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \Rightarrow (n+1)D = \frac{1}{b} - \frac{1}{a} \Rightarrow D = \frac{a-b}{ab(n+1)}$$

so $H_2 = \dfrac{1}{a} + \dfrac{2(a-b)}{ab(n+1)}$

$$H_2 = \frac{ab(n+1)}{2a + b(n-1)}$$

$$A_2 H_2 = \frac{(2b + a(n-1))ab}{(2a + b(n-1))}$$

31. **The correct answer is (D).**

32. **The correct answer is (D).**

Combined Solution to 31–32

$$f(t) = \sum_{k=0}^{\infty} \frac{1}{(t+2)^k} = \frac{1}{1 - \dfrac{1}{(t+2)}} = \frac{t+2}{t+1}$$

$$f(0) = 2$$

$$S = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \ldots$$

$$\frac{S}{2} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots$$

$$\frac{S}{2} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

$$\Rightarrow \quad \frac{S}{2} = \frac{1}{1 - \dfrac{1}{2}} \quad \Rightarrow \quad S = 4$$

$$f(k) = \frac{k+2}{(k+1)}$$

$$\frac{f(k)}{k(k+2)} = \frac{1}{k(k+1)}$$

$$\sum_{k=1}^{\infty} \frac{f(k)}{k(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1$$

Matrix-Match Type

1. A → r; B → q; C → q; D → p

(A) $a_1, a_2, a_3, \ldots a_n, \ldots$

$$a_1 = \frac{5}{2}; a_{10} = 16$$

$$\frac{5}{2} + 9d = 16 \quad \Rightarrow \quad 9d = \frac{27}{2} \quad \Rightarrow \quad d = \frac{3}{2}$$

given $a_1 + a_2 + \ldots + a_n = 110$

$$\frac{n}{2}[2a + (n-1)d] = 110$$

$$\Rightarrow \quad \frac{n}{2}\left[5 + (n-1)\frac{3}{2}\right] = 110$$

$$\Rightarrow \quad n[10 + 3n - 3] = 440$$

$$\Rightarrow \quad n(3n + 7) = 440 \quad \Rightarrow \quad 3n^2 + 7n - 440 = 0$$

$$\Rightarrow \quad 3n^2 + 40n - 33n - 440 = 0$$

$$\Rightarrow \quad n(3n + 40) - 11(3n + 40) = 0$$

$$\Rightarrow \quad n = 11$$

(B) $a, a + d, \ldots, a + 8d$

$\therefore \quad 9a + 36d = 7(180^\circ)$ or $a + 4d = 140^\circ$

Also $a + 8d < 180^\circ$

$\therefore \quad 4d < 40^\circ, d < 10^\circ$

i.e., the common difference is less than 10°

Ans. ⇒ (Q)

(C) $a_1, a_2, a_3, \ldots, a_n, \ldots$ is increasing G.P.

Let $a_1 = a; a_2 = ar$ etc $r > 1$

$$a_6 = 4a_4;\ ar^5 = 4ar^3$$

$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$ ($\because$ it is an increasing G.P.)

$\because \quad a_9 - a_7 = 192$

$\therefore \quad ar^8 - ar^6 = 192$ i.e., $256a - 64a = 192$

$\Rightarrow \quad 192a = 192 \quad \Rightarrow \quad a = 1$

$$\sum_{i=4}^{n} a_i = 1016$$

$$\Rightarrow \quad (a_1 + a_2 + a_3 + \ldots + a_n) - (a_1 + a_2 + a_3) = 1016$$

$$\Rightarrow \quad \frac{1(2^n - 1)}{2 - 1} = 1016 + 1 + 2 + 4 = 1023$$

$\therefore \quad 2^n = 1024 \quad \Rightarrow \quad n = 10 \quad \Rightarrow \quad$ (Q)

(D) $6, a, b$ in H.P. $\Rightarrow \frac{1}{6}, \frac{1}{a}, \frac{1}{b}$ are in A.P.

$$\Rightarrow \quad \frac{2}{a} = \frac{1}{6}, \frac{1}{b} \Rightarrow \frac{1}{b} = \frac{2}{a} - \frac{1}{6} \Rightarrow \frac{1}{b} = \frac{12 - a}{6a}$$

$$\Rightarrow \quad b = \frac{6a}{12 - a}$$

$a \in \{3, 4, 6, 8, 9, 10, 11\} \quad \Rightarrow \quad 7$

2. A → p; B → s; C → r; D → q

(A) $S_{2n} - S_n = \frac{2n}{2}(2a + (2n-1)d) - \frac{n}{2}(2a + (n-1)d)$

$$= \frac{n}{2}[2a + (3n-1)d] = \frac{1}{3}\frac{3n}{2}[2a + (3n-1)d] = \frac{1}{3}S_{3n}]$$

(B) The sum equals $\frac{n(n+1)(n+2)}{6} = 220$

which is true for $n = 10$

(C) Let first terms and common ratio be a and r respectively.

$$\text{Now,} \sum_{n=1}^{100} g_{2n} = ar + ar^3 + ar^5 + \ldots + ar^{199} = \frac{10}{3} \qquad \ldots(1)$$

$$\text{Also,} \sum_{n=1}^{100} g_{2n-1} = a + ar^2 + ar^4 + \ldots + ar^{198} = \frac{5}{9} \qquad \ldots(2)$$

$$\text{Now,} \frac{\text{Equation(1)}}{\text{Equation(2)}} \Rightarrow \frac{r(a + ar^2 + \ldots + ar^{198})}{(a + ar^2 + \ldots + ar^{198})} = \frac{\frac{10}{3}}{\frac{5}{9}}$$

$\Rightarrow \quad r = 6.$

(D) Let a and r be first term and common ratio of G.P.

Also, total number of terms $= 2n$

so, G.P. is $a, ar, ar^2, ar^3, ar^4, \ldots ar^{2n-1}$

$$\therefore \quad S_n = \frac{a(r^{2n} - 1)}{(r-1)} = \text{sum of all terms}$$

Also, sum of terms occupy the odd places $= \frac{a(r^{2n} - 1)}{(r^2 - 1)}$

Now, given that $\frac{a(r^{2n} - 1)}{(r-1)} = \frac{5a(r^{2n} - 1)}{(r-1)(r+1)}$

$\Rightarrow \quad r + 1 = 5 \quad \therefore \quad r = 4$

3. A → r; B → s; C → p; D → q

(A) $\left(5 + \sqrt{2}\right)x^2 - \left(4 + \sqrt{5}\right)x + 8 + 2\sqrt{5} = 0$ has roots x_1 and x_2.

$$\text{H.M.} = \frac{2x_1x_2}{x_1 + x_2} = \frac{2(8 + 2\sqrt{5})}{4 + \sqrt{5}} = 4$$

(B) $a_1 + 9d = 3$ and $\frac{1}{h_{10}} = \frac{1}{h_1} + 9d_1$

$$2 + 9d = 3 \quad \frac{1}{3} = \frac{1}{2} + 9d_1$$

$$d = \frac{1}{9} \quad 9d_1 = -\frac{1}{6} \quad \Rightarrow \quad d_1 = -\frac{1}{54}$$

$$\therefore \quad a_4 = 2 + 3d = 2 + \frac{1}{3} = \frac{7}{3}$$

$$\frac{1}{h_7} = \frac{1}{2} + 6\left(-\frac{1}{54}\right) = \frac{1}{2} - \frac{1}{9} = \frac{7}{18}$$

$$\therefore \quad a_4h_7 = \frac{7}{3} \times \frac{18}{7} = 6$$

(C) $a = 3^{1+x} + 3^{1-x} + 9^x + 9^{-x}$

$3(3^x + 3^{-x}) + (9^x + 9^{-x})$

$\geq 6 + 2 \geq 8$

(D) Product of n geometric means between two numbers is equal to n^{th} power of single geometric mean between them. So $n = \frac{6}{2} = 3$

4. A → r; B → p; t, C → s; D → q

(A) $F(n+1)=\frac{2F(n)+1}{2}=F(n)+\frac{1}{2}$

$F(1), F(2), F(3), \ldots$ is an AP with common difference $\frac{1}{2}$

(B) $a_1+2d+a_1+4d+a_1+10d+a_1+16d+a_1+18d=5a,+50d$

$=5(a_1+10d)=10$

i.e. $a_1+10d=2$

Now, $\sum_{i=1}^{21} a_i=\frac{21}{2}[2a_1+20d]=21(a_1+10d)=42$

(C) $S=1+5+13+29+\ldots+t_{10}$

$S=1+5+13+\ldots+t_9+t_{10}$

Subtrating

$t_{10}=1+4+8+16+\ldots$ up to 10 terms

$=1+(4+8+16+\ldots$ up to 9 terms$)$

$=2045$

(D) Sum of all two digit numbers $=\frac{90}{2}(10+99)=(45)(109)$

Sum of all two digit numbers is divisible by $2=\frac{45}{2}(10+98)=(45)(54)$

Sum of all two digit numbers is divisible by $3=\frac{30}{2}$ $(12+99)=15(111)$

Sum of all two digit numbers divisible by $6=\frac{15}{2}(12+96)=15(54)$

The required sum is $45(109)+15(54)-(45)(54)-15(111)=1620$

5. A → r; B → q; C → q; D → p

(A) $A_p=a+(p-1)d$ …(1)

$A_q=a+(q-1)d$ …(2)

$A_r=a+(r-1)d$ …(3)

$A_s=a+(s-1)d$ …(4)

$A_q=kA_p$

$A_r=k^2A_p$

$A_s=k^3A_p$ $\quad(\because A_p, A_q, A_r, A_s \text{ in} G.P)$

$(p-q)=\frac{A_p-A_q}{d}=A_p\frac{(1-k)}{d}$ from (1) and (2)

$(q-r)=A_pK\frac{(1-k)}{d}$ from (2) and (3)

$(r-s)=A_pk^2\frac{(1-k)}{d}$

$\Rightarrow$ $p-q, q-r, r-s$ are in G.P

(B) $\ln x, \ln y, \ln z$ are in G.P

$\Rightarrow$ $\ln(\ln x), \ln(\ln y), \ln(\ln z)$ are in A.P.

$\Rightarrow$ $2x+\ln(\ln x), 3x+\ln(\ln y), 4x+\ln(\ln z)$ are in A.P.

(C) $n!3\times n!$ and $(n+1)!$ are in G.P.

$\Rightarrow$ $9(n!)^2=n!(n+1)! \quad \Rightarrow (n+1)=9 \Rightarrow n=8$

$\therefore$ $n!=8!$

$5\times n!=5\times 8!$

$(n+1)!=9!$

$\frac{9!+8!}{2}=5\times 8!$

$\Rightarrow$ $n!, 5\times n!$ and $(n+1)!$ are in A.P.

(D) $\frac{(b-c)^2+(a-b)^2+(c-a)^2}{3}$

$=\frac{(b-c-2a)^2+(c+a-2b)^2+(a+b-2c)^2}{3}$

$\Rightarrow$ $(b+c-2a)^2-(b-c)^2+(c+a-2b)^2-$

$(c-a)^2+(a+b-2c)^2-(a-b)^2=0$

$\Rightarrow$ $a=b=c$

6. A → r; B → r; C → r; D → q

(A) $(b-a)=(c-b)$ and $(c-b)^2=a(b-a)$

$\Rightarrow$ $(b-a)^2=a(b-a)\Rightarrow b=2a, c=3a$

$\therefore$ $a:b:c=1:2:3 \quad \Rightarrow \quad \frac{a^3+b^3+c^3}{3abc}=2$

(B) $x=\frac{a+b}{2}, b=ar^3 \Rightarrow r=\left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$\frac{y^3+z^3}{xyz}=\frac{a^2b+ab^2}{\left(\frac{a+b}{2}\right)a\left(\frac{b}{a}\right)^{1/3}a\left(\frac{b}{a}\right)^{2/3}}=2$$

(C) $a_1{}^2-a_2{}^2+a_3{}^2\ldots a_{50}{}^2$

$=(a_1+a_2)(a_1-a_2)+(a_3+a_4)$

$(a_3-a_4)+\ldots+(a_{49}+a_{50})(a_{49}-a_{50})$

$=-d[a_1+a_2+---+a_{50}]$

$=-\frac{25}{49}(a_{50}-a_1)(a_{50}+a_1)$

$=\left(\frac{25}{49}\right)(a_1{}^2-a_{50}{}^2)\Rightarrow n=2$

(D) $1+a=2-b-c=(1-b)+(1-c)\geq 2\sqrt{(1-b)(1-c)}$

$1+b\geq 2\sqrt{(1-a)(1-c)}$, $(1+c)\geq 2\sqrt{(1-a)(1-b)}$

$\Rightarrow(1+a)(1+b)(1+c)\geq 8(1-a)(1-b)(1-c)$

7. A → q; B → r; C → s; D → t

(A) $a+b=12$

$ab+\frac{6ab}{a+b}=48$

$ab+\frac{ab}{2}=48 \quad \therefore \quad ab=32$

(B) $S=\frac{5}{1^2\cdot 4^2}+\frac{11}{4^2.7^2}+\frac{17}{7^2.10^2}+\ldots$

$$\Rightarrow \quad 3S = \frac{3.5}{1^2.4^2} + \frac{3.11}{4^2.7^2} + \frac{3.17}{7^2.10^2} + \ldots$$

$$\Rightarrow \quad 3S = \frac{(4-1).(4+1)}{1^2.4^2} + \frac{(7-4)(7+4)}{4^2.7^2} + \frac{(10-7)(10+7)}{7^2.10^2} + \ldots$$

$$\Rightarrow \quad 3S = \frac{4^2-1^2}{1^2.4^2} + \frac{7^2-4^2}{4^2.7^2} + \frac{10^2-7^2}{7^2.10^2} + \ldots$$

$$\Rightarrow \quad 3S = 1 - \frac{1}{4^2} + \frac{1}{4^2} - \frac{1}{7^2} + \frac{1}{7^2} - \frac{1}{10^2} + \ldots$$

$$\Rightarrow \quad 3S = 1 \qquad S = \frac{1}{3}$$

(C) H.M of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ is $\dfrac{4}{\frac{1}{1/2} + \frac{1}{1/3} + \frac{1}{1/4} + \frac{1}{1/5}} = \dfrac{4}{2+3+4+5}$

$$= \frac{2}{7}$$

(D) Since G.M. lies between the numbers

$$G.M. = -\sqrt{(-4)\times(-9)} = -6$$

Integer/Numerical Answer Type Questions

1. a, ar, ar^2

$2ar = a + 3 \Rightarrow a = \dfrac{3}{2r-1}$

$ar^2 = 4$

Solve, $r = 2/3$, $a = 9$

Hence, the correct answer is (9).

2. Let the series be $a, ax, ax^2, ax^3 \ldots$ given that $|x| < 1$ and $x \neq 0$

Also, $\dfrac{T_4}{T_2} = \dfrac{ax^3}{ax} = \dfrac{1}{16} \qquad \Rightarrow x^2 = \dfrac{1}{16}$

$$\Rightarrow \quad x = \pm\frac{1}{4}$$

But since it is a decreasing G.P. $\Rightarrow x = \dfrac{1}{4}$

Also, $\dfrac{T_3}{T_2^2} = \dfrac{ax^2}{(ax)^2} = \dfrac{1}{9} \qquad \Rightarrow \dfrac{1}{a} = \dfrac{1}{9} \Rightarrow a = 9$

$$S_\infty = \frac{a}{1-r} = \frac{9}{1-\frac{1}{4}} = \frac{9\times 4}{3} = 12 \text{ Ans.}$$

Hence, the correct answer is (12).

3. $\displaystyle\sum_{\alpha=4}^{n+3} 4(\alpha - 3) = An^2 + Bn + C$

$$\Rightarrow \sum_{\alpha=1}^{n} 4\alpha = An^2 + Bn + C$$

$\Rightarrow \quad 2n(n+1) = An^2 + Bn + C \quad \Rightarrow \quad A = 2, B = 2, C = 0$

$\Rightarrow \quad A + B + C = 4$

Hence, the correct answer is (4).

4. $(1-P)\left(1+3x+9x^2+27x^3+81x^4+243x^5\right) = 1-P^5$

$\Rightarrow \quad (1-P)\dfrac{1-(3x)^6}{1-3x} = 1-P^6$ which is possible only. If $P = 3x$

$\therefore \quad \dfrac{P}{x} = 3$ Ans.

Hence, the correct answer is (3).

5. $\left(1^2+2^2+\ldots+n^2\right) - \left(a_1+a_2+\ldots+a_n\right) = \dfrac{1}{3}n\left(n^2-1\right) \qquad \ldots(1)$

Replacing n by $(n-1)$, then

$\left(1^2+2^2+\ldots+(n-1)^2\right) - \left(a_1+a_2+\ldots+a_{n-1}\right)$

$= \dfrac{1}{3}(n-1)\left((n-1)^2-1\right) \qquad \ldots(2)$

Subtracting (2) from (1)

$$n^2 - a_n = n^2 - n$$

$\Rightarrow \quad a_n = n \Rightarrow a_7 = 7$ Ans.

Hence, the correct answer is (7).

6. A.M. $\geq$ H.M.

$$\Rightarrow (\Sigma x_i)\left(\Sigma\frac{1}{x_i}\right) \geq 3^2 = 9$$

Hence, the correct answer is (6).

7. General term $(t_r^2) = 1 + \dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} = \dfrac{n^2(n+1)^2 + (n+1)^2 + 2}{n^2(n+1)^2}$

$$\Rightarrow \quad t_r^2 = \frac{n^4+2n^3+3n^2+2n+1}{n^2(n+1)^2} = \left\{\frac{n^2+n+1}{n(n+1)}\right\}^2$$

$$\therefore \quad t_r = 1 + \frac{1}{n(n+1)};$$

Required sum $= \displaystyle\sum_{r=1}^{999}\left(1+\frac{1}{r}-\frac{1}{r+1}\right) = 999 + 1 - \frac{1}{1000} = \frac{10^6-1}{10^3}$,

$k = 6$

Hence, the correct answer is (6).

8. $\dfrac{(b-d)+b+(b+d)}{3} \geq \sqrt[3]{abc}$

$$b \geq 4^{1/3} \Rightarrow b \geq 2^{2/3}$$

$$k = 2$$

Hence, the correct answer is (2).

9. Given $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{2^{n-1}} = \dfrac{1.\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)}$

$$\Rightarrow \quad S_n = 2 - \frac{1}{2^{n-1}}$$

$$\therefore \quad 2 - S_n = \frac{1}{2^{n-1}} < \frac{1}{100}$$

Hence, least value of ‘n’ is 8

Hence, the correct answer is (8).

10. $(a+b)^2 = (a+c)(b+c)$

$(a+b)^2 = ab + c(a+b) + c^2$ … (1)

a, c, b are in H.P $\Rightarrow c = \dfrac{2ab}{a+b}$

$\Rightarrow \quad ab = \dfrac{c(a+b)}{2} = \dfrac{kc^2}{8}$

From (1) $\dfrac{k^2c^2}{16} = \dfrac{3kc^2}{8} + c^2$

$\Rightarrow \quad k^2 - 6k - 16 = 0 \Rightarrow k = 8 \text{ or } -2$

$\therefore \quad k = 8(k > 0)$

Hence, the correct answer is (8).

11. $\dfrac{1}{3}\log_2 x + \log_2 y = 5, \log_2 x + \dfrac{1}{3}\log_2 y = 7$

Solving both

$\log_2 x = 6 \text{ and } \log_2 y = 3$

$\therefore \quad x = 2^6,\ y = 2^3$

$\therefore \quad x^2 + y^2 = 4096 + 64 = 4160$

Hence, the correct answer is (2).

12. $a_1 = 2,\ \dfrac{1}{h_1} = \dfrac{1}{2}$

$2 + 9d = 3, \dfrac{1}{2} + 9d_1 = \dfrac{1}{3}$

$\therefore \quad d = \dfrac{1}{9}, d_1 = -\dfrac{1}{54}$

$\therefore \quad a_4 h_7 = \dfrac{a_1 + 3d}{\dfrac{1}{h_1} + 6d_1} = 6$

Hence, the correct answer is (6).

13. Let 150th term be n. Then $1 + 2 + 3 + \ldots + (n-1) < 150 < 1 + 2 + 3 + \ldots + n$

$\Rightarrow \quad n = 17$

Hence, the correct answer is (7).

14. $\log\left(1 - \dfrac{1}{x}\right) = \dfrac{-1}{x} - \dfrac{1}{2x^2} - \dfrac{1}{3x^3} \ldots$

$\dfrac{1}{2} + \dfrac{1}{2.2^2} + \dfrac{1}{3.2^3} + \ldots = -\log(1 - \dfrac{1}{2}) = \log 2 = \dfrac{1}{2}\log 4$

Hence, the correct answer is (4).

15. $\left[\sqrt{n+1}\right] \neq \left[\sqrt{n}\right]$ …(1)

$0 < \sqrt{n+1} - \sqrt{n} < 1$

$\Rightarrow \quad \left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] \text{ or } \left[\sqrt{n}\right] + 1$ …(2)

from (1) and (2)

$\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] + 1$

$k^2 \le n + 1 < (k+1)^2 \Rightarrow \left[\sqrt{n+1}\right] = k$

$\Rightarrow \quad n \ge k^2 - 1$ …(3)

$\left[\sqrt{n}\right] = k - 1 \Rightarrow n < k^2$ …(4)

from (3) and (4) $n = k^2 - 1$

$1 \le k^2 - 1 \le 100 \Rightarrow k = 2, 3, \ldots, 10.$

Hence, the correct answer is (9).

16. $n^2 + n + 2 = m^2$

$(2m)^2 - (2n+1)^2 = 4(m^2 - n^2 - n) - 1 = 7$

$(2m - 2n - 1)(2m + 2n + 1) = 1 \times 7$

Case-1; $2m - 2n - 1 = 1$ or $2m + 2n + 1 = 7$

$\Rightarrow \quad 4n + 2 = 6 \Rightarrow n = 1$

Case-2: $2m - 2n - 1 = 7$ or $2m + 2n + 1 = 1$

$\Rightarrow \quad 4n + 2 = -6$

$n = -2$

$2n + 1 = \pm 3$

$n = 1, -2.$

Hence, the correct answer is (1).

17. Let roots are $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$ then $\dfrac{a}{r^2} + \dfrac{a}{r} + a + ar + ar^2 = 40$

and $\dfrac{r^2}{a} + \dfrac{r}{a} + \dfrac{1}{a} + \dfrac{1}{ar} + \dfrac{1}{ar^2} = 10$

$$\Rightarrow \quad \frac{a\left[\dfrac{1}{r^2} + \dfrac{1}{r} + 1 + r + r^2\right]}{\dfrac{1}{a}\left[r^2 + r + 1 + \dfrac{1}{r} + \dfrac{1}{r^2}\right]} = \frac{40}{10}$$

$\Rightarrow \quad a^2 = 4$

$a = \pm 2$

Product of roots $= \dfrac{a}{r^2} \times \dfrac{a}{r} \times a \times ar \times ar^2 = -S$

$a^5 - S = 0$

$S = -32$

$|2S| = 64$

$\sqrt{|64|} = 8.$

Hence, the correct answer is (8).

18. For the A.P, $a = \dfrac{5}{2}, d = -\dfrac{7}{12}$

nth term is $\dfrac{37 - 7n}{12} > 0 \Rightarrow n \le 5$

$\therefore \quad$ 5th term of the A.P is $\dfrac{1}{6}$

The largest positive value of the H.P is 6

Hence, the correct answer is (6).

19. $17^2 = (m - n^2)(m + n^2)$

$m + n^2 = 17^2 \qquad m + n^2 = 17$

$m - n^2 = 1 \qquad m - n^2 = 17$

there are 4 such values of m possible

Hence, the correct answer is (4).

20. $a^2 + 1 \ge 2a,\ b^2 + 1 \ge 2b,\ c^2 + 1 \ge 2c$

$\Rightarrow \quad \dfrac{a^2+1}{b+c} + \dfrac{b^2+1}{c+a} + \dfrac{c^2+1}{a+b} \ge \dfrac{2a}{b+c} + \dfrac{2b}{c+a} + \dfrac{2c}{a+b}$

Let $b + c = x,\ c + a = y,\ a + b = z$

$$\sum \frac{2a}{b+c} = \sum \frac{y+z-x}{x} = \sum \left(\frac{x}{y}+\frac{y}{x}\right) - 3 \geq 6-3=3$$

Hence, the correct answer is (3).

21. $\sum_{m=1}^{5^{99}} T_{2m} = \sum_{m=1}^{5^{99}} \left(T_{2m-1}+d\right)$ (where d = common difference of A.P.)

$$\Rightarrow \quad 5^{100} = \sum_{m=1}^{5^{99}} T_{2m-1} + \sum_{m=1}^{5^{99}} d$$

$$\Rightarrow \quad 5^{100} = 5^{99} + \left(5^{99}\right)d$$

$$\Rightarrow \quad d = \frac{5^{100}-5^{99}}{5^{99}} = \frac{5^{99}(5-1)}{5^{99}} = 4.$$

Hence, the correct answer is (4).

22. $a_1 + a_2 + \ldots + a_n = 2n^2\left(3 + a_{n+1} + a_{n+2} + \ldots + a_{50}\right)$

$n = 1,$

$\Rightarrow \quad a_1 = 2\left(3 + a_2 + \ldots + a_{50}\right)$

$n = 50,$

$a_1 + a_2 + \ldots + a_{50} = 3.2.50^2$

$\Rightarrow$ By solving we get $a_1 = 10002$

Lowest odd prime factor of $10002 = 3$.

Hence, the correct answer is (3).

23. $f(x) = 2^x$ for all $x \in N$

$$\therefore \quad \sum_{k=1}^{n} f(a+k) = 16\left(2^n-1\right)$$

$$\Rightarrow \quad \sum_{K=1}^{n} 2^{a+k} = 16\left(2^n-1\right)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^a.2^k = 16\left(2^n-1\right)$$

$$\Rightarrow \quad 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16\left(2^n-1\right)$$

$$\Rightarrow \quad 2^a\left(2+2^2+\ldots.+2^n\right) = 16\left(2^n-1\right)$$

$$\Rightarrow \quad 2^a\left\{2\left(\frac{2^n-1}{2-1}\right)\right\} = 16\left(2^n-1\right)$$

$$\Rightarrow \quad 2^{a+1}\left(2^n-1\right) = 16\left(2^n-1\right)$$

$$\Rightarrow \quad 2^{a+1} = 2^4 \Rightarrow a+1=4 \Rightarrow a=3$$

Hence, the correct answer is (3).

24. $1, 2^2, 3^2, \ldots, 44^2, 45^2 = 2025, 46^2 = 2116$

Since $\left[\sqrt{2046}\right] = \left[\sqrt{2047}\right] = \left[\sqrt{2048}\right] = \left[\sqrt{2049}\right] = 45$

$\therefore$ 2003rd term is $2003 + 45 = 2048$

Hence remainder is 0

Hence, the correct answer is (0).

25. $a = \frac{6}{9}(10^n-1) = \frac{2}{3}(10^n-1)$

$$b = \frac{8}{9}(10^n-1)$$

$$c = \frac{4}{9}(10^{2n}-1)$$

$$a^2 + b = \frac{4}{9}(10^n-1)^2 + \frac{8}{9}(10^n-1)$$

$$= \frac{4}{9}\left[10^{2n} - 2\cdot 10^n + 1 + 2\cdot 10^n - 2\right]$$

$$= \frac{4}{9}(10^{2n}-1) = c$$

Hence, the correct answer is (0).

26. $T_r = \frac{1}{2}\left(\frac{1}{1\times 3\times 5\times\ldots\times 2r-1} - \frac{1}{1\times 3\times 5\times\ldots\times 2r+1}\right)$

$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left(1 - \frac{1}{1\times 3\times 5\times\ldots\times 2n+1}\right)$$

$$\therefore \quad \frac{k}{2} = 1 \Rightarrow k^2 = 4$$

Hence, the correct answer is (4).

27. $\dfrac{a+\dfrac{2ac}{a+c}}{2a-\dfrac{2ac}{a+c}} + \dfrac{c+\dfrac{2ac}{a+c}}{2c-\dfrac{2ac}{a+c}}$

$$= 1 + \frac{3}{2}\left(\frac{a}{c}+\frac{c}{a}\right) \geq 1 + \frac{3}{2}(2) \geq 4$$

so greatest $\lambda = 4$ $\quad \sqrt{\lambda\sqrt{\lambda\sqrt{\lambda\ldots\infty}}} = \lambda^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots} = \lambda$

Hence, the correct answer is (4).

28. Using A.M. $\geq$ H.M.

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Hence, the correct answer is (9).

29. $\left[\frac{-2010\times 2011}{-1004\times 1005}\right] = 4$

Hence, the correct answer is (4).

30. $T_r = r(n-r+1)$

$$T_r = nr - r^2 + r$$

$$S_n = \sum_{r=1}^{n} T_r = n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum r$$

$$= \frac{n\times n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[n - \frac{(2n+1)}{3} + 1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n-2n-1+3}{3}\right] = \frac{n(n+1)(n+2)}{6}$$

$$S_n = \frac{n(n+1)(n+2)}{6}, S_{25} = \frac{25\times 26\times 27}{6} = 25\times 13\times 9$$

$S_{25} = 325\lambda \Rightarrow \lambda = 9$

Hence, the correct answer is (9).

31. a, b, c are in G.P.

$b^2 = ac$

$\log_6 a + \log_6 b + \log_6 c = 6$

$a\,b\,c = 6^6$

$b^3 = 6^6$

$b = 6^2 = 36$

$ac = 36 \times 36 = 2^4 \times 3^4$

$b - a = N^2$

$36 - a = N^2$

a is factor of $2^4\,3^4$

$a = 27$ is possible value

$b - a = 36 - 27 = 9 = (3)^2$

$\Rightarrow \quad a = 27, b = 36, c = 48$

$a + b + c = 111$

Hence, the correct answer is (6).

32. $a^2 - b = b^2 - c \Rightarrow a + b = \dfrac{b-c}{a-b}$ similarly $b + c = \dfrac{c-a}{b-c}$ and $c + a = \dfrac{a-b}{c-a}$

$\therefore$ Product is equal to 1.

Hence, the correct answer is (1).

33. Since 1, abc, $a^2b^2c^2$ are in A.P

$2abc = 1 + a^2b^2c^2$

$(abc - 1)^2 = 0 \Rightarrow abc = 1$

Again $A.M \geq G.M \Rightarrow \dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

$\Rightarrow \quad a + b + c \geq 3$

Hence, the correct answer is (1).

34. If a, x, y, z, b to are in A.P then the common difference d of the A.P. is given by

$$b = a + 4d \Rightarrow d = \frac{b-a}{4}$$

$$\therefore \quad x = a + d = a + \frac{b-a}{4} = \frac{b+3a}{4}$$

$$y = a + 2d = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$$z = a + 3d = a + 3\left(\frac{b-a}{4}\right) = \frac{a+3b}{4}$$

$$\therefore \quad xyz = \frac{b+3a}{4} \times \frac{a+b}{2} \times \frac{a+3b}{4}$$

$$\Rightarrow \quad 55 = \frac{(3a+b)(a+b)(a+3b)}{32}$$

$$\Rightarrow (3a+b)(a+b)(a+3b) = 55 \times 32$$

When a, x, y, z, b are in H.P . Then $\dfrac{1}{a}, \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{b}$ are in A.P.

Let D be the common different of this A.P. Then

$$\frac{1}{b} = \frac{1}{a} + 4D \Rightarrow D = \frac{a-b}{4ab}$$

$$\therefore \quad \frac{1}{x} = \frac{1}{a} + D = \frac{1}{a} + \frac{a-b}{4ab} = \frac{3b+a}{4ab}$$

$$\frac{1}{y} = \frac{1}{a} + 2D = \frac{1}{a} = \frac{a-b}{2ab} = \frac{a+b}{2ab}$$

$$\frac{1}{z} = \frac{1}{a} + 3D = \frac{1}{a} = \frac{3(a-b)}{4ab} = \frac{3a+b}{4ab}$$

$$\therefore \quad \frac{1}{x}.\frac{1}{y}.\frac{1}{z} = \frac{(3a+b)(a+b)(3a+b)}{32a^3\,b^3}$$

$$\Rightarrow \quad \frac{1}{xyz} = \frac{(3a+b)((a+b)(a+3b))}{32a^3\,b^3}$$

$$\Rightarrow \quad \frac{55}{343} = \frac{55 \times 32}{32\,a^3b^3}$$

$$\Rightarrow \quad (ab)^3 = 7^3 \quad \Rightarrow ab = 7$$

$a = 1, b = 7$, or $a = 7, b = 1$

Hence, the correct answer is (8).

35. We have

$S = 1^3 + 3.2^2 + 3^3 + 3.4^2 + 5^3 + 3.6^2 + \ldots$

$S = \left(1^3 + 3^3 + 5^3 + \ldots\right) + 3.\left(2^2 + 4^2 + 6^2 + \ldots\right)$

$S = \left(1^3 + 3^3 + 5^3 + \ldots\right) + 12\left(1^2 + 2^2 + 3^2 + \ldots\right)$

Where $S_1 = 1^3 + 3^3 + 5^3 + \ldots$ and $S_2 = 1^2 + 2^2 + 3^2 + \ldots$

Now case arise

When n is, say even, say $n = 2m\ m \in N$

In this case S_1, and S_2 both contain m terms

$\therefore S_1 = 1^3 + 3^3 + 5^3 + \ldots + (2m-1)^3$

$$\sum_{r=1}^{M} (2r-1)^3 = \sum_{r=1}^{m} \left(8r^3 - 12r^2 + 6r - 1\right)$$

$$= 8\sum_{r=1}^{m} r^3 - 12\sum_{r=1}^{m} r^2 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 8\left\{\frac{m(m+1)}{2}\right\}^2 - 12\left\{\frac{m(m+1)(2m+1)}{6}\right\} + \frac{6m(m+1)}{2} - m$$

$$= 8\left\{\frac{n(n+2)}{8}\right\}^2 - \frac{12}{6}\left\{\frac{n}{2}\left(\frac{n+2}{2}\right) + (n+1)\right\} +$$

$$3\frac{n}{2}\left(\frac{n+2}{3}\right) - \frac{n}{2}$$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + 3\frac{n(n+2)}{4} - \frac{n}{2}$$

$$S_2 = 1^2 + 2^2 + 3^2 + \ldots + m^2$$

$$= \frac{m(m+1)(2m+1)}{6} = \frac{n(n+2)(n+1)}{24}$$

$\therefore \quad S = S_1 + 12S_2$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + \frac{3}{4}n\,(n+2) - \frac{n}{2} + \frac{n(n+1)(n+2)}{2}$$

$$= \frac{n^2(n+2)^2}{8} + \frac{3}{4}n(n+2) - \frac{n}{2}$$

$$= \frac{n}{8}\left(n^3 + 4n^2 + 10n + 8\right) = \frac{n}{k}\left(n^3 + an^2 + bn + c\right)$$

Now $b - a + c - k = 10 - 4 + 8 - 8 = 6$

Hence, the correct answer is (6).

36. Given x, y, z, w are 4 numbers such that x, y, z are in A.P. and y, z, w are in G.P.

Let $x = \frac{2a}{r} – a;\ y = \frac{a}{r};\ z = a;\ w = ar$

Now, $x + w = 16$ and $y + z = 8$

$\Rightarrow \quad \frac{2a}{r} - a + ar = 16$ and $\frac{a}{r} + a = 8$

$\Rightarrow \quad a\left(\frac{2}{r} - 1 + r\right) = 16$ and $\frac{a(1+r)}{r} = 8$

$\Rightarrow \quad 8r\left(\frac{2}{r} - 1 + r\right) = 16(1 + r)$

$\Rightarrow \quad 2 - r + r^2 = 2 + 2r$

$\therefore \quad r^2 = 3r$

$r = 0$ (reject) or 3

$\therefore \quad a = 6$

$\therefore \quad x = –2;\ y = 2;\ z = 6;\ w = 18$

Hence, $x^2 - y^2 + 2z - w = (–2)^2 - (2)^2 + 2(6) - 18$

$= 4 - 4 + 12 - 18$

$= - 6 \Rightarrow |- 6| = 6$

Aliter : Given x, y, z, w are 4 numbers such that x, y, z are in A.P. and y, z, w are in G.P.

Let $x = a - d$, $y = a$ and $z = a + d$

$\therefore \quad w = \frac{(a + d)^2}{a}$

Now, given $x + w = 16$

$\Rightarrow \quad (a - d) + \frac{(a+d)^2}{a} = 16 \qquad \ldots(1)$

and $y + z = 8 \Rightarrow 2a + d = 8 \qquad \ldots(2)$

$\therefore \quad$ (1) and (2)

$\Rightarrow \quad a^2 - ad + a^2 + d^2 + 2ad = 16a$

$2a^2 + d^2 + ad = 16a$

$2a^2 + (8 - 2a)^2 + a(8 - 2a) = 16a$

$\Rightarrow \quad 2a^2 + 4(4 - a)^2 + 8a - 2a^2 = 16a$

$\Rightarrow \quad (4 - a)^2 + 2a = 4a$

$\Rightarrow \quad a^2 - 10a + 16 = 0$

$\Rightarrow \quad (a - 8)\ (a - 2) = 2$

$\therefore \quad a = 2$ or $a = 8$

If $\begin{matrix} a = 2 \\ d = 4 \end{matrix}$ and $\left.\begin{matrix} if\ a = 8 \\ d = –8 \end{matrix}\right]$ (Reject as $z = a + d = 0$)

$\therefore \quad$ Numbers are – 2, 2, 6, 18

$\therefore \quad x^2 - y^2 + 2z - w = (–2)^2 - (2)^2 + 2(6) - 18$

$= 4 - 4 + 12 - 18$

$= - 6$

$\Rightarrow \quad |- 6| = 6$

Hence, the correct answer is (6).

37. Given $S_k = \underbrace{\frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \ldots\infty}_{Infinite\,G.P.}$, where $k \in N$ and $k \geq 2$

For $k > 1$,

$$S_k = \frac{\frac{1}{k}}{1 - \frac{1}{k}} = \frac{1}{k-1};\ E = \frac{S_k \cdot S_{k+2}}{(S_{k+1})^2}$$

Hence, $E = \left(\frac{1}{k-1} \cdot \frac{1}{k+1}\right) \div \frac{1}{k^2} = \frac{k^2}{k^2 - 1}$

So, $E = 1 + \frac{1}{k^2 - 1} \quad \Rightarrow \quad E_{\max} = 1 + \frac{1}{4-1} = \frac{4}{3} = \frac{p}{q}$ (Given)

Hence, $p = 4$ and $q = 3 \quad \Rightarrow \quad (p + q) = 7.$

Hence, the correct answer is (7).

38. $x = \frac{1}{9}[999\ldots99] = \frac{1}{9}[10^{20} - 1] \qquad \ldots(1)$

$y = \frac{1}{3}[999\ldots99] = \frac{1}{3}[10^{10} - 1]$

$z = \frac{2}{9}[10^{10} - 1]$

$$\therefore \quad \frac{x - y^2}{z} = \frac{(10^{20} - 1) - (10^{10} - 1)^2}{2(10^{10} - 1)}$$

$$= \frac{(10^{10} + 1) - (10^{10} - 1)}{2} = 1$$

Hence, the correct answer is (1).

39. $ax^2 + bx + c = 0$ has roots α and 4α.

$5\alpha = \frac{-b}{a}$ and $4\alpha^2 = \frac{c}{a}$

$\therefore \quad b = –5a\alpha$ and $c = 4a\alpha^2$

Now $3a = 2(c - b)$

$\therefore \quad 8\alpha^2 + 10\alpha - 3 = 0$

$\alpha = \frac{1}{4}$ or $\alpha = \frac{-3}{2}$ (rejected)

hence $\beta = 1$

Now $S = \beta(1 + \alpha + \ldots + \infty)$

$S = \frac{\beta}{1 - \alpha} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

$18S = 24.$

Hence, the correct answer is (24).

40. Given $a_1,\ a_2,\ a_3,\ \ldots\ldots,\ a_n$ be real number in arithmetic progression.

Let d = common difference of arithmetic progression

Also, $\sum_{r=1}^{10} (a_r)^2 = 1185$

$\Rightarrow \quad a_1^2 + a_2^2 + \ldots + a_{10}^2 = 1185$

so, $(15)^2 + (15+d)^2 + (15+2d)^2 + \ldots (15+9d)^2 = 1185$

$\Rightarrow \quad 2250 + 30d(1+2+\ldots+9) + d^2(1^2+2^2+3^2+\ldots+9^2) = 1185$

$\Rightarrow \quad 57d^2 + 270d + 213 = 0$

$\Rightarrow \quad (d+1)\,(57d+213) = 0$

$\Rightarrow \quad d = -1$ or $\dfrac{-213}{57}$. But a_2 is an integer, so $d = \dfrac{-213}{57}$ (rejected)

Hence, $d = -1$

$\Rightarrow \quad a_{16} = a_1 + 15d = 15 + 15(-1) = 0$

$\Rightarrow \quad S(n)$ will increase till $n = 16$.

Hence, the correct answer is (16).

41. $f(x) = x^3 - ax^2 + 14x - 8 = 0$ has roots α, β and γ

All are positive and real

$$\frac{\alpha+\beta+\gamma}{3} \ge (\alpha.\beta\gamma)^{1/3}$$

$$\frac{a}{3} \ge (8)^{1/3}$$

$$a \ge 6$$

Hence, the correct answer is (6).

42. We have $x_1 + x_2 + x_3 = -p$ …(1)

$x_1x_2 + x_1x_3 + x_2x_3 = q$ …(2)

$x_1x_2x_3 = -r$ …(3)

$x_1^2 = x_2x_3$ …(4)

from (2)

$$x_1x_2 + x_1x_3 + x_1^2 = q$$

$$x_1(x_1+x_2+x_3) = q$$

$$x_1 = -\frac{q}{p}$$

$\Rightarrow \quad px_1 + q = 0 \Rightarrow K_1 = 1$ …(5)

from (1)

$$x_2 + x_3 = \frac{q-p^2}{p} \quad \ldots(6)$$

$$x_2x_3 = \frac{rp}{q} \quad \ldots(7)$$

Hence, x_2, x_3 satisfy the equation

$$pqx^2 - (q-p^2)qx + p^2r = 0$$

$\Rightarrow \quad K_2 = 1$ …(8)

From (5) and (8)

$$K_1 + K_2 = 2$$

Hence, the correct answer is (2).

43. Using A.M. ≥ G.M.

$\Rightarrow \quad p+q+r \ge 3\sqrt[3]{pqr}$

But $p+q+r \le 3\sqrt[3]{pqr}$

$\Rightarrow \quad p+q+r \ge 3\sqrt[3]{pqr}$

$\Rightarrow \quad$ A.M. = G.M. $\Rightarrow p = q = r$

$\Rightarrow \quad \dfrac{p+2q+3r}{p+q+r} = 2$

Hence, the correct answer is (2).

44. Let roots of above equation are $a-d$, a and $a+d$

$\Rightarrow \quad 3a = -k \quad \Rightarrow \quad a = -\dfrac{k}{3}$

$\Rightarrow \quad -\dfrac{k^3}{27} + \dfrac{k^3}{9} - \dfrac{k^2}{3} - 15 = 0$

$\Rightarrow \quad 2k^3 - 9k^2 = 405$

Hence, the correct answer is (405).

45. $x^3 - 12x^2 + ax - 48 = 0$, roots are in A.P.

$\therefore \quad$ Let $b-d, b, b+d$ are the roots

$\therefore \quad 3b = 12 \quad \Rightarrow \quad b = 4$

and $\; b(b^2 - d^2) = 48$

$\Rightarrow \quad d = \pm 2$

$\therefore \quad$ roots are 2, 4, 6

$\Rightarrow \quad a = 44 \quad \Rightarrow \quad \dfrac{a}{11} = 4$

Hence, the correct answer is (4).

46. $T_r = \dfrac{8r}{(4r^2-1)^2} = \dfrac{8r}{(2r+1)^2(2r-1)^2}$

$$= \frac{(2r+1)^2 - (2r-1)^2}{(2r+1)^2(2r-1)^2}$$

$$T_r = \frac{1}{(2r-1)^2} - \frac{1}{(2r+1)^2}$$

$$T_1 = \frac{1}{1^2} - \frac{1}{3^2}$$

$$T_2 = \frac{1}{3^2} - \frac{1}{5^2}$$

$$\vdots$$

$$S_\infty = 1$$

Hence, the correct answer is (100).

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1. $1 + 49 + 49^2 + \ldots + 49^{125} = (49^{63}+1)\,\dfrac{(49^{63}-1)}{(48)}$

So greatest value of $k = 63$

Hence, the correct answer is (D).

2. Let the A.P is $a-2d, a-d, a, a+d, a+2d$

$\because$ sum $= 25 \Rightarrow a = 5$

Product $= (a-2d)(a-d)a(a+d)(a+2d) = 2520$

$\Rightarrow \quad (25-4d^2)\,(25-d^2) = 504$

$\Rightarrow \quad 4d^4 - 125d^2 + 121 = 0 \Rightarrow d^2 = 1, \dfrac{121}{4} \Rightarrow d = \pm 1, \pm\dfrac{11}{2}$

$d = \pm 1$ is rejected because none of the term can be $\dfrac{-1}{2}$

$d = \pm\frac{11}{2}$ is acceptable

Required AP $is - 6, 5, \frac{21}{2}, 16$hence Largest term is 16.

Hence, the correct answer is (C).

3. Sum of the 40 terms of $3 + 4 + 8 + 9 + 13 + 14 + 18 + 19\ldots$
$= (3 + 8 + 13 + \ldots \text{upto 20 terms}) + [4 + 9 + 15 + \ldots \text{ upto 20 terms}]$
$= 10[\{6 + 19 \times 5\} + \{8 + 19 \times 5\}] = 10 \times 204 = 20 \times 102$
Hence, the correct answer is (A).

4. $a_1 + a_2 = 4 \Rightarrow r^2a_1 + r^2a_2 = 16$
$\Rightarrow \quad r^2 = 4 \Rightarrow r = -2$ as $a_1 < 0$
and $a_1 + a_2 = 4$
$a_1 + a_1(-2) = 4 \Rightarrow a_1 = -4$

$$\Rightarrow \quad 4\lambda = (-4)\left(\frac{(-2)^9 - 1}{-2 - 1}\right) = (-4) \times \frac{513}{3} \Rightarrow \lambda = -171$$

Hence, the correct answer is (D).

5. $$f(x) = \frac{2(2^x + 2^{-x}) + (3^x + 3^{-x})}{2} \geq 3$$

$(A.M \geq G.M)$

Hence, the correct answer is (B).

6. $T_{10} = \frac{1}{20} = a + 9d$ …(1)

and $T_{20} = \frac{1}{10} = a + 19d$ …(2)

$a = \frac{1}{200} = d$

Hence, $S_{200} = \frac{200}{2}\left[\frac{2}{200} + \frac{199}{200}\right] = \frac{201}{2}$

Hence, the correct answer is (B).

7. $2^{\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots.} = 2^{\frac{1}{2}} = \sqrt{2}$

Hence, the correct answer is (B).

8. $$x = \sum_{n=0}^{\infty}(-1)^n \tan^{2n}\theta = 1 - \tan^2\theta + \ldots \quad \Rightarrow \quad x = \cos^2\theta$$

$$y = \sum_{n=0}^{\infty}\cos^{2n}\theta \quad \Rightarrow \quad y = 1 + \cos^2\theta + \cos^4\theta + \ldots$$

$$\Rightarrow \quad y = \frac{1}{\sin^2\theta} \quad \Rightarrow \quad y = \frac{1}{1 - x} \quad \Rightarrow \quad y(1 - x) = 1$$

Hence, the correct answer is (C).

9. $$\sum_{n=1}^{100} a_{2n+1} = 200 \Rightarrow a_3 + a_5 + a_7 + \ldots + a_{201} = 200$$

$$\Rightarrow \quad ar^2\frac{(r^{200} - 1)}{(r^2 - 1)} = 200$$

$$\sum_{n=1}^{100} a_{2n} = 100 \Rightarrow a_2 + a_4 + a_6 + \ldots + a_{200} = 100$$

$$\Rightarrow \quad ar\frac{(r^{200} - 1)}{(r^2 - 1)} = 100$$

On dividing $r = 2$
on adding $a_2 + a_3 + a_4 + a_5 + \ldots + a_{200} + a_{201} = 300$

$$\Rightarrow \quad r(a_1 + a_2 + a_3 + \ldots + a_{200}) = 300 \Rightarrow \sum_{n=1}^{200} a_n = 150$$

Hence, the correct answer is (D).

10. $a_1 + a_2 + a_3 + \ldots + a_{11} = 0 \Rightarrow (a_1 + a_{11}) \times \frac{11}{2} = 0$

$\Rightarrow \quad a_1 + a_{11} = 0 \Rightarrow a_1 + a_1 + 10d = 0$ where d is common difference
$\Rightarrow \quad a_1 = -5d$
$a_1 + a_3 + a_5 + \ldots + a_{23} = (a_1 + a_{23}) \times \frac{12}{2}$

$$= (a_1 + a_1 + 22d)6 = \left(2a_1 + 22\left(\frac{-a_1}{5}\right)\right)6 = -\frac{72}{5}a_1 = ka_1$$

$\Rightarrow \quad k = \frac{-72}{5}$

Hence, the correct answer is (B).

11. $S = (x + ka + 0) + (x^2 + ka + 2a) + (x^3 + ka + 4a) + (x^4 + ka + 6a) + \ldots$ 9 terms
$\Rightarrow \quad S = (x + x^2 + x^3 + x^4 + \ldots \text{ 9 terms}) + (ka + ka + ka + ka + \ldots \text{ 9 terms}) + (0 + 2a + 4a + 6a + \ldots \text{ 9 terms})$

$$\Rightarrow \quad S = x\left[\frac{x^9 - 1}{x - 1}\right] + 9ka + 72a = \frac{(x^{10} - x) + (9k + 72)a(x - 1)}{(x - 1)}$$

Compare with given sum, then we get, $(9k + 72) = 45$
$\Rightarrow \quad k = -3$
Hence, the correct answer is (C).

12. $|x| < 1, |y| < 1, x \neq y$
$(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$
By multiplying and dividing $x - y$

$$\frac{(x^2 - y^2) + (x^3 - y^3) + (x^4 - y^4) + \ldots}{x - y} =$$

$$\frac{(x^2 - x^3 + x^4 + \ldots) - (y^2 + y^3 + y^4 + \ldots)}{x - y}$$

$$= \frac{\frac{x^2}{1 - x} - \frac{y^2}{1 - y}}{x - y} = \frac{(x^2 - y^2) - xy(x - y)}{(1 - x)(1 - y)(x - y)} = \frac{x + y - xy}{(1 - x)(1 - y)}$$

Hence, the correct answer is (A).

13. Let three terms of G.P. are $\frac{a}{r}, a, ar$

product $= 27\ \frac{a}{r}.a.ar \quad \Rightarrow \quad a^3 = 27 \quad \Rightarrow \quad a = 3$

$$S = \frac{3}{r} + 3r + 3$$

For $r > 0$

$$\frac{\frac{3}{r}+3r}{2}\geq\sqrt{3^2}\quad\text{(By A.M.} \geq \text{G.M.)}$$

$$\Rightarrow\quad \frac{3}{r}+3r\geq 6 \qquad \ldots(1)$$

For $r<0$

$$\frac{3}{r}+3r\leq -6 \qquad \ldots(2)$$

From (1) and (2)

$S\in(-\infty-3]\cup[9,\infty)$

Hence, the correct answer is (D).

14. Sum of 1st 25 terms = sum of its next 15 terms

$\Rightarrow\quad (T_1+\ldots+T_{25})=(T_{26}+\ldots+T_{40})$

$\Rightarrow\quad (T_1+\ldots+T_{40})=2(T_1+\ldots+T_{25})$

$$\Rightarrow\quad \frac{40}{2}[2\times 3+(39d)]=2\times\frac{25}{2}[2\times 3+24d]\quad\Rightarrow\quad d=\frac{1}{6}$$

Hence, the correct answer is (D).

15. $a_n=a_1+(n-1)d\quad\Rightarrow\quad 300=1+(n-1)d$

$\Rightarrow\quad (n-1)d=299=13\times 23$

since, $n\in[15,50]$

$\therefore\quad n=24$ and $d=13$

$a_{n-4}=a_{20}=1+19\times 13=248\quad\Rightarrow\quad a_{n-4}=248$

$$S_{n-4}=\frac{20}{2}\{1+248\}=2490$$

Hence, the correct answer is (C).

16. Using A.M. ≥ G.M.

$$\Rightarrow\quad \frac{2^{\sin x}+2^{\cos x}}{2}\geq\sqrt{2^{\sin x}\cdot 2^{\cos x}}$$

$$\Rightarrow\quad 2^{\sin x}+2^{\cos x}\geq 2^{1+\left(\frac{\sin x+\cos x}{2}\right)}$$

$$\Rightarrow\quad \min(2^{\sin x}+2^{\cos x})=2^{1-\frac{1}{\sqrt{2}}}$$

Hence, the correct answer is (A).

17. $S=\frac{100}{5}+\frac{98}{5}+\frac{96}{5}+\frac{94}{5}+\ldots n$

$$S_n=\frac{n}{2}\left(2\times\frac{100}{5}+(n-1)\left(-\frac{2}{5}\right)\right)=488$$

$\Rightarrow\quad n(100-n+1)=488\times 5$

$\Rightarrow\quad n^2-101n+488\times 5=0$

$\Rightarrow\quad n=61$(accepted), 40(rejected)

$$T_n=a+(n-1)d=\frac{100}{5}-\frac{2}{5}\times 60=20-24=-4$$

Hence, the correct answer is (C).

18. $x^2-3x+p=0$ has roots α,β

$\alpha,\beta,\gamma,\delta$ in G.P.

$$\alpha+ar=3 \qquad \ldots(1)$$

$x^2-6x+q=0$ has roots γ,δ

$$\alpha r^2+\alpha r^3=6 \qquad \ldots(2)$$

Ratio of relation (2) and (1)

$r^2=2$

$$\text{So, }\frac{2q+p}{2q-p}=\frac{2r^5+r}{2r^5-r}=\frac{2r^4+1}{2r^4-1}=\frac{9}{7}$$

Hence, the correct answer is (C).

19. $1+(1-2^2.1)+(1-4^2.3)+\ldots+(1-20^2.19$

$=11-(2^2.1+4^2.3+\ldots+20^2.19)$

$$=11-2^2.\sum_{r=1}^{10}r^2(2r-1)$$

$$=11-4\left(\frac{110^2}{2}-35\times 11\right)=11-220(103)=\alpha-220\beta,$$

$\Rightarrow\quad \alpha=11,\ \beta=103$

Hence, the correct answer is (B).

20. $460=\log_7 x\cdot(2+3+4+\ldots+20+21)$

$$\Rightarrow\quad 460=\log_7 x\cdot\left(\frac{21\times 22}{2}-1\right)$$

$\Rightarrow\quad 460=230\cdot\log_7 x$

$\Rightarrow\quad \log_7 x=2\quad\Rightarrow\quad x=49$

Hence, the correct answer is (D).

21. Let first term $=a>0$ and Common ratio $=r>0$

$$ar+ar^2+ar^3=3 \qquad \ldots(1)$$

$$ar^5+ar^6+ar^7=243 \qquad \ldots(2)$$

$r^4(ar+ar^2+ar^3)=243\quad\Rightarrow\quad r^4(3)=243$

$\Rightarrow\quad r=3$ as $r>0$

from (1)

$$3a+9a+27a=3\quad\Rightarrow\quad a=\frac{1}{13}$$

$$S_{50}=\frac{a(r^{50}-1)}{(r-1)}=\frac{1}{26}(3^{50}-1)$$

Hence, the correct answer is (B).

22. Given that

$3^{4-2\sin 2}\alpha+3^{2\sin 2}\alpha^{-1}=28$ Let $3^{2\sin 2}\alpha=t$

$$\frac{81}{t}+\frac{t}{3}=28\quad\Rightarrow\quad t=81,3$$

$3^{2\sin 2}\alpha=3^1,3^4\quad\Rightarrow\quad 2\sin 2\alpha=1,4$

$$\Rightarrow\quad \sin 2\alpha=\frac{1}{2}\text{ or }\sin 2\alpha=2\text{ (rejected)}$$

First term $a=3^{2\sin 2}\alpha^{-1}=1$

Second term $=14$

$\therefore$ common difference $d=13$

$T_6=a+5d=1+5\times 13=66$

Hence, the correct answer is (A).

23. $a=2^{10};\ r=\frac{3}{2};\ n=11\qquad$ (G.P.)

$$S'=(2^{10})\frac{\left(\left(\frac{3}{2}\right)^{11}-1\right)}{\frac{3}{2}-1}=2^{11}\left(\frac{3^{11}}{2^{11}}-1\right)$$

$\Rightarrow\quad S'=3^{11}-2^{11}=S-2^{11}\quad$ (Given)

$\therefore\quad S=3^{11}$

Hence, the correct answer is (C).

24. $a_1\, a_2, \ldots, a_n$ (let common difference $= d$)

$b_1\, b_2, \ldots, b_n$ (let common difference $= d + 2$)

$$a_{40} = a + 39d = -159 \qquad \ldots(1)$$

$$a_{100} = a + 99d = -399 \qquad \ldots(2)$$

Subtract : $60d = -240 \;\Rightarrow\; d = -4$

using equation (1)

$a + 39(-4) = -159 \;\Rightarrow\; a = 156 - 159 = -3$

$a_{70} = a + 69d = -3 + 69(-4) = -279 = b_{100}$

$b_1 + 99(d + 2) = -279 \;\Rightarrow\; b_1 - 198 = -279 \;\Rightarrow\; b_1 = -81$

Hence, the correct answer is (B).

25. $(a^2 + b^2 + c^2)p^2 + 2(ab + bc + cd)p + b^2 + c^2 + d^2 = 0$

$\Rightarrow\; (a^2p^2 + 2abp + b^2) + (b^2p^2 + 2bcp + c^2) + (c^2p^2 + 2cdp + d^2) = 0$

$\Rightarrow\; (ap + b)^2 + (bp + c)^2 + (cp + d)^2 = 0$

This is possible only when

$ap + b = 0$ and $bp + c = 0$ and $cp + d = 0$

$$p = -\frac{b}{a} = -\frac{c}{b} = -\frac{d}{c} \quad \text{or} \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$\therefore\;$ a, b, c, d are in G.P.

Hence, the correct answer is (C).

26. $$\sum_{k=1}^{20} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{20} \frac{k(k+1)(k+2) - (k-1)k(k+1)}{3}$$

$$= \frac{1}{6} \times 20 \times 21 \times 22 = 1540.00$$

Hence, the correct answer is (1540).

27. $$\sum_{n=1}^{7} \frac{n(n+1)(2n+1)}{4} = \frac{1}{4}\left(\sum_{n=1}^{7} 2n^3 + \sum_{n=1}^{7} 3n^2 + \sum_{n=1}^{7} n\right)$$

$$= \frac{1}{4}\left(2\left(\frac{7\times 8}{2}\right)^2 + 3\left(\frac{7\times 8\times 15}{6}\right) + \frac{7\times 8}{2}\right) = 504$$

Hence, the correct answer is (504).

28. Common term are : 23, 51, 79, … T_n

$T_n \le 407$

$\Rightarrow\; 23 + (n - 1)28 \le 407$

$\Rightarrow\; n \le 14.71$

$n = 14$

Hence, the correct answer is (14).

29. 3, A_1, A_2 … A_m, 243 are in A.P. then

$$d = \frac{243 - 3}{m + 1} = \frac{240}{m + 1}$$

Now 3, G_1, G_2, G_3, 243 are in G.P. then $r = \left(\dfrac{243}{3}\right)^{\frac{1}{3+1}} = 3$

$\therefore\; A_4 = G_2 \;\Rightarrow\; a + 4d = ar^2$

$\Rightarrow\; 3 + 4\left(\dfrac{240}{m+1}\right) = 3(3)^2 \;\Rightarrow\; m = 39$

Hence, the correct answer is (39).

30. Let 3 distinct numbers are a, ar, ar^2

given $a + b + c = xb \;\Rightarrow\; a + ar + ar^2 = xar$

$\Rightarrow\; 1 + r + r^2 = xr \;\Rightarrow\; r^2 + r(1 - x) + 1 = 0$

$D \ge 0$

$(1 - x)^2 - 4 \ge 0 \;\Rightarrow\; ((1 - x) - 2)\,(1 - x + 2) \ge 0$

$\Rightarrow\; (x + 1)\,(x - 3) \ge 0 \;\Rightarrow\; x \le -1$ or $x \ge 3$

Hence, the correct answer is (B).

31. $S = a_1 + a_2 + \ldots a_{30} \;\Rightarrow\; S = \dfrac{30}{2}(a_1 + a_{30})$

$T = a_1 + a_3 + a_5 + \ldots a_{29} \;\Rightarrow\; T = \dfrac{15}{2}[a_1 + a_{29}]$

$\Rightarrow\; 2T = 15(a_1 + a_{29})$

$S - 27 = 75$

$15(a_1 + a_{30}) - 15(a_1 + a_{29}) = 75$

$a_{30} - a_{29} = 5$

$d = 5$

$a_5 = 27$

$a_1 + 4d = 27 \;\Rightarrow\; a_1 + 20 = 27 \;\Rightarrow\; a_1 = 7$

$a_{10} = a_1 + 9d = 7 + 9(5) = 7 + 45 = 52$

Hence, the correct answer is (C).

32. $$1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + \ldots + 5^2)}{11} + \ldots \text{ 15 terms}$$

$$= \frac{3(1)^2}{3} + \frac{6(1^2 + 2^2)}{5} + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12}{9}(1^2 + 2^2 + 3^2 + 4^2) + \ldots$$

$$T_r = \frac{3r}{2r + 1}(1^2 + 2^2 + \ldots + r^2)$$

$$= \frac{3r}{2r + 1}\,\frac{r(r + 1)(2r + 1)}{6} = \frac{1}{2}r^2\,(r + 1)$$

$$\text{Sum of } n \text{ terms} = \sum_{r=1}^{n} T_r = \frac{1}{2}\sum_{r=1}^{n}(r^3 + r^2)$$

$$= \frac{1}{2}\left[\frac{n^2(n + 1)^2}{4} + \frac{n(n + 1)(2n + 1)}{6}\right] \qquad (\text{put } n = 15\,)$$

$$= \frac{1}{2}\left(\frac{(15\times 16)^2}{4} + \frac{15\times 16\times 31}{6}\right) = 7820$$

Hence, the correct answer is (C).

33. $a = A + 6d, \quad b = A + 10d, \quad c = A + 12d$

a, b, c are in G.P.

$\Rightarrow\; (A + 10d)^2 = (A + 6d)\,(a + 12d)$

$\Rightarrow\; \dfrac{A}{d} = -14$

$$\frac{a}{c} = \frac{A + 6d}{A + 12d} = \frac{6 + \frac{A}{d}}{12 + \frac{A}{d}} = \frac{6 - 14}{12 - 14} = 4$$

Hence, the correct answer is (B).

34. Two digit numbers of the form $7\lambda + 2$ are 16, 23 …, 93

Two digit numbers of the form $7\lambda + 5$ are 12, 19, …, 96

Sum of all the above numbers equals to $\frac{12}{2}(16+93) = 654 + 702 = 1356$

Hence, the correct answer is (D).

35. $\frac{a}{1-r} = 3;\ \frac{a^3}{1-r^3} = \frac{27}{19}$

$\Rightarrow\ \frac{27(1-r)^3}{1-r^3} = \frac{27}{19}\ \Rightarrow\ 6r^2 - 13r + 6 = 0$

$\Rightarrow\ r = 2/3$ as $|r| < 1$

Hence, the correct answer is (C).

36. $\frac{a_3}{a_1} = \frac{a_1r^2}{a_1} = r^2\ \Rightarrow\ r^2 = 25$

Now $\frac{a_9}{a_5} = \frac{a_1r^8}{a_1r^4} = r^4 = (25)^2 = 5^4$

Hence, the correct answer is (A).

37. $a + 18d = 0 \Rightarrow a = -18d$

$\frac{t_{49}}{t_{29}} = \frac{a+48d}{a+28d} = \frac{-18d+48d}{-18d+28d} = \frac{30d}{10d} = 3$

Hence, the correct answer is (C).

38. $D = (1+\sin\theta\cos\theta)^2 - 4\sin\theta\cos\theta = (1-\sin\theta\cos\theta)^2$

roots are $\beta = \operatorname{cosec}\theta$ and $\alpha = \cos\theta$

$$\sum_{n=0}^{\infty}\left\{\alpha^n + \left(-\frac{1}{\beta}\right)^n\right\} = \sum_{n=0}^{\infty}(\cos\theta)^n + \sum_{n=0}^{\infty}(-\sin\theta)^n$$

$$= \frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$$

Hence, the correct answer is (B).

39. $$\frac{x^my^n}{(1+x^{2m})(1+y^{2n})} = \frac{x^my^n}{x^m\left(\frac{1}{x^m}+x^m\right)y^n\left(\frac{1}{y^n}+y^n\right)}$$

$$= \frac{1}{\left(\frac{1}{x^m}+x^m\right)\left(\frac{1}{y^n}+y^n\right)}$$

maximum value $= \frac{1}{\left(\frac{1}{x^m}+x^m\right)\left(\frac{1}{y^n}+y^n\right)} = \frac{1}{(2)(2)} = \frac{1}{4}$

Hence, the correct answer is (C).

40. Let the numbers be $\frac{a}{r}, a, ar$

Given $a^3 = 512\ \Rightarrow\ a = 8$

Now given $\frac{8}{r} + 4, 12, 8r$ are in A.P.

$\Rightarrow\ 2r^2 - 5r + 2 = 0\ \Rightarrow\ r = \frac{1}{2}$ or 2

Numbers are 4, 8, 16 or 16, 8, 4

Sum of numbers = 4 + 8 + 16 = 28

Hence, the correct answer is (D).

41. $S_k = \frac{k+1}{2}$

$$\sum_{k=1}^{10}\left(\frac{k+1}{2}\right)^2 = \frac{5}{12}A$$

$\Rightarrow 2^2 + 3^2 + \ldots 11^2 = \frac{5A}{3}\ \Rightarrow\ \frac{11\times12\times23}{6} - 1 = \frac{5A}{3}$

$\Rightarrow\ 505 \times \frac{3}{5} = A \Rightarrow A = 303$

Hence, the correct answer is (C).

42. $S = \left(\frac{3}{4}\right)^3 + \left(\frac{6}{4}\right)^3 + \left(\frac{9}{4}\right)^3 + \left(\frac{12}{4}\right)^3 + \ldots$ 15 terms

$$= \frac{27}{64}\sum_{r=1}^{15} r^3 = \frac{27}{64}.\left[\frac{15(15+1)}{2}\right]^2$$

$= 225\ K$ (Given in question)

$K = 27$

Hence, the correct answer is (B).

43. S_A = sum of numbers between 100 and 200 which are divisible by 7.

$\Rightarrow\ S_A = 105 + 112 + \ldots + 196$

$S_A = \frac{14}{2}[105 + 196] = 2107$

S_B = Sum of numbers between 100 and 200 which are divisible by 13.

$S_B = 104 + 117 + \ldots + 195 = \frac{8}{2}[104 + 195] = 1196$

S_C = Sum of numbers between 100 and 200 which are divisible by 7 and 13.

$S_C = 182$

$\Rightarrow$ H.C.F. $(91, n) > 1 = S_A + S_B - S_C = 3121$

Hence, the correct answer is (D).

44. $S = \sum_{k=1}^{20}\frac{k}{2^k}$

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{20}{2^{20}}$$

$$\frac{S}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \ldots + \frac{19}{2^{20}}. + \frac{20}{2^{21}}$$

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{20}} - \frac{20}{2^{21}}$$

$$\frac{S}{2} = \frac{\frac{1}{2}\left(1 - \frac{1}{2^{20}}\right)}{\frac{1}{2}} - \frac{20}{2^{21}}$$

$\Rightarrow\ S = 1 - \frac{11}{2^{19}}$

Hence, the correct answer is (A).

45. $b^2 = ac$

Also roots of $ax^2 + 2bx + c = 0$ are equal then $x = \frac{-b}{a}$ is a common root

$$\Rightarrow \quad d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$

$$\Rightarrow \quad db^2 - 2aeb + fa^2 = 0,\ b^2 = ac \quad \Rightarrow \quad dac - 2eab + fa^2 = 0$$

$$\Rightarrow \quad dc - 7eb + fa = 0 \Rightarrow \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0$$

$$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = 2.\frac{e}{b}$$

Hence $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

Hence, the correct answer is (A).

46. $S_n = 50n + \frac{n(n-7)}{2}A$

$T_n = S_n - S_{n-1} = 50n + \frac{n(n-7)}{2}A - 50(n-1) - \frac{(n-1)(n-8)}{2}A = 50 + A(n-4)$

$d = T_n - T_{n-1} = 50 + A(n-4) - 50 - A(n-5) = A$

$T_{50} = 50 + 46A$

$(d_1\ A_{50}) = (A, 50 + 46A)$

Hence, the correct answer is (A).

47. $\frac{n(n+1)}{2} + 99 = (n-2)^2$

$$\Rightarrow \quad n^2 + n + 198 = 2n^2 - 8n + 8 \quad \Rightarrow \quad n^2 - 9n - 190 = 0$$

$$\Rightarrow \quad (n-19)(n+10) = 0 \quad \Rightarrow \quad n = 19$$

$\therefore$ Number of balls is $\frac{19 \times 20}{2} = 190$

Hence, the correct answer is (A).

48. Let the three numbers in A.P. are $a - d, a, a + d$

Given that $a - d + a + a + d = 33 \quad \Rightarrow \quad a = 11$

$$(a-d)a(a+d) = 1155$$

$$\Rightarrow \quad a(a^2 - d^2) = 1155$$

$$\Rightarrow \quad 11(121 - d^2) = 1155 \quad \Rightarrow \quad d^2 = 16 \quad \Rightarrow \quad d = \pm 4$$

If $d = 4$ then first term $a - d = 7$, $T_{11} = 7 + 10(4) = 47$

If $d = -4$ then first term $a - d = 15$, $T_{11} = 15 + 10(-4) = -25$

Hence, the correct answer is (A).

49. $S = 1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + \ldots +$ upto 11 terms

n^{th} term of the series is $T_n = n(2n-1)$

$$\Rightarrow \quad S = \sum_{n=1}^{11} T_n = \sum_{n=1}^{11} (2n^2 - n)$$

$$\Rightarrow \quad S_n = \frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \qquad \text{Put } n = 11$$

$$\Rightarrow \quad S_{11} = \frac{2(11)(12)(23)}{6} - \frac{11(12)}{2} \quad \Rightarrow \quad S_{11} = 946$$

Hence, the correct answer is (B).

50. $T_n = \frac{(2n+1)(1^3 + 2^3 + \ldots + n^3)}{(1 + 2^2 + \ldots + n^2)} = \frac{(2n+1)\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)(2n+1)}{6}}$

$$= \frac{3}{2}n(n+1)$$

$$S_n = \sum T_n = \frac{3}{2}\sum n(n+1) = \frac{1}{2}\sum n(n+1)[(n+2) - (n-1)]$$

On solving $S_n = \frac{n(n+1)(n+2)}{2} \quad \Rightarrow \quad S_{10} = 660$

Hence, the correct answer is (B).

51. $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 114$

$$\Rightarrow \quad \frac{6}{2}(a_1 + a_{16}) = 114 \quad \Rightarrow \quad a_1 + a_{16} = 38$$

So, $a_1 + a_6 + a_{11} + a_{16} = \frac{4}{2}(a_1 + a_{16}) = 2 \times 38 = 76$

Hence, the correct answer is (A).

52. Let a is first term and d is common difference then, $a + 5d = 2$ (given) …(1)

$$f(d) = (2 - 5d)(2 - 2d)(2 - d)$$

$$f(d) = -10d^3 + 34d^2 - 32d + 8$$

$$f'(d) = -30d^2 + 68d - 32 \quad \Rightarrow \quad d = \frac{2}{3}, \frac{8}{5}$$

$f''(d) < 0$ at $d = \frac{8}{5}$ hence for maximize product common difference is $d = \frac{8}{5}$

Hence, the correct answer is (B).

53. Sum $= \sum_{n=1}^{15} \frac{1^3 + 2^3 + \ldots n^3}{1 + 2 + \ldots + n} - \frac{1}{2}.\frac{15.16}{2} = \sum_{n=1}^{15} \frac{n(n+1)}{2} - 60$

$$= \sum_{n=1}^{15} \frac{n(n+1)(n+2-(n-1))}{6} - 60$$

$$= \frac{15.16.17}{6} - 60 = 620$$

Hence, the correct answer is (A).

54. $a = a, b = ar, c = ar^2$

$3a, 7b, 15c \longrightarrow$ A.P. we have $14b = 3a + 15(ar^2)$

$15r^2 - 14r + 3 = 0 \quad \Rightarrow \quad r = \frac{1}{3}, \frac{3}{5}$ (rejected)

Common difference $= 7b - 3a = 7ar - 3a = \frac{7a}{3} - 3a = -\frac{2}{3}a$

4^{th} term is $= 15c - \frac{2}{3}a = \frac{15}{9}a - \frac{2}{3}a = a$

Hence, the correct answer is (A).

55. $\alpha + \beta = \frac{25}{375};\ \alpha\beta = \frac{-2}{375}$

α and $\beta \in (-1, 1)$, then $\lim\limits_{n \to \infty} \sum_{r=1}^{n} (\alpha^r + \beta^r)$

$$= (\alpha + \alpha^2 \ldots \text{ infinite }) + (\beta + \beta^2 \ldots \text{ infinite})$$

$$= \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} = \frac{\alpha(1-\beta) + \beta(1-\alpha)}{(1-\alpha)(1-\beta)} = \frac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha+\beta) + \alpha\beta}$$

$$= \frac{25 + 4}{375 - 25 - 2} = \frac{29}{348} = \frac{1}{12}$$

Hence, the correct answer is (C).

56. $S_4 = 2\{2a + 3d\} = 16 \quad \Rightarrow \quad 2a + 3d = 8$

$S_6 = 3(2a + 5d) = -48 \quad \Rightarrow \quad 2a + 5d = -16$

We have $a = 22$, $d = -12$

$S_{10} = 5\{44 - 9 \times 12\} = -320$

Hence, the correct answer is (C).

57. $a_1, a_2, \ldots a_n$ are A.P.

$a_1 + a_7 + a_{16} = 40 \quad \Rightarrow \quad a + a + 6d + a + 15d = 40$

$$\Rightarrow \quad 3a + 21d = 40 \Rightarrow a + 7d = \frac{40}{3}$$

$$S_{15} = \frac{15}{2}[2a + 14d] = 15[a + 7d] = 15 \times \frac{40}{3} = 200$$

Hence, the correct answer is (A).

58. α, β, γ are in G.P.

$\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common roots. Both roots will be common

$$\frac{\alpha}{1} = \frac{2\beta}{1} = \frac{\gamma}{-1} = \lambda$$

$$\alpha = \lambda,\ B = \frac{\lambda}{2},\ \gamma = -\lambda$$

$$\Rightarrow \quad \alpha(\beta + \lambda) = \lambda\left(\frac{\lambda}{2} - \lambda\right) = \frac{-\lambda^2}{2} = \beta\gamma$$

Hence, the correct answer is (D).

59. $$\because \sum_{k=0}^{12} a_{4k+1} = 416 \quad \Rightarrow \quad \frac{13}{2}[2a_1 + 48d] = 416$$

$\Rightarrow \quad a_1 + 24d = 32$ …(1)

Now, $a_9 + a_{43} = 66 \Rightarrow 2a_1 + 50d = 66$ …(2)

From eq. (1) and (2) we get; $d = 1$ and $a_1 = 8$

$$\text{Also,} \quad \sum_{k=0}^{17} a_r^2 = \sum_{r=1}^{17} [8 + (r-1)1]^2 = 140\ m$$

$$\Rightarrow \quad \sum_{k=1}^{17} (r+7)^2 = 140\ m$$

$$\Rightarrow \quad \sum_{k=1}^{17} (r^2 + 14r + 49) = 140\ m$$

$$\Rightarrow \quad \left(\frac{17 \times 18 \times 35}{6}\right) + 14\left(\frac{17 \times 18}{2}\right) + (49 \times 17) = 140$$

$\Rightarrow \quad m = 34$

Hence, the correct answer is (B).

60. $$\text{Here, } B - 2A = \sum_{n=1}^{40} a_n - 2\sum_{n=1}^{20} a_n = \sum_{n=21}^{40} a_n - 2\sum_{n=1}^{20} a_n$$

$$B - 2A = (2l^2 + 2.22^2 + 23^2 + 2.24^4 + \ldots + 40^2) - (1^2 + 2.2^2 + 3^2 + 2.4^2 \ldots + 20^2)$$

$$= 20[22 + 2.24 + 26 + 2.28 + \ldots + 60]$$

$$= 20\left[\underbrace{(22 + 24 + 26 \ldots 60)}_{\text{20 terms}} + \underbrace{(24 + 28 + \ldots + 60)}_{\text{10 terms}}\right]$$

$$= 20\left[\frac{20}{2}(22 + 60) + \frac{10}{2}(24 + 60)\right]$$

$$= 10[20.82 + 10.84] = 100[164 + 84] = 100.248$$

Hence, the correct answer is (A).

61. We have

$$9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$$

$$\Rightarrow \quad 225a^2 + 9b^2 + 25c^2 - 75ac = 45ab + 15bc$$

$$\Rightarrow \quad (15a)^2 + (3b)^2 + (5c)^2 - 75\,ac - 45ab - 15\,bc = 0$$

$$\Rightarrow \quad \frac{1}{2}[(15a - 3b)^2 + (3b - 5c)^2 + (5 - 15a)^2] = 0$$

It is possible when $15a - 3b = 0$, $3b - 5\,c = 0$ and $5c - 15a = 0$

$$\Rightarrow \quad = 15a = 3b = 5c \quad \Rightarrow \quad b = \frac{5c}{3}, a = \frac{c}{3}$$

$$\Rightarrow \quad a + b = \frac{c}{3} + \frac{5c}{3} = \frac{6c}{3} \quad \Rightarrow \quad a + b = 2c$$

$\Rightarrow \quad b, c, a$ are in A.P.

Hence, the correct answer is (C).

62. Let the GP be a are and ar^2 then $a = A + d$; $ar = A + 4d$; $ar^2 = A + 8d$

$$\Rightarrow \quad \frac{ar^2 - ar}{ar - a} = \frac{(A + 8d) - (A + d)}{(A + 4d) - (A + d)}$$

$$r = \frac{4}{3}$$

Hence, the correct answer is (D).

63. $$\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 \ldots + \left(\frac{44}{5}\right)^2$$

$$S = \frac{16}{25}(2^2 + 3^2 + 4^2 + \ldots + 11^2)$$

$$= \frac{16}{25}\left(\frac{11(11+1)(22+1)}{6} - 1\right)$$

$$= \frac{16}{25} \times 505 = \frac{16}{5} \times 101$$

$$\Rightarrow \quad \frac{16}{5} m = \frac{16}{5} \times 101$$

$\Rightarrow \quad m = 101.$

Hence, the correct answer is (D).

64. $$n^{\text{th}} \text{ term of series} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{n^2} = \frac{1}{4}(n+1)^2$$

$$\text{Sum of } n \text{ term} = \Sigma \frac{1}{4}(n+1)^2$$

$$= \frac{1}{4}\left[\Sigma n^2 + 2\Sigma n + n\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n\right]$$

Sum of 9 terms

$$= \frac{1}{4}\left[\frac{9 \times 10 \times 19}{6} + \frac{18 \times 10}{2} + 9\right] = \frac{384}{4} = 96$$

Hence, the correct answer is (D).

65. $M = \dfrac{l+n}{2}$ and common ration of G.P. $= r = \left(\dfrac{n}{l}\right)^{\frac{1}{4}}$

$\therefore \quad G_1 = l^{3/4}, G_2 = l^{1/2}n^{1/2}, G_3 = l^{1/4}n^{3/4}$

$$G_4^4 + 2G_2^4 + G_3^4 = l^3n + 2l^2n^2 + \ln^3$$
$$= ln\ (l+n)^2$$
$$= ln \times (2m)^2$$
$$= 4lm^2n$$

Hence, the correct answer is (D).

66. Let $10^9 + 2.\ (11)(10)^8 + 3(11)^2\ (10)^7 + \ldots + 10(11)^9 = k(10)^9$

Let $x = 10^9 + 2.(11)(10)^8 + 3(11)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9$

Multiplied by $\dfrac{11}{10}$ on both the sides

$$\frac{11}{10}x = 11.10^8 + 2(11)^2.(10)^7 + \ldots + 9(11)^9 + 11^{10}$$

$$x\left(1 - \frac{11}{10}\right) = 10^9 + 11(10)^8 + 11^2 \times (10)^7 + \ldots\ldots + 11^9 - 11^{10}$$

$$\Rightarrow \quad -\frac{x}{10} = 10^9\left[\frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1}\right] - 11^{10}$$

$$\Rightarrow \quad -\frac{x}{10} = (11^{10} - 10^{10}) - 11^{10} = -10^{10}$$

$\Rightarrow \quad x = 10^{11} = k.\ 10^9$ Given $\Rightarrow \quad k = 100$

Hence, the correct answer is (A).

67. Let a, ar, ar^2 are in G.P.

According to the question $a, 2ar, ar^2$ are in A.P.

$\Rightarrow \quad 2 \times 2ar = a + ar^2$

$\Rightarrow \quad 4r = 1 + r^2 \quad \Rightarrow \quad r^2 - 4r + 1 = 0$

$$r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Since $r > 1$

$\therefore \quad r = 2 - \sqrt{3}$ is rejected

Hence, $r = 2 + \sqrt{3}$

Hence, the correct answer is (B).

68. Given sequence can be written as

$$\frac{7}{10} + \frac{77}{100} + \frac{111}{10^3} + \ldots + \text{up to 20 terms}$$

$$= 7\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{10^3} + \ldots + \text{up to 20 terms}\right]$$

Multiple and divide by 9

$$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots + \text{up to 20 terms}\right]$$

$$= \frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots\text{upto 20 terms}\right]$$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left(1 - \left(\frac{1}{10}\right)^{20}\right)}{1 - \frac{1}{10}}\right]$$

$$= \frac{7}{9}\left[\frac{179}{9} + \frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}[179 + (10)^{-20}]$$

Hence, the correct answer is (C).

69. n^{th} term of the given series

$$= T_n = (n-1)^2 + (n-1)\ n + n^2$$

$$= \frac{(n-1)^3 - n^3}{(n-1) - n} = n^3 - (n-1)^3$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n}[k^3 - (k-1)^3] \quad \Rightarrow \quad 8000 = n^3$$

$\Rightarrow \quad n = 20$ which is a natural number.

Now, put $n = 1, 2,3,\ldots 20$

$T_1 = 1^3 - 0^3$

$T_2 = 2^3 - 1^3$

|

$T_{20} = 20^3 - 19^3$

Now, $T_1 + T_2 + \ldots + T_{20} = S_{20}$

$\Rightarrow \quad S_{20} = 20^3 - 0^3 = 8000$

Hence, both the given statements are true and statement 2 supports statement 1.

Hence, the correct answer is (B).

70. Till 10th minute number of counted notes = 1500

$$3000 = \frac{n}{2}\ [2 \times 148 + (n-1)\ (-2)] = n\ [148 - n + 1]$$

$$n^2 - 147\ n + 3000 = 0$$

$\Rightarrow \quad n = 125, 24$

But $n = 125$ is not possible

$\therefore \quad$ total time = 24 + 10 = 34 minutes.

Hence, the correct answer is (A).

71. Let required number of months $= n$

$\therefore \quad 200 \times 3 + (240 + 280 + 320) + \ldots + (n-3)^{\text{th}}$ term$) = 11040$

$\Rightarrow \quad \dfrac{n-3}{2}[2 \times 240 + (n-4) \times 40] = 11040 - 600$

$\Rightarrow \quad (n-3)\ [240 + 20n - 80] = 10440$

$\Rightarrow \quad (n-3)\ (20n + 160) = 10440$

$\Rightarrow \quad (n-3)\ (n + 8) = 522$

$\Rightarrow \quad n^2 + 5n - 546 = 0 \Rightarrow (n + 26)\ (n - 21) = 0$

$\therefore \quad n = 21$

Hence, the correct answer is (C).

72. We have

$$S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots\infty \qquad \ldots(1)$$

Multiplying both sides by $\dfrac{1}{3}$ we get

$$\frac{1}{3}S=\frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\ldots\infty \qquad \ldots(2)$$

Subtracting equation (2) from equation (1) we get

$$\frac{2}{3}S=1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots\infty$$

$$\Rightarrow \quad \frac{2}{3}S=\frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots\infty$$

$$\Rightarrow \quad \frac{2}{3}S=\frac{\frac{4}{3}}{1-\frac{1}{3}}+\frac{4}{3}\times\frac{3}{2}\Rightarrow S=3$$

Hence, the correct answer is (A).

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Single Correct Choice Type Problems

1. $\log_c b_1, \log_c b_2, \ldots, \log_c b_{101}$ are in A.P. with common difference ln 2

$b_1, b_2, b_3, \ldots, b_{101}$ are in G.P. with common ratio 2

Also $a_1, a_2, \ldots a_{101}$ are in A.P.

Where $a_1 = b_1$ are $a_{51} = b_{51}$.

$\therefore$ $b_2, b_3, \ldots, b_{50}$ are GM's and $a_2, a_3, \ldots a_{50}$ are AM's between b_1 and b_{51}.

$\because$ GM < AM $\Rightarrow$ $b_2 < a_2, b_3 < a_3, \ldots b_{50} < a_{50}$

$\because$ $b_1 + b_2 + \ldots + b_{51} < a_1 + a_2 + \ldots + a_{51}$

$\Rightarrow$ $t < s$

Also a_1, a_{51}, a_{101} are in AP

b_1, b_{51}, b_{101} are in GP

$\therefore$ $a_1 = b_1$ and $a_{51} = b_{51} \Rightarrow a_1 + 50d = a_1 2^{50}$

$$a_{101} = a_1 + 100d = a_1 + 2(a_1 2^{50} - a_1)$$

$$a_{101} = a_1 2^{51} - a_1 < a_1 2^{51} < a_1 2^{100}$$

$$a_{101} < b_{101}$$

$\therefore$ $b_{101} > a_{101}$

Hence, the correct answer is (B).

2. Given that for an A.P., $S_n = cn^2$

Then $T_n = S_n - S_{n-1} = cn^2 - c(n-1)^2$

$= (2n - 1)c$

$\therefore$ Sum of squares of n terms of this A.P.

$$=\sum T_n^2 = \sum (2n-1)^2.c^2 = c^2\left[4\sum n^2 - 4\sum n + n\right]$$

$$= c^2\left[\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n\right]$$

$$= c^2 n\left[\frac{2(2n^2+3n+1)-6(n+1)+3}{3}\right]$$

$$= c^2 n\left[\frac{4n^2-1}{3}\right]=\frac{n(4n^2-1)c^2}{3}$$

Hence, the correct answer is (C).

3. $\because$ $a_1, a_2, a_3, \ldots$ Are in H.P.

$\therefore$ $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ … are in A.P.

$\therefore$ $\frac{1}{a_1}=\frac{1}{5}$ *and* $\frac{1}{a_{20}}=\frac{1}{25}$

$$\frac{1}{a_1}+19d=\frac{1}{a_{20}} \quad\Rightarrow\quad \frac{1}{5}+19d=\frac{1}{25} \quad\Rightarrow\quad d=\frac{-4}{475}$$

Now $\frac{1}{a_n}=\frac{1}{5}+(n-1)\left(\frac{-4}{475}\right)$

Clearly $a_n < 0$ if $\frac{1}{a_n} < 0 \Rightarrow \frac{1}{5}-\frac{4n}{475}+\frac{4}{475}<0$

$$\Rightarrow \quad -4n < -99 \text{ or } n > \frac{99}{4} = 24\frac{3}{4} \quad \therefore \quad n \geq 25$$

Hence least value of n is 25.

Hence, the correct answer is (D).

Multiple Correct Choice Type Problems

1. Clearly we have $\alpha + \beta = 1$ and $\alpha\beta = -1$, $\alpha >$ b

$$\alpha=\frac{1+\sqrt{5}}{2}, \beta=\frac{1-\sqrt{5}}{2}; \alpha-\beta=\sqrt{5}$$

$$b_n = a_{n-1}+a_{n+1}=\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}+\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

$$=\frac{\alpha^{n-1}\left(1+\alpha^2\right)-\beta^{n-1}\left(1+\beta^2\right)}{\alpha-\beta}$$

$$=\frac{\alpha^{n-1}\left(\frac{5+\sqrt{5}}{2}\right)-\beta^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)}{\alpha-\beta}$$

$$=\frac{\sqrt{5}\left(\alpha^n+\beta^n\right)}{\alpha-\beta} : \text{As } \alpha-\beta=\sqrt{5}$$

$$b_n = \alpha^n + \beta^n$$

$$\text{Now, } \sum_{n=1}^{\infty}\frac{a_n}{10^n}=\sum_{n=1}^{\infty}\frac{\alpha^n-\beta^n}{(\alpha-\beta)10^n}$$

$$=\frac{1}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n-\sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n\right]$$

$$=\frac{\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}}-\frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}}{(\alpha-\beta)}$$

$$=\frac{\alpha(10-\beta)-\beta(10-\alpha)}{(10-\alpha)(10-\beta)(\alpha-\beta)}=\frac{10}{(10-\alpha)(10-\beta)}=\frac{10}{89}$$

$$\text{Let compute } \sum_{n=1}^{\infty}\frac{b_n}{10^n}=\sum_{n=1}^{\infty}\frac{\alpha^n+\beta^n}{10^n}=\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}}+\frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}$$

$$= \frac{\alpha(10-\beta)+\beta(10-\alpha)}{(10-\alpha)(10-\beta)}$$

$$= \frac{10(\alpha+\beta)-2\alpha\beta}{100-(\alpha+\beta)10+\alpha\beta} = \frac{12}{89}$$

Now $a_1+a_2+a_3+\ldots+a_n = \sum_{r=1}^{n} a_r = \sum_{r=1}^{n} \frac{\alpha^r-\beta^r}{\alpha-\beta}$

$$= \frac{\frac{\alpha(1-\alpha^n)}{1-\alpha} - \frac{\beta(1-\beta^n)}{1-\beta}}{\alpha-\beta}$$

$$= \frac{(\alpha-\alpha\beta)(1-\alpha^n)-\beta(1-\alpha)(1-\beta^n)}{(\alpha-\beta)(1-\alpha)(1-\beta)}$$

using $\alpha+\beta=1$ and $\alpha\beta=-1$

$$= \frac{(\alpha+1)(1-\alpha^n)-(\beta+1)(1-\beta^n)}{(\alpha-\beta)(1-(\alpha+\beta)+\alpha\beta)}$$

$$= \frac{(\alpha-\beta)-\alpha^n(1+\alpha)+\beta^n(1+\beta)}{-(\alpha-\beta)}$$

$$= \frac{(\alpha-\beta)-\alpha^{n+2}+\beta^{n+2}}{-(\alpha-\beta)} \qquad (\because 1+x=x^2)$$

$$= -1+\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta} = -1+a_{n+2}$$

Hence, (A), (B) and (C) are correct.

2. $S_n = -1^2-2^2+3^2+4^2-5^2-6^2+\ldots = (3^2+7^2+11^2+\ldots)+(4^2+8^2+12^2+\ldots)-(1^2+5^2+9^2+\ldots)-(2^2+6^2+10^2+\ldots)$

$$= \sum_{r=1}^{n}(4r-1)^2+\sum_{r=1}^{n}(4r)^2-\sum_{r=1}^{n}(4r-3)^2-\sum_{r=1}^{n}(4r-2)^2$$

$$= \left[\sum_{r=1}^{n}(4r-1)^2-(4r-3)^2\right]+4\left[\sum_{r=1}^{n}(2r)^2-(2r-1)^2\right]$$

$$= 8\sum_{r=1}^{n}(2r-1)+4\sum_{r=1}^{n}(4r-1)$$

$$= 8\left[2\frac{n(n+1)}{2}-n\right]+4\left[4\frac{n(n+1)}{2}-n\right]$$

$= 8n^2+8n^2+4n = 16n^2+4n$

For $n=8$, $16n^2+4n=1056$

And for $n=9$, $16n^2+4n=1332$

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. $V_1+V_2+\ldots+V_n = \sum_{r=1}^{n} V_r = \sum_{r=1}^{n}\left(r^3-\frac{r^2}{2}+\frac{r}{2}\right)$

$$= \sum n^3-\frac{\sum n^2}{2}+\frac{\sum n}{2}$$

$$= \frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left[n(n+1)-\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)(3n^2+n+2)}{12}$$

Hence, the correct answer is (B).

2. $T_r = V_{r+1}-V_r-2$

$$= \left[(r+1)^3-\frac{(r+1)^2}{2}+\frac{r+1}{2}\right]-\left[r^3-\frac{r^2}{2}+\frac{r}{2}\right]-2$$

$= 3r^2+2r-1$

$T_r = (r+1)(3r-1)$

For each r, T_r has two different factors other then 1 and itself.

$\therefore$ T_r is always a composite number.

Hence, the correct answer is (D).

3. $Q_{r+1}-Q_r = T_{r+2}-T_{r+1}-(T_{r+1}-T_r)$

$= T_{r+2}-2T_{r+1}+T_r$

$= (r+3)(3r+5)-2(r+2)(3r+2)+(r+1)(3r-1)$

$\because Q_{r+1}-Q_r = 6(r+1)+5-6r-5 = 6$ (constant)

$\therefore$ $Q_1, Q_2, Q_3, \ldots$ Are in A.P. with common difference 6.

Hence, the correct answer is (B).

4. Given $A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab}, H_1 = \frac{2ab}{a+b}$

Also $A_n = \frac{A_{n-1}+H_{n-1}}{2}, G_n = \sqrt{A_{n-1}H_{n-1}}$

$$H_n = \frac{2A_{n-1}H_{n-1}}{A_{n-1}H_{n-1}}$$

$\Rightarrow$ $G_n^2 = A_nH_n \Rightarrow A_nH_n = A_{n-1}H_{n-1}$

Similarly we can prove

$$A_nH_n = A_{n-1}H_{n-1} = A_{n-2}H_{n-2} = \ldots = A_1H_1$$

$\Rightarrow$ $A_nH_n = ab$

$\therefore$ $G_1^2 = G_2^2 = G_3^2 \ldots = ab$

$\Rightarrow$ $G_1 = G_2 = G_3 \ldots = \sqrt{ab}$

Hence, the correct answer is (C).

5. We have $A_n = \frac{A_{n-1}+H_{n-1}}{2}$

$\therefore$ $A_n-A_{n-1} = \frac{A_{n-1}+H_{n-1}}{2}-A_{n-1}$

$$= \frac{H_{n-1}-A_{n-1}}{2} < 0 \qquad (\because A_{n-1} > H_{n-1})$$

$\Rightarrow$ $A_n < A_{n-1}$ or $A_{n-1} > A_n$

$\therefore$ we can conclude that $A_1 > A_2 > A_3 > \ldots$

Hence, the correct answer is (A).

6. WE have $A_nH_n = ab \Rightarrow H_n = \frac{ab}{A_n}$

$\because \frac{1}{A_{n-1}} < \frac{1}{A_n} \Rightarrow H_{n-1} < H_n$ $\quad\therefore$ $H_1 < H_2 < H_3 < \ldots$

Hence, the correct answer is (B).

Reasoning Based Questions

1. Given a_1, a_2, a_3, a_4 are in G.P.

Then b_1, b_2, b_3, b_4, are the numbers

$$a_1, a_1+a_2, a_1, a_1+a_2+a_3, a_1+a_2+a_3+a_4$$

or $\quad a, a+ar, a+ar+ar^2, a+ar+ar^2+ar^3$

Clearly above numbers are neither in AP nor in G.P. and hence statement 1 is true.

Also $\frac{1}{a}, \frac{1}{a+ar}, \frac{1}{a+ar+ar^2}, \frac{1}{a+ar+ar^2+ar^3}$ are not in A.P.

$\therefore \quad b_1, b_2, b_3, b_4$, are not in H.P.

$\therefore \quad$ Statement 2 is false.

Hence, the correct answer is (C).

Integer/Numerical Answer Type Questions

1. As we know that $3^x > 0$

Now using A.M. ≥ G.M.

$$\Rightarrow \frac{3^{y_1}+3^{y_2}+3^{y_3}}{3} \ge \left(3^{y_1}\cdot 3^{y_2}\cdot 3^{y_3}\right)^{\frac{1}{3}}$$

$$\Rightarrow 3^{y_1}+3^{y_2}+3^{y_3} \ge 3^1 \cdot 3^{\frac{y_1+y_2+y_3}{3}}$$

$$\Rightarrow 3^{y_1}+3^{y_2}+3^{y_3} \ge 3^4$$

$$\Rightarrow 3^{y_1}+3^{y_2}+3^{y_3} \ge 3^4$$

$\Rightarrow \log_3(3^{y_1}+3^{y_2}+3^{y_3}) \ge 4$ we get $m = 4$

$$\log_3 x_1 + \log_3 x_2 + \log_3 x_3 = \log_3 x_1 \times x_2 \times x_3$$

$$\text{A.M.} \ge \text{G.M.} \Rightarrow \frac{x_1+x_2+x_3}{3} \ge (x_1x_2x_3)^{\frac{1}{3}}$$

$$\Rightarrow 3 \ge (x_1x_2x_3)^{\frac{1}{3}}$$

$\Rightarrow 27 \ge (x_1x_2x_3) \quad \Rightarrow \quad 3 \ge \log_3(x_1x_2x_3)$ we get $M = 3$

$$\log_2 m^3 + \log_3 M^2 = \log_2 4^3 + \log_3 3^2$$

$$= \log_2(2^2)^3 + 2\log_3 3 = 6 + 2 = 8$$

Hence, the correct answer is (8).

2. $2(a_1+a_2+\ldots+a_n) = b_1+b_2+\ldots+b_n$

$$\Rightarrow 2\left[\frac{n}{2}[2a_1+(n-1)d]\right] = b_1\left(\frac{r^n-1}{r-1}\right)$$

$$\Rightarrow n[2c+(n-1)2] = c\left(\frac{2^n-1}{2-1}\right) = c(2^n-1)$$

$$\Rightarrow 2nc + 2n(n-1) = c2^n - c$$

$$\Rightarrow 2n(n-1) = c[2^n - 2n - 1]$$

$$\Rightarrow c = \frac{2n(n-1)}{2^n-2n-1}$$

$n = 2 \qquad c = -4$

$n = 3 \qquad c = 12$

$n = 4 \qquad c = \frac{24}{7}$

$n = 5 \qquad c = \frac{40}{21}$

$n = 6 \qquad c = \frac{60}{51}$

$n = 7 \qquad c = \frac{84}{113}$

$n = 8, 9, 10, \ldots \qquad c < 1 \quad \text{but } c \in \text{N}$

$\therefore \quad n = 3, c = 12$ is the only accepted solution

Hence, the correct answer is (1).

3. A.P. $(1, 3) = \{1, 4, 7, 10, \ldots.\} = \{n \mid n = 3k+1, k \in \text{W}\}$

A.P. $(2, 5) = \{2, 7, 12, \ldots.\} = \{n \mid n = 5k+2, k \in \text{W}\}$

A.P. $(3, 7) = \{3, 10, 17, \ldots\} = \{n \mid n = 7k+3, k \in \text{W}\}$

First common term is 52,

Common difference is 3.5.7 =105

so $a = 52$, $d = 105$ and $a + d = 157$

Hence, the correct answer is (157).

4. $n(\text{X} \cup \text{Y}) = n(\text{X}) + n(\text{Y}) - n(\text{X} \cap \text{Y})$

1, 6, 11, … 2018 term

For first series $T_n = 1 + (n-1)5 = 5n - 4$

For second series $U_k = 9 + (k-1)7 = 7k + 2$

for common terms

$$5n - 4 = 7k + 2 \Rightarrow n = \frac{7k+6}{5} \le 2018, k \le 1440$$

$k = 2, 7, \ldots, 1440$; let terms are r

$\Rightarrow \quad 2 + (r-1)5 \le 1440 \quad \therefore \; r_{\max} = 288$

$\therefore \quad$ number of common term $= 288 = n(\text{X} \cup \text{Y})$

$$n(\text{X} \cup \text{Y}) = 2018 + 2018 - 288 = 3748.$$

Hence, the correct answer is (3748).

5. Let sides be $a-d, a, a+d, (d > 0)$, as they are AP

They are forming right angle triangle

$$\Rightarrow a^2 + (a-d)^2 = (a+d)^2$$

$$\Rightarrow a = 4d$$

sides are $3d, 4d, 5d$

Triangle area is 24

$$\Rightarrow \frac{1}{2} \times 3d \times 4d = 24 \Rightarrow d = 2$$

sides are 6, 8, 10, smallest side is 6.

Hence, the correct answer is (6).

6. Given ratio $= \dfrac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \dfrac{6}{11} \Rightarrow a = 9d$

$$a_7 = a + 6d = 15d$$

$\because \quad 130 < 15d < 140 \quad \Rightarrow \quad d = 9$

($\because$ All terms are natural numbers $\therefore d \in N$

Hence, the correct answer is (9).

7. $\therefore \quad a, b, c$ are in G.P.

$\therefore \quad b = ar$ and $c = ar^2$

Also $\frac{b}{a}$ is an integer

$\Rightarrow$ r is an integer

$\therefore$ A.M. of a, b, c is $b + 2$

$$\Rightarrow \quad \frac{a+b+c}{3} = b + 2$$

$$\Rightarrow \quad a + ar + ar^2 = 3ar + 6$$

$$\Rightarrow \quad a(r-1)^2 = 6$$

$\because$ a and r are integers

$\therefore$ The only possible values of a and r can be 6 and 2 respectively.

$$\text{Then } \frac{a^2 + a - 14}{a+1} = \frac{36+6-14}{6+1} = \frac{28}{7} = 4$$

Hence, the correct answer is (4).

8. Let $k, k+1$ be removed from pack.

$$\therefore \quad (1 + 2 + 3 + \ldots + n) - (k + k + 1) = 1224$$

$$\frac{n(n+1)}{2} - 2k = 1225$$

$$k = \frac{n(n+1) - 2450}{4}$$

for $n = 50, k = 25$

$\therefore$ $k - 20 = 5$

Hence, the correct answer is (5).

9. We have $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}[2\times 15 + 10(5n-1)d]}{\frac{n}{2}[6 + (n-1)d]}$

$$= \frac{5[6-d] + 5nd}{(6-d) + nd]}$$

Which will be independent of n if $d = 6$ or $d = 0$

For a proper A.P. we take $d = 6$

Then $a_2 = 3 + 6 = 9$

Hence, the correct answer is (9).

10. Using AM $\geq$ GM

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8}$$

$$\geq (a^{-5}.a^{-4}.a^{-3}.a^{-3}.a^{-3}.1.a^8.a^{10})^{1/8} = 1$$

Hence we have least value of $a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}$ as 8

Hence, the correct answer is (8).

11. Using $S_\infty = \frac{a}{1-r}$, we get

$$S_k = \begin{cases} 0, & k = 1 \\ \frac{1}{(k-1)!}. & k \geq 2 \end{cases}$$

$$\text{Now } \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = \sum_{k=2}^{100} |(k^2 - 3k + 1)| \frac{1}{(k-1)!}$$

$$= |-1| + \sum_{k=3}^{100} \frac{(k^2 - 1) + 1 - 3(k-1) - 2}{(k-1)!}$$

as $k^2 - 3k + 1 > 0 \ \forall k \geq 3$

$$= 1 + \sum_{k=3}^{100} \left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!} \right)$$

$$= 1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1!} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + \ldots + \left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99!}\right)$$

$$= 3 - \frac{1}{98!} - \frac{1}{99!} = 3 - \frac{9900}{100!} - \frac{100}{100!}$$

$$= 3 - \frac{10000}{100!} = 3 - \frac{(100)^2}{100!}$$

$$\therefore \quad \frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)\, S_k\right| = 3$$

Hence, the correct answer is (3).

12. Given that $a_k = 2a_{k-1} - a_{k-2}$

$$\Rightarrow \quad \frac{a_{k-2} + a_k}{2} = a_{k-1}, 3k \leq 11$$

$\Rightarrow$ $a_1, a_2, a_3, \ldots, a_{11}$ are in A.P.

If a is the first term and D the common difference then

$$a_1^2 + a_2^2 + \ldots. + a_{11}^2 = 990$$

$$\Rightarrow 11a_1^2 + d^2(1^2 + 2^2 + \ldots + 10^2) + 2a_1 ad(1 + 2 + \ldots + 10)$$
$$= 990$$

$$\Rightarrow \quad 11a_1^2 + \frac{10\times 11\times 21}{6} d^2 + 2a_1 d \times \frac{10\times 11}{2} = 990$$

$$\Rightarrow \quad a_1^2 + 35d^2 + 150d = 90$$

using $a_1 = 15$, we get

$$35d^2 + 150d + 135 = 0 \quad \text{or} \quad 7d^2 + 30d + 27 = 0$$

$$\Rightarrow \quad (d+3)\,(7d+9) = 0 \quad \Rightarrow \quad d = -3 \text{ or } -9/7$$

$$\text{Then } a_2 = 15 - 3 = 12 \quad \text{or} \quad 15 - \frac{9}{7} = \frac{96}{7} > \frac{27}{2}$$

$\therefore$ $d \neq -9/7$

$$\text{Hence } \frac{a_1 + a_2 + \ldots + a_{11}}{11} = \frac{\frac{11}{2}[2\times 15 + 10(-3)]}{11} = 0$$

Hence, the correct answer is (0).

CHAPTER 4: COMPLEX NUMBERS

Test Your Concepts-I

1. Refer Theory

2. (A) $4i + 15i + 6i - 25i = 0$ True

(B) $1 + i^2 + i^2 = -2$ True

(C) Summation of any four consecutive numbers = 0 True

(D) $6(i^4)^{13}i^2 + 5(i^4)^9 i - (i^4)^2 i^3 + 6(i^{17})^4$

$= -6 + 5i + i + 6 = 6i$ False

3. $$\frac{i^{10}(i^{582} + i^{580} + i^{578} + i^{576} + i^{574})}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$$

$$= -1 - 1 = -2$$

4. $$(1 + i^{4n+3})(1 - i)$$

$$= (1 - i)(1 - i)$$

$$= 1 + i^2 - 2i = -2i$$

5. $i^2 + i^4 + i^6 + i^8 + \ldots.$ (2n terms)

$= -1 + 1 - 1 + 1 + \ldots.$ (2n terms)

$= 0$

6. $$\sum_{K=1}^{4m+1} (i^k + i^{2k} + i^{3k} + i^{4k})$$

$$= (i + i^2 + i^3 + i^4 + \ldots.. + i^{4m+1}) + (i^2 + i^4 + i^6 + i^8 + \ldots.. + i^{2(4m+1)})$$
$$+ (i^3 + i^6 + i^9 + i^{12} + \ldots.. + i^{3(4m+1)}) + (i^4 + i^8 + \ldots.. + i^{4(4m+1)})$$
$$= i + (-1) + (-i) + (4m + 1)$$
$$= 4m.$$

7. $$\left(i^2 + i^4 + i^6 + \ldots. + i^{2(2n+1)}\right)^{\left(i + i^2 + i^3 + \ldots. + i^{107}\right)}$$

$$= \left(i^2\right)^{i + i^2 + i^3} = (-1)^{-1} = -1$$

8. (i) $1 + i^2 + i^4 + i^6 + \ldots\ldots.. + i^{2(4n)}$

$= 1 + (-1) + 1 + (-1) + \ldots\ldots.. + 1 = 1$

(ii) $= 1 + i^3 + i^6 + i^9 + \ldots\ldots + i^{3(4n+1)}$

$= 1 + (-i) + (-1) + (i) + \ldots\ldots.. + (-i)$

$= 1 - i = \dfrac{2}{1 + i}$

9. $$\left(\frac{1+i}{1-i}\right)^{10001} + \left(\frac{1-i}{1+i}\right)^{10001}$$

$$= \left\{\frac{(1+i)(1+i)}{(1-i)(1+i)}\right\}^{10001} + \left\{\frac{(1-i)(1-i)}{(1+i)(1-i)}\right\}^{10001}$$

$$= \left(\frac{2i}{2}\right)^{10001} + \left(\frac{-2i}{2}\right)^{10001}$$

$$= i - i = 0$$

Test Your Concepts-II

1. $$\frac{1-2i}{2+i} + \frac{4-i}{3+2i}$$

$$= \frac{(1-2i)(2-i)}{4+1} + \frac{(4-i)(3-2i)}{9+4}$$

$$= -i + \left(\frac{10-11i}{13}\right) = \frac{10}{13} - \frac{24}{13}i$$

2. $$\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$$

$$= \frac{1}{2}\left(\frac{1+2i}{5} + \frac{3(1-i)}{2}\right)\left(\frac{(3+4i)(1+2i)}{(1-2i)(1+2i)}\right)$$

$$= \frac{1}{20}(2 + 4i + 15 - 15i)\ \frac{(3+6i+4i-8)}{5}$$

$$= \frac{1}{100}(17 - 11i)(-5 + 10i); = \frac{1}{20}(17 - 11i)(-1 + 2i)$$

$$= \frac{1}{20}(-17 + 34i + 11i + 22); = \frac{1}{20}(5 + 45i)$$

$$= \frac{1}{4} + \frac{9}{4}i$$

3. $$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$$

$$R_1 \to R_1 + R_2$$

$$\Rightarrow \begin{vmatrix} 6i+4 & 0 & 0 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$$

$$\Rightarrow (6i + 4)(-3 + 3) = x + iy \Rightarrow 0 = x + iy \Rightarrow x = y = 0$$

4. Required term $= \dfrac{1}{3+\sqrt{7}i} = \dfrac{3-\sqrt{7}i}{9+7} = \dfrac{3}{16} - \dfrac{\sqrt{7}}{16}i$

5. $$\left\{\frac{(1-i)(1-i)}{(1+i)(1-i)}\right\}^{100} = a + ib$$

$$\Rightarrow \left(\frac{1-1-2i}{2}\right)^{100} = a + ib$$

$$\Rightarrow 1 = a + ib$$

$$\Rightarrow a = 1 \text{ and } b = 0$$

Test Your Concepts-III

1. $$z = \sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$$

$$z = 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i\,2\sin^2\frac{\pi}{10}$$

$$z = 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

$$\text{amp } z = \frac{\pi}{10}$$

2. $Z = \frac{(1+i)(1+2i)(1+3i)}{(1-i)(1-2i)(1-3i)}$

$|Z| = \frac{\sqrt{2}.\sqrt{5}.\sqrt{10}}{\sqrt{2}\sqrt{5}\sqrt{10}} = 1$

$\arg(z) = \arg(1 + i) + \arg(1 + 2i) + \arg(1 + 3i) - \arg(1 - i)$

$- \arg(1 - 2i) - \arg(1 - 3i) + 2k\pi, k \in I$

$\arg(z) = 2\left(\frac{\pi}{4} + \tan^{-1}2 + \tan^{-1}3\right) + 2k\pi, k \in I$

$(\because \tan^{-1}{}_1 + \tan^{-1}2 + \tan^{-1}3 = \pi)$

$\arg(z) = 2\pi + 2k\pi, k \in I$

let $k = -1$

So, $\arg(z) = 0$

3. $z = (\tan 1 - i)^2$

$|z| = 1 + \tan^2 1 = \sec^2 1$

$\arg(z) = -2.\tan^{-1}\left(\frac{1}{\tan 1}\right) = -2\tan^{-1}(\cot 1) = -2\left(\frac{\pi}{2} - 1\right)$

$\arg(z) = 2 - \pi$

4. $z = 1 + \cos\frac{10\pi}{9} + \text{i} \sin\frac{10\pi}{9}$

$z = 2\cos^2\frac{5\pi}{9} + 2\text{i}\sin\frac{5\pi}{9}\cos\frac{5\pi}{9}$

$z = 2\cos\left(\frac{5\pi}{9}\right)\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right)$

$z = -2\cos\left(\frac{4\pi}{9}\right)\left(-\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$

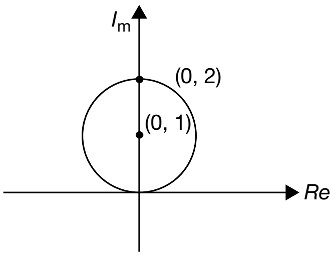
$z = 2\cos\frac{4\pi}{9}\left(\cos\frac{4\pi}{9} - i\sin\frac{4\pi}{9}\right)$

$|z| = 2\cos\left(\frac{4\pi}{9}\right)$; $\arg(z) = -\frac{4\pi}{9}$

5. let $z = yi$; $y \in R$

$$\arg(z) = \begin{cases} \pi/2, & y > 0 \\ \text{not defined}; & y = 0 \\ -\pi/2, & y < 0 \end{cases}$$

6.



$|z - i| = 1$

$\arg(z) = \frac{\pi}{2}$

$z = 0 + 2i$

7. $z = \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-1-4i} = \frac{1+7i}{3-4i}$

$z = \frac{(1+7i)(3+4i)}{25} = \frac{3+4i+21i-28}{25}$

$z = -1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

8. $\arg(z^{50}) = 50\arg(z) + 2k\pi, k \in I$

$= 50.100^\circ + 2k\pi, k \in I$

$= 50 \times 90^\circ + 500 + 2k\pi, k \in I$

$= 50 \times 90^\circ + 90 \times 5 + 40 + 2k\pi$; (take $k = 14$)

$= -40^\circ$

9. $z = \frac{1+i\sqrt{3}}{2i\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

$|z| = \frac{\sqrt{2}}{2.1.1} = 1$

$\arg(z) = \arg(1 + i\sqrt{3}) - \arg(2i) - \arg\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$+ 2k\pi, k \in I$

$\arg(z) = \frac{\pi}{3} - \frac{\pi}{2} - \frac{\pi}{3} + 2k\pi = -\frac{\pi}{2}$

10. $z = \frac{\left(\sqrt{3}+i\right)^{4n+1}}{\left(1-i\sqrt{3}\right)^{4n}} = \frac{\left(\sqrt{3}i-1\right)^{4n+1}}{i(1-i\sqrt{3})^{4n}} = -\frac{1}{i}(1 - \sqrt{3}\,i) = \sqrt{3} + i$

$\arg(z) = \frac{\pi}{6}$

11. (i) let $z = x + iy$; $z - \bar{z} = 2iy$

amp $(z - \bar{z}) = \pm\frac{\pi}{2}, y \neq 0$

(ii) amp$(-x - iy)$ – amp$(x + iy)$; let z in first quadrant

$= -\pi + \tan^{-1}\frac{y}{x} - \tan^{-1}\left(\frac{y}{x}\right) + 2\text{k}\pi, k \in I$

$= \pi$

12. Required term $= \left(\frac{2\cos^2\frac{\pi}{16} + i2\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{2\cos^2\frac{\pi}{16} - 2i\sin\frac{\pi}{16}\cos\frac{\pi}{16}}\right)^8 = \frac{e^{i\frac{\pi}{16}\cdot 8}}{e^{-i\frac{\pi}{16}\cdot 8}}$

$= e^{i\pi} = -1$

Test Your Concepts-IV

1. $\left|(1+i)\frac{2+i}{(3+i)}\right| = \frac{\sqrt{2}.\sqrt{5}}{\sqrt{10}} = 1$

2. $-3 + ix^2y = \overline{x^2 + y + 4i} \Rightarrow -3 + ix^2y = x^2 + y - 4i$

$\Rightarrow x^2 + y = -3$ and $x^2y = -4$

$\Rightarrow x^2 = -3 - y$ and $(-3 - y)y = -4$

$\Rightarrow y^2 + 3y - 4 = 0 \Rightarrow (y + 4)(y - 1) = 0$

if $y = -4 \Rightarrow x = \pm 1$

if $y = 1 \Rightarrow x \in \phi$

So, $(x, y) \equiv (\pm 1, -4)$

3. $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{1-\bar{\alpha}\left(\frac{1}{\bar{\beta}}\right)}\right|$

$$= |\bar{\beta}| \left|\frac{\beta-\alpha}{\bar{\beta}-\bar{\alpha}}\right| = 1.1 = 1;\ |\beta|^2 = \beta\bar{\beta} = 1$$

4. $iz^2 = \bar{z}$

$$|iz^2| = |\bar{z}| \quad \Rightarrow \quad |z|^2 = |z|$$

$$|z| = 0 \quad \text{or} \quad |z| = 1$$

5. $(a_1 + ib_1)(a_2 + ib_2) \ldots. (a_n + ib_n) = A + iB \qquad \ldots\text{(i)}$

$$\overline{(a_1+ib_1)}\ \overline{(a_2+ib_2)}\ \ldots\ldots\ \overline{(a_n+ib_n)} = \overline{A+iB}$$

$$(a_1 - ib_1)(a_2 - ib_2) \ldots\ldots (a_n - ib_n) = A - iB \qquad \ldots\text{(ii)}$$

multiply (i) and (ii) using $z\bar{z} = |z|^2$

$$\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right) \ldots\ldots\ldots \left(a_n^2+b_n^2\right) = A^2 + B^2$$

6. $\left|\dfrac{z_1 - z_2}{1-\bar{z}_1 z_2}\right| < 1$

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < (1 - \bar{z}_1 z_2)(1 - z_1\bar{z}_2)$$

$$|z_1|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 + |z_2|^2 < 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + |z_1|^2|z_2|^2$$

$$|z_1|^2 - 1 + |z_2|^2(1 - |z_1|^2) < 0$$

$$(1 - |z_1|^2)(|z_2|^2 - 1) < 0$$

$$(|z_1|^2 - 1)(|z_2|^2 - 1) > 0$$

So, either $|z_1| > 1$ and $|z_2| > 1$ or $|z_1| < 1$ and $|z_2| < 1$

7. (a) $\bar{z} = iz^2 \quad \Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$

$$\Rightarrow x = -2xy \quad \text{and} \quad -y = x^2 - y^2$$

$$\Rightarrow x = 0 \quad \text{and} \quad y = -\frac{1}{2}$$

$$\text{if } x = 0 \quad \Rightarrow \quad y^2 - y = 0 \quad \Rightarrow \quad y = 0, 1$$

$$\text{if } y = -\frac{1}{2} \quad \Rightarrow \quad x^2 = \frac{1}{4} + \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{\sqrt{3}}{2}$$

complex number are $\left(\pm\sqrt{3}/2, -1/2\right)$ and (0, 1)

(b) $|z_1| = |z_2| = |z_3| = \ldots\ldots\ldots = |z_n| = 1$

$$|z_1 + z_2 + \ldots\ldots + z_n|$$

$$= \left|\overline{z_1 + z_2 + \ldots.. + z_n}\right|$$

$$= |\bar{z}_1 + \bar{z}_2 + \ldots.. + \bar{z}_n|$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots.. + \frac{1}{z_n}\right|$$

8. $\omega = \dfrac{1 - iz_1}{z - i};\ |\omega| = 1$

$$|1 - iz| = |z - i|$$

$$(1 - iz_1)(1 + i\bar{z}_1) = (z - i)(\bar{z} + i)$$

$$1 + i\bar{z} - iz + |z|^2 = |z|^2 + zi - \bar{z}i + 1$$

$2i(\bar{z} - z) = 0 \quad \Rightarrow \quad z$ is purely red.

9. $z = \dfrac{9 + 40i + 9 - 40i + 2\sqrt{9^2 + 40^2}}{9 + 40i - (9 - 40i)}$

$$= \frac{2(9+41)}{80i} = -\frac{5}{4}i$$

$$|z| = \frac{5}{4},\ \arg(z) = -\frac{\pi}{2}$$

10. $\operatorname{Re}\left(\dfrac{1}{z}\right) < \dfrac{1}{2} \quad \Rightarrow \quad \operatorname{Re}\left(\dfrac{\bar{z}}{|z|^2}\right) < \dfrac{1}{2}$

let $z = x + iy$

$$\operatorname{Re}\left(\frac{x - iy}{x^2 + y^2}\right) < \frac{1}{2} \quad \Rightarrow \quad \frac{x}{x^2 + y^2} < \frac{1}{2}$$

$$\Rightarrow \quad x^2 + y^2 - 2x > 0$$

11. $|z| = 1$ and $w = \dfrac{z-1}{z+1}$,

$$\Rightarrow \quad \frac{w+1}{w-1} = \frac{z-1+z+1}{z-1-z-1} \quad \Rightarrow \quad z = \frac{w+1}{1-w}$$

$$|w + 1| = |1 - w|$$

$$\Rightarrow \quad (w + 1)(\bar{w} + 1) = (1 - w)(1 - \bar{w})$$

$$|w|^2 + w + \bar{w} + 1 = 1 - \bar{w} - w + |w|^2$$

$$\Rightarrow \quad w + \bar{w} = 0$$

$$\operatorname{Re}(w) = 0$$

Test Your Concepts-V

1. $\left|z - \dfrac{4}{z}\right| = 2$

put $z = re^{i\theta}$

$$\left|re^{i\theta} - \frac{4}{r}e^{-i\theta}\right| = 2$$

$$\left|r(\cos\theta + i\sin\theta) - \frac{4}{r}(\cos\theta - i\sin\theta)\right| = 2$$

$$\left|\left(r - \frac{4}{r}\right)\cos\theta + i\left(r + \frac{4}{r}\right)\sin\theta\right| = 2$$

$$\left(r - \frac{4}{r}\right)^2\cos^2\theta + \left(r + \frac{4}{r}\right)^2\sin^2\theta = 4$$

$$\left(r^2 + \frac{16}{r^2} - 8\right)\cos^2\theta + \left(r^2 + \frac{16}{r^2} + 8\right)\sin^2\theta = 4$$

$$r^2 + \frac{16}{r^2} - 8\cos 2\theta = 4$$

$$r^4 - (4 + 8\cos 2\theta)r^2 + 16 = 0$$

$$r^2 = \frac{4 + 8\cos 2\theta \pm 4\sqrt{1 + 4\cos^2 2\theta + 4\cos 2\theta - 4}}{2}$$

$$r^2 = 2 + 4\cos 2\theta \pm 2\sqrt{(2\cos 2\theta + 1)^2 - 4}$$

$$r_{\max}^2 = 6 + 2\sqrt{5} \quad \Rightarrow \quad r_{\max} = \sqrt{5} + 1$$

$$r_{\max}^2 = 6 - 2\sqrt{5} \quad \Rightarrow \quad r_{\min} = \sqrt{5} - 1$$

2. (i) $|z_1 + z_2| = |z_1 - z_2|$

$$(z_1 + \bar{z}_2)(\bar{z}_1 + \bar{z}_2) = (z_1 + z_2)(\bar{z}_1 - \bar{z}_2)$$

$$\Rightarrow |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_1|^2$$

$$|z_1|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 - |z_2|^2$$

$$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0$$

$$z_1\bar{z}_2 = -z_2\bar{z}_1$$

$$\frac{z_1}{z_2} = -\left(\overline{\frac{z_1}{z_2}}\right)$$

$\frac{z_1}{z_2}$ is purely imaginary

Alternate: $\left|\frac{z_1}{z_2}+1\right| = \left|\frac{z_1}{z_2}-1\right|$

$\frac{z_1}{z_2}$ is purely imaginary

(ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2$$

$$\Rightarrow |z_1|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 + |z_2|^2 = |z_1|^2 + |z_2|^2$$

$$\Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0$$

$$\Rightarrow z_1\bar{z}_2 = -z_2\bar{z}_1$$

$$\Rightarrow \frac{z_1}{z_2} = -\frac{\bar{z}_1}{\bar{z}_2}$$

$\frac{z_1}{z_2}$ is purely imaginary

(iii) $|z_1 + z_2| = |z_1| + |z_2|$

$$\frac{\lambda z_2 - z_1}{\lambda + 1} = 0 \quad \Rightarrow \quad \frac{z_1}{z_2} = \lambda$$

So, $\frac{z_1}{z_2}$ is purely real

(iv) $|z_1 + z_2| = ||z_1| - |z_2||$

$$\frac{\lambda z_2 + z_1}{\lambda + 1} = 0$$

$$\frac{z_1}{z_2} = -\lambda$$

$\frac{z_1}{z_2}$ is purely real

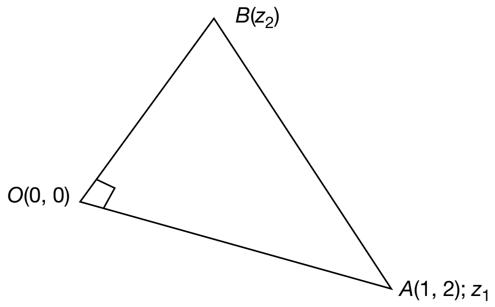
3. $|z + 4| \le 3,$

$$|z + 1| = |z + 4 - 3| \le |z + 4| + 3$$

$$\Rightarrow \quad |z + 1| \le 3 + 3$$

$$\Rightarrow \quad |z + 1| \le 6$$

4.



(i) when $\theta = \frac{\pi}{2}$

$$\frac{OB}{OA} = \frac{|OB|}{|OA|}e^{i\pi/2}$$

Given $|OB| = |OA|$

$$OB = OA\ e^{i\pi/2}$$

$$z_2 - O = (1 + 2i)(i)$$

$$z_2 = i + 2(-1)$$

$z_2 = -2 + i$; $B(-2, 1)$

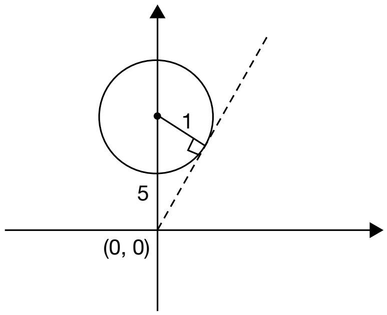
(ii) $\theta = -\frac{\pi}{2}$

$$z_2 = (1 + 2i)(-i)$$

$z_2 = -i + 2$; $(2, -1)$

$B(2, -1)$ or $(-2, 1)$

5.



$$|z - 5i| \le 1$$

equation of circle $x^2 + (y - 5)^2 = 1$

equation of tangent $y = mx$

$$x^2 + m^2x^2 + 25 - 10\,mx = 1$$

$$(1 + m^2)x^2 - 10mx + 24 = 0$$

$$D = 0 \quad \Rightarrow \quad (10m)^2 - 4(1 + m^2).24 = 0$$

$$100m^2 - 96 - 96m^2 = 0$$

$$\Rightarrow \quad m^2 = 24 \quad \Rightarrow \quad m = \pm 2\sqrt{6}$$

so target is $y = 2\sqrt{6}\,x$

$$x^2 + (2\sqrt{6}\,x - 5)^2 = 1$$

$$x^2 + 24x^2 + 25 - 20\sqrt{6}\,x = 1$$

$$25x^2 - 20\sqrt{6}\,x + 24 = 0$$

$$x = \frac{20\sqrt{6}}{50}$$

$$x = \frac{2\sqrt{6}}{5} \text{ and } y = \frac{24}{5}$$

point is $\left(\frac{2\sqrt{6}}{5} + \frac{24}{5}i\right)$

6. $|3z - 1| = 3|z - 2|$

$$\left|z - \frac{1}{3}\right| = |z - 2|$$

Angle bisector of points $\left(\frac{1}{3}, 0\right)$ and $(2, 0)$

So, parallel to y-axis.

7. $\left|z - \frac{5}{2}\right| + \left|z + \frac{5}{2}\right| = 5$

$PA + PB = AB$; P : z and $A\left(\frac{5}{2}, 0\right)$; $B\left(-\frac{5}{2}, 0\right)$

Line segment between point $\left(\frac{5}{2}, 0\right)$ and $\left(-\frac{5}{2}, 0\right)$

Test Your Concepts-VI

1. let $z = x + iy$

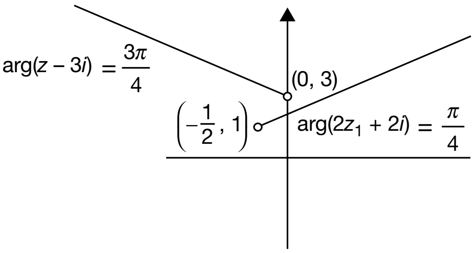
$$x^2 + (y + 3)^2 - \{(x - 3)^2 + y^2\} = 6$$
$$(x - x + 3)\ (x + x - 3) + (y + 3 - y)\ (y + 3 + y) = 6$$
$$2x - 3 + 2y + 3 = 2$$
$$\Rightarrow \quad x + y = 1$$

2. $\arg[(1 + 2i)z - 1 + 3i] = \dfrac{\pi}{4}$

$$z = x + iy;\ \arg(x + 2ix + iy - 2y - 1 + 3i) = \frac{\pi}{4}$$
$$\arg(x - 2y - 1 + i(2x + y + 3)) = \frac{\pi}{4}$$
$$\frac{2x + y + 3}{x - 2y - 1} = 1$$
$$2x + y + 3 = x - 2y - 1$$
$$x + 3y + 4 = 0$$

passing through (2, – 2) and (5, –3)

3.



$\arg(z - 3i) = \dfrac{3\pi}{4}$
(0, 3)
$\left(-\dfrac{1}{2}, 1\right)$
$\arg(2z_1 + 2i) = \dfrac{\pi}{4}$

No point of intersection

4. $\left|\dfrac{z - 2}{z + 2}\right| = \dfrac{\pi}{6}$

$$\left|\frac{z - z_1}{z - z_2}\right| = \alpha\ ;\ \alpha \neq 0$$

locus of z is perpendicular bisector of z_1, z_2 if $\alpha = 1$
if $\alpha = 1$ locus of z is a circle

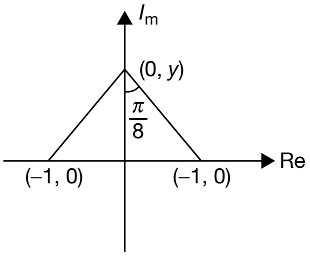
5. $\arg\left(\dfrac{z - z_1}{z_2 - z}\right) = \dfrac{\pi}{2}$

$$\tan^{-1}\left(\frac{y - 1}{x - 6}\right) - \tan^{-1}\left(\frac{y + 3}{x - 4}\right) = \frac{\pi}{2}$$
$$\frac{\dfrac{y - 1}{x - 6} - \dfrac{y + 3}{x - 4}}{1 + \dfrac{(y - 1)(y + 3)}{(x - 6)(x - 4)}} = \frac{1}{0}$$
$$\Rightarrow \quad (x - 6)\ (x - 4) + (y - 1)\ (y + 3) = 0$$
$$\Rightarrow \quad x^2 + y^2 - 10x + 2y + 21 = 0$$

centre (5 – i), radius = $\sqrt{5}$

$$|z - (5 - i)| = \sqrt{5}$$

6.



$$|z + 1| = |z - 1|$$

z lies on perpendicular bisector of (–1, 0) and (1, 0)
i.e. y axis.

$$\text{amp}\left(\frac{z - 1}{z + 1}\right) = \frac{\pi}{4}$$

let $z = yi$

$$\tan\frac{\pi}{8} = \frac{1}{y}$$
$$\Rightarrow \quad y = \sqrt{2} + 1$$

So, $x = 0$; $y = \sqrt{2} + 1$

7. $z_1 + z_3 = z_3 + z_4$

⇒ mid points of diagonals is same

$$|z_1 - z_3| = |z_2 - z_4|$$

⇒ length of diagonal is same

So, quadrilateral is rectangle or square

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pm\frac{\pi}{2}$$

Test Your Concepts-VII

1. $\because\ w^3 = 1$ and $1 + w + w^2 = 0$

$$w^{1994} + w^{1995} = w^{1994}\ (1 + w)$$
$$= -w^{1996} = -w \text{ or } -w^4$$

2. $y^3 - 3y^2 + 3y + 26 = 0$

$$\Rightarrow \quad y^3 - 3y^2 + 3y + 26 = 0$$
$$\Rightarrow \quad (y - 1)^3 = -27$$
$$\Rightarrow \quad (y - 1)^3 = (-3)^3$$
$$y - 1 = -3;\ y - 1 = -3w,\ y - 1 = -3w^2$$
$$\Rightarrow \quad a - 1 = -3,\ b - 1 = -3w,\ c - 1 = -3w^2$$
$$\frac{a - 1}{b - 1} + \frac{b - 1}{c - 1} + \frac{c - 1}{a - 1} = \frac{-3}{-3w} + \frac{-3w}{-3w^2} + \frac{-3w^2}{-3}$$
$$= \frac{1}{w} + \frac{1}{w} + \omega^2 = 3\omega^2$$

3. $R_1 \to R_1 + R_3$

$$\begin{vmatrix} 1 - i & -1 & w^2 - 1 \\ 1 - i & -1 & w^2 - 1 \\ -i & -1 + w - i & -1 \end{vmatrix} = 0$$

4. $z^2 + z + 1 = 0$,

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots..\left(z^6 + \frac{1}{z^6}\right)^2$$
$$= 2\left(z^2 + \frac{1}{z^2} + 2 + 2^4 + \frac{1}{z^4} + 2 + 4\right)$$
$$= 2\left(z^2 + \frac{1}{z^2} + 2 + \frac{1}{2} + 8\right)$$
$$= 2(-1 + (-1) + 8)$$
$$= 12$$

5. $$\sum_{k=1}^{14}\left(\sin\frac{2\pi k}{15}-i\cos\frac{2\pi k}{15}\right)$$

$$=\sin\frac{2\pi}{15}+\sin\frac{4\pi}{15}+\ldots..+\sin\frac{28\pi}{15}-i\left(\cos\frac{2\pi}{15}+\ldots..+\cos\frac{28\pi}{15}\right)$$

$$=\frac{\sin(\pi)\sin\left(\frac{4\pi}{15}\right)}{\sin\left(\frac{\pi}{15}\right)}-i\frac{\cos(\pi)\sin\left(\frac{14\pi}{15}\right)}{\sin\left(\frac{\pi}{15}\right)}$$

$$=0+i$$

6. $$z_r=\cos\frac{r\alpha}{n^2}+i\sin\frac{r\alpha}{n^2}=e^{i\frac{r\alpha}{n^2}}$$

$$\lim_{n\to\infty}(z_1z_2z_3\ \ldots\ldots\ z_n)=\lim_{n\to\infty}e^{i\frac{r\alpha}{n^2}(1+2+\ldots.+n)}$$

$$=\lim_{n\to\infty}e^{i\alpha\frac{n(n+1)}{2n^2}}=e^{i\frac{\alpha}{2}}=\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}$$

7. $z^n=1$

has roots $1, w, w^2, \ldots\ldots, w^{n-1}$

replace z by $7+z$

$$(7+z)^n=1$$

$$^nC_0z^n+{}^nC_1\,7.z^{n-1}+\ldots\ldots+{}^nC_n\,7^n=1$$

product of roots $=7^n-1$

$$(7-1)\,(7-w)\ \ldots\ldots\ (7-w^{n-1})=7^n-1$$

$$(7-w)\,(7-w^2)\ \ldots\ldots\ (7-w^{n-1})=\frac{7^n-1}{6}$$

8. $S=1+3\alpha+5\alpha^2+\ldots.+T_n$

$$\alpha S=0+\alpha+3\alpha^2+\ldots\ldots+T_{n-1}+T_n$$

$$(1-\alpha)S=1+2\alpha+2\alpha^2+2\alpha^{n-1}-(2_{n-1})\alpha^n$$

$$S=\frac{-1+2(1+\alpha+\alpha^2+\ldots..+\alpha^{n-1})-(2n-1)\alpha^n}{(1-\alpha)}$$

$$S=\frac{-1-(2n-1)}{1-\alpha}=-\frac{2n}{1-\alpha}$$

Single Correct Choice Type Questions

1. We know that, $\frac{z_2-z_1}{z_4-z_1}=\frac{|z_2-z_1|}{|z_4-z_1|}\,e^{i\frac{\pi}{2}}=i$

(as $|z_2-z_1|=|z_4-z_1|$)

$\Rightarrow\quad (z_4-z_1)^2+(z_2-z_1)^2=0$

Similarly $\frac{z_4-z_3}{z_2-z_3}=i$

$\Rightarrow\quad (z_4-z_3)^2+(z_2-z_3)^2=0$

On adding equations (1) and (2), we get

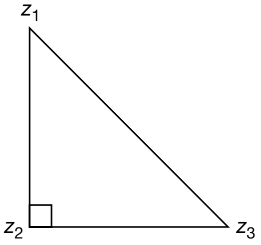
$$2[z_1^2+z_2^2+z_3^2+z_4^2-z_1z_2-z_4z_1-z_4z_3-z_2z_3]=0$$

$$z_1^2+z_2^2+z_3^2+z_4^2+=z_1z_2+z_2z_3+z_3z_4+z_4z_1$$

Hence, the correct answer is (A).

2. We know that, $\frac{z_3-z_2}{z_1-z_2}=\frac{|z_3-z_2|}{|z_1-z_2|}.e^{-\frac{i\pi}{2}}$

$\Rightarrow\quad z_3-z_2=-i(z_1-z_2)$



$\Rightarrow\quad (z_3-z_2)^2+(z_1-z_2)^2=0$

Hence, the correct answer is (A).

3. $C+iS=e^{i\theta}+e^{i(2\theta)}+e^{i(4\theta)}$, where $\theta=\frac{2\pi}{7}$

i.e., $C+iS=\alpha+\alpha^2+\alpha^4$, where $\alpha=e^{i\theta}$ …(1)

So,

$$C-iS=\bar{\alpha}+(\bar{\alpha}^2)+(\bar{\alpha}^4)=\alpha^6+\alpha^5+\alpha^3 \quad\ldots(2)$$

$$\left(\text{Since }\bar{\alpha}=\frac{\alpha\bar{\alpha}}{\alpha}=\frac{1}{\alpha}=\frac{\alpha^7}{\alpha}=\alpha^6,\text{etc.}\right)$$

From equations (1) and (2), we get

$$2C=\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6=\frac{\alpha(\alpha^6-1)}{\alpha-1}$$

$$\Rightarrow\quad 2C=\frac{1-\alpha}{\alpha-1}\quad\Rightarrow\quad C=-\frac{1}{2}\qquad(\because\ \alpha^7=1)$$

Again, $(C+iS)\,(C-iS)$

$$=(\alpha+\alpha^2+\alpha^4)\,(\alpha^6+\alpha^5+\alpha^3)$$

$(\alpha^7=1)$

$$\Rightarrow\quad\left(-\frac{1}{2}\right)^2+S^2$$

$$=2+(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\alpha^6)$$

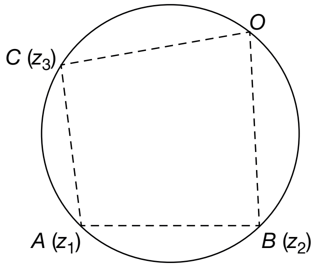
$$\Rightarrow\quad S^2=2+0-\frac{1}{4}=\frac{7}{4}\quad\Rightarrow\quad S=\frac{\sqrt{7}}{2}$$

Hence, the correct answer is (D).

4. $$\frac{2}{z_1}=\frac{1}{z_2}+\frac{1}{z_3}$$

$$\Rightarrow\quad \text{Arg}\left(\frac{z_2-z_1}{z_3-z_1}\right)=\text{Arg}\left(-\frac{z_2}{z_3}\right)=\text{Arg}\left(\frac{z_2}{z_3}\right)\pm\pi$$

$$\Rightarrow\quad \text{Arg}\left(\frac{z_2-z_1}{z_3-z_1}\right)-\text{Arg}\left(\frac{z_2-0}{z_3-0}\right)=\pm\pi$$



Sum of angles at A and origin is $\pm\pi$. Hence, the points O, B, A and C are concyclic.

Hence, the correct answer is (C).

5. Let $z = x + iy = r(\cos\theta + i\sin\theta)$, then the equation is

$$|2(x-2) + 2i(y-1)| = r\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)$$

$$= \frac{1}{\sqrt{2}}(r\cos\theta - r\sin\theta)$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}\left(\frac{x-y}{\sqrt{2}}\right)$$

It is an ellipse with focus at (2, 1) and directrix $x - y = 0$ and eccentricity $= \frac{1}{\sqrt{2}}$

Hence, the correct answer is (A).

6. Let $a = e^{i\alpha}$, $b = e^{i\beta}$, $c = e^{i\gamma}$

Clearly, $a + b + c = 0 \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

$$a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow \quad \sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

Hence, the correct answer is (D).

7. The given equation is $(z^2 + z + 1)(z^2 + 1) = 0$.

$z = \pm i, w, w^2$, w being an imaginary cube root of unity. Thus, $|z| = 1$.

Hence, the correct answer is (B).

8. Centroid of ΔABC is at $z_0 = \frac{z_1 + z_2 + z_3}{3}$.

Centroid (z_0) divides orthocentre (z) and circumcentre (0) in 2:1, internally.

Hence, the correct answer is (D).

9. The given equation is $\left|\frac{z - \frac{i}{2}}{z+1}\right| = \frac{m}{2}$, which does not represent a circle when $\frac{m}{2} = 1$.

Hence, the correct answer is (C).

10. The new complex number is

$$2(3 + 4i)e^{i\frac{\pi}{4}} = \sqrt{2}\ (-1 + 7i)$$

Hence, the correct answer is (B).

11. $(a + ib)^5 = \alpha + i\beta$

By taking complex conjugate,

$$(a - ib)^5 = \alpha - i\beta \Rightarrow (-i^2 a - ib)^5 = a - i\beta$$

$$\Rightarrow \quad (-i)^5\,(b + ai)^5 = \alpha - i\beta$$

$$(b + ai)^5 = -\frac{\alpha}{i} + \beta = \alpha i + \beta$$

Hence, the correct answer is (B).

12. Diagonals of parallelogram intersect at midpoint. So, by equating the midpoints of diagonals,

$$\frac{a+1}{2} = \frac{a+1}{2} \text{ and } \frac{-1-a}{2} = \frac{1+a}{2}$$

$$\Rightarrow \quad 2a = -2 \Rightarrow a = -1$$

Hence, the correct answer is (B).

13. $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = \alpha + i\beta$

$$(1 - i)(1 - 2i)(1 - 3i) \ldots (1 - ni) = \alpha - i\beta$$

By multiplying both equations,

$$2 \times 5 \times 10 \ldots (1 + n^2) = \alpha^2 + \beta^2$$

Hence, the correct answer is (C).

14. $z = a + \bar{a}$ is always real $\Rightarrow$ Arg(z) = 0 or π.

Hence, the correct answer is (B).

15. $3z_1 = 5z_2 - 2z_3 \Rightarrow z_1 = \frac{5z_2 - 2z_3}{5 - 2}$

$\Rightarrow$ z_1 divides the line joining z_2 and z_3 externally in the ratio 5:2.

$\Rightarrow$ z_1, z_2 and z_3 are collinear.

Hence, the correct answer is (C).

16. $$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1 \Rightarrow \left|\frac{\frac{z_1}{z_2} + 1}{\frac{z_1}{z_2} - 1}\right| = 1 \Rightarrow \left|\frac{z_1}{z_2} + 1\right| = \left|\frac{z_1}{z_2} - 1\right|$$

Locus of $\frac{z_1}{z_2}$ is the perpendicular bisector of the points joining (1, 0) and (–1, 0). It means, it lies on imaginary axis.

Therefore, $\frac{z_1}{z_2}$ is zero or purely imaginary.

Hence, the correct answer is (C).

17. $|4z_2z_3 + 9z_3z_1 + 16z_1z_2|$

$$= |z_1\bar{z}_1 z_2z_3 + \bar{z}_2 z_2z_3z_1 + \bar{z}_3 z_3z_1z_2|$$

$$||z_1||z_2||z_3||\ |z_1 + z_2 + z_3| = 120$$

Hence, the correct answer is (D).

18. $$A = \sin\left(\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^z\right) = \sin\ln e^{\frac{iz\pi}{2}}$$

$$= \sin\left(\frac{z\pi}{2}i\right)$$

Again, $|z - 2i| = 1$ is a circle centred at (0, 2) with radius 1. Therefore, a point on circle of least modulus is $z = i$.

$\therefore$ By equation, $A = \sin\left(-\frac{\pi}{2}\right) = -1$.

Hence, the correct answer is (C).

19. Since, ω is cube root of unity, $1 + \omega + \omega^2 = 0$

$$\Rightarrow \quad 1 + \omega = -\omega^2$$

Now, if $(1 + \omega)^7 = l + m\omega$

$$\Rightarrow \quad (-\omega^2)^7 = l + m\omega \quad \Rightarrow \quad -\omega^{14} = l + m\omega$$

$$\Rightarrow \quad -\omega^2 = l + m\omega \Rightarrow 1 + \omega = l + m\omega$$

From comparison, $l = 1$ and $m = 1$.

Hence, the correct answer is (C).

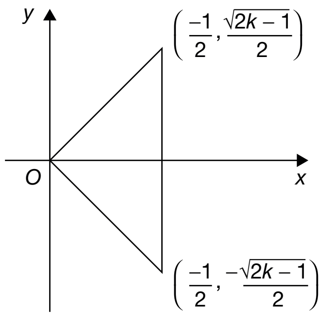
20. $2z^2 + 2z + k = 0$

$$z = \frac{-2 \pm \sqrt{4 - 8k}}{4}$$

Since ‘z’ is a complex number,
$4 - 8k$ will be negative

$$\Rightarrow \quad k > \frac{1}{2}$$

$$(0, 0), \left(\frac{-1}{2}, \frac{\sqrt{2k-1}}{2}\right)\left(\frac{-1}{2}, \frac{-1}{2}\sqrt{2k-1}\right)$$



y
x
O
$\left(\frac{-1}{2}, \frac{\sqrt{2k-1}}{2}\right)$
$\left(\frac{-1}{2}, \frac{-\sqrt{2k-1}}{2}\right)$

Since, the triangle is equilateral.

Therefore, $\frac{1}{4}(2k-1) + \frac{1}{4} = (2k-1)$

$$\Rightarrow \quad k = \frac{2}{3}.$$

Hence, the correct answer is (C).

21. $$\log_{\tan 30^o}\left(\frac{2|z|^2 + 2|z| - 3}{|z|+1}\right) < -2$$

$$\Rightarrow \quad \frac{2|z|^2 + 2|z| - 3}{|z|+1} > 3$$

$$\Rightarrow \quad ((|z| - 2)(2|z|) + 3) > 0 \Rightarrow |z| > 2$$

Hence, the correct answer is (C).

22. $x^3 + 2x^2 + 2x + 1 = 0$

$\Rightarrow \quad x = -1, w, w^2$

However, $x = w, w^2$ will only satisfy

$x^3 + 2x^2 + 2x + 1 = 0$ and $x^{2012} + x^{2014} + 1 = 0$.

Hence, the correct answer is (B).

23. If $|z| = |z-1|$, then $|z|^2 = |z-1|^2$

$\Rightarrow \quad z + \bar{z} = 1$

If $|z| = |z+1|$, then

$|z|^2 = |z+1|^2$

$\Rightarrow \quad z + \bar{z} = -1 \Rightarrow |z + \bar{z}| = 1$

Hence, the correct answer is (A).

24. $(z - i)(z^2 + 2iz - 2) = 0$

$\Rightarrow \quad z = i, 1 - i, -1 - i$

Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = 2$ square units.

Hence, the correct answer is (D).

25. Given that, $z = \cos\theta + i\sin\theta = e^{i\theta}$

Therefore, $\sum_{m=1}^{15}(z^{2m-1}) = \sum_{m=1}^{15}\mathrm{Im}(e^{i\theta})^{2m-1}$

$$= \sum_{m=1}^{15}\mathrm{Im}e^{i(2m-1)\theta}$$

$$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin 29\theta$$

$$= \frac{\sin\left(\frac{\theta + 29\theta}{2}\right)\sin\left(\frac{15 \times 2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)}$$

$$= \frac{\sin(15\theta)\sin(15\theta)}{\sin\theta} = \frac{1}{4\sin 2^\circ}$$

Hence, the correct answer is (D).

26. $a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \ldots + a_{n-1}z + a_n = 3$

$$|3| = |a_0z^n + az^{n-1} + a_2z^{n-2} + \ldots + a_{n-1}z + a_n|$$

$$3 \le |a_0||z|^n + |a_1||z|^{n-1} + \ldots + a_{n-1}|z| + |a_n|$$

$$3 < 2(|z|^n + |z|^{n-1} + \ldots + |z| + 1)$$

$$\Rightarrow \quad \frac{3}{2} < 1 + |z| + |z|^2 + \ldots + |z|^n \qquad (\because |z| < 1)$$

$$\Rightarrow \quad \frac{1 - |z|^{n+1}}{1 - |z|} > \frac{3}{2} \Rightarrow 2 - 2|z|^{n+1} > 3(1 - |z|)$$

$$\Rightarrow \quad 2 > 3(1 - |z|) \Rightarrow |z| > \frac{1}{3}$$

Hence, the correct answer is (A).

27. $$x = \frac{x^3}{x^2} = \frac{2 + 11i}{3 + 4i} = \frac{(2 + 11i)(3 - 4i)}{25}$$

$$\therefore \quad a + ib = \frac{6 + 44 + 25i}{25} = 2 + i$$

$$\Rightarrow \quad a = 2, b = 1 \Rightarrow a + b = 3$$

Hence, the correct answer is (C).

28. Let $z = a + ib$

$$\Rightarrow \quad a + ib + \sqrt{a^2 + 64} = 2 + 8i$$

$$\Rightarrow \quad b = 8, a + \sqrt{a^2 + 64} = 2$$

$$a^2 + 64 = a^2 - 4a + 4$$

$$\Rightarrow \quad -4a = 60 \Rightarrow a = -15$$

$$\therefore \quad |z| = \sqrt{a^2 + b^2} = \sqrt{225 + 64} = \sqrt{289} = 17$$

Hence, the correct answer is (B).

29. $$|3z + 9 - 7i| = |3z + 6 - 3i + 3 - 4i|$$

$$\le |3(z + 2 - i)| + |3 - 4i|$$

$$\le 3|z + 2 - i| + \sqrt{3^2 + 4^2} = 3(5) + 5 = 20$$

Hence, the correct answer is (A).

30. $|x + iy - 4| = x$

$$(x - 4)^2 + y^2 = x^2$$

$$y^2 - 8x + 16 = 0$$

z lies on the parabola with vertex (2, 0) focus (4, 0) and tangents from (0, 0), i.e., a point on the directrix in x always include 90°.

Therefore, greatest Arg(z) is $45^o = \frac{\pi}{4}$

Hence, the correct answer is (D).

31. $\bar{z} + i\bar{w} \quad 0 \Rightarrow z - iw = 0 \Rightarrow z = iw$

$$\mathrm{Arg}(zw) = \pi \Rightarrow \mathrm{Arg}(z) + \mathrm{Arg}(w) = \pi$$

$$\Rightarrow \quad \mathrm{Arg}(iw) + \mathrm{Arg}(w) = \pi$$

$$\text{Arg}(i) + 2\text{Arg}(w) = \pi \Rightarrow \frac{\pi}{2} + 2\text{Arg}(w) = \pi$$

$$2\text{Arg}(w) = \frac{\pi}{2} \Rightarrow \text{Arg}(w) = \frac{\pi}{4}$$

$\Rightarrow$ $\text{Arg}(z) = \dfrac{3\pi}{4}$

Hence, the correct answer is (C).

32. $|z_2 + iz_1| = |z_1| + |z_2| \Rightarrow z_2, iz_1, 0$ are collinear.

Therefore, $\text{Arg}(iz_1) = \text{Arg}(z_2)$

$\Rightarrow$ $\text{Arg}(i) + \text{Arg}(z_1) = \text{Arg}(z_2)$

$\Rightarrow$ $\text{Arg}(z_2) - \text{Arg}(z_1) = \dfrac{\pi}{2}$

$$z_3 = \frac{z_2 - iz_1}{1-i} \Rightarrow (1-i)z_3 = z_2 - iz_1$$

$$z_3 - z_2 = i(z_3 - z_1)$$

$$\frac{z_3 - z_2}{z_3 - z_1} = i \Rightarrow \text{Arg}\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \frac{\pi}{2} \text{ and}$$

$$|z_3 - z_2| = |z_3 - z_1|$$

$\therefore$ $AB = BC,$

$\therefore$ $AB^2 = AC^2 + BC^2$

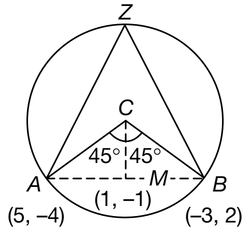
$\Rightarrow$ $25 = 2AC^2 \Rightarrow AC = \dfrac{5}{\sqrt{2}}$

Required area $= \dfrac{1}{2} \times \dfrac{5}{\sqrt{2}} \times \dfrac{5}{\sqrt{2}} = \dfrac{25}{4}$ square units.

Hence, the correct answer is (D).

33. $A(5, -4)$ $B(-3, 2)$ subtends an angle $\dfrac{\pi}{4}$ at $C(z)$ on the circle hence $\dfrac{\pi}{2}$ at centre.

$\therefore$ $AM = \dfrac{AB}{2} = \dfrac{\sqrt{64+36}}{2} = \dfrac{10}{2} = 5$



$$\text{Radius} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Hence, the correct answer is (A).

34. Let $x = \dfrac{3-5i}{2}$; $2x = 3 - 5i$

$$(2x-3)^2 = (5i)^2$$

$$4x^2 - 12x + 9 = 25i^2$$

$\Rightarrow$ $4x^2 - 12x + 34 = 0 \Rightarrow 2x^2 - 6x + 17 = 0$

$$2x^3 + 2x^2 - 7x + 72 = (2x^2 - 6x + 17)(x+4) + 4$$

Hence, the correct answer is (D).

35. Let $x = \text{cis}\alpha$ $y = \text{cis}\beta$ $z = \text{cis}\gamma$

Clearly, $x + y + z = 0$, $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 0$

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

$\Rightarrow$ $\text{cis}2\alpha + \text{cis}2\beta + \text{cis}2\gamma = 0$

$\Rightarrow$ $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$$1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$$

$\Rightarrow$ $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{3}{2}$

Hence, the correct answer is (B).

36. (Using $z\bar{z} = |z|^2$)

$$|z - |z+1||^2 = |z + |z-1||^2$$

$\Rightarrow$ $(z + \bar{z})(|z+1| + |z-1| - 2) = 0$

$\Rightarrow$ z lies on y-axis or Z lies on line segment joining the points $(-1, 0)$ and $(1, 0)$

Hence, the correct answer is (D).

37. $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$

$$|z_1 + z_2| + |z_1 - z_2|$$

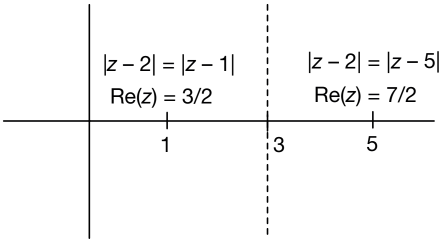
$$= \sqrt{2 + 2\cos(\alpha - \beta)} + \sqrt{2 - 2\cos(\alpha - \beta)}$$

Let $\alpha - \beta = \theta$

$$= 2\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2} = 2\sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

Hence, the correct answer is (B).

38.



draw the locus Z in Argand plane.

$$\text{Re}(z) \in \left\{\frac{3}{2}, \frac{7}{2}\right\}$$

Hence, the correct answer is (C).

39. Let $z = a + ib$, $b \neq 0$ where $\text{Im}(z) = b$

$$z^5 = (a+ib)^5 = a^5 + {}^5C_1 a^4 bi + {}^5C_2 a^3 b^2 i^2 + {}^5C_3 a^2 b^3 i^3 + {}^5C_4\, ab^4 i^4 + i^5 b^5$$

$$\text{Im}(z^5) = 5a^4 b - 10a^2 b^3 + b^5$$

$$y = \frac{\text{Im}(z^5)}{\text{Im}^5(z)} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$$

Let $\left(\dfrac{a}{b}\right)^2 = x$ (say), $x \in R^+$;

$$y = 5x^2 - 10x + 1$$

$$= 5(x^2 - 2x) + 1 = 5[(x-1)^2] - 4$$

Hence $y_{\min} = -4$.

Hence, the correct answer is (C).

40. We have

$$\left|\frac{z_1}{2} + \frac{z_2}{3} + \frac{z_1}{4} + \frac{z_1}{5}\right| = \frac{k}{60}\,|z_1 z_2 z_3 z_4|$$

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4}\right|$$

Now, $z_1\bar{z}_1 = 2$, $z_2\bar{z}_2 = 3$, $z_3\bar{z}_3 = 4$ and $z_4\bar{z}_4 = 5$

So, $k = \dfrac{60}{|z_1 z_2 z_3 z_4|} = \dfrac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}}$

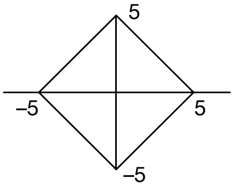
$= \sqrt{30} = |z_4 z_1 z_2|$

Note for objective take

$z_1 = \sqrt{2}\,;\ z_2 = \sqrt{3}\,;\ z_3 = 2;\ z_4 = \sqrt{5}$

Hence, the correct answer is (C).

41.



$z_1 = \bar{z}$

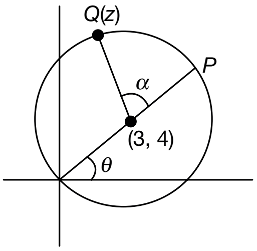
$|z - \bar{z}| + |z + \bar{z}| = 10 \Rightarrow 2|y| + 2|x| = 10$

$\Rightarrow \quad |x| + |y| = 5$

Area $= 4\left(\dfrac{1}{2} 5.5\right) = 50$

Hence, the correct answer is (C).

42. $|z - 3 - 4i| = 5 \Rightarrow (x-3)^2 + (y-4)^2 = 25$



$p(10e^{i\theta})$ rotation theorem about centre (3, 4)

$\tan\theta = \dfrac{3}{4}$

$p\left(10\left(\dfrac{3}{5} + \dfrac{4i}{5}\right)\right) \quad \dfrac{z_Q - (3+4i)}{R} = \dfrac{z_P - 3 - 4i}{R} e^{i\alpha}$

$\Rightarrow \quad z_P = 6 + 8i$

$z_Q = (3+4i) + (3+4i)e^{i\alpha}$

$= (3+4i)\left(1 + \dfrac{4}{5} + \dfrac{3}{5}i\right) = 3 + 9i$

R is $(3 + 7i)$

Hence, the correct answer is (B).

43. $|8z_2z_3 + 27z_3z_1 + 64z_1z_2|$

$= |z_1|\,|z_2|\,|z_3| \left|\dfrac{8}{z_1} + \dfrac{27}{z_2} + \dfrac{64}{z_3}\right|$

$= (2)(3)(4)\left|\dfrac{8\bar{z}_1}{|z_1|^2} + \dfrac{27\bar{z}_2}{|z_2|^2} + \dfrac{64\bar{z}_3}{|z_3|^2}\right|$

$= 24|\,2\bar{z}_1 + 3\bar{z}_2 + 4\bar{z}_3\,| = 24\left|\overline{2z_1 + 3z_1 + 4z_1}\right|$

$= 24|2z_1 + 3z_2 + 4z_3|$

$= (24)\,(4) = 96$

Hence, the correct answer is (D).

44. $0 = \dfrac{1-z}{1+z} + \dfrac{1-\bar{z}}{1+\bar{z}} \Rightarrow \dfrac{(1-z)(1+\bar{z}) + (1-\bar{z})(1+z)}{(1+z)(1+\bar{z})} = 0$

$\Rightarrow \quad \dfrac{2(1-|z|^2)}{|1+z|^2} = 0 \Rightarrow |z| = 1$

Hence, the correct answer is (B).

45. $\left|\dfrac{1}{z_2} + \dfrac{1}{z_2}\right| = \left|\dfrac{1}{z_2} - \dfrac{1}{z_1}\right|$

$\Rightarrow \quad |z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad z_1\bar{z}_2 + z_2\bar{z}_1 = 0 \Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary

$\Rightarrow \quad \text{Arg}\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2} \Rightarrow \angle POQ = \dfrac{\pi}{2}$

Circumcentre of ΔPOQ is the midpoint of PQ.

Hence, the correct answer is (B).

46. Let $A = 1 + 3\alpha + 5\alpha^2 + 7\alpha^3 + \ldots + 11\alpha^5 + 13\alpha^6$

$\alpha A = \alpha + 3\alpha^2 + 5\alpha^3 + 7\alpha^4 + \ldots + 11\alpha^6 + 13\alpha^7$

$(1-\alpha)\,A = 1 + 2\alpha + 2\alpha^2 + 2\alpha^3 + \ldots + 2\alpha^6 - 13\alpha^7$

$= -12 + 2(\alpha + \alpha^2 + \ldots + \alpha^6) = -14$

$A = -\dfrac{14}{1-\alpha}$

Hence, the correct answer is (C).

47. It is given that $|z+4| \le 3$. Hence the greatest value of $|z + 1|$ is 6.

$(\because |z+1| = |z+4-3| \le |z+4| + |3|) \Rightarrow |z+1| \le 6$

Since the least value of the modulus of a complex number is zero, therefore

$|z+1| = 0 \Rightarrow z = -1 \Rightarrow |z+4| = |-1+4| = 3$

$\Rightarrow \quad |z + 4| \le 3$ is satisfied by $z = -1$. Therefore the least value of $|z+1|$ is 0

Hence, the correct answer is (C).

48. $\left|\dfrac{1+x^2}{2x}\right| \le 1$

$\Rightarrow \quad \dfrac{1+x^2}{2|x|} \le 1 \Rightarrow \dfrac{1+|x^2|}{2|x|} - 1 \le 0 \Rightarrow \dfrac{(|x|-1)^2}{|x|} \le 0$

$\therefore \quad |x| > 0,$

$\therefore \quad (|x| - 1)^2 \le 0$

$\Rightarrow \quad (|x| - 1)^2 = 0$

$\therefore \quad |x| = 1 \Rightarrow x = \pm 1$

$\therefore \quad x = 1$ $\qquad (\because x > 0)$

$\left(\dfrac{1+i}{1-i}\right)^n = \dfrac{2}{\pi} \times \sin^{-1}(1) = \dfrac{2}{\pi} \times \dfrac{2}{\pi} = 1$

$\Rightarrow \quad \left(\dfrac{(1+i)^2}{2}\right)^n = 1 \quad \Rightarrow \quad (i)^n = 1$

$\therefore \quad n = 4$ (least positive value)

Hence, the correct answer is (B).

49. $az^2 + z + 1 = 0$ $\qquad$ …(1)

Taking conjugate of both sides, $\overline{az^2 + z + 1} = 0$

$\Rightarrow \quad \bar{a}(\bar{z})^2 + \bar{z} + \bar{1} = 0$

$\bar{a}\,z^2 - z + 1 = 0$

(since $\bar{z} = -z$ as ‘z’ is purely imaginary) $\qquad$ …(2)

Eliminating ‘z’ from both the equations, we get $(\bar{a}-a)^2+2(a+\bar{a})=0$

Let $a=\cos\theta+i\sin\theta$

(since $|a|=1$) so that $(-2i\sin\theta)^2+2(2\cos\theta)=0$

$\Rightarrow \quad \cos\theta=\dfrac{-1\pm\sqrt{1+4}}{2}$

Only feasible value of $\cos\theta$ is $\dfrac{\sqrt{5}-1}{2}$

Hence $a=\cos\theta+i\sin\theta$, where $\theta=\cos^{-1}\left(\dfrac{\sqrt{5}-1}{2}\right)$

Hence, the correct answer is (A).

50. Required data

$$=-i\left(\cos\frac{2k\pi}{11}+i\sin\frac{2k\pi}{11}\right)=-i\left(e^{i\frac{2\pi}{11}}\right)^k$$

Let $e^{i\frac{2\pi}{11}}=z$

$$\therefore \quad \sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}-i\cos\frac{2k\pi}{11}\right)=-i\sum_{k=1}^{10}z^k$$

$$=(-i)[z+z^2+z^3+\ldots+z^{10}]$$

$$=-i\left[\frac{z(z^{10}-1)}{z-1}\right]=-i\left[\frac{z^{11}-z}{z-1}\right]=i \qquad (\because z^{11}=1)$$

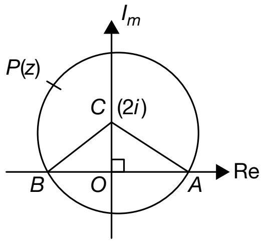
Hence, the correct answer is (D).

51. $CA=CB=2\sqrt{2}\times OC=2$

$\Rightarrow \quad OA=OB=2$

$\Rightarrow \quad A=2+0i,\ B=-2+0i$

Clearly, $\angle BCA=\dfrac{\pi}{2}$



$\Rightarrow \quad \angle BPA=\dfrac{\pi}{4}$

$\Rightarrow \quad \text{Arg}\left(\dfrac{z-2}{z+2}\right)=\dfrac{\pi}{4}$

Hence, the correct answer is (B).

52. $|a^2-2a|<3$

$\Rightarrow \quad -3<a^2-2a<3$

$\Rightarrow \quad -3+1<a^2-2a+1<3+1$

$\Rightarrow \quad -2<(a-1)^2<4$

$\therefore \quad 0\le(a-1)^2<4\Rightarrow -2<a-1<2$ or $-1<a<3$

But $a\in R^+$

$\therefore \quad 0<a<3\Rightarrow a\in(0,3)$

Hence, the correct answer is (D).

53. The given equation simplifies $x^2+2x(a+b)+3ab=0$, whose roots are given table ω, ω^2

Hence $a+b=\dfrac{1}{2}$, $ab=\dfrac{1}{3}$.

So a,b are roots of $x^2-x\left(\dfrac{1}{2}\right)+\dfrac{1}{3}=0$.

Hence, the correct answer is (B).

54. Since $|z-1|=1\Rightarrow z-1=\text{cis}\theta$

$\Rightarrow \quad z=(1+\cos\theta)+i\sin\theta=2\cos\dfrac{\theta}{2}\,\text{cis}\,\dfrac{\theta}{2}$

$$\therefore \quad \frac{1}{z}-\frac{1}{2}=\frac{\text{cis}\left(-\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}}-\frac{1}{2}=-\frac{i}{2}\tan\frac{\theta}{2}$$

which is purely imaginary

Hence, the correct answer is (B).

55. If z_1, z_2, z_3 are collinear and $z_1+bz_2+cz_3=0$, then $a+b+c=0$. Hence

$$1+|\sin\theta|+|\cos\theta|-1-\sqrt{2}=0$$

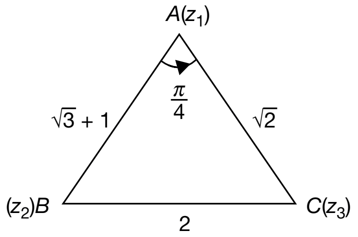
$\Rightarrow \quad |\sin\theta|+|\cos\theta|=\sqrt{2}$

$\Rightarrow \quad \theta=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}$

Hence, the correct answer is (B).

56. $\cos A=\dfrac{2+(\sqrt{3}+1)^2-4}{2\cdot\sqrt{2}(\sqrt{3}+1)}=\dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos A=+\dfrac{1}{\sqrt{2}}\Rightarrow A=\dfrac{\pi}{4}$



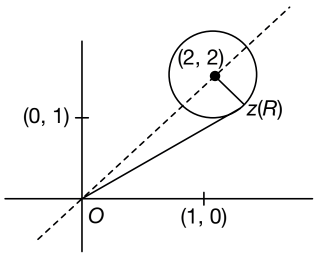
$$\therefore \quad \frac{z_3-z_1}{z_2-z_1}=\left|\frac{z_3-z_1}{z_2-z_1}\right|e^{\frac{i\pi}{4}}$$

$$\Rightarrow \quad \left(\frac{z_3-z_1}{z_2-z_1}\right)^4=\left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^4 e^{i\pi}$$

$$\Rightarrow \quad \left(\frac{z_3-z_1}{z_2-z_1}\right)^4=-\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4$$

Hence, the correct answer is (A).

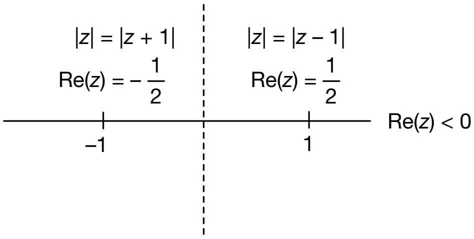
57. $|z|=OR=\sqrt{8-1}=\sqrt{7}$



Hence, the correct answer is (C).

58. Let $|z| = |z-1|$ if $\text{Re}(Z) > 0 \Rightarrow$ lies line $z = \frac{1}{2}$

$$\Rightarrow \quad z + \bar{z} = \frac{1}{2} + \frac{1}{2} = 1$$



$$z + \bar{z} = \frac{-1}{2} - \frac{1}{2} = -1$$

Hence, the correct answer is (A).

59. $|z|^2 + 2(z+\bar{z}) + 3i(z-\bar{z}) + 4 = 0$

put $z = x + iy$

$$x^2 + y^2 + 4x - 6y + 4 = 0$$

$$c(-2, 3)\ R = 3,\ |z + 2 - 3i| = 2$$

Let $w = (z + 3 + 2i) = z + 2 - 3i + 1 + 5i$

$$\Rightarrow \quad |w - (1+5i)| = |z + 2 - 3i| = 3.$$

Hence, the correct answer is (C).

60. Let $i\log\frac{x-i}{x+i} - \pi + 2\tan^{-1}x = k$

$$\Rightarrow \quad \log\left(\frac{x+i}{x-i}\right) = (k + \pi - 2\tan^{-1}x)i = i\theta$$

$$\Rightarrow \quad \frac{x+i}{x-i} = e^{i\theta} \Rightarrow x = \cot\frac{\theta}{2} \Rightarrow \theta = 2\cot^{-1}x$$

$$\therefore \quad k + \pi - 2\tan^{-1}x = 2\cot^{-1}x$$

$$\Rightarrow \quad k = 0$$

Hence, the correct answer is (A).

61. $\left|\frac{z_1}{z_1}\right| = 1 \Rightarrow |z_1| = |z_2| = r_1$ as $z_1 = r_1(\cos\theta_1 + \sin\theta_1)$

$$\text{Arg}(z_1z_2) = 0 \Rightarrow z_2 = r_1\,(\cos(-\theta_1) + i\sin(-\theta_1))$$

$$\Rightarrow \quad z_2 = \bar{z}_1 \Rightarrow \bar{z}_2 = z_1 \Rightarrow |z_2|^2 = z_1 \times z_2$$

Hence, the correct answer is (B).

62. $$\begin{vmatrix} z & \bar{z} & 1 \\ \bar{z} & z & 1 \\ 1 & z & \bar{z} \end{vmatrix} \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

Hence, the correct answer is (C).

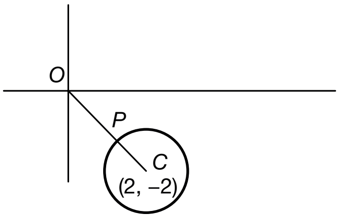
63. $a = |\omega z - \omega^3 - \omega^2| = |\omega|\ |z - \text{w}^2 - \omega|$

$$\Rightarrow \quad |\omega|\ |z+1| = a$$

$$\Rightarrow \quad |z-1| + 2 \geq a$$

Hence, the correct answer is (D).

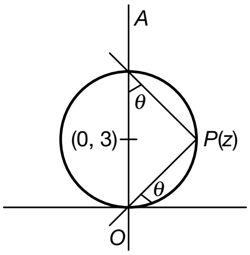
64. $OP = OC - CP$



$$P(2\sqrt{2}-1)e^{i\left(-\frac{\pi}{4}\right)} = (2\sqrt{2} - 1)\left(\frac{1-i}{\sqrt{2}}\right)$$

Hence, the correct answer is (A).

65.



$$r = OA\sin\theta = 6\sin\theta$$

$$z = 6\sin\theta\,(\cos\theta + i\sin\theta)$$

$$\frac{6}{z} = \frac{e^{-i\theta}}{\sin\theta} = \cot\theta - i \quad \Rightarrow \quad \cot\theta - \frac{6}{z} = i$$

Hence, the correct answer is (A).

66. $|z + 16|^2 = 16|z+1|^2$

$$\Rightarrow \quad (z+16)\,(\bar{z}+16) = 16(z+1)(\bar{z}+1)$$

$$\Rightarrow \quad z\bar{z} + 16z + 16\bar{z} + 256$$

$$= 16z\bar{z} + 16z + 16\bar{z} + 16$$

$$\Rightarrow \quad z\bar{z} = 16 \Rightarrow |z|^2 = 16 \Rightarrow |z| = 4$$

Hence, the correct answer is (A).

67. If $z_1 + \sqrt{z_1^2 - z_2^2} = u$ and $z_1 - \sqrt{z_1^2 - z_2^2} = v$,

We have

$$|u|^2 + |v|^2 = \frac{1}{2}|u+v|^2 + \frac{1}{2}|u-v|^2$$

$$= 2|z_1|^2 + 2|z_1^2 - z_2^2|$$

and so

$$(|u| + |v|)^2 = 2\{|z_1|^2 + |z_1^2 - z_2^2| + |z_2|^2\}$$

$$= |z + z_2|^2 + |z_1 - z_2|^2 + 2|z_1^2 - z_2^2|$$

$$= (|z_1 + z_2| + |z_1 - z_2|)^2$$

Hence, the correct answer is (D).

68. Let $z_1 = \cos\phi_1 + i\sin\phi_1$ and $z_2 = \cos\phi_2 + i\sin\phi_2$ be the roots of $z^2 + az + b = 0$

$$z_1 + z_2 = (-a) \text{ and } z_1z_2 = b$$

$$-2\cos\left(\frac{\phi_1+\phi_2}{2}\right)\left(\cos\frac{\phi_1+\phi_2}{2} + i\sin\frac{\phi_1+\phi_2}{2}\right) = a$$

$$\Rightarrow \quad \text{Arg}(a) = \frac{\phi_1+\phi_2}{2}$$

$$\text{Arg}(b) = \phi_1 + \phi_2$$

$$\therefore \quad \text{Arg}(b) = 2\,\text{Arg}(a)$$

Also $|z_1z_2| = |b| = 1$ and $|a| \leq 2$

Hence, the correct answer is (A).

69. $z\bar{z} - 1 = \left(\frac{z_1 - z_2}{z_1\bar{z}_2 - 1}\right)\left(\frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 z_2 - 1}\right) - 1$

$$= \frac{z_1\bar{z}_1 + z_2\bar{z}_2 - z_1z_2\bar{z}_1\bar{z}_2 - 1}{z_1\bar{z}_2\bar{z}_1z_2 - z_1\bar{z}_2 - \bar{z}_1z_2 + 1} = \frac{(1 - z_2\bar{z}_2)(z_1\bar{z}_1 - 1)}{(z_1\bar{z}_2 - 1)(\bar{z}_1z_2 - 1)}$$

$$\Rightarrow \quad |z|^2-1 = = \frac{(1-|z_2|^2)(|z_1|^2-1)}{|z_1\bar{z}_2-1|^2}$$

$$\therefore \quad |z_1|^2-1 \le 0 \Rightarrow |z_1| \le 1$$

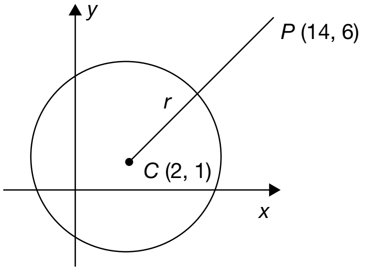
Hence, the correct answer is (B).

70. $|z-14-6i|$

$$= |(z-2-i)-(12+5i)| \le |z-2-i| + |12+5i|$$

$$\Rightarrow \quad |z-14-6i| \le 5+13 = 18$$

Therefore, the complete solution can be obtained geometrically.



$$|z-14-6i|_{\max} = CP + r$$
$$= 13+5 = 18$$
$$|z-14-6i|_{\min} = CP - r$$
$$= 13-5 = 8$$

Hence, the correct answer is (A).

71. $z_1^3 - 3z_1z_2^2 + 3iz_1^2z_2 - iz_2^3 = 2+11i$

$$\Rightarrow \quad (z_1+iz_2)^3 = 2+11i$$

Similarly, $(z_1-iz_2)^3 = 2-11i$

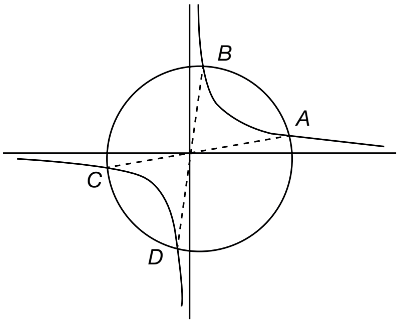
$$\left|z_1^2+z_2^2\right| = |(z_1+iz_2)(z_1-iz_2)|$$

$$= \left|(2+11i)^{\frac{1}{3}}(2-11i)^{\frac{1}{3}}\right| = 5$$

Hence, the correct answer is (C).

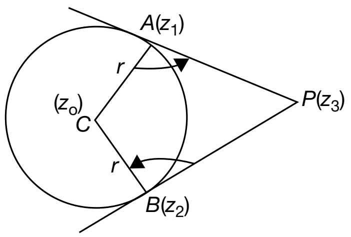
72. $z^2 = (\bar{z})^2 + 4i \Rightarrow \left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right) = 1$

or $\quad xy = 1$ (where $z = x+iy$)



The circle $x^2+y^2=4$ intersects the rectangular hyperbola in four points, which are symmetrical about the origin in parts.
Hence, the correct answer is (B).

73.



$$\frac{z_3-z_1}{z_0-z_1} = \left(\frac{PA}{AC}\right)i \text{ and } \frac{z_0-z_2}{z_3-z_2} = \left(\frac{BC}{BP}\right) \qquad \ldots(1)$$

$$\frac{(z_3-z_1)(z_0-z_2)}{(z_0-z_1)(z_3-z_2)} = \left(\frac{PA}{AC}\times\frac{BC}{PB}\right)(-1) = -1.$$

Hence, the correct answer is (B).

74. $Z^{2009} = \bar{z} \Rightarrow |z| = 0$ or $|z| = 1$

$$Z^{2010} = z\bar{z} = |z|^2$$

$z^{2010} = 0;\ z^{2010} = 1$; 2011roots.

Hence, the correct answer is (D).

75. $\cos\alpha + i\sin\alpha$ is a root of

$$\alpha_n\left(\frac{1}{z}\right)^n + \alpha_{n-1}\left(\frac{1}{z}\right)^{n-1} + \ldots + \alpha_2\left(\frac{1}{z}\right)^2 + \alpha_1\left(\frac{1}{z}\right) + 1 = 0$$

put $z = \cos\alpha + i\sin\alpha$

By equating the real parts on both sides,

$$\alpha_n\cos n\alpha + \alpha_{n-1}\cos(n-1)\alpha + \ldots + \alpha_1\cos\alpha + 1 = 0$$

Hence, the correct answer is (C).

76. $\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\ldots = \frac{1}{2}\left(1+\frac{3}{4}+\frac{9}{16}+\ldots\infty\right)$

$$= \frac{1}{2}\left(\frac{1}{1-\frac{3}{4}}\right) = \frac{1}{2}\times 4 = 2$$

$$\Rightarrow \quad \omega+\omega^2 = -1 = i^2$$

Hence, the correct answer is (B).

77. $z_1\left(\frac{\bar{z}_2z_3-z_2\bar{z}_3}{2i}\right)+z_2\left(\frac{\bar{z}_3z_1-z_3\bar{z}_1}{2i}\right)+z_3\left(\frac{\bar{z}_1z_2-z_1\bar{z}_2}{2i}\right)$

$$= \frac{1}{2i}\times 0 = 0$$

Hence, the correct answer is (A).

78. Let $z = i\beta\ (\beta\in R)$ be a root, then

$$-\beta^2+i\beta+\alpha = 0 \Rightarrow \alpha = \beta^2-i\beta$$

Now, as $|\alpha| = 1 \Rightarrow \beta^4+\beta^2 = 1 \Rightarrow \beta^2 = \frac{-1+\sqrt{5}}{2}$.

Hence, the correct answer is (D).

79. $|z(\theta,2\theta)| = |\cos\theta + e^{i2\theta}\sin\theta|$

$$= |\cos\theta+\sin\theta\cos2\theta+i\sin\theta\sin2\theta|$$

$$= \sqrt{(\cos\theta+\sin\theta\cos2\theta)^2+\sin^2\theta\sin^2 2\theta}$$

$$= \sqrt{1+\frac{1}{2}\sin4\theta} \in \left[\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}}\right]$$

Hence, the correct answer is (B).

80. $\operatorname{Arg}(z) - \operatorname{Arg}(z+1) - \operatorname{Arg}(z-1)$

$$= \operatorname{Arg}\left(\frac{z}{z^2-1}\right) = \operatorname{Arg}\left(\frac{z}{z^2 - z\bar{z}}\right)$$

$$= \operatorname{Arg}\left(\frac{1}{z-\bar{z}}\right) = \operatorname{Arg}\text{ (purely imaginary number).}$$

Hence, the correct answer is (C).

81. $\dfrac{a}{|z_2 - z_3|} = \lambda \Rightarrow \dfrac{a^2}{z_2 - z_3} = \lambda^2(\bar{z}_2 - \bar{z}_3)$, etc.

so $\displaystyle\sum \frac{a^2}{z_2 - z_3} = 0$

Hence, the correct answer is (D).

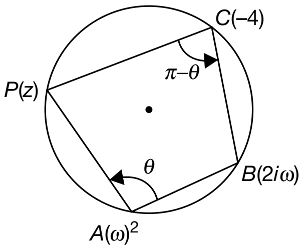
82. By applying rotation formula at A and C,

$$\frac{z-\omega^2}{2i\omega - \omega^2} = \frac{PA}{AB}e^{i\theta}\,;\ \frac{2i\omega+4}{z+4} = \frac{BC}{PC}e^{i(\pi-\theta)}$$

By multiplying we get,

$$\frac{z-\omega^2}{2i\omega - \omega^2} \times \frac{2i\omega+4}{z+4} = -1$$

$$\Rightarrow \quad z = -2i\omega$$



Hence, the correct answer is (B).

83. Given equation can be written as $(144z^2 + 60z + 4)(144z^2 + 60z + 6) = 48 \Rightarrow t(t+2) = 48$,

where $t = 144z^2 + 60z + 4$.

Therefore, $t = 6$ or -8.

Hence, $z = \dfrac{-5 \pm \sqrt{33}}{24}, \dfrac{-5 \pm i\sqrt{23}}{24}$

Hence, the correct answer is (C).

84. $z' = \dfrac{1+z^2}{z} \cdot \dfrac{\bar{z}+z}{z\bar{z}} = z + \bar{z}$,

which is purely real $\Rightarrow \operatorname{Im}(z') = 0$.

Hence, the correct answer is (D).

85. Let $x^{2007} - 1 = (x^2+1)(x^2+x+1)\,p(x) + f(x)$.

$f(x)$ in the form of $ax^3 + bx^2 + cx + d$

Substitute $x = \pm i, w, w^2$ for getting $f(x)$.

Hence, the correct answer is (D).

86. Let $\alpha = \operatorname{cis}\dfrac{2K\pi}{7}$ $(K = 0 \text{ to } 6)$

$(\alpha^7 = 1)$

$$S = \alpha^{2009} + 3\alpha^{2010} + 5\alpha^{2011} + \ldots + 13\alpha^{2015}$$

$$S = 1 + 3\alpha + 5\alpha^2 + \ldots + 13\alpha^6 \text{(AGP)}$$

$$\alpha S = \alpha + 3\alpha^2 + \ldots + 11\alpha^6 + 13\alpha^7$$

$$S(1-\alpha) = 1 + 2\alpha + \ldots + 2\alpha^6 - 13\alpha^7$$

$$S(1-\alpha) = 1 + \frac{2\alpha(1-\alpha^6)}{1-\alpha} - 13\alpha^7$$

$$S = \frac{-14}{1-\alpha} = \frac{-14}{1-\operatorname{cis}\dfrac{2K\pi}{7}} = -7\left[1 + i\cot\frac{K\pi}{7}\right]$$

Hence, the correct answer is (C).

87. $\dfrac{z^{10}}{(1-z)^{10}} = 1 \Rightarrow \dfrac{z}{1-z} = 1^{\frac{1}{10}} = \cos\dfrac{2K\pi}{10}$ $(K = 0 \text{ to } 9)$

$$\Rightarrow \quad z = \frac{\operatorname{cis}\dfrac{2K\pi}{10}}{1+\operatorname{cis}\dfrac{2K\pi}{10}} = \frac{1}{2} + \frac{i}{2}\tan\frac{K\pi}{10}$$

Hence, the correct answer is (A).

88. Let $P = \displaystyle\sum_{i=0}^{n-1} \frac{\alpha_i}{3-\alpha_i} = -\sum_{i=0}^{n-1} \frac{(3-\alpha_i)-3}{(3-\alpha_i)}$

$$= 3\sum_{i=0}^{n-1} \frac{1}{3-\alpha_i} - \sum_{i=0}^{n-1} 1 \qquad \ldots(1)$$

$$z^n - 1 = \prod_{i=0}^{n-1}(z-\alpha_i) \Rightarrow \log(z^n - 1) = \sum_{i=0}^{n-1} \ln(z-\alpha_i)$$

By differentiating both sides with respect to Z,

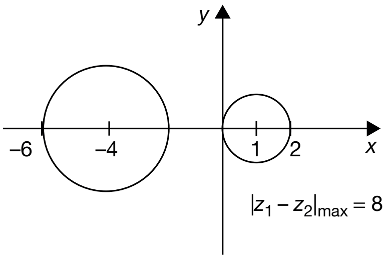
$$\frac{nZ^{n-1}}{Z^n - 1} = \sum_{i=0}^{n-1} \frac{1}{z-\alpha_i}$$

Put $Z = 3 \Rightarrow \dfrac{n3^{n-1}}{3^{n-1}} = \displaystyle\sum_{i=0}^{n-1} \frac{1}{3-\alpha_i}$

$$P = \frac{3n3^{n-1}}{3^n - 1} - n = \frac{n3^n}{3^n - 1} - n = \frac{n}{3^n - 1}$$

Hence, the correct answer is (A).

89. Maximum distance between two curves lies along their common normal.



Hence, the correct answer is (A).

90. $|iz + 1 + 2i| = |i(z - (1+i)) + 3i| \le |z - (1+i)| + 3 < 2 + 3$

Hence, the correct answer is (C).

91. $x = \omega, \omega^2 \Rightarrow (x^3 - 1)^2 = 0 \Rightarrow (x-1)^2(x^2+x+1)^2 = 0$

$$\omega^r + \omega^{2r} = \begin{cases} 2, & \text{if } r \text{ is a multiple of 3} \\ -1, & \text{if } r \text{ is not a multiple of 3} \end{cases}$$

so $\sum_{r=1}^{50}(x^r + x^{2r})^3 = 8\times 16 - 1\times 34 = 94$

Hence, the correct answer is (D).

92. $\frac{z_1 - z_2}{z_3 - z_2} = \sqrt{2}e^{\frac{i\pi}{4}}$

$= 1 + i \Rightarrow z_1 - z_2 = (1 + i)\,(z_3 - z_2)$

$\Rightarrow \quad z_1 - (1 + i)\,z_3 = -\,iz_2$

$\Rightarrow \quad z_2 = iz_1 + (1 - i)\,z_3 = z_3 + i(z_1 - z_3)$

Hence, the correct answer is (C).

93. Take $z_1 = i, z_2 = \omega,\ \text{Arg}(z_1) = \frac{\pi}{2},\ \text{Arg}(z_2) = \frac{2\pi}{3}$

$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{7\pi}{6}$ should be equivalent to $\frac{7\pi}{6} - 2\pi$

Hence, the correct answer is (C).

94. $\left(\frac{x}{y} + \frac{y}{x} - \frac{i}{m}\right)^2 = \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16}$

$\text{LHS} = \left(\frac{x}{y} + \frac{y}{x}\right)^2 - \frac{2i}{m}\left(\frac{x}{y} + \frac{y}{x}\right) - \frac{1}{m^2}$

$= \frac{x^2}{y^2} + \frac{y^2}{x^2} + 2 + \frac{4}{m}\cdot\frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) - \frac{1}{m^2}$

$m = 4$

Hence, the correct answer is (C).

95. Clearly, origin is the centre of the polygon.

Let $z_1 = re^{i\theta},\ \overline{z}_1 = re^{-i\theta}$

$\text{Re}(z) = r\cos\theta, \text{Im}\,(\overline{z}_1) = -\,r\sin\theta$

$\Rightarrow \quad -\frac{\sin\theta}{\cos\theta} = 1 - \sqrt{2} \Rightarrow \tan\theta = \sqrt{2} - 1$

$\Rightarrow \quad \theta = \frac{\pi}{8}$. If n be the number of sides, then $\theta = \frac{\pi}{n}$

$\Rightarrow \quad n = 8.$

Hence, the correct answer is (B).

96. $z^2 + az + b = 0$. Let z_0 is the purely imaginary root of the equation.

Then $z_0^2 + az_0 + b = 0; z_0 + \overline{z}_0 = 0 \Rightarrow \overline{z}_0 = -z_0$

We have $z_0^2 + az_0 + b = 0; \Rightarrow \overline{z}_0^2 - a\overline{z}_0 + \overline{b} = 0.$

Now, $z_0^2 + az_0 + b = 0$ and $z_0^2 - a\overline{z}_0 + \overline{b} = 0.$

We should have a common root. Find common root.

Hence, the correct answer is (A).

97. Let $z = x + iy$

$\frac{2z + 1}{iz + 1} = \frac{2(x + iy) + 1}{i(x + iy) + 1} = \frac{(2x + 1) + 2iy}{(1 - y) + ix}$

$= \frac{\left[(2x + 1) + 2iy\right]\left[(1 - y) - ix\right]}{(1 - y)^2 + x^2}$

Since, $\text{Im}\left(\frac{2z + 1}{iz + 1}\right) = -4$, we get

$\frac{2y(1 - y) - x(2x + 1)}{x^2 + (1 - y)} = -4$

$\Rightarrow \quad 2x^2 + 2y^2x - 6y + 4 = 0$, which represents a circle.

Hence, the correct answer is (C).

98. Let $z_k = w^k$ where, $w = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$

Therefore, $z_1z_2z_3z_4 = ww^2w^3w^4 = w^{10}$

$= \cos\frac{10\pi}{10} + i\sin\frac{10\pi}{10}$

(Since, de Moivre's theorem)

$= \cos\pi + i\sin\pi = -1.$

Hence, the correct answer is (A).

99. We have $(1 + i)^2 = \left(\frac{p}{a} + \frac{q}{b} + \frac{r}{c}\right)^2$

$\Rightarrow \quad 1 - 1 + 2i = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + 2\left(\frac{qr}{bc} + \frac{rp}{ca} + \frac{pq}{ab}\right)$

$\Rightarrow \quad 2i = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + 2\frac{pqr}{abc}\left(\frac{a}{p} + \frac{q}{b} + \frac{r}{c}\right)$

$\Rightarrow \quad 2i = \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} + \frac{2abc}{pqr}(0) \Rightarrow \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 2i$

Hence, the correct answer is (C).

100. Since $|z_1| = |z_2| = |z_3| = 1,$

we get $z_1\overline{z}_1 = z_2\overline{z}_2 = z_3\overline{z}_3 = 1$

Therefore,

$1 = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| \Rightarrow \overline{z}_1 = \frac{1}{z_1}, \overline{z}_2 = \frac{1}{z_2}, \overline{z}_3 = \frac{1}{z_3}$

$= |\overline{z}_1 + \overline{z}_2 + \overline{z}_3| = |z_1 + z_2 + z_3|$

Hence, the correct answer is (A).

101. $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right| \Rightarrow \frac{|z_1 - z_3|}{|z_2 - z_3|} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\Rightarrow \quad |z_1 - z_3| = |z_2 - z_3|$

Again $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

$\frac{z_1 - z_3}{z_2 - z_3} - 1 = \frac{1 - i\sqrt{3}}{2} - 1 \Rightarrow \frac{z_1 - z_3}{z_2 - z_3} = \frac{-1 - i\sqrt{3}}{2}$

$\left|\frac{z_1 - z_2}{z_2 - z_3}\right| = \left|\frac{-1}{2}\frac{-i\sqrt{3}}{2}\right|$

$\Rightarrow \quad \frac{|z_1 - z_2|}{|z_2 - z_3|} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\Rightarrow \quad |z_1 - z_2| = |z_2 - z_3|$

Therefore, z_1, z_2 and z_3 are the vertices of an equilateral triangle.

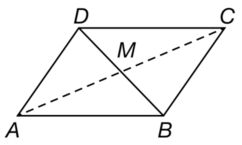
Hence, the correct answer is (C).

102. Let the affix of A be Z.

$M = \text{Midpoint of } BD = \left(\frac{5}{2}, 0\right)$

$\angle AMB = 90°$

$\text{Arg}\left(\frac{1 - 2i - \frac{5}{2}}{z - \frac{5}{2}}\right) = \frac{\pi}{2}$



$$\Rightarrow \quad \frac{1-2i-\frac{5}{2}}{z-\frac{5}{2}} = \left|\frac{1-2i-\frac{5}{2}}{z-\frac{5}{2}}\right| \text{cis}\,\frac{\pi}{2}$$

$$= \frac{|BM|}{|AM|}i = \frac{\left|\frac{BD}{2}\right|}{\left|\frac{AC}{2}\right|}i = \frac{|BD|}{|AC|}i$$

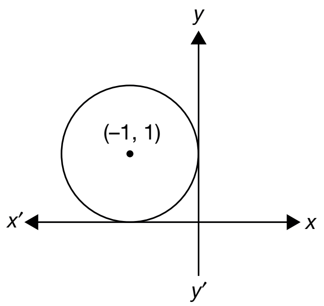
$$= \frac{1}{2}i \quad \left(\text{Since}\,|AC| = 2|BD|\right)$$

Therefore,

$$\left(\frac{-3}{2}-2i\right)\frac{2}{i} = z - \frac{5}{2} \Rightarrow z = \frac{5}{2} - \frac{3}{i} - 4 = \frac{-3}{2} + 3i$$

Hence, the correct answer is (C).

103. $|z+1-i| = 1 \Rightarrow |z-(-1+i)| = 1$. Therefore, locus of z is a circle with centre (–1, 1) and radius as 1.



The circle touches both the axes in the second quadrant. All points on this circle lie in the region,

+ve y-axis corresponding to $\text{Arg}\left(z = \frac{\pi}{2}\right)$

and –ve x-axis corresponding to $\text{Arg}(z) = \pi$.

Therefore, $\frac{\pi}{2} \le \text{Arg}(z) \le \pi$.

Hence, the correct answer is (A).

104. Let Z be a complex number satisfying

$$z^5 = |z-1|^5 \Rightarrow |z^5| = |(z-1)^5|$$

$$\Rightarrow \quad |z|^5 = |z-1|^5 \Rightarrow |z| = |z-1|$$

Thus, Z lies on the perpendicular bisector of the segment joining the origin and $A(1+i\,0)$,

i.e., Z lies on $\text{Re}(Z) = \frac{1}{2}$.

Hence, the correct answer is (A).

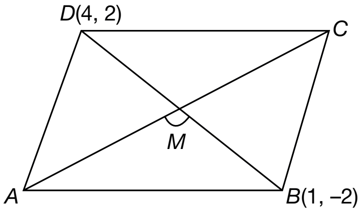
105. $||z|-|5-12i|| \le |z-(5-12i)| \Rightarrow ||z|-13| \le 1$

$$\Rightarrow \quad -1 \le |z| - 13 \le 1$$

$$\Rightarrow \quad 12 \le |z| \le 14$$

$m = 12$, $n = 14$

$$\frac{x^2+24x+1}{x} = x + \frac{1}{x} + 24 \ge 26 \quad \left(\text{Since, } x + \frac{1}{x} \ge 2\right)$$



$l = 26 = m + n$

Hence, the correct answer is (B).

106. $x = 2 + 5i \Rightarrow x - 2 = 5i \Rightarrow (x-2)^2 = (5i)^2$

$$x^2 - 4x + 4 = -25 \Rightarrow x^2 - 4x + 29 = 0 \qquad \ldots(1)$$

Therefore, $x^3 - 5x^2 + 33x - 19 = (x-1)(x^2 - 4x + 29) + 10$

$= (x-1)0 + 10$ (Since, from (1)).

Hence, the correct answer is (D).

107. Since the triangle with z_1, z_2 and z_3 as vertices is an equilateral triangle, its circumcentre and centroid will coincide.

Therefore, $z_0 = \frac{z_1+z_2+z_3}{3}$

$$\Rightarrow (3z_0)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_2z_3 + z_3z_1 + z_1z_2) \qquad \ldots(1)$$

Since the triangle is equilateral, we have

$$z_1^2 + z_2^2 + z_3^2 = (z_1z_2 + z_2z_3 + z_3z_1) \qquad \ldots(2)$$

From (1) and (2) we get,

$$9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_1^2 + z_2^2 + z_3^2)$$

$$\Rightarrow \quad 3z_0^2 = z_1^2 + z_2^2 + z_3^2$$

Hence, the correct answer is (B).

108. Let $Z_1 = \sin x + i\cos 2x$, $Z_2 = \cos x - i\sin 2x$

$$\bar{z}_1 = z_2 \Rightarrow \sin x - i\cos 2x = \cos x - i\sin 2x$$

$$\Rightarrow \quad \sin x = \cos x \text{ and } \cos 2x = \sin 2x$$

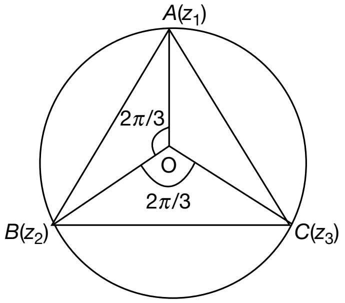
$$\Rightarrow \quad \tan x = 1 \text{ and } \tan 2x = 1$$

$$\Rightarrow \quad x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8}$$

which is not possible. Hence, there is no value for x.

Hence, the correct answer is (D).

109.



Let z_1, z_2 and z_3 are the vertices A, B and C of equilateral triangle ABC inscribed in a circle. $|z| = 2$ with centre (0, 0) and radius 2. Given $Z_1 = 1 + i\sqrt{3}$ Rotating OA about O by an angle $\frac{2\pi}{3}$ we have,

$$\frac{z-0}{1+i\sqrt{3}-0} = \frac{|z-0|}{\left|1+\sqrt{3}i-0\right|}e^{\pm\frac{2\pi}{3}}$$

$$z = \left(1+i\sqrt{3}\right)\left(\cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3}\right)$$

$$= \left(1+i\sqrt{3}\right)\left(\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

On solving, $z = -2$ or $1 - i\sqrt{3}$

Hence, the correct answer is (C).

110. $a = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} \Rightarrow a^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7$

$= \cos 2\pi + i\sin 2\pi = 1$ ...(1)

Sum of roots $= \alpha + \beta = a + a^2 + a^4 + a^3 + a^5 + a^6$

$= \frac{a(1-a^6)}{1-a} = \frac{a-a^7}{1-a} = \frac{a-1}{1-a} = -1$ ...(2)

Product of roots $= \alpha\beta = (a + a^2 + a^4)(a^3 + a^5 + a^6)$

$= a^4 + a^5 + a^7 + a^6 + a^7 + a^9 + a^7 + a^8 + a^{10}$

$= a^4 + a^5 + 1 + a^6 + 1 + a^2 + 1 + a + a^3$ (from (1))

$= 3 + a + a^2 + a^3 + a^4 + a^5 + a^6$

$= 3 + (-1) = 2$ (from (2))

Therefore, the required equation is $x^2 - x(-1) + 2 = 0$

$\Rightarrow \quad x^2 + x + 2 = 0.$

Hence, the correct answer is (A).

111. Let Arg$(w) = \theta$

Therefore, Arg$(z) = \pi - \theta$

$w = |w|(\cos\theta + i\sin\theta)$ and

$z = |z|[\cos(\pi - \theta) + i\sin(\pi - \theta)]$

$= |w|(-\cos\theta + i\sin\theta)$

$= -|w|(\cos\theta - i\sin\theta) = -\overline{w}$

Hence, the correct answer is (C).

112. $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$

$\leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$

$\leq 1 + 2 + 3 + 6 \leq 12$

Greatest value of $|z_1 + z_2 + z_3| = 12$

Hence, the correct answer is (D).

113. Let $z = (1 + \cos 2\alpha) + i\sin 2\alpha$

$= 2\cos^2\alpha + 2i\sin\alpha\cos\alpha$

$= 2\cos\alpha(\cos\alpha + i\sin\alpha)$

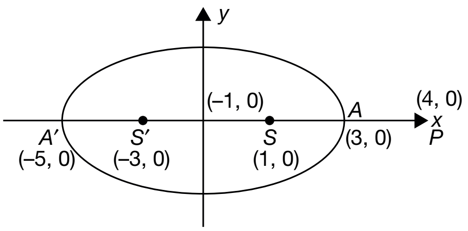
$= -2\cos\alpha(-\cos\alpha - i\sin\alpha)$

$= -2\cos\alpha[\cos(\alpha - \pi) + i\sin(a - \pi)]$ $\left(\because \frac{\pi}{2} < \alpha < \frac{3\pi}{2}\right)$

$\therefore \quad |z| = -2\cos\alpha$ and Arg$(z) = \alpha - \pi$

Hence, the correct answer is (B).

114.



Given $|z - 1| + |z + 3| \leq 8$

Therefore, z lies inside or on the ellipse whose foci are (1, 0) and (–3, 0) and vertices are (–5, 0) and (3, 0). Clearly, the minimum and maximum values of $|z - 4|$ are 1 and 9, respectively representing the distances PA and PA'. Therefore, $1 \leq |z - 4| \leq 9$.

Hence, the correct answer is (A).

115. If $|z - 25i| \leq 15$, then z lies either in the interior and or on the boundary of the circle with centre at $C(0, 25)$ and radius equal to 15.

The least argument is for point A and the greatest argument is for point B from right.

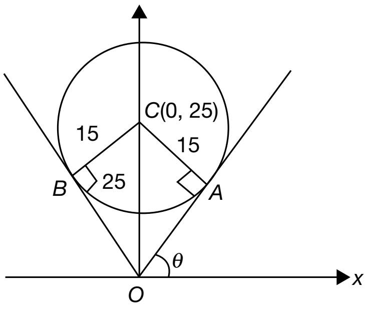
ΔOAC, $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{OA}{OC} = \frac{20}{25} = \frac{4}{5}$;

$\frac{\pi}{2} - \theta = \cos^{-1}\left(\frac{4}{5}\right)$

Now for $|z - 25i| \leq 15$

$|$Maximum Arg(z) – Minimum Arg$(z)|$

$= |\text{Arg}(B) - \text{Arg}(A)| = \angle BOA$



$= \angle BOX - \angle AOX = \frac{\pi}{2} + \frac{\pi}{2} - \theta - \theta$

$= \pi - 2\theta = 2\cos^{-1}\frac{4}{5}$

Hence, the correct answer is (B).

116. Let $\frac{z_1}{z_2} = z$, then $z + \frac{1}{z} = 1 \Rightarrow z^2 - z + 1 = 0$

$\Rightarrow \quad z = \frac{1 \pm i\sqrt{3}}{2}$

Therefore, $\frac{z_1}{z_2} = \frac{1 \pm i\sqrt{3}}{2}$.

If z_1 and z_2 are represented by A and B, respectively and O be the origin, then

$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \left|\frac{1 \pm i\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \Rightarrow OA = OB$

Also, $\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left|1 - \frac{z_1}{z_2}\right| = \left|1 - \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right|$

$\left|\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \Rightarrow AB = OB.$

Thus, $OA = OB = AB$

$\therefore$ ΔAOB is an equilateral triangle.

Hence, the correct answer is (C).

117. Let $z = x + iy$, then $x = \lambda + 3$, $y = -\sqrt{5 - \lambda^2}$

$\Rightarrow \quad (x - 3)^2 = \lambda^2$ and $y^2 = 5 - \lambda^2$

On solving, $(x - 3)^2 = 5 - y^2$

$\Rightarrow \quad (x - 3)^2 + y^2 = 5$

Clearly, it is a semicircle as $y < 0$. Hence, part of the circle lies below the x-axis.

Hence, the correct answer is (B).

118. $x^2 + x + 1 = 0 \Rightarrow x = w$ or w^2

Let $x = w$, then $x + \frac{1}{x} = w + \frac{1}{w} = w + w^2 = -1$,

$x^2 + \frac{1}{x^2} = w^2 + \frac{1}{w^2} = w^2 + w = -1$,

$$x^3+\frac{1}{x^3}=w^3+\frac{1}{w^3}=1+1=2,$$

$$x^4+\frac{1}{x^4}=w^4+\frac{1}{x^3}=w+\frac{1}{w}=-1\text{, etc.}$$

$$\therefore\quad\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2$$

$$+\ldots\left(x^{27}+\frac{1}{x^{27}}\right)^2$$

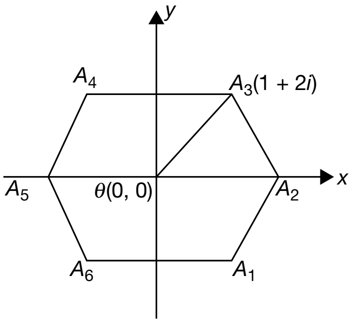
$$=18+9(2)^2=54$$

Hence, the correct answer is (D).

119. Let the vertices be z_1, z_2, z_3, z_4, z_5 and z_6 w.r.t centre O at origin $|z_3|=\sqrt{5}$.

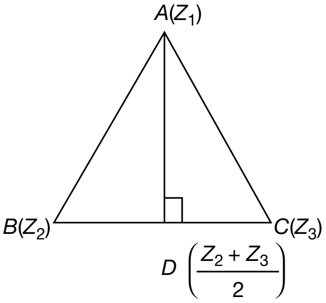
Now ΔOA_2A_3 is equilateral

$\Rightarrow\quad OA_2=OA_3=A_2A_3=\sqrt{5}$, perimeter $=6\sqrt{5}$.



Hence, the correct answer is (D).

120.



$$\text{Arg}\left(\frac{z_2+z_3-2z_1}{z_3-z_2}\right)$$

$$=\text{Arg }2\left\{\frac{\left(\frac{z_2+z_3}{2}-z_1\right)}{z_3-z_2}\right\}$$

$$=\text{Arg}\left\{\frac{\left(\frac{z_2+z_3}{2}-z_1\right)}{z_3-z_2}\right\}=\frac{\pi}{2}$$

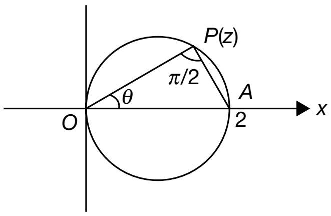
Clearly, $AD\perp BC$.

Hence, the correct answer is (C).

121. Let $w=5(z-i)-6\Rightarrow|w+1|=5|z-1-i|=5$.

Hence, the correct answer is (B).

122.



$$\text{Arg}\left(\frac{2-z}{0-z}\right)=\frac{\pi}{2}\Rightarrow\frac{z-2}{z}=\frac{AP}{OP}i$$

$$\tan\theta=\frac{AP}{OP}\text{, then }\frac{z-2}{z}=i\tan\theta$$

Hence, the correct answer is (B).

123. $x^n+ax+b=0$

$$=(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)$$

$$\Rightarrow\quad(x-x_2)(x-x_3)\ldots.(x-x_n)=\frac{x^n+ax+b}{x-x_1}$$

$$\therefore\quad(x-x_2)(x-x_3)(x-x_4)\ldots(x-x_n)$$

$$=\lim_{x\to x_1}\left(\frac{x^n+ax+b}{x-x_1}\right)=nx_1^{\,n-1}+a$$

Hence, the correct answer is (B).

124. Circumcentre of triangle ABC is origin. Let $G(Z_G)$ be its centriod, then $Z_G=\frac{1}{3}(z_1+z_2+z_3)$ the points $O(0)$, $G(z_G)$, $H(z_H)$ are collinear and $OG:GH=1:2$

$$z_G=\frac{2\times0+1\times Z_H}{3}=Z_H=3Z_G=z_1+z_2+z_3$$

Hence, the correct answer is (C).

125. Tangent $xx_1+yy_1=4$

$$\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z_1+\overline{z_1}}{2}\right)+\left(\frac{z+\bar{z}}{2i}\right)\left(\frac{z_1+\overline{z_1}}{2i}\right)=4$$

$$z\bar{z}_1+\bar{z}z_1=8$$

Equation of tangent at $A(z_1)$ is $\frac{z}{z_1}+\frac{\bar{z}}{\bar{z}_1}=2$

$$\frac{z}{z_1^2}+\frac{\bar{z}}{4}=\frac{2}{z_1}$$

Equation of tangent at $B(z_2)$ is $\frac{z}{z_2^2}+\frac{\bar{z}}{4}+\frac{2}{z_2}$

$$\Rightarrow\quad z\left(\frac{1}{z_1^2}-\frac{1}{z_2^2}\right)=2\left(\frac{1}{z_1}-\frac{1}{z_2}\right)$$

$$\Rightarrow\quad z=\frac{2z_1z_2}{z_1+z_2}$$

Hence, the correct answer is (C).

126. $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$

$$(1+\omega)^n=c_0+c_1\omega+c_2\omega^2+\ldots+c_n\omega^n$$

$$(1+\omega^2)^n=c_0+c_1\omega^4+\ldots+c_n\omega^{2n}$$

$$2^n=c_0+c_1+c_2+\ldots+c_n$$

$$2^n+(-\omega)^n+(-\omega^2)^n=3c_3+\ldots+3^nc_n$$

$$c_0+c_3+c_6+\ldots+c_n=\frac{1}{3}[2^n+(-1)\omega^n+(-1)^n\omega^{2n}]$$

$$= \frac{1}{3}[2^n + (-1)^{3P}\omega^{3P} + (-1)^{3P}\omega^{6P}]$$

$$= \frac{1}{3}[2^n - 1 - 1] = \frac{1}{3}[2^n - 2]$$

Hence, the correct answer is (A).

127. The quadrilateral $ABXC$ is a parallelogram. If z is the affix of x,

$$\frac{1}{2}(z_1 + z) = \frac{1}{2}(z_2 + z_3) \Rightarrow z = z_2 + z_3 - z_1$$

Similarly affix of y is $z_1 + z_3 - z_2$ and that of z is $z_1 + z_2 - z_3$, centroid of ΔXYZ is

$$\frac{1}{3}(z_2 + z_3 - z_4 + z_1 + z_3 - z_2 + z_1 + z_3 - z_3)$$

$$= \frac{1}{3}(z_1 + z_2 + z_3) = z_0$$

Hence, the correct answer is (C).

128. The given equation is $(1 + z)(1 + z^3) = 0$ the distinct roots being $-1, -\omega, -\omega^2$ which if be represented by points a, b and c in that order,

$$ab = |1 - \omega| = |\omega|\ |\omega^2 - 1| = |\omega^2 - 1|$$

$$bc = |\omega - \omega^2| = |\omega^2|\ |\omega^2 - 1| = |\omega^2 - 1|$$

$$ca = |\omega^2 - 1|$$

The three points represent the vertices of an equilateral triangle.

Hence, the correct answer is (B).

129. $\dfrac{6+10i}{(1+i)^2} = \dfrac{6+10i}{2i} = 5 - 3i$

Put $z = x + iy \Rightarrow i(x - iy) = x + iy$

$\Rightarrow \quad ix + y = x + iy$

$\Rightarrow \quad (x - y) - i(x - y) = 0 \Rightarrow x - y = 0$

Reflection is $(-3 + 5i)$

Hence, the correct answer is (A).

130. $\text{Arg}\left(\dfrac{z^4 - 1}{z^3 + 1}\right) = \text{Arg}(z^4 - 1) - \text{Arg}(z^3 + 1)$

$$= \left(2\alpha - \frac{\pi}{2} + \pi\right) - \frac{3\alpha}{2} = \frac{\pi + \alpha}{2}$$

$\Rightarrow \quad z^4 - 1 = e^{4\alpha} - 1$

$$= \cos 4\alpha - 1 + i\sin 4\alpha$$

$$= -2\sin^2 2\alpha + 2i\sin 2\alpha\cos 2\alpha$$

$$= -2\sin 2\alpha[\sin 2\alpha - i\cos 2\alpha]$$

$$= 2i\sin 2\alpha\, e^{i2\alpha}$$

$$= 2\sin 2\alpha\ e^{i\left(\frac{\pi}{2} + 2\alpha\right)}$$

$$Z^3 + 1 = e^{i3\alpha} + 1.$$

$$= 2\cos^2\frac{3\alpha}{2} + 2i\sin\frac{3\alpha}{2}\cos\frac{3\alpha}{2}$$

$$= 2\cos\frac{3\alpha}{2}\ e^{i\frac{3\alpha}{2}}$$

Hence, the correct answer is (A).

131. Since, $|x| + |y| = 2$ …(1)

$|z + 2i| + |z - 2i| = 4$ …(2)

Equation (1) represents square and equation (2) represents line segment solution are $z = \pm 2i$.

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. Let $x = \left|a + \sqrt{a^2 - b^2}\right| + \left|a - \sqrt{a^2 - b^2}\right|$

Then,

$$x^2 = 2|a|^2 + 2|a^2 - b^2| + 2|a^2 - (a^2 - b^2)|$$

$$x^2 = 2|a|^2 + 2|a^2 - b^2| + 2|a^2 - (a^2 - b^2)|$$

$$= 2|a|^2 + 2|b|^2 + 2|a^2 - b^2|$$

$$= \left(|a + b|^2 + |a - b|^2 + 2|a + b|\ |a - b|\right)$$

$$= \left(|a + b| + |a - b|\right)^2$$

Therefore, $x = |a + b| + |a - b| \le 2|a| + 2|b|$

Hence, (A) and (D) are correct.

2. $|z + 1| < |z - 2| \Rightarrow z + \bar{z} < 1$

So, $\omega + \bar{\omega} = 3(z + \bar{z}) + 4 \Rightarrow \omega + \bar{\omega} < 7$

Now, $|\omega + 1| < |\omega - 8|$ if $\omega + \bar{\omega} < 7$, which is true.

$|\omega + 1 + i| < |\omega - 8 + i|$ if $\omega + \bar{\omega} < 7$, which is true.

$|\omega + 5| < |\omega - 4|$ if $\omega + \bar{\omega} < -1$, which is not true for every ω possible.

$|\omega - 12 + i| < |\omega - 3 + i|$ if $\omega + \bar{\omega} > 15$, which is not true.

Hence, (A) and (B) are correct.

3. $(\alpha\beta)^5 = \alpha^5\beta^5 = -1$, which implies (A) is true.

$(\alpha^2\beta)^5 = \alpha^{10}\beta^5 = -1$, which implies (B) is false.

Since $(\alpha\beta)^5 = -1$

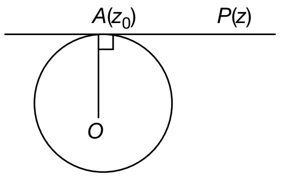
$\therefore \quad |\alpha\beta| = 1$, which implies (C) is true.

Finally, $|\alpha + \beta| < |\alpha| + |\beta| = 1 + 1$, which implies ($D$) is true. (Equality is not possible. Since, α and β do not have the same arg.)

Hence, (A), (C) and (D) are correct.

4. $\angle OAP = \dfrac{\pi}{2}$ complex slope of OA and AP is ω_1 and ω_2 respectively

$$\omega_1 + \omega_2 = \frac{z - z_0}{z_0} + \frac{\bar{z} - \overline{z_0}}{\overline{z_0}} = 0;\ \frac{z}{z_0} + \frac{\bar{z}}{\bar{z}_0} = 2 \qquad \text{…(1)}$$



$$\text{Re}\left(\frac{z}{z_0}\right) = 1;\ \text{from (1)}$$

$$z\bar{z}_0 + \bar{z}z_0 = 2|z_0|^2 = 2r^2$$

Hence, (A) and (B) are correct.

5. $z_2{}^2 = z_1z_3$ and $z_1z_2z_3 = 1 \Rightarrow z_2{}^3 = 1$

$\Rightarrow \quad z_2 = 1, w, w^2$

$1 - b + 3 - 1 = 0 \Rightarrow b = 3$

If z_1, z_2, z_3 and z_4 are the vertices of square.

Hence, (A) and (C) are correct.

6. $z_1 = \frac{5-4z}{4z-2} \Rightarrow z = \frac{5+2z_1}{4(z_1+1)} \Rightarrow |z| = \left|\frac{5+2z_1}{4(z_1+1)}\right| = 1$

$\Rightarrow \quad \left|z_1 + \frac{5}{2}\right| = 2|z_1+1|$

$\Rightarrow \quad \left(x+\frac{5}{2}\right)^2 + y^2 = 4\left((x+1)^2+y^2\right)$

$\Rightarrow \quad 3x^2+3y^2+3x-\frac{9}{4} = 0 \Rightarrow x^2+y^2+x-\frac{3}{4}=0$

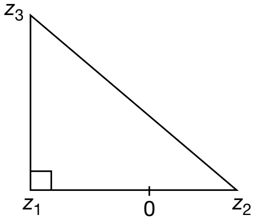
$\Rightarrow$ Centre $= \left(-\frac{1}{2}, 0\right)$, radius = 1, circle passes through all 4 quadrants.

Hence, (C) and (D) are correct.

7. $\text{Arg}\left(\frac{z_1}{z_2}\right) = \pm\pi$ and

$|z_1 + i(z_3-z_1)| = |z_1| + |z_3 - z_1|$ iff $\text{Arg}\left(\frac{z_1}{z_3-z_1}\right) = \frac{\pi}{2}$.

Therefore, centre of circle $= \left(\frac{z_2+z_3}{2}\right)$ and ΔABC is right angle triangle.

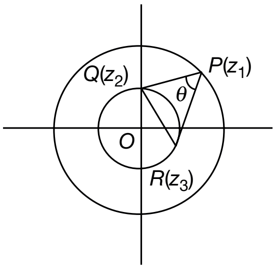


Hence, (A) and (B) are correct.

8. As ΔPQR is equilateral, orthocentre, circumcentre and centroid will coincide.

Circumcircle $\angle PQR \left|\frac{z_1+z_2+z_3}{3}\right| = 1$

$\Rightarrow \quad |z_1+z_2+z_3|^2 = 9$



$\Rightarrow \quad (z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3) = 9$

$\Rightarrow \quad \left(\frac{4}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)+\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) = 9$

$\left(\text{Since, } |z_1|^2 = 4, |z_2|^2 = 1, |z_1|^2 = 1\right)$ and $\angle QOR = 120^\circ$

Hence, (A), (B), (C) and (D) are correct.

9. $\left|z+\frac{1}{z}\right|^2 = r^2 + \frac{1}{r^2} + 2\cos 2\theta = 4$

$r^4 + (2\cos 2\theta - 4)r^2 + 1 = 0$

$r_1^2 + r_2^2 = 4 - 2\cos 2\theta; r_1^2 r_2^2 = 1 \Rightarrow r_1 r_2 = 1$

$(r_1-r_2)^2 = 2 - 2\cos 2\theta; (r_1-r_2)^2 = 4\sin^2\theta$

$|r_1 - r_2| \in (0, 2)$

Let $|z_1| = r_1, |z_2| = r_2;\ |z_1| = r_1, |z_2| = r_2;$

$|z_1+z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1-\theta_2)$

Let $z = re^{i\theta}, \frac{1}{z} = \frac{1}{r}e^{-i\theta}; |r_1-r_2| = \left|r-\frac{1}{r}\right| = 2\sin\theta$

Hence, (A), (B) and (D) are correct.

10. $|z_1| = 1, |z_2| = 2$

(A) $||z_1| - |2z_2|| \le |z_1 - 2z_2| \le |z_1| + |-2z_2|$

$\Rightarrow \quad |1-2(2)| \le |z_1 - 2z_2| \le |1+2(2)| \Rightarrow 3 \le |z_1-2z_2| \le 5$

(B) $|z_1| - |z_2| \le |z_1+z_2| \le |z_1| + |z_2|$

$\Rightarrow \quad |1-2| \le |z_1+z_2| \le 1+2 \Rightarrow 1 \le |z_1+z_2| \le 3$

(C) $||z_1| - 3|z_2|| \le |z_1 - 3z_2| \le |z_1| + 3|z_2|$

$\Rightarrow \quad 5 \le |z_1 - 3z_2| \le 7$

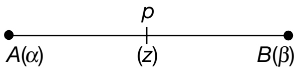
(D) $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$

$\Rightarrow \quad 1 \le |z_1 - z_2| \le 3$

Hence, (A), (B), (C) and (D) are correct.

11. $AP + PB = AB$

$|z-\alpha| + |z-\beta| = |\beta-\alpha| \Rightarrow A$ is true



Now $z = \alpha + t(\beta - \alpha)$

$= (1-t)\alpha + t\beta$ where $t \in (0,1) \Rightarrow B$ is true

Again $\frac{z-\alpha}{\beta-\alpha}$ is real $\Rightarrow \frac{z-\alpha}{\beta-\alpha} = \frac{\bar{z}-\bar{\alpha}}{\bar{\beta}-\bar{\alpha}}$

$\Rightarrow \quad \begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0 \Rightarrow C$ is true

Also $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ iff $\begin{vmatrix} z-\alpha & \bar{z}-\bar{\alpha} & 0 \\ \alpha & \bar{\alpha} & 1 \\ \beta-\alpha & \bar{\beta}-\bar{\alpha} & 0 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} (z-\alpha) & \bar{z}-\bar{\alpha} \\ \beta-\bar{\alpha} & \bar{\beta}-\bar{\alpha} \end{vmatrix} = 0 \Rightarrow D$ is true

Hence, (A), (B), (C) and (D) are correct.

12. Here, $|z_1| = |z_2| = |z_3| = 1$

However, $|z| = 1$ and $z\bar{z} = 1$

Also given $z_1^3 + z_2^3 + z_3^3 + z_1z_2z_3 = 0$

$\Rightarrow \quad z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 = -4z_1z_2z_3$

$\Rightarrow \quad (z_1+z_2+z_3)\left(z_1^2+z_2^2+z_3^2 - \sum z_1z_2\right) = -4z_1z_2z_3$

Let $z_1 + z_2 + z_3 = z$.

$\Rightarrow \quad \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \bar{z} = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}$

$\therefore \quad \sum z_1\left(\sum z_1^2 - \sum z_1z_2\right) = -4z_1z_2z_3$

$\Rightarrow \quad \sum z_1\left(\left(\sum z_1\right)^2 - 3\sum z_1z_2\right) = -4z_1z_2z_3$

$\Rightarrow \quad z\left(z^2 - 3\sum z_1z_2\right) = -4z_1z_2z_3$

$$\Rightarrow \quad z^3 = z_1 z_2 z_3 \left(3z\Sigma \frac{1}{z_1} - 4 \right)$$

$$z^3 = z_1 z_2 z_3 \left(3z\bar{z} - 4 \right)$$

$$\Rightarrow \quad |z|^3 = \left| 3|z|^2 - 4 \right|$$

If $|z|^3 = \frac{2}{\sqrt{3}}$, then $|z|^3 = 3|z|^2 - 4$

$$\Rightarrow \quad |z|^3 - 3|z|^2 + 4 = 0$$

$$\Rightarrow \quad (|z|-2)^2(|z|+1) = 0$$

$$\Rightarrow \quad |z| = 2 \Rightarrow |z_1 + z_2 + z_3| = 2$$

If $0 \le |z| \le \frac{2}{\sqrt{3}}$, then $|z|^3 = 4 - 3|z|^2$

$$\Rightarrow \quad |z|^3 + 3|z|^2 - 4 = 0$$

$$\Rightarrow \quad (|z|-1)(|z|+2)^2 = 0$$

$$\Rightarrow \quad |z| = 1 \Rightarrow |z_1 + z_2 + z_3| = 1$$

Hence, (C) and (D) are correct.

13. $x^{10} + (13x-1)^{10} = 0 \Rightarrow x^{10} = -(13x-1)^{10}$

$$\Rightarrow \quad \left(\frac{13x-1}{x} \right)^{10} = -1 \Rightarrow 13 - \frac{1}{x} = (-1)^{\frac{1}{10}}$$

$$\Rightarrow \frac{1}{x} = 13 - (-1)^{\frac{1}{10}} = 13 - \text{cis}\left(\frac{2k\pi + \pi}{10} \right), k = 0, 1, 2, \ldots, 9$$

$$\frac{1}{z_1} = 13 - cis\left(\frac{\pi}{10} \right), \frac{1}{\bar{z}_1} = 13 - cis\left(\frac{-\pi}{10} \right)$$

$$\Rightarrow \quad \frac{1}{|z_1|^2} = 170 - 26\cos\frac{\pi}{10}$$

Therefore, $N = \sum_{r=1}^{5} \frac{1}{|z_r|^2} = 5(170) - 26(0) = 850$

$$\left(\cos\frac{\pi}{10} + \cos\frac{3\pi}{10} + \cos\frac{5\pi}{10} + \cos\frac{7\pi}{10} + \cos\frac{9\pi}{10} \right) = 0$$

Hence, (A), (B), (C) and (D) are correct.

14. Let $z_1 = x_1 + iy_1 \;\; z_2 = x_2 + iy_2$, then

$$(z_1 + z_2) = |x_1 + x_2| + |y_1 + y_2| \le (z_1) + (z_2)$$

Hence, (A) and (D) are correct.

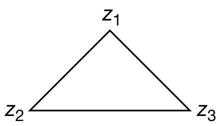
15. $z_3 = z_1 + (z_2 - z_1)e^{\pm i\frac{\pi}{3}}$

$$= (1+i) - 2(1+i)\left(\frac{1 \pm \sqrt{3}i}{2} \right) = (1+i)\left(1 - \left(1 \pm \sqrt{3}i\right)\right)$$

$$= \pm(1+i)\left(\sqrt{3}i\right) = \pm\sqrt{3}(i-1)$$

$$z_3 = \sqrt{3}(1-i) \text{ or } \sqrt{3}(i-1)$$

Hence, (B) and (C) are correct.



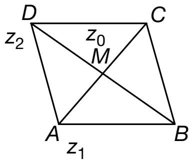
16. Let $A = z_1, z_0 = 2 - i, z_2 = (1 + \mathrm{i})$

Then, $\frac{z_1 - z_0}{z_2 - z_0} = \frac{1}{2}e^{\pm i\frac{\pi}{2}} = \pm\frac{i}{2}$

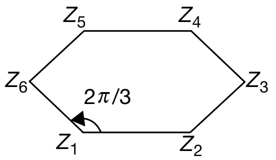
$$\Rightarrow \quad z_1 = z_0 \pm (z_2 - z_0)\frac{i}{2}$$

$$= (2-i) \pm \frac{i}{2}(-1+2i) \;\; \Rightarrow z_1 = 1 - \frac{3i}{2} \text{ or } 3 - \frac{i}{2}$$

Hence, (A) and (D) are correct.



17.



$$Z_6 - Z_1 = (Z_2 - Z_1)e^{i\frac{2\pi}{3}}$$

$$\Rightarrow \quad Z_6 = (1-w)Z_1 + wZ_2$$

$$Z_3 - Z_2 = (Z_1 - Z_2)e^{i\frac{4\pi}{3}} = (Z_1 - Z_2)w^2$$

$$\Rightarrow \quad z_3 = z_2(1-\omega^2) + z_1\omega^2$$

$$Z_4 - Z_1 = 2(Z_2 - Z_1)e^{i\frac{\pi}{3}}; Z_4 = Z_1\left(1 - 2e^{i\frac{\pi}{3}}\right) + 2Z_2e^{i\frac{\pi}{3}}$$

$$= -\sqrt{3}iZ_1 + Z_2\left(1 + \sqrt{3}i\right)$$

$$z_5 - z_1 = \sqrt{3}(z_2 - z_1)e^{i\pi/2}$$

$$z_5 = z_1 + \sqrt{3}iz_2 - \sqrt{3}iz_1$$

$$\Rightarrow \quad z_5 = (1 - \sqrt{3}i)z_1 + \sqrt{3}iz_2$$

Hence, (A), (B), (C) and (D) are correct.

18. $z_1 = 5 + 12i, |z_2| = 4$

(A) $|z_1 + iz_2| \le |z_1| + |iz_2| = 13 + 4$

(B) $|z_1 + (1+i)z_2| \ge \left| |z_1| - |(1+i)z_2| \right| \ge 13 - 4\sqrt{2}$

(C) $\left| z_2 + \frac{4}{z_2} \right| \le |z_2| + \frac{4}{|z_2|} = 5$ and

$$\left| z_2 + \frac{4}{z_2} \right| \ge \left| |z_2| - \left| \frac{4}{z_2} \right| \right| = |4 - 1| = 3$$

$\therefore \quad \left| \frac{z_1}{z_2 + \frac{4}{z_2}} \right|$ has min $\frac{13}{5}$ and max $\frac{13}{3}$.

Hence, (A), (B) and (D) are correct.

19. $iz^3 + z^2 + i^2z + i = 0$

$$\Rightarrow \quad (z^2 + i)(iz + 1) = 0$$

$$\Rightarrow \quad z^2 = -i \text{ or } z = -i \Rightarrow |z| = 1$$

Hence, (B) and (D) are correct.

20. $P_k = e^{i\left(\frac{2k\pi}{n}\right)}, k \in Z$

$= e^{i\theta}$, where $\theta = \frac{2k\pi}{n}, k = 1, 2, 3, \ldots.n$

$$A_k = a\cos\theta + ib\sin\theta$$

Let $A_k = (x, y) = x + iy$

$x = a\cos\theta, y = b\sin\theta$

Eliminating θ, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $\sum_{k=1}^{n} A_k S = \sum_{k=1}^{n} (a - ex_k) = na - e.\sum_{k=1}^{n} \cos\theta = na$

$$\left(\begin{array}{l} \text{Since, } \sum P_k = 0 = \sum \cos\theta + i\sum \sin\theta \\ \Rightarrow \sum \cos\theta = \sum \sin\theta = 0 \end{array} \right)$$

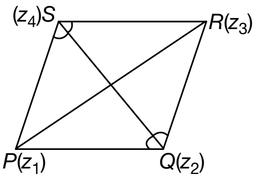
(ii) $\sum_{k=1}^{n}(A_kS)^2=\sum a^2+e^2\sum x_k^2-2ae\sum x_k$

$$=na^2+e^2a^2\sum\frac{1+\cos 2\theta}{2}$$

$$=na^2+\frac{ne^2a^2}{2}+\frac{e^2a^2}{2}\sum_{k=1}^{n}\cos 2\theta=\frac{n}{2}(3a^2-b^2)+0$$

Hence, (A), (C) and (D) are correct.

21.



(A) Since, $\overrightarrow{PS}\,||\,\overrightarrow{QR}$,

$$\text{Amp}\left(\frac{z_1-z_4}{z_2-z_3}\right)\text{ is zero }\Rightarrow\frac{z_1-z_4}{z_2-z_3}\text{ is purely real.}$$

(B) Since, diagonals are angular bisectors

$$\text{Amp}\left(\frac{z_1-z_4}{z_2-z_4}\right)=\text{Amp}\left(\frac{z_2-z_4}{z_3-z_4}\right)$$

(C) Since, diagonals are perpendicular,

$$\text{Amp}\left(\frac{z_1-z_3}{z_2-z_4}\right)\text{ is }\frac{\pi}{2}\Rightarrow\frac{z_1-z_3}{z_2-z_4}\text{, is purely imaginary.}$$

(D) Since, diagonals are not equal, $|z_1-z_3|\neq|z_2-z_4|$

Hence, (A), (B) and (C) and (D) are correct.

22. Locus is $xy=32$ which is rectangular hyperbola for $xy=c^2$.

$$L.R=2\sqrt{2}C=2\sqrt{2}\sqrt{32}=16$$

$e=\sqrt{2}\Rightarrow e^2=2$ and foci are

$(c\sqrt{2},c\sqrt{2})$ and $(-c\sqrt{2},-c\sqrt{2})=(8, 8)$ and $(-8, -8)$.

Hence, (A), (B) and (D) are correct.

23. $|z_1+z_2|^2=(z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$$=z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+\bar{z}_1z_2$$

$$=|z_1|^2+|z_2|^2$$

We have,

$$z_1\bar{z}_2+\bar{z}_1z_2=0\ \Rightarrow z_1\bar{z}_2=-\bar{z}_1z_2\Rightarrow\frac{z_1}{z_2}=-\left(\frac{\bar{z}_1}{\bar{z}_2}\right).$$

So, $\frac{z_1}{z_2}$ is purely imaginary.

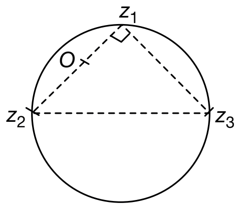
Hence, (B) and (C) are correct.

24. $|z_1+z_2|=|z_1|-|z_2|\Rightarrow z_1, z_2$ and O are collinear

$\text{Arg}\left(\frac{z_1}{z_2}\right)=\pm\pi$ and $|z_1+i(z_3-z_1)|=|z_1|+|z_3-z_1|$

iff $\text{Arg}\left(\frac{z_1}{z_3-z_1}\right)=\frac{\pi}{2}$

So, centre of circle $=\left(\frac{z_2+z_3}{2}\right)$ and ABC is right angle triangle.



Hence, (A) and (B) are correct.

25. $\frac{|z|^2-|z|+1}{2+|z|}>\left(\frac{1}{\sqrt{3}}\right)^2$ and $\frac{|z|^2-|z|+1}{2+|z|}<\left(\frac{1}{\sqrt{3}}\right)^{-2}$

$$|z|^2-4|z|-5<0$$

$\Rightarrow$ $(|z|-5)(|z|+1)<0\Rightarrow -1<|z|<5$

$$3|z|^2-4|z|+1>0\Rightarrow(3|z|-1)(|z|-1)>0$$

Therefore, $|z|\in\left(0,\frac{1}{3}\right)\cup(1,5)$

Hence, (B) and (D) are correct.

26. $Z'=Ze^{i\alpha}$; $Z''=Ze^{-i\alpha}$

$Z'Z''=Z^2$; Z', Z, Z'' are in GP

Also

$$\frac{1}{Z'}+\frac{1}{Z''}=\frac{1}{Z}\left(e^{-i\alpha}+e^{i\alpha}\right)$$

$$\frac{1}{Z'}+\frac{1}{Z''}=\frac{2\cos\alpha}{Z}$$

$$\Rightarrow\ \frac{1}{Z'}+\frac{1}{Z''}\neq\frac{2}{Z}$$

$$Z'+Z''=2Z\cos\alpha$$

$$z'^2+z''^2=z^2(e^{i2\alpha}+e^{-i2\alpha})=2z^2\cos 2\alpha$$

Hence, (A), (C) and (D) are correct.

27. Let the four points represented by $Z_1Z_2Z_3$ and Z_4 are A, B, C and D.

Midpoint of AC = Midpoint of BD

$$Z_1+Z_3=Z_2+Z_4$$

$$AB=BC=CD=DA$$

$$|Z_1-Z_2|=|Z_2-Z_3|=|Z_3-Z_4|=|Z_4-Z_1|$$

Diagonlas are equal $|Z_1-Z_3|=|Z_2-Z_4|$

Since, AC perpendicular to BD $\frac{Z_1-Z_3}{Z_2-Z_4}$ is purely imaginary.

Hence, (A), (B) and (C) are correct.

28. $\alpha=(1,1)=A$, $\beta=(0,-1)=B$

(A) Least value $=\overline{AB}=\sqrt{1+4}=\sqrt{5}$

(using triangular inequality)

(B) Least value $=\overline{AB}=\sqrt{5}$ (occurs when, $z=\beta$)

(C) Least value $=\overline{AB}=\sqrt{5}$ (occurs when $z=\alpha$)

(D) Greatest value $=\overline{AB}=\sqrt{5}$

(Since, $|z_1|+|z_2|\geq|z_1-z_2|$)

Hence, (A), (B), (C) and (D) are correct.

29. Given that, $1, \alpha_1, \alpha_2, \ldots, \alpha_8$ are roots of the equation $x^9=1$.

$\Rightarrow$ $1+\alpha_1+\alpha_2+\ldots+\alpha_8=0$ and …(1)

$$\alpha_1^9=\alpha_2^9=\alpha_3^9=\ldots\ldots\ldots\alpha_8^9=1$$

Required $= \frac{1}{\alpha_1^8} + \frac{1}{\alpha_2^8} + \frac{1}{\alpha_3^8} + \ldots\ldots + \frac{1}{\alpha_8^8}$

$= \frac{\alpha_1}{\alpha_1^9} + \frac{\alpha_2}{\alpha_2^9} + \frac{\alpha_3}{\alpha_3^9} + \ldots\ldots + \frac{\alpha_8}{\alpha_8^9}$

$= \alpha_1 + \alpha_2 + \ldots\ldots.. + \alpha_8 = -1$

$1^8 + \alpha_1^8 + \ldots + \alpha_8^8 = 0 \Rightarrow \alpha_1^8 + \alpha_2^8 + \ldots + \alpha_8^8 = -1$

Hence, (A), (C) and (D) are correct.

30. $|z_1 z_2| = \left|\frac{c}{a}\right| = 1$ and $|z_1 + z_2| = \left|-\frac{b}{a}\right| = 1$

So, $(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = 1 \Rightarrow 2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 = 1$

$\Rightarrow \quad 2 + \frac{z_2}{z_1} + \frac{z_1}{z_2} = 1 \Rightarrow \frac{(z_1 + z_2)^2}{z_1 z_2} = 1 \Rightarrow \frac{b^2}{a^2} = \frac{c}{a}$

$\Rightarrow \quad b^2 = ac$

Now, $z_2 = z_1 e^{i\theta} \Rightarrow |z_1 + z_2| = |z_1||1 + e^{i\theta}| = 1$

$= 2\cos\frac{\theta}{2}\left|\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right|$

$|z_1 + z_2| = 2\cos\frac{\theta}{2} = 1 \Rightarrow \frac{\theta}{2} = \frac{\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$

Now, $PQ = |z_2 - z_1| = |z_1||e^{i\theta} - 1| = \left|2\sin\frac{\theta}{2}\right|$

We know, $\theta = \frac{2\pi}{3}$.

Therefore, PQ$= |z_2 - z_1| = \sqrt{3}$

Hence, (A) and (B) are correct.

31. $\text{Arg}\left(\frac{z - (3 + 4i)}{z - (1 - 2i)}\right) = \frac{\pi}{2}$

So $z_c = \frac{3 + 4i + 1 - 2i}{2} = 2 + i = z_1$

$\Rightarrow \quad |z_1^8| = 625 \Rightarrow \text{Arg}(z_1 - 2) = \frac{\pi}{2}$

Hence, (A), (B) and (D) are correct.

32. $|z_3| = \frac{|z_2||z_1 - z_4|}{|\vec{z}_1 z_4 - 1|} = \frac{|z_1 - z_4|}{|\vec{z}_1 z_4 - 1|} \le 1$

Iff $|z_1 - z_4|^2 \le |\vec{z}_1 z_4 - 1|^2$

$\Rightarrow \quad (z_1 - z_4)(\vec{z}_1 - \vec{z}_4) \le (\vec{z}_1 z_4 - 1)(z_1 \vec{z}_4 - 1)$

$|z_1|^2 + |z_4|^2 - |z_1|^2 |z_4|^2 - 1 \le 0$

$\Rightarrow \quad (|z_1|^2 - 1)(|z_4|^2 - 1) \ge 0$

as $|z_1| < 1 \Rightarrow |z_4| \le 1$.

Hence, (B) and (C) are correct.

33. $z_n - z_1 = z_n - z_{n-1} + \ldots.. + z_3 - z_2 + z_2 - z_1$

$\Rightarrow \quad |z_n - z_1| \le |z_n - z_{n-1}| + |z_{n-1} - z_{n-2}| + \ldots +$
$|z_3 - z_2| + |z_2 - z_1|$

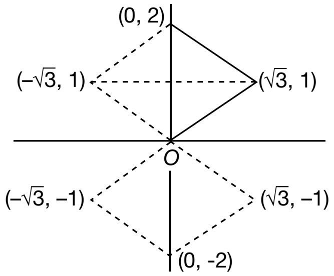
$\frac{|z_2 - z_1|^2 + |z_3 - z_2|^2 + \ldots + |z_n - z_{n-1}|^2}{(n-1)} \ge$

$\left(\frac{|z_n - z_{n-1}| + |z_{n-1} - z_{n-2}| + \ldots + |z_2 - z_1|}{n-1}\right)^2 \ge \frac{|z_n - z_1|^2}{(\text{n}-1)^2}$

$\Rightarrow \quad \frac{|z_2 - z_1|^2 + |z_3 - z_2|^2 + \ldots.. + |z_n - z_{n-1}|^2}{|z_n - z_1|^2} \ge \frac{1}{n-1}$

Hence, (A), (B) and (C) are correct.

34.



Substitute $z = x + iy$ in $|z^2 - 1| = |z|^2 + 1$ and solve, result is $x = 0$. Which means, z is purely imaginary. Hence, $z = 0 \pm 2i$. Then, the third vertex will be as per the figure.

Hence, (A), (B), (C) and (D) are correct.

35. Clearly, the inscribed triangle is equilateral.
One vertex is z_0 $(0 + 0i)$, second is z_1 $(2 + 2i)$
Now third veretex we get after rotating $z_1 - z_0$ by clockwise or anticlockwise

$\Rightarrow \quad \frac{z_2 - z_0}{z_1 - z_0} = e^{i\frac{2\pi}{3}}, \frac{z_3 - z_0}{z_1 - z_0} = e^{-i\frac{2\pi}{3}}$

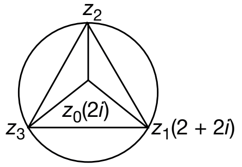
$\Rightarrow \quad z_2 = -1 + i(2 + \sqrt{3})$ and $z_3 = -1 + i(2 - \sqrt{3})$

Hence, (A) and (C) are correct.

36. $|z_1 - z_2|^2 = |1 - \bar{z}_1 z_2|^2 \Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2)$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 = 1 + |z_1|^2 |z_2|^2 \Rightarrow |z_1|^2 = |z_2|^2 = 1$



Hence, (A), (B), (C) and (D) are correct.

37. Let $\log_{|z+i|}^{5} < \log_{|z+i|}^{13} \Rightarrow |z + i| > 1 \Rightarrow x^2 + y^2 + 2y > 0$

$|z + i| > 1$

$C(0, -1)\ r = 1$

$z = -i + e^{i\theta}$ outside it

Hence, (B) and (D) are correct.

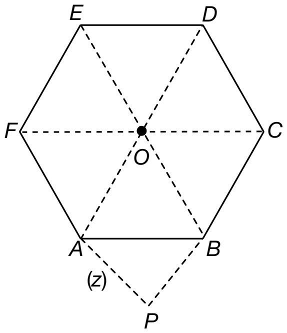
38. $G_{\Delta OAC} = \frac{0 + z + z\text{cis}\frac{2\pi}{3}}{3} = \frac{z}{3}(1 + \omega)$

$A = z; B = z\text{cis}\frac{\pi}{3}, P = z + z\text{cis}\frac{\pi}{3}$

$G_{\Delta ABP} = \frac{A + B + P}{3}$

$= \frac{2}{3}\left(1 + cis\frac{\pi}{3}\right)z = \frac{2}{3}(1 - \omega^2)z$

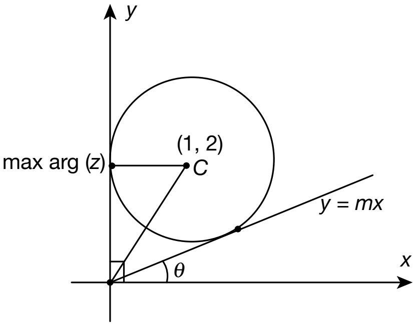
Area of hexagon $= 6 \times \frac{\sqrt{3}}{4}|z|^2 = \frac{3\sqrt{3}}{2}|z|^2$



Area of rectangle = $4\times\frac{\sqrt{3}}{4}|z|^2=\sqrt{3}\,|z|^2$

Hence, (A), (B), (C) and (D) are correct.

39.



$\therefore\quad m=\frac{3}{4}$

min Arg(z) $z=\tan^{-1}\left(\frac{3}{4}\right)$; Max Arg $(z)=\frac{\pi}{2}$

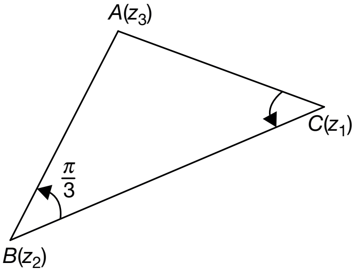
$$|z-1-2i|\geq\big||z|-\sqrt{5}\big|$$

$\therefore\quad \big||z|-\sqrt{5}\big|\leq 1; -1\leq|z|-\sqrt{5}\leq 1;\ \sqrt{5}-1\leq|z|\leq 1+\sqrt{5}$

$$\text{Min}(|z|)=\sqrt{5}-1;\ \text{Max}(|z|)=\sqrt{5}+1$$

Hence, (A), (B), (C) and (D) are correct.

40.



$$(z_1-z_2)=(z_3-z_2)\,e^{\frac{i\pi}{3}}\ \text{(about B rotation)}\qquad\ldots(1)$$

$$(z_2-z_3)=(z_1-z_3)\,e^{\frac{i\pi}{3}}\ \text{(about C rotation)}\qquad\ldots(2)$$

$$\frac{(1)}{(2)}\Rightarrow\frac{z_1-z_2}{z_2-z_3}=\frac{z_3-z_2}{z_1-z_3}$$

$$(z_1-z_2)(z_1-z_3)=-(z_3-z_3)^2$$

$\Rightarrow\quad z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

$\Rightarrow\quad \sum(z_1-z_2)^2=0$

$\Rightarrow\quad (z_1-z_2)^2=(z_2-z_3)(z_3-z_1);$

$$\sum(z_1-z_2)(z_2-z_3)=0$$

After simplification, $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$

Hence, (A), (B), (C) and (D) are correct.

41. $a^2+b^2=c^2+d^2=0\qquad\ldots(1)$

$$\text{Re}(z_1\bar{z}_2)=0\Rightarrow ac+bd=0\Rightarrow\frac{a}{b}=\frac{-d}{c}=\alpha\qquad\ldots(2)$$

From (1) and (2), $b^2\alpha^2+b^2=c^2\alpha^2+c^2\Rightarrow b^2=c^2$

$$|\omega_1|=\sqrt{a^2+c^2}=\sqrt{c^2+d^2}=1$$

$$|\omega_2|=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1$$

$$\text{Re}\,(\omega_1\bar{\omega}_2)=ab+cd=b\alpha\,b+c(-c\alpha)=\alpha(b^2-c^2)=0$$

Hence, (A), (B) and (C) are correct.

42. $x^2+y^2=8;\ \tan^{-1}\left(\frac{y-1}{x-5}\right)-\tan^{-1}\left(\frac{y-1}{x-1}\right)=\frac{\pi}{4}$

$$\tan^{-1}\left(\frac{\frac{y-1}{x-5}-\frac{y-1}{x-1}}{1+\left(\frac{y-1}{x-5}\right)\left(\frac{y-1}{x-1}\right)}\right)=\frac{\pi}{4}$$

$\Rightarrow\quad 4y-4=x^2+y^2-6x-2y+6$

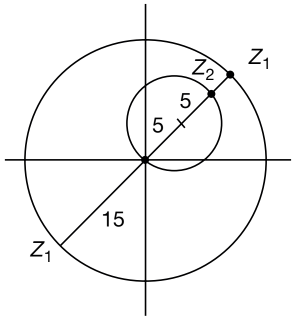
$\Rightarrow\quad x^2+y^2-6x-6y+10=0$

solving $x^2+y^2=8$ and $x^2+y^2-6x-6y+10=0$

we get $x=\frac{3\pm\sqrt{7}}{2}$ and $y=\frac{3\mp\sqrt{7}}{2}$

Hence, (A) and (B) are correct.

43.



Min. value of $|z_1-z_2|=5$

Max. value of $|z_1-z_2|=25$

Hence, (A) and (D) are correct.

44. $|z_1+z_2|=|z_1-z_2|\qquad\ldots(1)$

$\Rightarrow\quad (z_1+z_2)\left(\overline{z_1}+\overline{z_2}\right)=(z_1-z_2)\left(\overline{z_1}-\overline{z_2}\right)$

$$z_1\overline{z_2}+z_2\overline{z_1}=0\ \Rightarrow\ \frac{z_1}{z_2}=-\left(\frac{\overline{z_1}}{z_2}\right)\Rightarrow\frac{z_1}{z_2}$$

is purely imaginary

Also from (1), $|z_1-z_2|=|z_1|^2+|z_2|^2$

ΔOAB is a right angled triangle, we get right angle at O.

Hence, circumcentre $=\frac{z_1+z_2}{2}$

Hence, (A), (B) and (C) are correct.

45. Let $z = \left(1+\sqrt{2}\right) + i, \bar{z} = \left(1+\sqrt{2}\right) - i$

For adjacent vertices,

$$z = \bar{z} cis \frac{2\pi}{n} \Rightarrow \text{cis}\frac{2\pi}{n} = \frac{1+i\left(\sqrt{2}-1\right)}{1-i\left(\sqrt{2}-1\right)}$$

$$= \frac{1+i\tan\frac{\pi}{8}}{1-i\tan\frac{\pi}{8}} = cis\frac{2\pi}{8}$$

$\Rightarrow \quad n = 8$

Hence, the correct answer is (A).

46. Let $|\alpha| = r > 1$ and $\alpha = r\text{cis}\theta$ then $z = x + iy$

$$= \alpha + \frac{1}{\alpha} = r\text{cis}\theta + \frac{\text{cis}(-\theta)}{r}$$

$$\Rightarrow \quad x = \left(r+\frac{1}{r}\right)\cos\theta \text{ and } y = \left(r-\frac{1}{r}\right)\sin\theta$$

Eliminating θ gives, $\dfrac{x^2}{\left(r+\frac{1}{r}\right)^2} + \dfrac{y^2}{\left(r-\frac{1}{r}\right)^2} = 1$

which is an ellipse. $a = r + \frac{1}{r}, b = r - \frac{1}{r}$

$(r = |a| > 1 \Rightarrow a > b)$

$$\therefore \quad e = \sqrt{1-\frac{b^2}{a^2}} = \frac{2}{r+\frac{1}{r}} = \frac{2|\alpha|}{1+\alpha^2}$$

Distance between foci $= 2ae = 4$

Distance between directrices $= \dfrac{2a}{e} = \dfrac{2\left(r+\frac{1}{r}\right)^2}{2}$.

$$= \left(|\alpha| + \frac{1}{|\alpha|}\right)^2$$

Hence, (A), (B) and (D) are correct.

47. $|a| = |b| = |c| = r$

$|c| = |-az^2 + bz| \le r|z|^2 + r|z|$

$\Rightarrow \quad |z|^2 + |z| - 1 \ge 0$ …(1)

and $az^2 = -(bz + c)$

$|a|\,|z|^2 = \le r|z| + r$

$|z|^2 - |z| - 1 \le 0$ …(2)

Solve (1) and (2)

Hence, (A) and (D) are correct.

48. $x^2 + ax + b = 0 \Rightarrow |a| \le 2$ and $|b| = 1$

$$y = \frac{-|a| \pm \sqrt{|a|^2 - 4|b|}}{2} = \frac{-|a| \pm i\sqrt{4-|a|^2}}{2}$$

$|y| = 1$

Hence, (A) and (B) are correct.

49. If $|z-1| \le |z-5|$

$\left||z|^2 - z - \bar{z} + 1 \le |z|^2 - 5z - 5\bar{z} + 25 \text{ and } |z-3| = |z-1|\right|$

$\Rightarrow \quad 4(z+\bar{z}) \le 24 \Rightarrow z\bar{z} - 3z = 3\bar{z} + 9$

$z + \bar{z} \le 6 = z\bar{z} - z - \bar{z} + 1$

$\text{Re}(z) \le 3$

$\Rightarrow \quad 2(z + \bar{z}) = 8$

$z + \bar{z} = 4$

$\text{Re}(z) = 2$

Hence, (A) and (D) are correct.

50. Let r be the common ratio of GP a, b, c we have,

$$z = \frac{\frac{a}{b}+i}{\frac{c}{b}-1} = \frac{\frac{1}{r}+i}{r-1} = \frac{i}{r}$$

$$z = \frac{ib}{c} \text{ or } \frac{ia}{b}$$

Hence, (A) and (B) are correct.

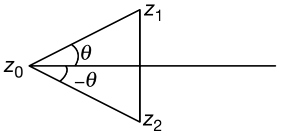
51. $|Z^2 - 9| + |Z^2| = 41 \Rightarrow |Z+3| + |Z-3| = 10$

Hence, (A) and (C) are correct.

52. $a\left(\dfrac{\bar{z}_1 + \bar{z}_2}{2}\right) + \bar{a}\left(\dfrac{\bar{z}_1 + \bar{z}_2}{2}\right) + b = 0$

$\Rightarrow \quad a(\bar{z}_1 - \bar{z}_2) + \bar{a}(z_1 + z_2) + 2b = 0$

$\Rightarrow \quad \text{Re}\, a(\bar{z}_1 + \bar{z}_2) = -b \Rightarrow \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} - \dfrac{a}{\bar{a}} = 0$



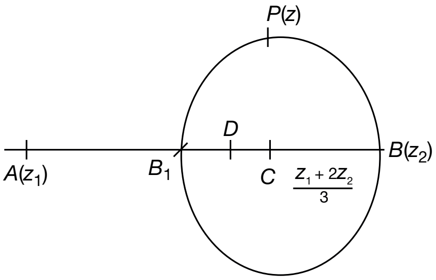
Hence, the correct answer is (D).

53. $\left|z - \dfrac{z_1 + 2z_2}{3}\right| = \dfrac{1}{3}|z_1 + z_2|$ and

$$\text{Arg}\left(\frac{z_1 - z_2}{z - (kz_1 + (1-k)z_2}\right) = \pm\frac{\pi}{2}$$

Angle between the line segment joining z_1 and z_2; z and $kz_1 + (1-k)z_2$ is $\frac{\pi}{2}$ $\frac{z_1 + 2z_2}{3}$ is a point on segment AB such that $AC{:}CS = 2{:}1$ and $D\ \left(kz_1 + (1-k)z_2\right)$ is a point on the segment AB such that $AD{:}DB = 1 - k : k$

$\Rightarrow \quad BD = k|(z_1 - z_2)|$



(A) For no solution, $BD > BB_1$

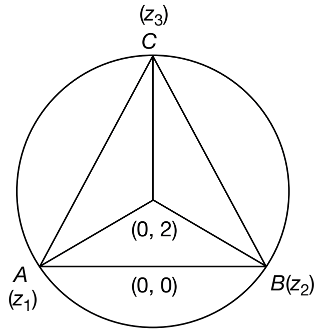
$\Rightarrow \quad k|z_1 - z_2| > \frac{2}{3}|z_1 - z_2| \Rightarrow k > \frac{2}{3}$

(B) For more than one solution, $0 < BD < BB_1$

$\Rightarrow \quad 0 < k|z_1 - z_2| < \frac{2}{3}|z_1 - z_2|\ 0 < k < \frac{2}{3}$

Hence, (A) and (B) are correct.

54. $\frac{z_2 - z_0}{z_1 - z_0} = e^{i\frac{2\pi}{3}}$; $\frac{z_3 - z_0}{z_1 - z_0} = e^{-i\frac{2\pi}{3}}$

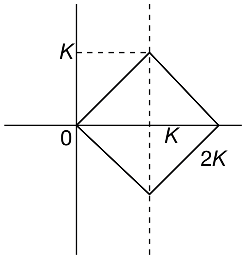


Hence, (A) and (C) are correct.

Linked Comprehension Type Questions

1. $z = x + iy$

$$|y| = K - |x - K| \Rightarrow K \geq |x - K|$$



$\Rightarrow \quad 0 \leq x \leq 2K.$; Area $= 4.\frac{1}{2}.K.K = 2K^2.$

Hence, the correct answer is (C).

2. Since the curve is symmetrical about $y = 0$; $x = K$ the centre of the circle $(K, 0)$ and it touches $y = |x|$.

$$\Rightarrow \text{Radius} = \frac{K}{\sqrt{2}}$$

Hence, the correct answer is (B).

3. B is $|Z - K| > |Z - 2K|$ is the region to the right of $x = \frac{K + 2K}{2} = \frac{3K}{2}$ points of contact of the circle $(x - K)^2 + y^2 = \frac{K^2}{2}$ with A are $\left(\frac{K}{2}, \pm\frac{K}{2}\right)\left(\frac{3K}{2}, \pm\frac{K}{2}\right)$ same $x > \frac{3K}{2}$ for B we have no points.

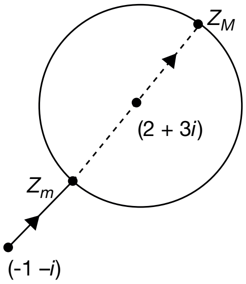
Hence, the correct answer is (A).

4. Distance between $(2 + 3i)$ and $(-1 - i)$

$$= \sqrt{(2+1)+(3+1)^2} = 5, \quad \text{if } 0 < \alpha < 5$$

Complex number $(-1 - i)$ lies outside the circle $|Z - 2 - 3i| = \alpha$

In this case, $\text{Arg}\left(\frac{Z_m + 1 + i}{Z_M - 2 - 3i}\right) = 0.$

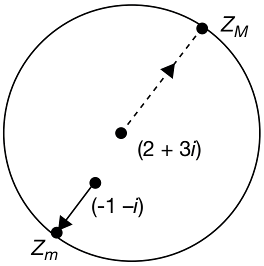


Hence, the correct answer is (A).

5. If complex number $(-1 - i)$ lies inside the circle $|Z - 2 - 3i| = \alpha$

$$\text{Arg}\left(\frac{Z_m + 1 + i}{Z_M - 2 - 3i}\right) = \pi$$

Because direction of rays will be in opposite direction as shown in the figure for this $\alpha > 5$.



Hence, the correct answer is (B).

6. Locus of Z is a circle having centre at $(2 + 3i)$ and radius $= \alpha$. If circle cuts the real axis at negative side, in that case, Arg(z) will be maximum, i.e., π.
This is possible when,
$\alpha >$ distance between origin and $(2 + 3i)$,
i.e., $\alpha > \sqrt{13}$.

Hence, the correct answer is (D).

7. $|Z - 2| \leq |Z - 4| \Rightarrow |Z - 2|^2 \leq |Z - 4|^2$

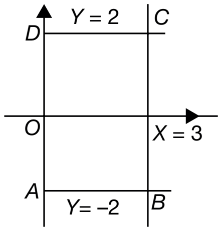
$\Rightarrow \quad (x - 2)^2 + y^2 \leq (x - 4)^2 + y^2$

$\Rightarrow \quad 8x - 4x \leq 16 - 4 \Rightarrow x \leq 3$

Similarly $|Z - 3| \leq |Z + 3| \Rightarrow x \geq 0$

$|Z - i| \leq |Z - 3i| \Rightarrow y \leq 2$

$|Z + i| \leq |Z + 3i| \Rightarrow y \leq -2$



So, the figure formed is a rectangle (boundary and interior) formed by the lines $x = 0$, $y = -2$, $x = 3$, $y = 2$.

$\therefore$ Max of $|Z| = OC$ or OB

$$= \sqrt{3^2 + 2^2} = \sqrt{13}$$

Hence, the correct answer is (D).

8. Largest circle that can be inscribed in ABCD is circle with small side as diameter.

So, radius = $\frac{3}{2}$. Hence, area of largest circle = $\frac{9\pi}{4}$.

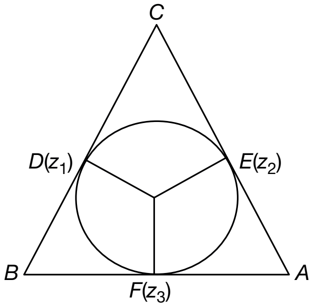
Hence, the correct answer is (B).

9. $|Z_1 - Z_2|_{\max}$ = Length of diagonal

$$= \sqrt{3^2 + 4^2} = 5$$

Hence, the correct answer is (A).

10.



Given that, $|z_1| = |z_2| = |z_3| = r$. Hence, z_1, z_2 and z_3 lies on a circle whose radius is r.

Clearly, $\text{Arg}\left(\frac{z_2 - z}{z_2}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \frac{z_2 - z}{z_2} + \frac{\bar{z}_2 - \bar{z}}{\bar{z}_2} = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad 2r^2 - \bar{z}_2 z - z_2 \bar{z} = 0 \qquad \ldots(1)$$

$$\text{and Arg}\left(\frac{z_3 - z}{z_3}\right) = -\frac{\pi}{2} \qquad \ldots(2)$$

$$\frac{z_3 - z}{z_3} + \frac{\bar{z}_3 - \bar{z}}{\bar{z}_3} = 0$$

$$\Rightarrow \quad 2r^2 - \bar{z}_3 z - z_3 \bar{z} = 0 \qquad \ldots(2)$$

Hence, the correct answer is (A).

11. In above solution, by eliminating $\bar{z}$ from equation (1) and equation (2) we get,

$$(\bar{z}_2 z_3 - \bar{z}_3 z_2) z - (z_3 - z_2) 2r^2 = 0$$

$$\Rightarrow \quad r^2\left(\frac{z_3}{z_2} - \frac{z_2}{z_3}\right) z - 2(z_3 - z_2) r^2 = 0$$

$$\Rightarrow \quad z = \frac{2z_2 z_3}{z_2 + z_3}$$

Hence, the correct answer is (B).

12. Intersection of AO and DE is the value of z that satisfied the equation (3) and (4). By substituting the value of z from (3) in (4),

$$(\bar{z}_1 - \bar{z}_2) z - (z_1 - z_2) z \sqrt{\frac{\bar{z}_2 \bar{z}_3}{z_2 z_3}} + (z_1 \bar{z}_2 + \bar{z}_1 z_2) = 0$$

$$\Rightarrow \quad r^2\left(\frac{1}{z_1} - \frac{1}{z_2}\right) z - (z_1 - z_2) z \sqrt{\frac{\bar{z}_2 \bar{z}_3}{z_2 z_3}} + r^2\left(\frac{z_1}{z_2} - \frac{z_2}{z_1}\right) = 0$$

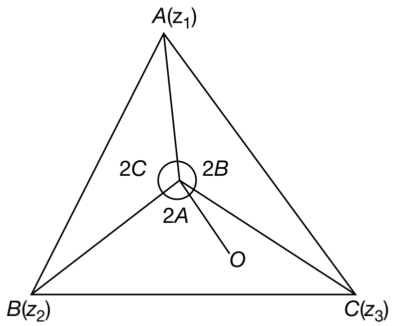
$$\Rightarrow \quad \left(\frac{z_2 - z_1}{z_1 z_2}\right) z - \left(\frac{z_1 - z_2}{z_2 z_3}\right) z + \frac{z_1^2 - z_2^2}{z_1 z_2} = 0$$

$$\text{or } z\left(\frac{1}{z_1 z_2} + \frac{1}{z_2 z_3}\right) = \frac{z_1 - z_2}{z_1 z_2}$$

which gives, $z = \frac{(z_1 + z_2) z_3}{z_1 + z_3}$

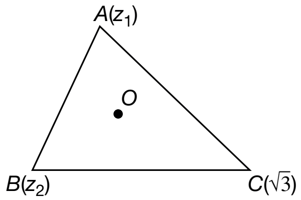
Hence, the correct answer is (C).

13. Obviously, origin coincides with circumcentre.



Hence, the correct answer is (B).

14. The angles of the triangle ABC are $C = 45°$, $B = 60°$, $A = 75°$



Using the concept of rotation in triangle AOC at 'O' we get,

$$\frac{z_1}{\sqrt{3}} = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad z_1 = \frac{-\sqrt{3}}{2} + i\frac{3}{2}$$

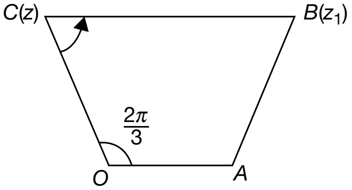
$$\text{and } z_2 = \sqrt{3}\text{cis}\left(-\frac{5\pi}{6}\right)$$

$$z_2 = \sqrt{3}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)$$

$$= \sqrt{3}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\frac{3}{2} - i\frac{\sqrt{3}}{2}$$

Hence, the correct answer is (D).

15. It can be seen that, $BC = 2OC$ using concept of rotation at C, we get



$$\frac{z - z_1}{z - 0} = 2e^{\frac{i\pi}{3}} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad 1-\frac{z_1}{z}=1+i\sqrt{3} \Rightarrow z=-\frac{z_1}{i\sqrt{3}} = i\frac{z_1}{\sqrt{3}}$$

Hence, the correct answer is (C).

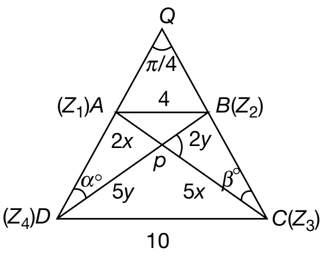
16. **Hence, the correct answer is (D).**

17. **Hence, the correct answer is (B).**

18. **Hence, the correct answer is (A).**

Combined Solution to 16–18

Triangles ABP and CDP are similar. Let $AP = 2x$, $BP = 2y$, then $CP = 5x$, $DP = 5y$



Area of trapezium $ABCD = \frac{49}{2}xy$.

Also, $\tan\alpha = \frac{2x}{5y}$, $\tan\beta = \frac{2x}{5y}$ Also, $a+b = 45°$

$$\Rightarrow \quad \frac{10\left(x^2+y^2\right)}{21xy}=1$$

$$\Rightarrow \quad xy=\frac{10}{21}\left(x^2+y^2\right)$$

Also, $AB^2 = AP^2 + BP^2 \Rightarrow x^2 + y^2 = 4$

$$\therefore \quad xy=\frac{40}{21}$$

Area of trapezium $ABCD = \frac{49}{2}\times\frac{40}{21}=\frac{140}{3}$

Area $\Delta PCB = \frac{1}{2}(2y)(5x) = 5xy$

$$|CP-DP| = 5|x-y|$$

$$|CP-DP|^2 = 25(x^2+y^2-2xy)$$

$$= 25\left(\frac{21}{10}xy-2xy\right)$$

$$|CP-DP|^2 = \frac{5}{2}xy = \frac{100}{21}$$

$$\Rightarrow \quad |CP-DP| = \frac{10}{\sqrt{21}}$$

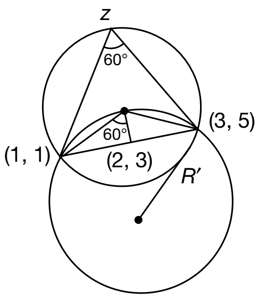
Combined Solution to 19–21

19. $\because \quad \frac{R}{\sqrt{5}} = \operatorname{cosec} 60° = \frac{2}{\sqrt{3}} \Rightarrow R = \frac{2\sqrt{5}}{\sqrt{3}}$

Hence, the correct answer is (A).

20. $$\frac{BC}{\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)} = 2R' \Rightarrow R' = \frac{BC}{2}\sec\left(\frac{A}{2}\right)$$

$$R' = \frac{2\sqrt{5}}{2}\sec 30° = \frac{2\sqrt{5}}{\sqrt{3}}$$



Hence, the correct answer is (D).

21. $$\frac{R'}{R} = \frac{2\sqrt{5}\sqrt{3}}{2\sqrt{5}}$$

$$\Rightarrow \quad \frac{R'}{R} = 1$$

Hence, the correct answer is (D).

22. **Hence, the correct answer is (D).**

23. **Hence, the correct answer is (C).**

24. **Hence, the correct answer is (C).**

Combined Solution to 22–24

$$az^2+z+1=0 \Rightarrow \bar{a}z^2 - z+1=0$$

On eliminating z,

$$\Rightarrow \quad \frac{z^2}{1+1}=\frac{2}{\bar{a}-a}=\frac{1}{-a-\bar{a}}$$

$$\Rightarrow \quad z^2=\frac{2}{a+\bar{a}} \text{ and } z=\frac{a-\bar{a}}{a+\bar{a}}$$

$$(\bar{a}-a)^2 + 2(a+\bar{a}) = 0$$

$$\Rightarrow \quad -\sin^2\theta+\cos\theta=0 \Rightarrow \cos\theta=\sin^2\theta$$

Now $f(x) = x^3 - 3x^2 + 3(1+\cos\theta)\,x + 5$;

$$f'(x) = 3x^2 - 6x + 3(1+\cos\theta)$$

Its discriminant is $-36\sin^2\theta < 0$.

Hence, $f(x)$ is increasing for all x. Also, $f(0) = 5$.

Therefore, $f(x) = 0$ has one negative root.

Now, $\cos\theta = \sin^2\theta$ and $\cos 2\theta = \cos\theta \Rightarrow \cos\theta = \frac{1}{3}$ which has 4 roots in $(0, 4\pi)$.

Combined Solution to 25–27

25. Clearly ω and ω^2 are roots of the cubic equation,

$$\frac{1}{x+a}+\frac{1}{b+x}+\frac{1}{c+x}=\frac{2}{x},$$

whose 3rd root is $x = 1$.

Therefore, $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=2$

Hence, the correct answer is (C).

26. $$\begin{vmatrix} -xw & -y & -zw^2 \\ -y & -zw^2 & -xw \\ -zw^2 & -xw & -y \end{vmatrix} = 0$$

$$\Rightarrow \quad x^3+y^3+z^3-3xyz=0$$

$$\Rightarrow \quad (x+yw+zw^2)(x^2+y^2w^2+z^2w-xyw-yz-zxw^2)=0.$$

The determinant vanishes for

$$x=y=z \text{ or } x+yw+zw^2=0.$$

Hence, the correct answer is (B).

27. $$\frac{1}{a+w}+\frac{1}{b+w}+\frac{1}{c+w}=\frac{1}{w};$$

$$\frac{1}{a+w^2}+\frac{1}{b+w^2}+\frac{1}{c+w^2}=\frac{1}{w^2}$$

Here, w^2 is the conjugate of w

Subtracting, $\sum \frac{1}{a^2-a+1}=1$.

Hence, the correct answer is (A).

Combined Solution to 28–30

28. $$\frac{\left|\frac{z_2}{z_1}+1-2i\right|}{\left|\frac{z_2}{z_1}+1+2i\right|}=1.$$

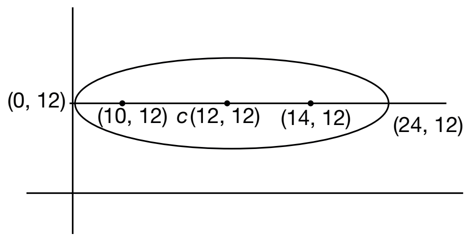
Let $\frac{z_2}{z_1}=p+iq$

$$\Rightarrow (p+1)^2+(q-2)^2=(p+1)^2+(q+2)^2 \Rightarrow q=0$$

Therefore, $\frac{z_2}{z_1}$ is purely real.

Hence, the correct answer is (A).

29.



From figure,

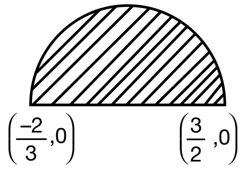
Max. Amp$(z)=\frac{\pi}{2}$

Hence, the correct answer is (B).

30. $$\text{Arg}\left(\frac{(z-1)(2z-3)}{(z-1)(3z+2)}\right)=\alpha$$

$$\Rightarrow \quad \text{Arg}\left(\frac{z-\frac{3}{2}}{z+\frac{2}{3}}\right)=\alpha$$

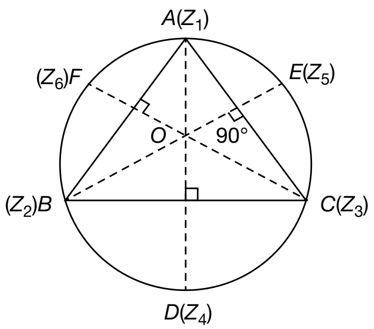
$$\text{Area}=\frac{\pi a^2}{2}=\frac{\pi}{2}\cdot\left(\frac{13}{12}\right)^2=\frac{169\pi}{288}>\frac{\pi}{2}$$



Hence, the correct answer is (C).

Combined Solution to 31–33

31.



32. Applying rotation in ΔOAC and OBD

$$\frac{z_1}{z_3}=e^{i2B};\ \frac{z_4}{z_2}=e^{i(\pi-2B)};$$

$$\frac{z_1z_4}{z_2z_3}=e^{i\pi}=-1 \Rightarrow z_4=\frac{-z_2z_3}{z_1}$$

Similarly, $z_5=\frac{-z_3z_1}{z_2}$ and $z_6=\frac{-z_1z_2}{z_3}$

Therefore, $z_4z_5z_6=-z_1z_2z_3$ or $z_4z_5z_6+z_1z_2z_3=0$

AD is the angle bisector $\Rightarrow z_4^2=z_2z_3$

As in previous part, $z_4=\frac{-z_2z_3}{z_1} \Rightarrow \left(\frac{-z_2z_3}{z_1}\right)^2$

$$=z_2z_3 \qquad \therefore\ z_2z_3\left(\frac{z_2z_3}{z_1^2}-1\right)=0$$

$$z_1^2=z_2z_3 \quad (\text{Since, } z_2z_3\neq 0)$$

Hence, the correct answer is (A).

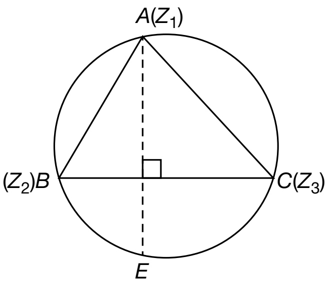
33. $\frac{BL}{DC}=\tan C$ applying rotation in ΔBLD

$$\frac{z_4-z}{z_2-z}=\left|\frac{z_4-z}{z_2-z}\right|e^{i\frac{\pi}{2}} \Rightarrow \frac{z_4-z}{z_2-z}=i\cot C=i$$

$$\Rightarrow \quad z=\frac{z_2+z_4-i(z_2-z_4)}{2}$$

Hence, the correct answer is (A).

Combined Solution to 34–36



34. Since, $AB=AC \Rightarrow |z_1-z_2|^2=|z_1-z_3|^2$

$$\Rightarrow \quad (z_1-z_2)\left(\overline{z_1}-\overline{z_2}\right)=(z_1-z_3)\left(\overline{z_1}-\overline{z_3}\right)$$

$$\Rightarrow \quad z_1\bar{z}_2+z_2\bar{z}_1=z_1\bar{z}_3+z_3\bar{z}_1$$

$$\Rightarrow \frac{z_1}{z_2}+\frac{z_2}{z_1}=\frac{z_1}{z_3}+\frac{z_3}{z_1}$$

$$\Rightarrow z_1\left(\frac{1}{z_2}-\frac{1}{z_3}\right)=\frac{1}{z_1}(z_3-z_2)\Rightarrow z_1^2=z_2z_3$$

Hence, the correct answer is (A).

35. In any triangle,

$$\frac{OG}{GM}=\frac{1}{2}\Rightarrow\frac{z_1+z_2+z_3}{3}=\frac{1\times h+2\times 0}{3}$$

(where h is orthocentre of triangle)

$\Rightarrow h=z_1+z_2+z_3$.

Hence, the correct answer is (C).

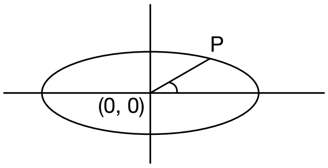
36. If ABC triangle is taken as is the figure, then, $z_1=1$, $z_2=-1$, $z_3=1$

$\Rightarrow D=-1$.

Therefore, only option (D) is satisfied.

Hence, the correct answer is (C).

37.



$$P=\left(0+\frac{r\sqrt{3}}{2},0+\frac{r}{2}\right)\text{ lies on ellipse}$$

$$\Rightarrow \frac{3r^2}{64}+\frac{r^2}{28}=1;r^2=\frac{448}{37}\quad\Rightarrow\quad[r^2]=12$$

Hence, the correct answer is (D).

38. $O=(0,0)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\qquad L=\left(ae,\frac{b^2}{a}\right)$$

$$OL^2=a^2e^2+\frac{b^4}{a^2}=a^2e^2+\frac{a^4(1-e^2)^2}{a^2}$$

$$=a^2\{e^2+e^4-2e^2+1\}=a^2(e^4-e^2+1)$$

$$OL=a\sqrt{e^4-e^2+1}=\frac{k}{2}\sqrt{e^4-e^2+1}$$

Hence, the correct answer is (B).

39. $P=z_3=(a\cos\theta, a\sin\theta)$

$S=z_1=(ae,0)\quad S'=z_2=(-ae,0)$

$$GE=SP^2+S'P^2=(ae-a\cos\theta)^2+a^2\sin^2\theta$$

$$+(ae+a\cos\theta)^2+a^2\sin^2\theta$$

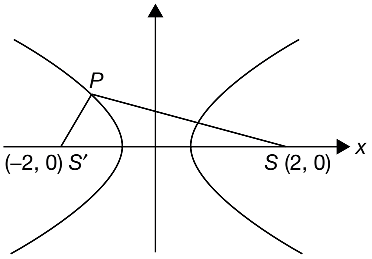
$$=2(a^2e^2+a^2\cos^2\theta)+2a^2\sin^2\theta$$

$$=\frac{4a^2}{2}(e^2+1)=\frac{k^2}{2}(e^2+1)$$

Hence, the correct answer is (B).

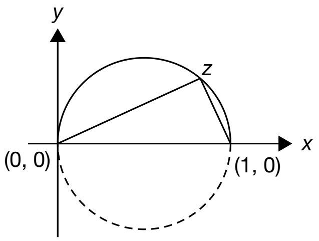
Combined Solution to 40–42

40. (A)



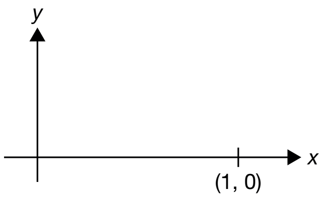
$SS'>k\quad\therefore z$ lies on hyperbola

(B)



z lies semicircle above x-axis

(C)



z lies on x-axis $x<1$

(D) $\text{Re}\left(\frac{z-1}{z+1}\right)=0\Rightarrow\frac{z-1}{z+1}+\frac{\bar{z}-1}{\bar{z}+1}=0$

$\Rightarrow z\bar{z}=1\Rightarrow|z|^2=1$

$x^2+y^2=1$

A, B and C are represented geometrically.

Clearly, $A\cap B\cap C=\phi$.

Hence, the correct answer is (A).

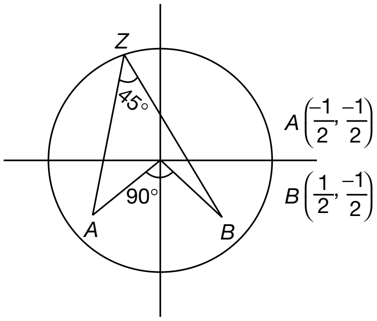
41. Clearly, S represents the set of complex number lying on the circle $|z|=1, z\neq-1$.

$$|z_1+z_2|^2+|z_2+z_3|^2+|z_3+z_1|^2$$

$$=3+(z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3)=3+|z_1+z_2+z_3|^2\geq 3$$

Hence, the correct answer is (B).

42. From the diagram,



$$\text{Arg}\left(\frac{\frac{1}{2}-\frac{1}{2}i-z}{-\frac{1}{2}-\frac{1}{2}i-z}\right)=\frac{\pi}{4} \Rightarrow \text{Arg}\left(\frac{2z+i-1}{2z+i-1}\right)=\frac{\pi}{4}$$

Hence, the correct answer is (B).

43. **Hence, the correct answer is (B).**

44. **Hence, the correct answer is (D).**

45. **Hence, the correct answer is (B).**

Combined Solution to 43–45

Let $\alpha = \text{cis}\frac{2\pi}{7}$, then $a = \alpha + \alpha^2 + \alpha^4$, $b = \alpha^3 + \alpha^5 + \alpha^6$

$\Rightarrow \quad a + b = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 = -1$

and $ab = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6) = 2$

Therefore, the equation whose roots are a and b is

$$x^2 - x(-1) + 2 = 0$$

$$f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + \ldots. + f(\alpha^6 x)$$

$$= (1 + 1 + 1 \ldots + 1) + 2x(\alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^6)$$

$$+ 3x^2(1 + \alpha^2 + \alpha^4 + \ldots + \alpha^2) + \ldots$$

$$+ x^6(1 + \alpha^6 + \alpha^{12} + \alpha^{18} + \ldots + \alpha^{36})7 \text{ times} = 7$$

Also, $\frac{x^7-1}{x-1}$

$$= (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)(x-\alpha^5)(x-\alpha^6)$$

Substituting $\alpha = \text{cis}\frac{2\pi}{7}$ by applying $x \to 1$ gives

$$7 = \left|\left(1-\text{cis}\frac{2\pi}{7}\right)\left(1-\text{cis}\frac{4\pi}{7}\right)\ldots\ldots.\left(1-\text{cis}\frac{12\pi}{7}\right)\right|$$

$$7 = \left(2\sin\frac{\pi}{7}\right)\left(2\sin\frac{2\pi}{7}\right)\ldots\ldots.\left(2\sin\frac{6\pi}{7}\right)$$

$$\left(\because \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7}\right)$$

$$= \sqrt{\frac{7}{64}} = \frac{\sqrt{7}}{8}$$

Combined Solution to 46–48

46. $\displaystyle\sum_{k=0}^{n-1}(z_1+\omega^k z_2)(\bar{z}_1+\bar{\omega}^k\bar{z}_2)$

$$= \sum_{k=0}^{n-1}\left(z_1\bar{z}_1 + (\omega\bar{\omega})^k z_2\bar{z}_2 + \bar{z}_1 z_2\omega^k + z_1\bar{z}_2\bar{\omega}^k\right)$$

$$= (|z_1|^2)\sum_{k=0}^{n-1}(1) + |z_2|^2$$

$$\sum_{k=0}^{n-1}(1) + (z_2\bar{z}_1)\frac{(1-\omega^n)}{1-\omega} + (z_1\bar{z}_2)\left(\frac{1-\bar{\omega}^n}{1-\bar{\omega}}\right)$$

$$= n(|z_1|^2 + |z_2|^2)$$

Hence, the correct answer is (A).

47. $x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\ \ldots.\ (x-\alpha^{n-1})$

$$(x-\alpha)(x-\alpha^2)(x-\alpha^3)\ \ldots.\ (x-\alpha^{n-1})$$

$$= 1 + x + x^2 + \ldots. + x^{n-1}$$

Taking log and then differentiating, we get

$$\frac{1}{x-\alpha}+\frac{1}{x-\alpha^2}+\ldots\ldots+\frac{1}{x-\alpha^{n-1}}$$

$$= \frac{1+2x+\ldots..+(n-1)x^{n-2}}{1+x+x^2+\ldots..+x^{n-1}}$$

$$\Rightarrow \quad \frac{1}{2-\alpha}+\frac{1}{2-\alpha^2}+\ldots\ldots.+\frac{1}{2-\alpha^{n-1}}$$

$$= \frac{1+2\times2+3\times2^2+\ldots..+(n-1)2^{n-2}}{1+2+2^2+\ldots\ldots+2^{n-1}}$$

$$= \frac{\sum_{r=1}^{n-1} r2^{r-1}}{\sum_{r=0}^{n-1} 2^r} = \frac{(n-2)2^{n-1}+1}{2^{n-1}}$$

Hence, the correct answer is (A).

48. $(x-\omega)(x-\omega^2)\ \ldots.\ (x-\omega^{n-1}) = 1 + x + x^2 + \ldots. + x^{n-1}$

By substituting $x = 1$, we get

$$(1-\omega)(1-\omega^2)\ldots.(1-\omega^{n-1}) = 1 + 1 + \ldots\ n \text{ times} = n$$

Hence, the correct answer is (C).

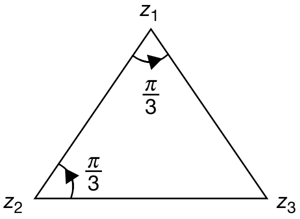
Matrix Match/Column Match Type Questions

1. A → (t); B → (r); C → (p, s); D → (q)

put $z = x + iy$

Simplify conditions written in column (II) and match them.

2. A → (s); B → (p); C → (q); D → (r)



(A) $\dfrac{z_3-z_1}{z_2-z_1} = \dfrac{z_1-z_2}{z_3-z_2} = e^{i\frac{\pi}{3}}$

$$\Rightarrow (z_3-z_1)(z_3-z_2) = (z_1-z_2)(z_2-z_1)$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3$$

(B) $\text{Re}\left(\dfrac{z_3-z_1}{z_3-z_2}\right) = 0 \Rightarrow \text{Arg}\left(\dfrac{z_3-z_1}{z_3-z_2}\right) = \pm\dfrac{\pi}{2}$

(C) $\text{Re}\left(\dfrac{z_3-z_1}{z_2-z_2}\right) < 0 \Rightarrow \text{Arg}\left(\dfrac{z_3-z_1}{z_3-z_2}\right) > \dfrac{\pi}{2}$

(D) $\left(\dfrac{z_3-z_1}{z_3-z_2}\right) = i = e^{i\frac{\pi}{2}}$ and $|z_3-z_1| = |z_3-z_2|$

3. A → (r); B → (q); C → (p)

(A) $$\left(\frac{w^{n+1}-1}{w-1}\right)^m = \begin{cases} 0, \text{ if } n+1=3k, k \in N \text{ and } m \in N \\ 1, \text{ if } n+1=3k+1, k \in N \text{ and } m \in N \\ (\omega+1)^m, \text{ if } n+1=3k+2, k \in N \end{cases}$$

Now, $(w + 1)^m = (-w^2)^m$

$(-w^2)^m \in \{-1, w, w^2, -w^2\}$, depending upon the value of m.

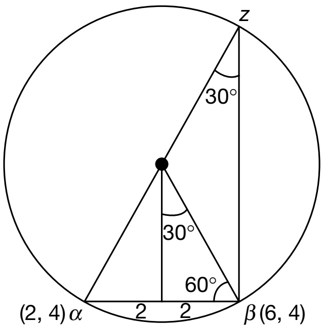
$\Rightarrow$ Answer is 6; $\{0, 1, -1, w, 1 - w^2, w^2\}$

(B) Coefficient of polynomial are real and roots are $2w$, $2w^2$, $2+3w$ and $2+3w^2$ are conjugate of each other and $2 - w - w^2 = 3$

so total 5 roots are possible

$\Rightarrow$ Roots are $2w$, $2w^2$, $2+3w$, $2+3w^2$ and 3

(C) Referring to the figure,



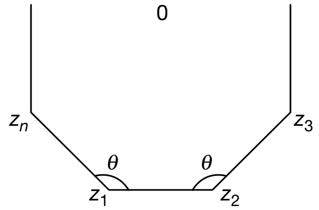
$$\cos 60° = \frac{2}{R} \Rightarrow R = 4$$

4. A → (r); B → (r); C → (q); D → (q)

Let θ be an interior angle of the polygon.

$z_n - z_1 = (z_2 - z_1)e^{i\theta}$, $z_3 - z_2 = (z_1 - z_2)e^{-i\theta}$

$\therefore$ $z_3 + z_n = z_2 + (z_2 - z_1)\, e^{i\theta} + z_1 + (z_1 - z_2)e^{-i\theta}$



$= (1 - 2i \sin \theta)z_1 + (1 + 2i \sin \theta)z_2$

Therefore, $A = 1 - 2i \sin \theta$

$\Rightarrow$ $|A| = \sqrt{3 - 2\cos 2\theta} = \sqrt{3 - 2\cos\frac{4\pi}{n}}$

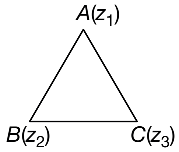
Therefore, $|A| = \sqrt{5}, 2, \sqrt{3}, \sqrt{2}$ according to $n = 4, 6, 8, 12$

5. A → (s); B → (r); C → (q); D → (p)

(A) $\overline{BA} = \overline{BC}\, e^{\frac{i\pi}{3}}$

$z_1 - z_2 = (z_3 - z_2)\, e^{i\frac{\pi}{3}}$ …(1)

$\overline{CA} = \overline{CB}\, e^{\frac{-i\pi}{3}}$



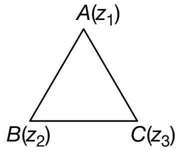
$\Rightarrow$ $(z_1 - z_3) = (z_2 - z_3)\; e^{\frac{-i\pi}{3}}$ ….(2)

From (1) and (2), $\dfrac{z_2 - z_1}{z_3 - z_2} = \dfrac{z_3 - z_2}{z_3 - z_1}$

$\Rightarrow$ $\sum z_1^2 = \sum z_1 z_2 \Rightarrow$ A → s

(B) $\overline{CA} = \overline{CB}\, e^{\frac{i\pi}{2}}$

$\Rightarrow$ $z_1 - z_3 = (z_2 - z_3)(-i)$ …(3)



$\overline{BA} = \sqrt{2}\,\overline{BC}\, e^{\frac{i\pi}{4}}$

$(z_1 - z_2)^2 = 2(z_3 - z_2)^2$ …(4)

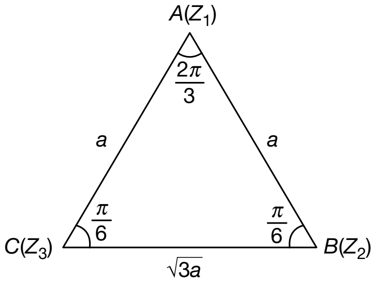
$\Rightarrow$ $(z_1 - z_2)^2 = 2(z_3 - z_2)^2 \dfrac{(-z_3 + z_1)}{-z_2 + z_3}$

$= 2(z_3 - z_2)(z_1 - z_3) \Rightarrow$ B → r

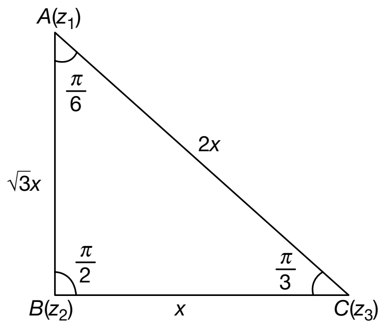
(C) $\overline{BA} = \dfrac{1}{\sqrt{3}}\overline{BC}\, e^{\frac{-i\pi}{6}}, \overline{CA} = \dfrac{1}{\sqrt{3}}\, e^{\frac{i\pi}{6}}, \overline{CB}$

$\Rightarrow$ $\overline{BC}^2 = -3\overline{BA}.\overline{CA} \Rightarrow (z_2 - z_3)^2$

$= -3(z_3 - z_1)\,(z_2 - z_1) = 3(z_1 - z_2)\,(z_3 - z_1)$



(D) $\overline{BA} = \overline{BC}\sqrt{3}\, e^{\frac{i\pi}{2}}$



By squaring we get,

$$(z_1 - z_2)^2 = 3(z_3 - z_2)^2 (-1)$$

$\Rightarrow$ D → p

6. A → (t); B → (r); C → (p); D → (q)

(A) $|z-1|^2 + |z+1|^2 = 4 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow$ $r = 1$

$\Rightarrow$ A → t

(B) Equation of diameter are (0, 2) (0, – 2)

$\Rightarrow$ radius = 2

$\Rightarrow$ B → r

(C) $z\bar{z} - z(1-2i) - \bar{z}(1+2i) - 4 = 0$

centre (1, 2), $|z_0|^2 - r^2 = -4 \Rightarrow r = 3$

$\Rightarrow$ C → p

(D) $|z+3| = 2|z-3| \Rightarrow x^2 + y^2 - 10x + 9 = 0$

$\Rightarrow$ $r = 4$

$\Rightarrow$ D → q

7. A → (s); B → (r); C → (p, q); D → (q)

(A) Let $z = x + iy \Rightarrow x^2 - y^2 = 2x \Rightarrow z$ lies on a hyperbola

(B) $|z - z_1| + |z - z_2| = \lambda$ and $\lambda > |z_1 - z_2| \Rightarrow z$ lies on an ellipse

(C) $\dfrac{\left|z - \dfrac{i}{2}\right|}{|z+1|} = \dfrac{m}{2}$, if $m = 2 \Rightarrow z$

lies on a straight line

if $m \neq 2 \Rightarrow z$ lies on a circle

(D) $|z| = 25$, let $-1 + 75\bar{z} = \omega$

$\Rightarrow$ $|\omega + 1| = 75|\bar{z}| = 75 \times 25$

$\Rightarrow$ ω lies on a circle

8. A → (p); B → (s); C → (p); D → (q)

(A) Let $f(x) = e^x - x - \dfrac{x^3}{3}$. Then $f'(x) = e^x - 1 - x^2$.

When $x < 0$,

$e^x < 1$ and $1 + x^2 > 1$, so $f'(x) = e^x - (1 + x^2) < 0$.

Thus, f is decreasing on $(-\infty, 0)$. When $x = 0$,

$f'(x) = f'(0) = e^o - 1 - 0^2 = 1 - 1 = 0$.

Finally, for $x > 0$, $f'(x) = e^x - 1 - x^2 > 0$

by a Maclaurin series expansion. So, f is increasing on $(0, \infty)$. Thus, f must attain its minimum when $x = 0$, at which point f has the value, $e^0 - 0 - \dfrac{0^3}{3} = 1$.

(B) $\dfrac{z_j}{1+z_j^2} = \dfrac{z_j^4}{1+z_j^3}$ and $\dfrac{z_j^2}{1+z_j^4} = \dfrac{z_j^3}{1+z_j}$

So, $w_j = 2\left(\dfrac{z_j}{1+z_j^2} + \dfrac{z_j^2}{1+z_j^4}\right)$

$$= 2\left(\frac{z_j\left(1+z_j^4\right) + z_j^2\left(1+z_j^2\right)}{\left(1+z_j^2\right)\left(1+z_j^4\right)}\right)$$

$$= 2\left(\frac{1+z_j+z_j^2+z_j^4}{1+z_j+z_j^2+z_j^4}\right)$$

= 2, for each j.

$$\Rightarrow \sum_{j=1}^{5} w_j = 10$$

(C) We have,

$$\frac{1+z+z^2}{1-z+z^2} = 1 + 2\frac{z}{1-z+z^2} \in \mathbb{R} \text{ iff } \frac{z}{1-z+z^2} \in \mathbb{R}$$

That is

$$\frac{z}{1-z+z^2} = \frac{1}{z} - 1 + z \in \mathbb{R}, \text{ i.e., } z + \frac{1}{z} \in \mathbb{R}.$$

The last relation is equivalent to

$$z + \frac{1}{z} = \bar{z} + \frac{1}{\bar{z}}, \text{ i.e., } (z - \bar{z})(1 - |z|^2) = 0.$$

We find $z = \bar{z}$ or $|z| = 1$.

As z is not a real number, it follows $|z| = 1$.

(D) Consider the complex plane such that the coordinates of points A, B and C are the cube roots of unity 1, ε and ε^2, respectively. Let z be the coordinate of point P.

Then $|z| = 1$ and we have

$$PA^2 + PB^2 + PC^2 = |z-1|^2 + |z-\varepsilon|^2 + |z-\varepsilon^2|^2$$

$$= (z-1)(\bar{z}-1) + (z-\varepsilon)(\bar{z}-\bar{\varepsilon}) + (z-\varepsilon^2)(\bar{z}-\bar{\varepsilon}^2)$$

$$= 3|z|^2 - (1+\varepsilon+\varepsilon^2)\bar{z} - (1+\bar{\varepsilon}+\bar{\varepsilon}^2)z + 1 + |\varepsilon|^2 + |\varepsilon^2|^2$$

$$= 3 - 0.\bar{z} - 0.z + 1 + 1 + 1 = 6$$

9. A → (q), (s); B → (r, s); C → (p, q, r, s); D → (p, r)

(A) $x^6 + 1 = 0 \Rightarrow (x^2+1)(x^4 - x^2 + 1) = 0$

$\Rightarrow x = \pm i, \pm iw, \pm iw^2$

(B) $x^8 + x^4 + 1 = 0 \Rightarrow (x^4 + x^2 + 1)(x^4 - x^2 + 1) = 0$

$\Rightarrow x = \pm w, \pm w^2, \pm iw, \pm iw^2$

(C) $x^{12} - 1 = (x^6 - 1)(x^6 + 1)$

(D) $x^6 - 1 = 0 \Rightarrow (x^3 - 1)(x^3 + 1) = 0$

$\Rightarrow x = 1, -1, \pm w, \pm w^2$,

$= (x^2 - 1)(x^4 + x^2 + 1)(x^2 + 1)(x^4 - x^2 + 1)$

$\because$ $x^8 + x^4 + 1 = (x^4 + x^2 + 1)(x^4 - x^2 + 1)$

The roots of $x^2 - 1 = 0$ are $\pm i$

The roots of $x^2 + 1 = 0$ are ± 1

The roots of $x^4 + x^2 + 1 = 0$ are $\pm w, \pm w^2$

The roots of $x^4 - x^2 + 1 = 0$ are $\pm iw, \pm iw^2$

10. A → (s); B → (p); C → (q); D → (r)

(A) $\big||z-w| - |z-\bar{w}|\big| \max = |w - \bar{w}| = \sqrt{3}$

(using triangular inequality)

(B) Applying rotation we get,

$$z_4 = \frac{-z_2 z_3}{z_1} \Rightarrow z_1 z_4 + z_2 z_3 = 0$$

(C) $Q = (-3, 0), PQ = \sqrt{16+1-5} = \sqrt{12}$

(D) I^{st} series $t_n = 3 + (n-1)4$

II^{nd} series last term is 47.

$\Rightarrow \quad t_n \le 47$

$\Rightarrow \quad 3 + (n-1)4 \le 47$

$\Rightarrow \quad n \le 12$

11. A → (p, q); B → (p, q, r, s, t); C → (r, s); D → (p, q)

(a) Ellipse $k > |7i - 2i|$ (or) $k > 5$

(b) $$\left|\frac{2z-3}{3z-2}\right| = k \Rightarrow \left|\frac{Z-\frac{3}{2}}{Z-\frac{2}{3}}\right| = \frac{3k}{2}$$

$$\Rightarrow \quad \frac{3k}{2} > 1 \Rightarrow k > \frac{2}{3}$$

(c) $|Z-3| - |Z-4i| = k$ is hyperbola

If $k < |3-4i| \Rightarrow 0 < k < 5$

(d) $$|Z-(3+4i)| = \frac{k}{5}\frac{(a\bar{z}+\bar{a}z+b)}{2|3+4i|}$$

Hyperbola is $\frac{k}{5} > 1 = k > 5$

Integer/Numerical Answer Type Questions

1. $z_1\bar{z}_1 = 4;\ z_2 \cdot \bar{z}_2 = 9,\ z_3 \cdot \bar{z}_3 = 16$

$|8z_2z_3 + 27z_3z_1 + 64z_1z_2| = |z_1||z_2||z_3|$

$$\left|\frac{8}{z_1}+\frac{27}{z_2}+\frac{64}{z_3}\right| = 24\left|\frac{8\bar{z}_1}{4}+\frac{27\bar{z}_2}{9}+\frac{64\bar{z}_3}{16}\right|$$

$$= 24|2z_1 + 3z_2 + 4z_3| = 96$$

Hence, the correct answer is (6).

2. $${}^nc_0 - \frac{{}^nc_2}{(2+\sqrt{3})^2} + \frac{{}^nc_4}{(2+\sqrt{3})^4} - \frac{{}^nc_6}{(2+\sqrt{3})^6} + \ldots..$$

$$= (-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n$$

$$= \text{real part of } \left(1+\frac{i}{2+\sqrt{3}}\right)^n$$

$$= \text{real part of } (1 + i(2-\sqrt{3}))^n$$

$$= \text{real part of } \left(1 + i\tan\frac{\pi}{12}\right)^n$$

$$= \text{real part of } \frac{\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)^n}{\cos^n\frac{\pi}{12}}$$

$$= \text{real part of } \left(\frac{\cos\frac{n\pi}{12} + i\sin\frac{n\pi}{12}}{\cos^n\frac{\pi}{12}}\right) = \frac{\cos\frac{n\pi}{12}}{\cos^n\frac{\pi}{12}} = \frac{\cos m\pi}{\cos^n\frac{\pi}{12}}$$

$$= (-1)^m\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)^n \qquad \left(\text{Since, } \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$P - Q = 8 - 3 = 5$

Hence, the correct answer is (5).

3. $z + \frac{1}{z} = 1 \Rightarrow z = -\omega, -\omega^2$

$z^{2005} + \frac{1}{z^{2005}} = -\omega - \omega^2 = 1;$ for $z = -\omega$ or ω^2

Therefore, $a = 1$, for $n > 1$, $2^{2^n} = 2^{4k}, k \in Z$, i.e., 16^k

16^k has last digit as 6. Therefore, $b = 6 - 1 = 5$

Therefore, $\frac{a^2+b^2}{13} = 2$

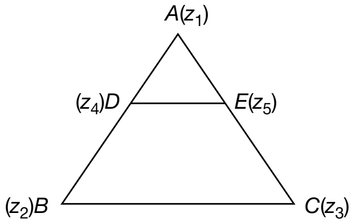
Hence, the correct answer is (2).

4. $Z = x + iy \Rightarrow 2xy = 16 \Rightarrow xy = 8 = c^2$

Latus rectum $= 2\sqrt{2}\,c = 8$

Hence, the correct answer is (8).

5.



$\because \quad DE \,||\, BC$

$\Rightarrow \quad \frac{AD}{AB} = \frac{DE}{BC}$ (using basic proportionalitytheorem)

$\Rightarrow \quad DE = \frac{2}{5}|z_2 - z_5|$

$\Rightarrow \quad DE = \frac{2}{5} \times 10 = 4$

Hence, the correct answer is (4).

6. $x + iy = 3\text{cis}\theta + \frac{1}{3}\text{cis}(-\theta)$

$x = \frac{10}{3}\cos\theta,\ y = \frac{8}{3}\sin\theta$

$$\Rightarrow \quad \frac{9x^2}{100} + \frac{9y^2}{64} = 1 \Rightarrow \frac{x^2}{100} + \frac{y^2}{64} = \frac{1}{9}$$

Therefore, $k = 9$

Hence, the correct answer is (9).

7. $\alpha = x + iy$

$(x+iy)i - (x-iy)i + 2 = 0 \Rightarrow y = 1$

$z(1+i) + \bar{z}(1-i) + 2 = 0 \Rightarrow x - y + 1 = 0$

$x^2 + (y-1)^2 = 4 \Rightarrow x = \pm\sqrt{2}$

$[|x|] = 1$

Hence, the correct answer is (1).

8. $iz^3 + z^2 + i^2z + i = 0 \Rightarrow (iz+1)(z^2+i) = 0$

$\Rightarrow \quad |z| = \left|\frac{-1}{i}\right|$ or $|\sqrt{i}| = 1$ (Since, $|z| = 1$ for any case)

Therefore, $[|z|] = 1$

Hence, the correct answer is (1).

9. From the given equation,

$x + y + z = (a + b\omega + c\omega^2) + (a\omega + b\omega^2 + c) + (a\omega^2 + b + c\omega)$

$= a(1 + \omega + \omega^2) + b(\omega + \omega^2 + 1) + c(\omega^2 + 1 + \omega)$

$= a \times 0 + b \times 0 + c \times 0 = 0$

$$x^3 + y^3 + z^3 = 3xyz \Rightarrow \frac{x^2 + y^3 + z^3}{xyz} = 3$$

Therefore, $\frac{x^2}{yz} + \frac{y^2}{zx} + \frac{z^2}{xy} = 3$

Hence, the correct answer is (3).

10. Let xi be the root, where $x \neq 0$ and $x \in R$

$$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$$

$$\Rightarrow \quad x^4 - a_2x^2 + a_4 = 0 \qquad \ldots(1)$$

and $a_1x^3 - a_3x = 0 \qquad \ldots(2)$

From equation (2), $a_1x^2 = a_3 \Rightarrow x^2 = \frac{a_3}{a_1}$ (as $x \neq 0$)

By substituting the value of x^2 in equation (1),

we get $\frac{a_3^2}{a_1^2} - \frac{a_2a_3}{a_1} + a_4 = 0 \Rightarrow a_3^2 + a_4a_1^2 = a_1a_2a_3$

$$\Rightarrow \quad \frac{a_3}{a_1a_2} + \frac{a_1a_4}{a_2a_3} = 1 \qquad \text{(dividing by } a_1a_2a_3\text{)}$$

Hence, the correct answer is (1).

11. We have,

$$z_1\left(z_1^2 - 3z_2^2\right) = 2 \qquad \ldots(1)$$

$$z_2\left(3z_1^2 - z_2^2\right) = 11 \qquad \ldots(2)$$

Multiplying (2) by i and adding it to (1), we get

$$z_1^3 - 3z_2^2z_1 + iz_2\left(3z_1^2 - z_2^3\right) = 2 + 11i$$

$$\Rightarrow \quad (z_1 + iz_2)^3 = 2 + 11i \qquad \ldots(3)$$

Multiplying (2) by i and subtracting it from (1), we get

$$z_1^3 - 3z_2^2z_1 - i\left(3z_1^2 - z_2^3\right) = 2 - 11i$$

$$\Rightarrow \quad \left(z_1 - iz_2\right)^3 = 2 - 11i \qquad \ldots(4)$$

Multiplying (3) and (4), we get

$$\left(z_1^2 + z_2^2\right)^3 = 2^2 - 121i^2 = 4 + 121 = 125$$

$$\Rightarrow \quad z_1^2 + z_2^2 = 5$$

Hence, the correct answer is (5).

12. Let the roots are $1 + i, 1 - i, 1 + 2i, 1 - 2i, r$,
Sum of the roots is $6 = 4 + r \Rightarrow r = 2$ and
product of the roots $-\delta$ is $= 2.5.2 \Rightarrow \delta = -20$
$\delta + 25 = 5$

Hence, the correct answer is (5).

13. z lies on an verticle ellipse whose foci are $(0, 1)$ $(0, -1)$. Therefore, $|z|_{\max} =$ length of semi major axis $= 4$.
Hence, the correct answer is (4).

14. We know that,

$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2$

$G\left(\frac{0+0+3}{3}\right), \frac{0+0+6}{3} = G(1, 2)$

min. of $PA^2 + PB^2 + PC^2$ exists at $P = G$.

$N = 5 + 8 + 17 = 30 \Rightarrow N - 28 = 2$

Hence, the correct answer is (2).

15. $|z_2 + iz_1| = |z_1| + |z_2| \Rightarrow \text{Arg}(iz_1) = \text{Arg}(z_2)$

$\Rightarrow \quad \text{Arg}(z_2) - \text{Arg}(z_1) = \frac{\pi}{2}$

Let $z_3 = \frac{z_2 - iz_1}{1 - i} \Rightarrow z_3 - z_2 = i(z_3 - z_1)$

Therefore, $\angle ACB = \frac{\pi}{2}$ as $\frac{z_2 - z_3}{z_1 - z_3} = \text{I}$ and $|z_2 - z_3|$

$= |z_1 - z_3| = \Rightarrow AC = BC$

Therefore, $AB^2 = AC^2 + BC^2$

$\Rightarrow \quad AC = \frac{5}{\sqrt{2}}$ (as $AB = 5$)

$$\Delta^2 = \frac{1}{2}AC.BC = \frac{25}{4} \Rightarrow [\Delta] = 2$$

Hence, the correct answer is (2).

16. If $PA^2 + PB^2 + PC^2$ is minimum, then 'P' coincide with mean position of A, B and C. (i.e., centroid with vertices $(1, 0)$, $(2, 0)$ and $(3, 0)$)
Therefore, $z = 2 \Rightarrow$ min of $|z - 1|^2 + |z - 2|^2 + |z - 3|^2$ is 2
Hence, the correct answer is (2).

17. $w^7 = 1$

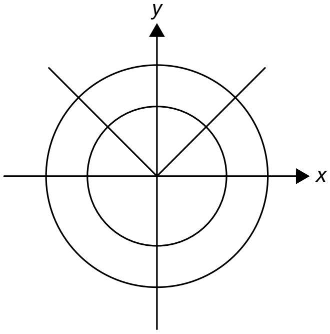
$\Rightarrow 1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

Therefore, $\alpha\beta = w^4 + w^6 + w^7 + w^5 + w^7 + w^8 + w^7 + w^9 + w^{10}$

$= w^4 + w^6 + 1 + w^5 + 1 + w + 1 + w^2 + w^3 = 2$

Hence, the correct answer is (2).

18. $\theta = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{2}$



$$\text{Area} = \frac{1}{2}r_1^2\theta - \frac{1}{2}r_2^2\theta = \frac{1}{2}\left(5^2 - 3^2\right)\frac{\pi}{2} = 4\pi$$

$\Rightarrow \quad k = 2$

Hence, the correct answer is (2).

19. $|\alpha|_{\max} = 2 + \sqrt{2}$, $|\beta|_{\max} = 3 + \sqrt{41}$

$2|\beta|_{\max} - 3|\alpha|_{\max} = \sqrt{164} - \sqrt{18}$

Therefore, $\left[\dfrac{k_1}{k_2}\right]=\left[\dfrac{164}{18}\right]=9$

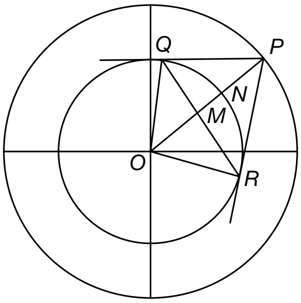
Hence, the correct answer is (9).

20. $OQ=1, OP=2$

$\sin\angle OPQ=\dfrac{1}{2}\Rightarrow\angle OPR=\dfrac{\pi}{3}$ and

and $PQ=PR\Rightarrow\Delta PQR$ is equilateral and $OM\perp QR$

From ΔOMQ, $OM=\dfrac{1}{2}$. Hence, $MN=\dfrac{1}{2}$



Centroid of ΔPQR lies on $|z|$ is 1.

$$\left|\frac{z_1+z_2+z_3}{3}\right|=1\Rightarrow|z_1+z_2+z_3|^2=9$$

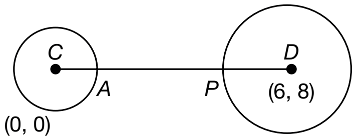
$$\Rightarrow\quad(z_1+z_2+z_3)(\bar{z}_1+\bar{z}_2+\bar{z}_3)=9$$

$$\Rightarrow\quad\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$$

Hence, the correct answer is (9).

21. $\text{Arg}\left(\dfrac{z-z_1}{z-z_2}\right)=\dfrac{\pi}{4}$ is circle with centre at (6, 8) radius $3\sqrt{2}$.

$|z|=1$ is a circle with centre (0, 0) and radius 1.



The distance between the two circles is

$10-(1+3\sqrt{2})\approx 10-(5.34)\Rightarrow[d]=4$

Hence, the correct answer is (4).

22. $(aw^2+bw)(aw+bw^2)=a^2-ab+b^2$

$$=\frac{(a+b)(a^2-ab+b^2)}{a+b}=\frac{a^3+b^3}{a+b}=\frac{105}{21}=5$$

Hence, the correct answer is (5).

23. $Z=\lim\limits_{n\to\infty}\dfrac{\pi}{2}.\dfrac{1}{n}\sum\limits_{r=0}^{n-1}\text{cis}\left(\dfrac{\pi}{2}\times\dfrac{r}{n}\right)$

$$=\frac{\pi}{2}\int_0^1 e^{i\frac{\pi}{2}x}dx$$

$$=\frac{\pi}{2}.\frac{2}{\pi}.\frac{1}{i}\left[e^{i\frac{\pi}{2}x}\right]_0^1$$

$$=\frac{1}{i}\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}-1\right]$$

$$=\frac{1}{i}(i-1)=1+i\Rightarrow|z|=\sqrt{2}$$

$$\Rightarrow\quad|z|^2=2$$

(OR)

$$Z=\lim_{n\to\infty}\frac{\pi}{2n}\left(\frac{a^n-1}{a-1}\right)$$

$$\lim_{n\to\infty}\frac{\pi}{2n}\frac{(\text{i}-1)}{-1+\cos\frac{\pi}{2n}+i\sin\frac{\pi}{2n}}$$

$$=\lim_{n\to\infty}\frac{\pi}{2n}\frac{1-i}{2\sin^2\frac{\pi}{4n}-2i\sin\frac{\pi}{4n}\cos\frac{\pi}{4n}}$$

$$=\lim_{n\to\infty}\frac{1-i}{\dfrac{\sin^2\frac{\pi}{4n}}{\left(\frac{\pi}{4n}\right)}-i\dfrac{\sin\left(\frac{\pi}{4n}\right)}{\frac{\pi}{4n}}.\cos\left(\frac{\pi}{4n}\right)}$$

$$=\frac{1-i}{0-i\times1}=1+i$$

Hence, the correct answer is (2).

24. First equation represents an ellipse and the second equation represents a line segment joining (–1, 0) and (1, 0), totally contained inside the ellipse. Hence, there is no common solution.

Hence, the correct answer is (0).

25. $x^{n-1}=(x-1)(x-\alpha)\ldots(x-\alpha^{n-1})$

$$\Rightarrow\quad S=\frac{(3-\alpha)(3-\alpha^2)\ldots\ldots(3-\alpha^{n-1})}{6}$$

If $n=6$, $\dfrac{S}{91}=\dfrac{3^6-1}{91\times2}=\dfrac{729-1}{182}=\dfrac{728}{182}=4$

Hence, the correct answer is (4).

26. Clearly, all vertices are at unit distance from origin. Let $z_1=\cos\theta+i\sin\theta=x+iy$

G.E. $\Rightarrow\dfrac{2xy}{x}=\dfrac{\sqrt{5}-1}{2}\Rightarrow y=\dfrac{\sqrt{5}-1}{4}$

$$\Rightarrow\quad\sin\theta=\frac{\sqrt{5}-1}{4}\Rightarrow\theta=18°=\frac{2\pi}{n}$$

Number of sides = 20

Therefore, $\sqrt{n}=4$

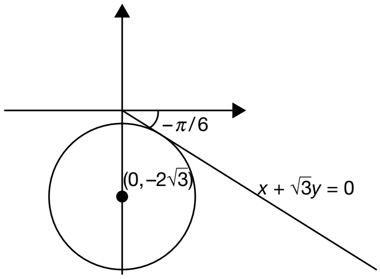
Hence, the correct answer is (4).

27. $\arg(z)=-\dfrac{\pi}{6}\quad\Rightarrow\quad\dfrac{y}{x}=-\dfrac{1}{\sqrt{3}}$

$$\Rightarrow\quad\sqrt{3}y+x=0$$

$$r\geq d$$

$$\Rightarrow \quad r \geq \frac{|0-\sqrt{3}(2\sqrt{3})|}{\sqrt{1+3}}$$



$$\Rightarrow \quad r \geq 3$$

Hence, the correct answer is (3).

28. $1 + Z + Z^2 + Z^3 + Z^4 = 0$

$$\log_{\sqrt{3}}\left|1+z+z^2+z^3-\frac{2}{z}\right|$$

$$= \log_{\sqrt{3}}\frac{\left|z+z^2+z^3+z^4+1-3\right|}{z} = \log_{\sqrt{3}} 3 = 2$$

Hence, the correct answer is (2).

29. $\lambda = 1(\alpha_1 + \alpha_{10}) + 2(\alpha_2 + \alpha_9) + \ldots.$

$$= 11[\alpha_1 + \alpha_2 + \ldots + \alpha_{10}] = 11(-1) = -11$$

$$\Rightarrow \quad \frac{\lambda + 11}{11} = 0$$

Hence, the correct answer is (0).

30. $\sqrt{(x-3)^2+(y-2)^2} = \sqrt{x^2+y^2}\left|\dfrac{\cos\theta+\sin\theta}{\sqrt{2}}\right|;$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x^2 + y^2 - 2xy - 6x - 4y + 13 = 0; a + b - 2h = 1 + 1 - 2(-1) = 4$$

Hence, the correct answer is (4).

31. $k = |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$$k = |z_1 - z_2|^2$$

$$\Rightarrow \quad k = |2 + 0 - i|^2 = 4$$

Hence, the correct answer is (4).

32. $\displaystyle\sum_{n=2}^{5}(n-1)(n-\omega)(n-\omega^2) = \sum_{n=2}^{5}(n^3-1) = 220 = 44k \Rightarrow k = 5$

Hence, the correct answer is (5).

33. $(x^2 - y^2 + 2ixy) + (x^2 - y^2 - 2ixy) + 2(x^2 + y^2) = 8i(-2iy)$

$$\Rightarrow \quad 4x^2 = 16y \Rightarrow x^2 = 4y \Rightarrow LR = 4$$

Hence, the correct answer is (4).

34. We have,

$$z^3 + 2z^2 + 2z + 1 = 0$$

$$\Rightarrow \quad (z^3 + 1) + 2z(z + 1) = 0$$

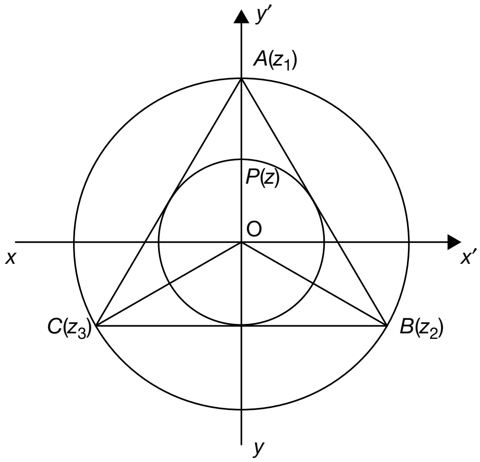
$$\Rightarrow \quad (z + 1)(z^2 + z + 1) = 0 \Rightarrow z = -1, \omega, \omega^2$$

Since, $z = -1$ does not satisfy $z^{1985} + z^{100} + 1 = 0$ while $z = \omega, \omega^2$ satisfy it. Hence, the sum is

$$\omega + \omega^2 = -1 \Rightarrow |k| = 1$$

Hence, the correct answer is (1).

35.



Let $z_1 = \dfrac{2}{\sqrt{3}}\, e^{i\frac{\pi}{2}}$, $z_2 = \dfrac{2}{\sqrt{3}}\, e^{i\frac{\pi}{6}}$ and $z_3 = \dfrac{2}{\sqrt{3}} e^{\frac{-i5\pi}{6}}$

Clearly, the points lie on the circle $|z| = \dfrac{2}{\sqrt{3}}$ and ΔABC is equilateral.

$\Rightarrow$ Centroid coincides with circumcentre.

$\Rightarrow$ $z_1 + z_2 + z_3 = 0$ and $\bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 0$

Since, the triangle is equilateral and inradius,

$$r = ID = \frac{1}{\sqrt{3}}$$

$\Rightarrow$ Equation of the incircle $|z| = \dfrac{1}{\sqrt{3}}$

Let P(z) be any point on the incircle.

Now, $AP^2 = |z - z_1|^2 = |z|^2 + |z_1|^2 - (\bar{z}z_1 + z\bar{z}_1)$

Similarly, $BP^2 = |z|^2 + |z_2|^2 - (\bar{z}z_2 + z\bar{z}_2)$

and $CP^2 = |z|^2 + |z_3|^2 - (\bar{z}z_3 + z\bar{z}_3)$

$$\Rightarrow \quad AP^2 + BP^2 + CP^2 = 3|z|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2$$
$$- z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) - \bar{z}(z_1 + z_2 + z_3) = 5$$

Hence, the correct answer is (5).

36. $\alpha = e^{\frac{i2\pi}{5}}, \alpha^2 = e^{\frac{i4\pi}{5}},$

$$|1 + \alpha + \alpha^2| = \left|1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\left(\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5}\right)\right|$$

$$= \left|1 + 2\cos\frac{2\pi}{5}\right|\left|\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right|$$

$$= 1 + 2\left(\frac{\sqrt{5}-1}{4}\right) = \frac{1+\sqrt{5}}{2} \Rightarrow a + b = 6$$

Hence, the correct answer is (6).

37. Substitute $x = \omega$ and ω^2

$$2^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots\ldots.. \qquad \ldots.(1)$$

$$2^{2010} = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots\ldots.. \qquad \ldots.(2)$$

Add (1) and (2)

$$2^{2011} = 2a_0 + a_1(\omega + \omega^2) + a_2(\omega^2 + \omega) + 2a_3 + \ldots.$$

$$2^{2010} = a_0 - \frac{a_1}{2} - \frac{a_2}{2} + a_3 + \ldots.$$

Therefore, $a = 2$ and $b = 2010 \Rightarrow a + b = 2012$

$$R = 2 + 0 + 1 + 2 = 5$$

Hence, the correct answer is (5).

38. Let $z = x + iy$

$$|z - 3 - i| = |z - 9 - i|$$

$\Rightarrow \quad (x - 3)^2 + (y - 1)^2 = (x - 9)^2 + (y - 1)^2$

$\Rightarrow \quad 12x = 72 \Rightarrow x = 6$

$$|z - 3 + 3i| = 3$$

$\Rightarrow \quad (x - 3)^2 + (y + 3)^2 = 9$

at $x = 6, y = -3$

Therefore, one point is $(6, -3)$.

Hence, the correct answer is (1).

39. $\left|\frac{z - 4i}{z - 2i}\right| = 1 \Rightarrow z = x + 3i$ using this in

$$\left|\frac{z - 8 + 3i}{z + 3i}\right| = \frac{3}{5} \Rightarrow 5|x - 8 + 6i| = 3|x + 6i|$$

$$25\left((x - 8)^2 + 6^2\right) = 9(x^2 + 6^2)$$

$\Rightarrow \quad x^2 - 25x + 136 = 0$

$\Rightarrow \quad x = 8, 17$

Two complex numbers $8 + 3i$, $17 + 3i$

Sum of real part $= 8 + 17 = 25$

Hence, the correct answer is (5).

40. $|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0| = |\lambda|$

$$\frac{z_3 - z_0}{z_2 - z_0} = \frac{e^{\frac{i7\pi}{11}}}{e^{\frac{i\pi}{4}}} = e^{\frac{i17}{44}} \Rightarrow \angle BSC = 17\frac{\pi}{44}$$

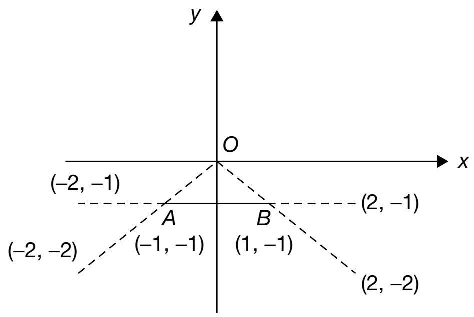
$\Rightarrow \quad \angle BSC = 17\frac{\pi}{88}$

Similarly, $\frac{z_2 - z_0}{z_1 - z_0} = e^{\frac{ip}{4}} \Rightarrow \angle ACB = \frac{\pi}{8}$

Therefore, $\angle ABC = \pi - \frac{\pi}{8} - \frac{17\pi}{88} = \frac{15\pi}{22} \Rightarrow k = 5$

Hence, the correct answer is (5).

41.



length $AB = 2$; $k = 2$

Hence, the correct answer is (2).

42. $Z = \frac{-1}{2} i\,(1 + i\sqrt{3}) = i\omega^2$

$\Rightarrow \quad Z^{101} = i\omega$

$$(Z^{101} + i^{109})^{106} = -(i\omega^2)^{106} = -\omega^2$$

$\therefore \quad -\omega^2 = (i\omega^2)^n = i^n\,\omega^{2n}$

$$\omega^{2n-2}\,i^n = -1$$

This is possible only when $n = 4r + 2$ and $2n - 2$ is multiple of 3,

i.e., $2(4r + 2) - 2$ is a multiple of 3,

i.e., $8r + 2$ is multiple of $3 \Rightarrow r = 2$

Hence, $n = 10$

Therefore, $\frac{2}{5}k = 4$

Hence, the correct answer is (4).

43. $z_n = e^{\frac{i\pi}{n(n+1)(n+2)}}$

$\Rightarrow \quad z = \lim\limits_{n\to\infty} e^{\left(i\pi\sum\limits_{n=1}^{n}\frac{1}{n(n+1)(n+2)}\right)}$

$$z = \lim_{n\to\infty} e^{\frac{i\pi}{2}\sum\limits_{n=1}^{n}\left\{\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right\}}$$

$\Rightarrow \quad z = e^{\frac{i\pi}{4}} \Rightarrow \text{Arg}(z) = \frac{\pi}{4} \Rightarrow k = 6$

Hence, the correct answer is (6).

44. Let x be the $(2009)^{\text{th}}$ root of unity $\neq 1$, then

$$x^{2009} - 1 = (x - 1)(x - w) \ldots. (x - w^{2008})$$

Taking log on both sides, we get

$\ln(x^{2009} - 1) = \ln(x - 1) + \ln(x - w) + \ln(x - w^2) \ldots + \ln(x - w^{2008})$

Therefore, on differentiating both the sides with respect to x, we get

$$\frac{(2009)x^{2008}}{x^{2009} - 1} = \frac{1}{x - 1} + \sum_{r=1}^{2008}\frac{1}{x - w^r} \qquad \ldots(2)$$

By substituting $x = 2$ in equation (2), we get

$\Rightarrow \quad 1 + \sum\limits_{r=1}^{2008}\frac{1}{2 - w^r} = \frac{2009(2^{2008})}{2^{2009} - 1}$

Multiplying both sides of above equation by $(2^{2009} - 1)$, we get

Therefore, $(2^{2009} - 1)\sum\limits_{r=1}^{2008}\frac{1}{2 - w^r} = 2009.2^{2008} - 2^{2009} + 1$

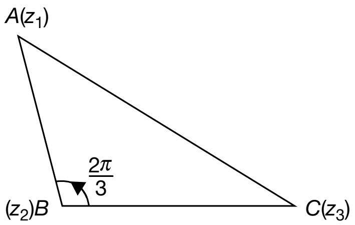
$$= 2^{2008}(2009 - 2) + 1 = 2^{2008}.2007 + 1$$

$$= [(a)(2^b) + c]$$

Therefore, $a = 2007$, $b = 2008$, $c = 1$

Hence, $a + b + c = 4016 \Rightarrow k = 2$

Hence, the correct answer is (2).

45. 

$$\cos B = -\frac{1}{2} \Rightarrow B = \frac{2\pi}{3}$$

By rotation, $\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\frac{2\pi}{3}}$

$$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_2} = \frac{3}{2} e^{\frac{i2\pi}{3}}$$

$$4(z_1 - z_2) = 3(z_3 - z_2)(-1 + i\sqrt{3})$$

Squaring to get the required result.

$$\left(\frac{4(z_1 - z_2)}{3(z_2 - z_2)} + 1\right)^2 = \left(i\sqrt{3}\right)^2$$

$$\Rightarrow \quad 4(z_1 - z_2)^2 + 9(z_3 - z_2)^2 + 6(z_1 - z_2)(z_3 - z_2) = 0$$

$(k = 0)$

Hence, the correct answer is (0).

46. Roots are $2w$, $(2 + 3w)$, $(2 + 3w^2)$, $(2 - w - w^2)$, $(2 + w)$

and $2 + 3w^2$ are conjugate to each other $2w$ is complex root, then other root must be $2w^2$ (as conjugate root occur in conjugate pair)

$2 - w - w^2 = 2 - (-1) = 3$ which is real. Hence, the least degree of the polynomial is 5.

Hence, the correct answer is (5).

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1. $$\left(\frac{1+i}{1-i}\right)^{\frac{m}{2}} = \left(\frac{1+i}{i-1}\right)^{\frac{n}{3}} = 1 \Rightarrow \left(\frac{(1+i)^2}{2}\right)^{m/2} = \left(\frac{(1+i)^2}{-2}\right)^{m/3} = 1$$

$\Rightarrow \quad (i)^{m/2} = (-i)^{n/3} = 1$

$\Rightarrow \quad \frac{m}{2} = 4k_1$ and $\frac{n}{3} = 4k_2$

$\Rightarrow \quad m = 8k_1$ and $n = 12k_2$

Least value of $m = 8$ and $n = 12$.

$\therefore \quad$ GCD = 4

Hence, the correct answer is (4).

2. $Re\left(\frac{z-1}{2z+i}\right) = 1$ Put $z = x + iy$

$$Re\left(\frac{(x+iy)-1}{2(x+iy)+i}\right) = 1$$

$$\Rightarrow \quad Re\left(\left(\frac{(x-1)+iy)}{2x+i(2y+1)}\right)\left(\frac{2x-i(2y+1)}{2x-i(2y+1)}\right)\right) = 1$$

$$Re\left(\frac{2x(x-1)+y(2y+1)+i(2xy-(x-1)(2y+1)))}{4x^2+(2y+1)^2}\right) = 1$$

$$\Rightarrow \quad \frac{2x(x-1)+y(2y+1)}{4x^2+(2y+1)^2} = 1 \Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$$

locus is a circle whose Centre is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$ and radius $\frac{\sqrt{5}}{4}$;

diameter $= \frac{\sqrt{5}}{2}$

Hence, the correct answer is (B).

3. $$\frac{3+i\sin\theta}{4-i\cos\theta} = \frac{3+i\sin\theta}{4-i\cos\theta}\,\frac{4+i\cos\theta}{4+i\cos\theta}$$
$$= \frac{12-\sin\theta\cos\theta+i(4\sin\theta+3\cos\theta)}{16+\cos^2\theta}$$

is a real number

$\Rightarrow \quad 3\cos\theta + 4\sin\theta = 0$

$\Rightarrow \quad \tan\theta = \frac{-3}{4}$ where θ may lie in II or IV quadrant

Case I $\theta \in [\pi/2, \pi]$

Complex number $z = \sin\theta + i\cos\theta$ lies in IVth quadrant

$$\arg(z) = -\tan^{-1}(-\cot\theta) = \tan^{-1}(-4/3) = -\tan^{-1}(4/3)$$

Case II $\theta \in [3\pi/2, 2\pi]$

Complex number $z = \sin\theta + i\cos\theta$ lies in IInd quadrant

$$\arg(z) = \pi - \tan^{-1}(-\cot\theta) = \pi + \tan^{-1}(-4/3) = \pi - \tan^{-1}(4/3)$$

Hence, the correct answer is (B).

4. Assuming z is a root of the given equation

$$z = \frac{-b \pm i\sqrt{180-b^2}}{2}$$

so, $\left(1 - \frac{b}{2}\right)^2 + \frac{180-b^2}{4} = 40 \Rightarrow -4b + 184 = 160 \Rightarrow b = 6$

Hence, the correct answer is (D).

5. $\alpha = \omega$

$$a = (1+\omega)(1+\omega^2+\omega^4+ \ldots + \omega^{200}) = (1+\omega)\frac{\left(1-(\omega^2)^{101}\right)}{1-\omega^2} = 1$$

$$b = 1 + \omega^3 + \omega^6 + \ldots + \omega^{300} = 101$$

$$x^2 - 102x + 101 = 0$$

Hence, the correct answer is (C).

6. $\left|\frac{z-i}{z+2i}\right| = 1 \Rightarrow |z-i| = |z+2i|$

z lies on perpendicular bisector of (0, 1) and (0, –2),

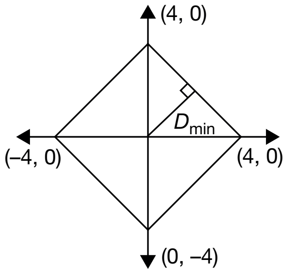
we have $\text{Im}(z) = -\frac{1}{2}$

Let $z = x - \frac{i}{2}$ and $|z| = \frac{5}{2} \Rightarrow x^2 = 6$

$$\therefore\quad |z+3i| = \left|x+\frac{5i}{2}\right| = \sqrt{x^2+\frac{25}{4}} = \sqrt{6+\frac{25}{4}} = \frac{7}{2}$$

Hence, the correct answer is (B).

7.



$z = x + iy$

$|x| + |y| = 4$

$|z| = \sqrt{x^2+y^2}$

$\Rightarrow\quad |z|_{\min} = \sqrt{8}$ and $|z|_{\max} = 4 = \sqrt{16}$

So, $|z|$ cannot be $\sqrt{7}$

Hence, the correct answer is (A).

8. $\left(3+2\sqrt{-54}\right) = 3 + 6\sqrt{6}\,i = \left(3+\sqrt{6}i\right)^2$

Similarly $\left(3-2\sqrt{54}\right) = \left(3-\sqrt{6}i\right)^2$

$$\left(3+2\sqrt{-54}\right)^{1/2} - \left(3-2\sqrt{-54}\right)^{1/2} = \pm\left(3+\sqrt{6}i\right) \pm \left(3-\sqrt{6}i\right)$$

$= 6, –6,\ 2\sqrt{6}i\,, -2\sqrt{6}i\,.$

Hence, the correct answer is (A).

9. The value of $\left(\dfrac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$

$$= \left(\frac{1+\sin\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)+i\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)}{1+\sin\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)-i\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)}\right)^3$$

$$= \left(\frac{1+\cos\frac{5\pi}{18}+i\sin\frac{5\pi}{18}}{1+\cos\frac{5\pi}{18}-i\sin\frac{5\pi}{18}}\right)^3 = \left(\frac{2\cos^2\frac{5\pi}{18}+2i\sin\frac{5\pi}{36}\cos\frac{5\pi}{36}}{2\cos^2\frac{5\pi}{36}-2i\sin\frac{5\pi}{36}.\cos\frac{5\pi}{36}}\right)^3$$

$$= \left(\frac{\cos\frac{5\pi}{36}+i\sin\frac{5\pi}{36}}{\cos\frac{5\pi}{36}-i\sin\frac{5\pi}{36}}\right)^3$$

$$= \left(\frac{e^{i5\pi/36}}{e^{-i5\pi/36}}\right)^3 = (e^{i5\pi/18})^3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

Hence, the correct answer is (B).

10. **(4)** $\alpha = \omega \qquad (\omega^3 = 1 \text{ and } 1+\omega+\omega^2 = 0)$

$(2+\omega)^4 = 2^4 + 4.2^3\omega + 6.2^2\omega^2 + 4.2.\omega^3 + \omega^4$

$= 16 + 32\omega + 24\omega^2 + 8 + \omega$

$= 24 + 24\omega^2 + 33\omega = 24\,(1+\omega^2) + 33\omega = -24\omega + 33\omega$

$= 9\omega = a + b\omega$

$a = 0, b = 9$

Hence, the correct answer is (D).

11. $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on it such that slope of

$$\text{AB} = \frac{y_2-y_1}{x_2-x_1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$\left(\arg(z_1 - z_2) = \dfrac{\pi}{6}\right)$

$\text{Re}(z) = |z-1|$ put $z = x + iy$

$\Rightarrow\quad x = \sqrt{(x-1)^2+(y-0)^2}$

$\Rightarrow\quad y^2 = 2x - 1$ point z_1 and z_2 lies on it

$y_1^2 = 2x_1 - 1$ and $y_2^2 = 2x_2 - 1$

$y_2^2 - y_1^2 = 2(x_2 - x_1)$

$$m_{\text{AB}} = \frac{y_2-y_1}{x_2-x_1} = = \frac{2}{y_1+y_2} = \frac{1}{\sqrt{3}} \Rightarrow y_1 + y_2 = 2\sqrt{3}$$

Hence, the correct answer is (D).

12. $$u = \frac{2z+i}{z-ki} = \frac{2x+i(1+2y)}{x+(y-k)i}.\frac{x-(y-k)i}{x-(y-k)i}$$

$$= \frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2} + i\,\frac{\left(x(2y+1)-2x(y-k)\right)}{x^2+(y-k)^2}$$

Since $\text{Re}(u) + \text{Im}(u) = 1$

$\Rightarrow 2x^2 + (2y+1)\,(y-k) + x(2y+1) - 2x(y-k) = x^2 + (y-\text{k})^2$

this curve cuts y axis at point $P(0, y_1)$ and $Q(0, y_2)$ so $x = 0$, we get $y^2 + y - k - k^2 = 0$ has roots y_1 and y_2

$\Rightarrow\quad y_1 + y_2 = -1, y_1y_2 = -k - k^2$

$\because\quad PQ = |y_1 - y_2| = 5 \Rightarrow (y_1 + y_2)^2 - 4y_1y_2 = 25$

$\Rightarrow\quad k^2 + k - 6 = 0$

$\Rightarrow\quad k = -3, 2$

So, $k = 2 (k > 0)$

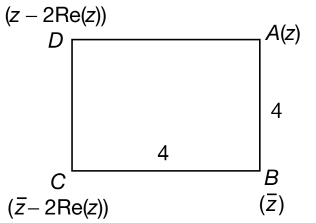
Hence, the correct answer is (C).

13. $\left(\dfrac{-1+i\sqrt{3}}{1-i}\right)^{30}$

$$= \left(\frac{2e^{i2\pi/3}}{\sqrt{2}e^{-i\pi/4}}\right)^{30} = 2^{15}\left(e^{i(2\pi/3+\pi/4)}\right)^{30} = 2^{15}e^{i55\pi/2} = 2^{15}(-i)$$

Hence, the correct answer is (C).

14.



Let $z = x + iy$

Then four point $A(x, y)$ $B(x, -y)$ $C(-x, -y)$ $D(-x, y)$

Length of side $= AB = 4 = 2|x| \Rightarrow |x| = 2$

$BC = 4 = 2|y| \Rightarrow |y| = 2$

$|z| = \sqrt{x^2+y^2} = \sqrt{4+4} = 2\sqrt{2}$

Hence, the correct answer is (D).

15. $z = x + iy$

$z^2 = i|z|^2$

$(x + iy)^2 = i(x^2 + y^2)$

$(x^2 - y^2) - i(x^2 + y^2 - 2xy) = 0$

$(x - y)(x + y) - i(x - y)^2 = 0$

$(x - y)((x + y) - i(x - y)) = 0$

$x = y$ or $(x + y) - i(x - y) = 0$ it gives origin but z is non zero

Hence z lies on $y = x$

Hence, the correct answer is (C).

16. $z = x + iy$

$|z| - \text{Re}(z) \le 1$

$\Rightarrow \sqrt{x^2+y^2} - x \le 1 \Rightarrow \sqrt{x^2+y^2} \le 1 + x$

$\Rightarrow x^2 + y^2 \le 1 + 2x + x^2 \Rightarrow y^2 \le 2x + 1 \Rightarrow y^2 \le 2\left(x + \frac{1}{2}\right)$

Hence, the correct answer is (D).

17. $|z_1| = 9 \Rightarrow x^2 + y^2 = 81$

$|z_2 - 3i - 4| = 4 \Rightarrow (x - 3)^2 + (y - 4)^2 = 16$

$C_1C_2 = \sqrt{3^2 + 4^2} = 5 ; r_1 = 9, r_2 = 4$

$C_1C_2 = r_1 - r_2$ given circles touch internally

Hence, the correct answer is (A).

18. $x^2 - 2x + 2 = 0 \Rightarrow (x - 1)^2 = -1$

$\Rightarrow x = 1 + i$ and $1 - i$

Let $\alpha = 1 + i$ and $\beta = 1 - i$

$$\frac{\alpha}{\beta} = \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{1+i^2+2i}{2} = i$$

$\left(\frac{\alpha}{\beta}\right)^n = (i)^n = 1$; Least value of $n = 4$

Hence, the correct answer is (C).

19. $z = \frac{\sqrt{3}}{2} + \frac{i}{2} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$(i + iz + z^5 + iz^8)^9$

$$= \left(1 + i\frac{\sqrt{3}}{2} - \frac{1}{2} + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + i\left(\cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6}\right)\right)^9$$

$$= \left(1 + i\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{i}{2} + i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)^9 = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^9$$

$$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9 = \cos 3\pi + \text{i} \sin 3\pi = -1$$

Hence, the correct answer is (B).

20. Let $\frac{\alpha+i}{\alpha-i} = z \Rightarrow \left|\frac{\alpha+i}{\alpha-i}\right| = |z|$

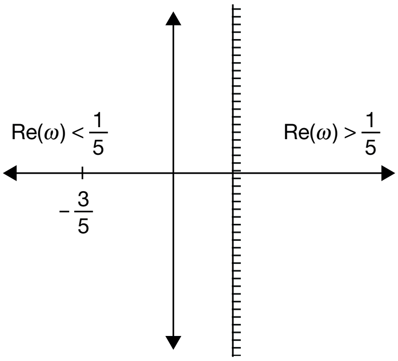
$\Rightarrow |z| = 1$

so Circle of radius = 1

Hence, the correct answer is (A).

21. $\omega = \frac{5+3z}{5(1-z)}$

$\Rightarrow 5\omega - 5\omega z = 5 + 3z \Rightarrow z = \frac{5\omega - 5}{3 + 5\omega}$



given $|z| < 1$

$\Rightarrow |5\omega - 5| < |3 + 5\omega|$

$\Rightarrow |\omega - 1| < \left|\frac{3}{5} + \omega\right|$

Required solution 5 Re $(\omega) > 1$

Hence, the correct answer is (B).

22. $z = \frac{(1+i)^2}{a-i} = \frac{2i(a+i)}{a^2+1}$

$|z| = \frac{2}{\sqrt{a^2+1}} = \sqrt{\frac{2}{5}} \Rightarrow a = 3$

$\therefore \bar{z} = \frac{-2i(3-i)}{10} = \frac{-1-3i}{5}$

Hence, the correct answer is (B).

23. $|zw| = 1 \Rightarrow |z|\ |w| = 1$

Let $w = \frac{1}{r}e^{i\theta}$ then $z = re^{i\left(\theta+\frac{\pi}{2}\right)}$

$\bar{z}w = e^{-i\left(\theta+\frac{\pi}{2}\right)}.e^{i\theta} = e^{-i(\pi/2)} = -i$

and $\bar{w}z = e^{i\left(\theta+\frac{\pi}{2}\right)}.\ e^{-i\theta} = e^{i(\pi/2)} = i$

Hence, the correct answer is (C).

24. $|z - i| = |z - 1|$ put $z = x + iy$

$|x + i(y - 1)| = |(x - 1) + iy| \Rightarrow x^2 + (y - 1)^2 = (x - 1)^2 + y^2$

$\Rightarrow y = x$ straight line having slope 1, pass origin (0, 0)

Hence, the correct answer is (D).

25. given $\frac{2z - n}{2z + n} = 2i - 1$ Let $z = x + 10i$

$\Rightarrow (2x - n) + 20i = (2i - 1)[(2x + n) + 20i]$

Comparing real and imaginary part

$\Rightarrow 2x - n = 2(-20) - (2x + n)$ and $20 = 2(2x + n) - 20$

We get $x = \text{Re}(z) = -10$ and $n = 40$

Hence, the correct answer is (C).

26. Equation $x^2 + 2x + 2 = 0$ has roots α and β.

$(x + 1)^2 + 1 = 0$

$x + 1 = \pm i \Rightarrow x = -1 \pm i$

Let $\alpha = -1 + i, \quad \beta = -1 - i$

$\alpha = \sqrt{2}e^{i\frac{3\pi}{4}}, \beta = \sqrt{2}e^{i\frac{5\pi}{4}}$

Therefore, $\alpha^{15}+\beta^{15}=\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^{15}+\left(\sqrt{2}e^{i\frac{5\pi}{4}}\right)^{15}$

$$=\left(\sqrt{2}\right)^{15}\left[\cos\frac{45\pi}{4}+i\sin\frac{45\pi}{4}+\cos\frac{75\pi}{4}+i\sin\frac{75\pi}{4}\right]$$

$$=2^{\frac{15}{2}}\left[\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]=-2^8=-256$$

Hence, the correct answer is (A).

27. Given $\theta\in\left(-\frac{\pi}{2},\pi\right)$

Let $f(\theta)=\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$

$$=\frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$f(\theta)=\frac{3+6i\sin\theta+2i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$$

$$=\frac{\left(3-4\sin^2\theta\right)+8i\sin\theta}{1+4\sin^2\theta}$$

Therefore, $f(\theta)$ is purely imaginary.

Hence, $3-4\sin^2\theta=0$

$\sin^2\theta=\frac{3}{4}\quad\Rightarrow\quad\sin\theta=\pm\frac{\sqrt{3}}{2}$

Therefore, $\theta=-\frac{\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}$

Hence, the correct answer is (C).

28. $1+x+x^2=0$

$$x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}$$

$z_0=w,\ w^2$

Now, $z=3+6iz_0^{\ 81}-3iz_0^{\ 93}$

$z=3+6iw^{81}-3iw^{93}$

$(w^{93}=w^{81}=1)$

$\Rightarrow\quad z=3+3i$

Then, $\mathrm{Arg}(z)=\tan^{-1}\left(\frac{3}{3}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

Hence, the correct answer is (D).

29. $3|z_1|=4|z_2|$

$\Rightarrow\quad 9|z_1|^2=16|z_2|^2$

$\Rightarrow\quad 9z_1\bar{z}_1=16z_2\bar{z}_2$

$\Rightarrow\quad \frac{3z_1}{2z_2}=\frac{8}{3}\left(\frac{\bar{z}_2}{\bar{z}_1}\right)\qquad$ let $\frac{3z_1}{2z_2}=re^{i\theta}$

$\Rightarrow\quad re^{i\theta}=\frac{4}{r}e^{+i\theta}$

$\Rightarrow\quad r^2=4\Rightarrow r=2$

$$z=\frac{3z_1}{2z_2}+\frac{2z_2}{3z_1}=re^{i\theta}+\frac{1}{re^{i\theta}}$$

$$z=2e^{i\theta}+\frac{1}{2}e^{i\theta}$$

$$z=\frac{5}{2}\cos\theta+i\,\frac{3}{2}\sin\theta$$

$$|z|=\sqrt{\frac{25}{4}\cos^2\theta+\frac{9}{4}\sin^2\theta}$$

$$=\sqrt{\frac{9}{4}+4\cos^2\theta}$$

Therefore, no options are matched.

30. $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$z=\left(e^{i\frac{\pi}{6}}\right)^5+\left(e^{-i\frac{\pi}{6}}\right)^5$

$z=e^{i\frac{5\pi}{6}}+e^{-i\frac{5\pi}{6}}$

$z=2\cos\frac{5\pi}{6}$

$z=2\left(-\frac{\sqrt{3}}{2}\right)=-\sqrt{3}$

It means $\mathrm{Im}(z)=0$

Hence, the correct answer is (C).

31. $\left(-2-\frac{1}{3}i\right)^3=\frac{x+iy}{27}$

$\Rightarrow\quad -1\left(2+\frac{i}{3}\right)^3=\frac{x+iy}{27}$

$\Rightarrow\quad -\left[8+\frac{i^3}{27}+3\cdot2\frac{i}{3}\left(2+\frac{i}{3}\right)\right]=\frac{x+iy}{27}$

$\Rightarrow\quad -\left[8-\frac{i}{27}+4i-\frac{2}{3}\right]=\frac{x+iy}{27}$

$\Rightarrow\quad -\frac{22}{3}-\frac{107i}{27}=\frac{x+iy}{27}$

$\Rightarrow\quad \frac{x}{27}=-\frac{22}{3}\quad\Rightarrow\quad x=-198$

and $-\frac{107}{27}=\frac{y}{27}\quad\Rightarrow\quad y=-107$

$y-x=91$

Hence, the correct answer is (D).

32. $|z|+z=3+i$

Let $z=x+iy$

$\Rightarrow\quad \sqrt{x^2+y^2}+x+iy=3+i$

$\Rightarrow\quad (x+\sqrt{x^2+y^2})+iy=3+i$

$y=1\qquad\Rightarrow x+\sqrt{x^2+y^2}=3$

$x+\sqrt{1+x^2}=3$

$$1 + x^2 = 9 + x^2 - 6x$$

$$\Rightarrow \quad 6x = 8 \quad \Rightarrow \quad x = \frac{4}{3}$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\frac{16}{9} + 1} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

Hence, the correct answer is (B).

33. If $\dfrac{z-\alpha}{z+\alpha}$ is purely imaginary,

then $\dfrac{z-\alpha}{z+\alpha} + \left(\overline{\dfrac{z-\alpha}{z+\alpha}}\right) = 0$

$$\frac{z-\alpha}{z+\alpha} + \left(\frac{\bar{z}-\alpha}{\bar{z}+\alpha}\right) = 0$$

$$z\bar{z} + z\alpha - \bar{z}\alpha - \alpha^2 + z\bar{z} + \alpha\bar{z} - 2\alpha - \alpha^2 = 0$$

$$\Rightarrow \quad 2|z|^2 - 2\alpha^2 = 0$$

$$\Rightarrow \quad |z|^2 = \alpha^2$$

$$\alpha^2 = 4 \quad \Rightarrow \quad \alpha = \pm 2 \qquad \because |z| = 2$$

According to option, $\alpha = 2$

Hence, the correct answer is (C).

34. $x^2 - x + 1 = 0$

Roots $(-\omega, -\omega^2)$, $\alpha = -\omega$, $\beta = -\omega^2$

$$\alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107} = 1$$

Hence, the correct answer is (D).

35. Determinant simplifies to

$$3k = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$\Rightarrow \quad 3k = -3(1 + 2\omega)$$

$$\Rightarrow \quad 3k = -3z$$

$$\Rightarrow \quad k = -z$$

Hence, the correct answer is (A).

36. Rationalising the given expression

$$\frac{(2 + 3i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$$

$$\Rightarrow \quad \frac{2 - 6\sin^2\theta}{1 + 4\sin^2\theta} = 0 \Rightarrow \sin^2\theta = \frac{1}{3}$$

$$\Rightarrow \quad \sin\theta = \pm\frac{1}{\sqrt{3}}$$

Hence, the correct answer is (A).

37. $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1, \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|$

$$\Rightarrow \quad (z_1 - 2z_2)\left(\overline{z_1 - 2z_2}\right)$$

$$= (2 - z_1\bar{z}_2)\left(\overline{2 - z_1\bar{z}_2}\right)$$

$$\Rightarrow \quad (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2)$$

$$= (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$$

$$\Rightarrow \quad (z_1\bar{z}_1) - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4z_2\bar{z}_2$$

$$= 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$$

$$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2|z_2|^2$$

$$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 - 4 - |z_1|^2|z_2|^2 = 0$$

$$(|z_1|^2 - 4)(1 - |z_2|^2) = 0$$

Since, $|z_2| \neq 1$. Therefore, $|z_1|^2 = 4 \Rightarrow |z_1| = 2$

$\Rightarrow$ Point z_1 lies on the circle of radius 2.

Hence, the correct answer is (A).

38. We know the minimum value of

$$|z_1 + z_2| \text{ is } \big||z_1| - |z_2|\big|.$$

Thus, minimum value of $\left|z + \dfrac{1}{2}\right|$ is

$$\left||z| - \frac{1}{2}\right| \leq |z| + \frac{1}{2}$$

Since, $|z| \geq 2$ therefore $2 - \dfrac{1}{2} < \left|z + \dfrac{1}{2}\right| < 2 + \dfrac{1}{2}$

$$\Rightarrow \quad \frac{3}{2} < \left|z + \frac{1}{2}\right| < \frac{5}{2}$$

Hence, the correct answer is (B).

39. Given $|z| = 1$, $\mathrm{Arg}(z) = \theta$

As we know, $\bar{z} = \dfrac{1}{z}$

Therefore, $\mathrm{Arg}\left(\dfrac{1+z}{1+\bar{z}}\right) = \mathrm{Arg}\left(\dfrac{1+z}{1+\dfrac{1}{z}}\right) = \mathrm{Arg}(z) = \theta$

Hence, the correct answer is (C).

40. Let $z = x + iy$. Therefore, $z^2 = x^2 - y^2 + 2ixy$

Now, $\dfrac{z^2}{z-1}$ is real $\Rightarrow \mathrm{Im}\left(\dfrac{z^2}{z-1}\right) = 0$

$$\Rightarrow \quad \mathrm{Im}\left(\frac{x^2 - y^2 + 2ixy}{(x-1) + iy}\right) = 0$$

$$\Rightarrow \quad \mathrm{Im}[(x^2 - y^2 + 2ixy)(x - 1 + iy)] = 0$$

$$\Rightarrow \quad 2xy(x-1) - y(x^2 - y^2) = 0$$

$$\Rightarrow \quad y(x^2 + y^2 - 2x) = 0$$

$$\Rightarrow \quad y = 0,\ x^2 + y^2 - 2x = 0$$

Therefore, z lies either on real axis or on a circle through origin.

Hence, the correct answer is (B).

41. Therefore, real part of roots is 1 s

Let roots are $1 + pi$ and $1 + qi$.

Therefore, the sum of roots $= 1 + pi + 1 + qi = -\alpha$ which is real.

$\Rightarrow \quad q = -p$ or root are

$1 + pi$ and $1 - pi$ product of roots $= 1 + p^2$

$= \beta \in (1, \infty)$ $p \neq 0$ as roots are distinct.

Hence, the correct answer is (B).

42. $(1+\omega)^7 = A + B\omega$, $(-\omega^2)^7 = A + B\omega$

$-\omega^2 = A + b\omega, \quad 1+\omega = A + B\omega$

$\Rightarrow \quad A = 1, B = 1.$

Hence, the correct answer is (D).

43. Let $z = x + iy$

$|z-1| = |z+1| \Rightarrow (x-1)^2 + y^2 = (x+1)^2 + y^2$

$\Rightarrow \quad \text{Re}(z) = 0 \Rightarrow x = 0$

$|z-1| = |z-i| \Rightarrow (x-1)^2 + y^2 = x^2 + (y-1)^2$

$\Rightarrow \quad x = y$

$|z+1| = |z-i| \Rightarrow (x+1)^2 + y^2 = x^2 + (y-1)^2$

$\Rightarrow \quad y = -x$

Only (0, 0) will satisfy all the conditions

$\Rightarrow$ Number of complex number, $z = 1$

Hence, the correct answer is (B).

44. $z^2 - z + 1 = 0 \Rightarrow z = -\omega \text{ or } -\omega^2$

$\therefore \quad \alpha^{2009} + \beta^{2009} = (-\omega)^{2009} + (-\omega^2)^{2009}$

$= -\omega - \omega^2 = 1$

Hence, the correct answer is (C).

45. $$\left||Z| - \frac{4}{|Z|}\right| \le \left|Z - \frac{4}{Z}\right| = 2$$

$-2 \le r - \frac{4}{r} \le 2$ (let $|z| = r$)

$r^2 + 2r - 4 \ge 0 \qquad r^2 - 2r - 4 \in 0$

$r \in (\sqrt{5} - 1, \infty) \qquad \sqrt{5} - 1, \le r \le 1 + \sqrt{5}$

$r_{\max} = 1 + \sqrt{5}$

Hence, the correct answer is (D).

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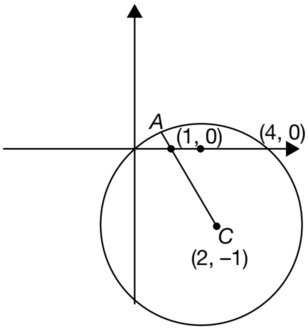
Single Correct Choice Type Problems

1.



$S: |z - 2 + i| \ge \sqrt{5}$

(1) $\frac{1}{|z_0 - 1|}$ is maximum

When $|z_0 - 1|$ is minimum $|z_0 - 1|_{\min}$ when z_0 is at 'A'.

'C' is the centre of the circle, i.e, AC is normal to the circle.

Equation of 'AC' in (x, y) is $x + y = 1$

$\Rightarrow \quad z_0 + \bar{z}_0 + i(z_0 - \bar{z}_0) = 2$

Therefore, $$\frac{4 - z_0 + \bar{z}_0}{z_0 - \bar{z}_0 + 2i} = \frac{4 - (2 + i(z_0 - \bar{z}_0))}{z_0 - \bar{z}_0 + 2i}$$

$$= \frac{2 - i(z_0 - \bar{z}_0)}{z_0 - \bar{z}_0 + 2i} = -i$$

Hence, the required argument is $-\frac{\pi}{2}$.

Hence, the correct answer is (A).

2. Given circles are

$(x - x_0)^2 + (y - y_0)^2 = r^2$ and

$(x - x_0)^2 + (y - y_0)^2 = 4r^2$

Or $|z - z_0| = r \qquad \ldots(1)$

And $|z - z_0| = 2r \qquad \ldots(2)$

Where $z_0 = x_0 + iy_0$

Now, α and $\frac{1}{\bar{\alpha}}$ lies on circle (1) and (2), respectively. Then $|\alpha - z_0| = r$ and $\left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$

$\Rightarrow \quad |\alpha - z_0| = r$ and $|1 - \bar{\alpha}z_0| = 2r|\bar{\alpha}|$

$\Rightarrow \quad |\alpha - z_0|^2 = r^2$ and $|1 - \bar{\alpha}z_0|^2 = 4r^2|\alpha|^2$

Subtracting, we get

$$|1 - \bar{\alpha}z_0|^2 - |\alpha - z_0|^2 = 4r^2|\alpha|^2 - r^2$$

$\Rightarrow \quad 1 + |\alpha z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0 - (|\alpha|^2 + |z_0|^2 - \bar{\alpha}z_0 - \alpha\bar{z}_0)$

$= 4r^2|\alpha|^2 - r^2$

$\Rightarrow \quad 1 + |\alpha|^2|z_0|^2 - |\alpha|^2 - |z_0|^2 = 4r^2|\alpha|^2 - r^2$

$\Rightarrow \quad (1 - |\alpha|^2)(1 - |z_0|^2) = 4r^2|\alpha|^2 - r^2$

Given $2|z_0|^2 = r^2 + 2$

$\Rightarrow \quad (1 - |\alpha|^2)\left(1 - \frac{r^2 + 2}{2}\right) = 4r^2|\alpha|^2 - r^2$

$\Rightarrow \quad (1 - |\alpha|^2)\left(\frac{-r^2}{2}\right) = 4r^2|\alpha|^2 - r^2$

$\Rightarrow \quad |\alpha|^2 - 1 = 8|\alpha|^2 - 2 \Rightarrow |\alpha|^2 = \frac{1}{7} \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

Hence, the correct answer is (C).

3. Given equation is $z^2 + z + 1 - a = 0$

Clearly, this equation do not have real roots if

$D < 0$

$\Rightarrow \quad 1 - 4(1 - a) < 0 \Rightarrow 4a < 3 \Rightarrow a < \frac{3}{4}$

Hence, the correct answer is (D).

4. The expression may not attain integral value for all a, b and c.

If we consider $a = b = c$, then

$$y = \alpha(1 + \omega + \omega^2) = \alpha(1 + i\sqrt{3})$$

$$z = \alpha(1 + w^2 + \omega) = \alpha(1 + i\sqrt{3})$$

Hence, $|x|^2 + |y|^2 + |z|^2 = 9|\alpha|^2 + 4|a|^2 + 4|a|^2$

$$= 17|\alpha|^2$$

Therefore, $\dfrac{|x|^2+|y|^2+|z|^2}{|\alpha|^2+|b|^2+|c|^2} = \dfrac{17}{3}$

5. $z = \cos\theta + i\sin\theta$

$$z^{2m-1} = (\cos\theta + i\sin\theta)^{2m-1}$$

$$= \cos(2m-1)\theta + i\sin(2m-1)\theta$$

Therefore, $\displaystyle\sum_{m=1}^{15} \operatorname{Im}\left(z^{2m-1}\right) = \sum_{m=1}^{15} \sin(2m-1)\theta$

$$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \text{upto 15 terms}$$

$$= \frac{\sin\left[15\left(\frac{2\theta}{2}\right)\right] \cdot \sin[\theta + 14 \times \theta]}{\sin\theta}$$

$$= \frac{\sin 15\theta \cdot \sin 15\theta}{\sin\theta} = \frac{\sin 30^\circ \cdot \sin 30^\circ}{\sin 2^\circ} = \frac{1}{4\sin 2^\circ}$$

Hence, the correct answer is (D).

6. $z\bar{z}\left(\bar{z}^2 + z^2\right) = 350$

By substituting $z = x + iy$, we have

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow \quad (x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$$

$$x^2 + y^2 = 25 \text{ and } x^2 - y^2 = 7$$

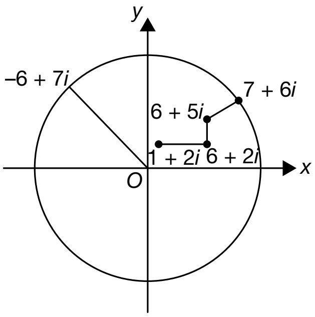
(as other combinations give non-integral values of x and y)

$\therefore \quad x = \pm 4, y = \pm 3 (x, y \in I)$

Hence, area is $8 \times 6 = 48$ square units.

Hence, the correct answer is (A).

7.

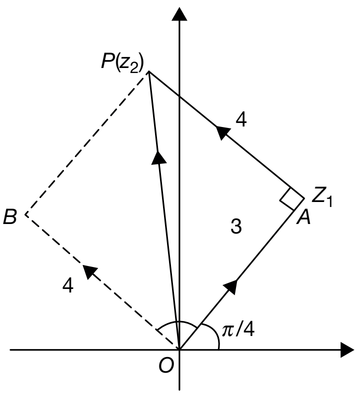


$$z_0 \equiv (1 + 2i),\ z_1 \equiv (6 + 5i) \text{ and } z_2 \equiv (-6 + 7i)$$

Hence, the correct answer is (D).

8. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

By rotating OA by an angle θ in anticlockwise direction to get OP, we have



$$\frac{z_2 - 0}{z_1 - 0} = \frac{|z_2|}{|z_1|} e^{i\theta}$$

$$\Rightarrow \quad \frac{z_2 - 0}{3e^{i\frac{\pi}{4}}} = \frac{5}{3}(\cos\theta + i\sin\theta)$$

$$\Rightarrow \quad \frac{z_2 - 0}{e^{i\frac{\pi}{4}}} = 5\left(\frac{5}{3} + i\frac{4}{5}\right)$$

$$\Rightarrow \quad z_2 = (3 + 4i)\, e^{\frac{i\pi}{4}}$$

Hence, the correct answer is (D).

9. Given $|z| = 1$ and $z \neq \pm 1$. To find locus of

$\omega = \dfrac{z}{(1 - z^2)}$. We have,

$$\omega = \frac{z}{1 - z^2} = \frac{z}{z\bar{z} - z^2} = \frac{1}{\bar{z} - z}$$

(Since $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar{z} = 1$)

which is a purely imaginary number. Therefore, ω must lie on the y-axis.

Hence, the correct answer is (D).

Multiple Correct Choice Type Problems

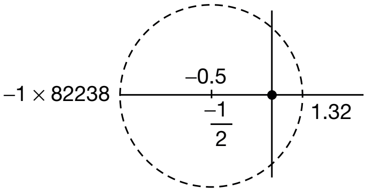
1. $z^2 + z + 1 = re^{i\theta}$ where $r = 1$

it has ∞ solution

$$z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$||z_1 - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

let $z_1 = \left(z + \dfrac{1}{2}\right)^2$ and $z_2 = \dfrac{3}{4}$



$$\left|\left|\left(z + \frac{1}{2}\right)^2\right| - \frac{3}{4}\right| \leq |z^2 + z + 1| \leq \left|z + \frac{1}{2}\right|^2 + \frac{3}{4}$$

$$\left|\left|z+\frac{1}{2}\right|^2-\frac{3}{4}\right|\le 1\le\left|z+\frac{1}{2}\right|^2+\frac{3}{4}$$

$$\left|\left|z+\frac{1}{2}\right|^2-\frac{3}{4}\right|\le 1 \text{ and } 1\le\left|z+\frac{1}{2}\right|^2+\frac{3}{4}\Rightarrow\left|z+\frac{1}{2}\right|\ge\frac{1}{2}$$

$$-1\le\left|z+\frac{1}{2}\right|^2-\frac{3}{4}\le 1$$

$$-\frac{1}{4}\le\left|z+\frac{1}{2}\right|^2\le\frac{7}{4}$$

$$\left|z+\frac{1}{2}\right|\le\frac{\sqrt{7}}{2}\approx 1\times 3228 \text{ hence } |Z|\le 2$$

Alternate Approach

$$z^2+z+1=re^{i\theta} \qquad r=1$$

$$z^2+z+1=e^{i\theta}=\cos\theta+i\sin\theta$$

$$z^2+z+1-\cos\theta-i\sin\theta=0$$

$$\Rightarrow\quad z+\frac{1}{2}=\pm\frac{\sqrt{4\cos\theta-3+4i\sin\theta}}{2}$$

$$\Rightarrow\quad \left|z+\frac{1}{2}\right|=\frac{1}{2}\sqrt{|4\cos\theta-3+4i\sin\theta|}$$

$$=\frac{1}{2}\sqrt{\sqrt{(4\cos\theta-3)^2+(4\sin\theta)^2}}$$

$$\Rightarrow\quad \left|z+\frac{1}{2}\right|=\frac{1}{2}\sqrt{\sqrt{16+9-24\cos\theta}}$$

$$\Rightarrow\quad \frac{(25-24)^{\frac{1}{4}}}{2}\le\left|z+\frac{1}{2}\right|\le\frac{1}{2}(25+24)^{\frac{1}{4}}$$

$$\Rightarrow\quad \frac{1}{2}\le\left|Z+\frac{1}{2}\right|\le\frac{\sqrt{7}}{2}$$

Hence, (B) and (C) are correct.

2. (A) $\operatorname{Arg}(-1-i)=-\dfrac{3\pi}{4}$

(B) $\operatorname{Arg}(-1+it)=\begin{cases}\pi-\tan^{-1}(t) & \text{if } t\ge 0\\ \tan^{-1}(t)-\pi & \text{if } t<0\end{cases}$

Not continuous at $t=0$

(C) $\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right)-\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)=0$

(D) $\operatorname{Arg}\left(\dfrac{z-z_1}{z_2-z_1}\right)+\operatorname{Arg}\left(\dfrac{z_2-z_3}{z-z_3}\right)=\pi$

z, z_1, z_2, z_3 are concyclic

Hence, (A), (B), (C) and (D) are correct.

3. $sz+t\bar{z}+r=0\Rightarrow\overline{sz}+\bar{t}z+\bar{r}=0$

By eliminating $\bar{z}$, we get $z=\dfrac{\bar{r}t-r\bar{s}}{|s|^2-|t|^2}$

(A) L has exactly one element, z should be defined

$|s|\ne|t|$

(B) $s=t=I$ and $r=i\Rightarrow L$ has no element

(C) Adding both equations

$(s+\bar{t})z+(\bar{s}+\bar{t})\bar{z}+(r+\bar{r})=0$ which is a line, if $s+\bar{t}\ne 0$

If $s+\bar{t}=0\Rightarrow t=-\bar{s},\ sz-\overline{sz}+r=0$

Which either represents no points or a straight line. In either case, number of points of intersection of given circle and L cannot be more than 2.

(D) If α and β both satisfy the given equation, then any number of form $\lambda\alpha+(1-\lambda)\beta$ satisfies the given equation whither $\lambda\in R$ infinite solutions

Hence, (A), (C) and (D) are correct.

4. $\operatorname{Im}\left(\dfrac{a(x+iy)+b}{x+iy+1}\right)=y$

$\Rightarrow\quad (x+1)^2+y^2=1$

$\Rightarrow\quad x=-1\pm\sqrt{1-y^2}$

Hence, (A) and (D) are correct.

5. $z=\dfrac{1}{a+ibt}=x+iy$

$\Rightarrow\quad x+iy=\dfrac{a-ibt}{a^2+b^2t^2}$

$\Rightarrow\quad x=\dfrac{a}{a^2+b^2t^2},\ y=\dfrac{-bt}{a^2+b^2t^2}$

$\Rightarrow\quad x^2+y^2=\dfrac{1}{a^2+b^2t^2}=\dfrac{x}{a}$

$\Rightarrow\quad x^2+y^2-\dfrac{x}{a}=0$

Therefore, locus of z is a circle with centre $\left(\dfrac{1}{2a},0\right)$ and radius

$=\dfrac{1}{2|a|}$ irrespective of a positive or negative

Also for $b=0, a\ne 0$ we get $y=0$

Therefore, locus is x-axis.

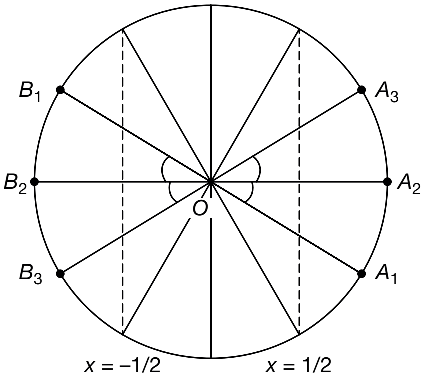
For $a=0, b\ne 0$, we get $x=0$

$\Rightarrow$ locus is y-axis

Therefore, a, c and d are the correct options.

Hence, (A), (C) and (D) are correct.

6. $w=\dfrac{\sqrt{3}+i}{2}=e^{\frac{i\pi}{6}}$, so $w^n=e^{i\left(\frac{i\pi}{6}\right)}$, $n=0, 1, 2, 3, \ldots\ldots 12$



Now, for z_1 $\cos\frac{n\pi}{6} > \frac{1}{2}$ and for z_2, $\cos\frac{n\pi}{6} < -\frac{1}{2}$

Possible positions of z_1 are A_1, A_2 and A_3, whereas of z_2 are B_1, B_2 and B_3 (as shown in the figure).

So, possible value of $\angle z_1Oz_2$ according to the given options is $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

Hence, (C) and (D) are correct.

7. Given, $z = (1-t)z_1 + tz_2$

$$\Rightarrow \quad z = \frac{(1-t)z_1 + tz_2}{(1-t)+t}$$

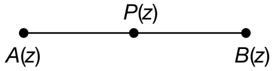
$\Rightarrow$ z divides the line segment joining z_1 and z_2 in the ratio $(1-t) : t$ internally as $0 < t < 1$

$\Rightarrow$ z, z_1 and z_2 are collinear

$\Rightarrow$ $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

$$\Rightarrow \quad \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$



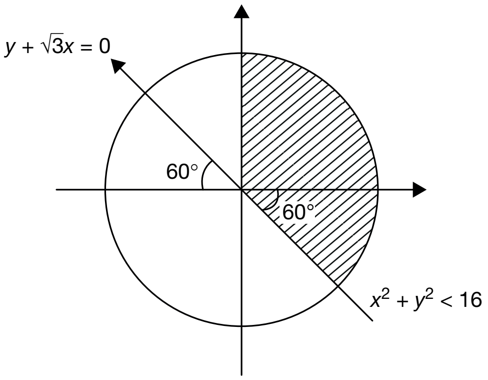
$$AP + PB = AB \Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$$

Hence, (A), (C) and (D) are correct.

Linked Comprehension Type Questions

1. S_1: $|z| < 4$, z lies inside the circle of radius 4.

S_2: $\sqrt{3}x + y > 0$, z lies above the line $\sqrt{3}x + y > 0$.



S_3: $\text{Re}(z) > 0$, z lies to the right of the imaginary axis.
Distance of $(1, -3)$ form

$$y + \sqrt{3}x = 0 \text{ is } \left|\frac{-3+\sqrt{3}}{2}\right| = \frac{3-\sqrt{3}}{2}$$

$$\Rightarrow \quad \min_{z \in S}|1 - 3i - z| = \frac{3-\sqrt{3}}{2}$$

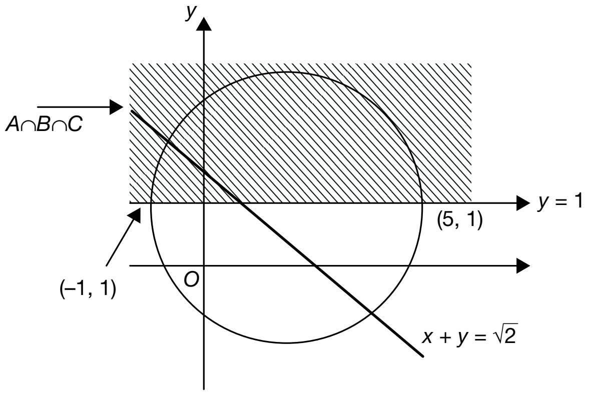
Hence, the correct answer is (C).

2. Area of region $S_1 \cap S_2 \cap S_3$ = Shaded area

$$\Rightarrow \quad \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6} = 4^2\pi\left\{\frac{1}{4} + \frac{1}{6}\right\} = \frac{20\pi}{3}$$

Hence, the correct answer is (B).

3.



A is the set of points on and above the line $y = 1$ in the Argand plane. B is the set of points on the circle.

$(x-2)^2 + (y-1)^2 = 9$ and

$C = \text{Re}(1-i)z = \text{Re}(1-i)(x+iy)$

$\Rightarrow \quad x + y = \sqrt{2}$

Hence, $A \cap B \cap C$ has only one point of intersection.

Hence, the correct answer is (B).

4. The points (–1, 1) and (5, 1) are the extremities of a diameter of the given circle. Hence,

$$|z + 1 - i|^2 + |z - 5 - i|^2 = 36$$

Hence, the correct answer is (C).

5. $||z| - |w|| < |z - w|$ and $|z - w|$ is the distance between z and w. Here, z is fixed. Hence, the distance between z and w would be maximum for diametrically opposite points. Therefore,

$|z - w| < 6$

$\Rightarrow \quad -6 < |z| - |w| < 6$

$\Rightarrow \quad -3 < |z| - |w| + 3 < 9$

Matrix Match/Column Match Type Questions

1. (P) → (1) : $z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}$, $k = 1$ to 9

Therefore, $z_k = e^{i\frac{2k\pi}{10}}$

Now, $z_k \cdot z_j = 1 \Rightarrow z_j = \frac{1}{z_k} = e^{-i\frac{2k\pi}{10}} = \bar{z}_k$

We know if z_k is 10th root of unity so it will be $\bar{z}_k$.

Therefore, for every z_k, there exist $z_i = \bar{z}_k$

Such that, $z_k \cdot z_j = z_k \cdot \bar{z}_k = 1$

Hence, the statement is true.

(Q) → (2) $zz_1 = z_k \Rightarrow z = \frac{z_k}{z_1}$ for $z_1 \neq 0$

Therefore, we can always find a solution to $z_1 . z = z_k$

Hence, the statement is false.

(R) → (3) : We know $z^{10} - 1 = (z - 1)(z - z_1) \ldots.(z - z_9)$

$$\Rightarrow \quad (z - z_1)(z - z_2) \ldots (z - z_9) = \frac{z^{10} - 1}{z - 1}$$

$$= 1 + z + z^2 + \ldots z^9$$

For $z = 1$ we get

$$(1 - z_1)(1 - z_2) \ldots (1 - z_9) = 10$$

Therefore, $\frac{|1 - z_1||1 - z_2|........|1 - z_9|}{10} = 1$

(S) → (4) : $z_1, z_2, \ldots z_9$ are 10th roots of unity.

Therefore, $z^{10} - 1 = 0$

From equation, $1 + z_1 + z_2 s + \ldots + z_9 = 0$

$$\text{Re}(1) + \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = 0$$

$$\Rightarrow \quad \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = -1$$

$$\Rightarrow \quad \sum_{k=1}^{9} \cos\frac{2k\pi}{10} = 1 \Rightarrow 1 - \sum_{k=1}^{9} \cos\frac{2k\pi}{10} = 2$$

Hence, the correct answer is (C).

2. A → q, r; B → p; C → p, s, t; D → q, r, s, t

(a) $\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|, z \neq 0$

$\frac{z}{|z|}$ is unimodular complex number and lies on the perpendicular bisector of i and $-i$

$$\Rightarrow \quad \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm |z|$$

$\Rightarrow$ z is real number $\Rightarrow \text{Im}(z) = 0$

(b) $|z + 4| + |z - 4| = 10$

z lies on an ellipse whose focus are (4, 0) and (–4, 0) and length of major axis is 10

$$\Rightarrow \quad 2ae = 8 \text{ and } 2a = 10 \Rightarrow e = \frac{4}{5}$$

$$|\text{Re}(z)| \leq 5.$$

(c) $|\omega| = 2 \Rightarrow \omega = 2(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad z = x + iy = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$$

$$= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$$

$$\Rightarrow \quad e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

$$|\text{Re}(z)| = \frac{3}{2}|\cos\theta| \leq \frac{3}{2}$$

$$|\text{Im}(z)| = \frac{5}{2}|\sin\theta| \leq \frac{5}{2}$$

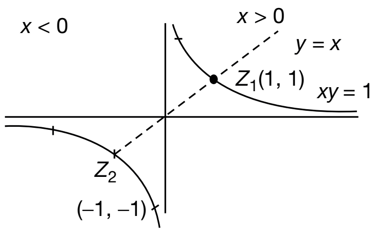
(d) $|\omega| = 1 \Rightarrow x + iy = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$$x + iy = 2\cos\theta$$

$$|\text{Re}(z)| \leq 1. \ \text{Im}(z) = 0.$$

Integer/Numerical Answer Type Questions

1.



$$Z^4 - |Z|^4 = 4iZ^2 \qquad (\because Z\bar{Z} = |Z|^2)$$

$$Z^4 - (Z\bar{Z})^2 = 4iZ^2$$

$$Z^2[Z^2 - \bar{Z}^2] = 4iZ^2 \qquad \text{put } Z = x + iy$$

$$(Z - \bar{Z})(Z + \bar{Z}) = 4i$$

$$\Rightarrow \quad (2yi)(2x) = 4i$$

$\Rightarrow$ $xy = 1$ it is rectangular hyperbola

Z_1 lies in first quadrant and Z_2 lies in second quadrant

For minimum distance $Z_1(1, 1)$ and $Z_2\,(-1, -1)$

min of $|Z_1 - Z_2|^2 = \left(\sqrt{2^2 + 2^2}\right)^2 = 8$

Hence, the correct answer is (8).

2. $|z|^2 = z\bar{z}$

$$\Rightarrow \quad |a + b\omega + c\omega^2|^2$$

$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$= (a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= \frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2\right)$$

$$a, b, c \in I - \{0\}$$

For minimum $(a - b)^2 = 1$

$$\Rightarrow \quad a = 2;\ b = 1;\ c = 3$$

Minimum of $\frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2)$

$$= \frac{1}{2}(1 + 4 + 1) = 3$$

Hence, the correct answer is (3).

3. $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{i\frac{\pi k}{7}}$

$$\alpha_{k+1} - \alpha_k = e^{i\frac{\pi(k+1)}{7}} - e^{i\frac{\pi k}{7}} = e^{i\frac{\pi k}{7}}\left(e^{i\frac{\pi}{7}} - 1\right)$$

$$|\alpha_{k+1} - \alpha_k| = \left|e^{i\frac{\pi}{7}} - 1\right|$$

$$\Rightarrow \quad \sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k| = 12\left|e^{i\frac{\pi}{7}}\right|$$

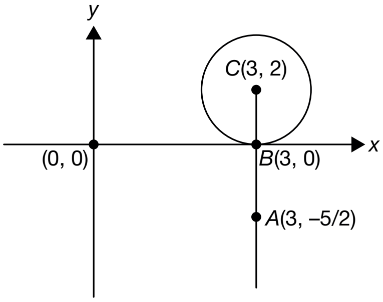
Similarly, $\sum_{k=1}^{3} |\alpha_{4k+1} - \alpha_{4k-2}| = 3\left|e^{i\frac{\pi}{7}} - 1\right|$

Therefore, $\dfrac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k+1}-\alpha_{4k-2}|}=4$

Hence, the correct answer is (4).

4. $|z-3-2i| \le 2$

$\Rightarrow$ z lies on or inside the circle of radius 2 and centre (3, 2).



$$|2z-6+5i|_{\min} = 2\left|z-3+\frac{5}{2}i\right|_{\min}$$

= 2(minimum distance of any point on the circle to the point $\left(3, \frac{-5}{2}\right)$) $= 2\left(\frac{5}{2}\right) = 5$

Hence, the correct answer is (5).

5. $\omega = e^{\frac{i2\pi}{3}}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying $(C_1 \to C_1 + C_2 + C_3)$

$$\Rightarrow \quad z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0 \Rightarrow z^3 = 0$$

$z = 0$ is only solution.

Hence, the correct answer is (1).

CHAPTER 5: BINOMIAL THEOREM

Test Your Concepts-I

1. Given, $\left(3x^2-\frac{1}{3}\right)^{10}$

To find 7th term in the expansion of $\left(3x^2-\frac{1}{3}\right)^{10}$.

We know that, the general term in the expansion of $(x+a)^n$ is

$$T_{r+1}={}^{n}C_r\, x^{n-r}\, a^r$$

Here, $n=10$, $x=3x^2$, $a=-\frac{1}{3}$

$$T_{6+1}={}^{10}C_6\,(3x^2)^4\left(-\frac{1}{3}\right)^6$$

$$={}^{10}C_6\,\frac{3^4}{3^6}\,x^8={}^{10}C_6\,\frac{1}{3^2}\,x^8$$

$$=\frac{10!}{6!4!}\times\frac{1}{9}\times x^8=\frac{210}{9}\,x^8=\frac{70}{3}\,x^8$$

2. Given, $\left(2x^2-\frac{3}{x^3}\right)^{25}$

To find Term independent of x in the expansion of $\left(2x^2-\frac{3}{x^3}\right)^{25}$.

Let $(r+1)$th term be independent of x.

$$\therefore\ T_{r+1}={}^{25}C_r\,(2x^2)^{25-r}\left(-\frac{3}{x^3}\right)^r$$

$$=(-1)^r\,{}^{25}C_r\,(2)^{25-r}\,3^r\,(x)^{50-5r} \qquad \dots (1)$$

For the term independent of x,

put $50-5r=0$

$\Rightarrow r=10$

Put $r=10$ in Eq. (1), we get

$$T_{11}={}^{25}C_{10}\,(2)^{15}\,(3)^{10}$$

$$=\frac{25!}{10!15!}(2)^{15}\,(3)^{10}$$

3. (i) Given, $\left(2x-\frac{1}{\sqrt{x}}\right)^{15}$

To find Term independent of x.

Let $(r+1)$th term be independent of x.

$$\therefore\ T_{r+1}={}^{15}C_r\,(2x)^{15-r}\left(-\frac{1}{\sqrt{x}}\right)^r$$

$$=(-1)^r\,{}^{15}C_r\,(2)^{15-r}\,x^{15-r-r/2}$$

$$T_{r+1}={}^{15}C_r\,(2)^{15-r}\,x^{\frac{30-3r}{2}} \qquad \dots(1)$$

To get term independent of x, put $\frac{30-3r}{2}=0$

$\Rightarrow 30-3r=0 \quad \Rightarrow r=10$

Put $r=10$ in Eq. (1), we get

$$T_{11}={}^{15}C_{10}\,(2)^5=\frac{15!}{10!5!}\times(2)^5$$

$$=3003\times 32=96096$$

(ii) Given, $\left(\frac{4x^2}{3}-\frac{3}{2x}\right)^9$

To find the term independent of x.

Let $(r+1)$th be independent of x.

$$\therefore\ T_{r+1}={}^{9}C_r\left(\frac{4x^2}{3}\right)^{9-r}\left(-\frac{3}{2x}\right)^r$$

$$=(-1)^r\,{}^{9}C_r\left(\frac{4}{3}\right)^{9-r}\left(\frac{3}{2}\right)^r x^{18-2r-r}$$

$$T_{r+1}=(-1)^r\,{}^{9}C_r\left(\frac{4}{3}\right)^{9-r}\left(\frac{3}{2}\right)^r x^{18-3r} \qquad \dots(1)$$

To get term independent of x, put $18-3r=0$

$\Rightarrow r=6$

Put $r=6$ in Eq. (1), we get

$$T_7={}^{9}C_6\left(\frac{4}{3}\right)^3\left(\frac{3}{2}\right)^6=\frac{9!}{6!3!}\times(3)^3$$

$$=84\times 27=2268$$

4. Given, $(1+x)^m\left(1+\frac{1}{x}\right)^n$

To find term independent of x.

Coefficient of x^{o} in $\left\{(1+x)^m\cdot\left(\frac{x+1}{x}\right)^n\right\}$

$\Rightarrow$ Coefficient of x^{o} in $\left\{\frac{(1+x)^{m+n}}{x^n}\right\}$

$\Rightarrow$ Coefficient of x^n in $(1+x)^{m+n}$

$\Rightarrow$ Coefficient of x^n in $\{{}^{m+n}C_r\,x^r)$

$\Rightarrow {}^{m+n}C_n$, put $r=n$.

$\therefore$ Term independent of $x={}^{m+n}C_n$ or ${}^{m+n}C_m$.

5. Given, $(1+x^2)^{40}\left(x^2+2+\frac{1}{x^2}\right)^{-5}$

$$\Rightarrow (1+x^2)^{40}\left\{\left(x+\frac{1}{x}\right)^2\right\}^{-5}$$

$$\Rightarrow (1+x^2)^{40}\left(x+\frac{1}{x}\right)^{-10}=x^{10}\,(1+x^2)^{40}\,(1+x^2)^{-10}$$

$$=x^{10}\,(1+x^2)^{30}$$

To find Coefficient of x^{20} in the expansions of

$(1+x^2)^{20}\left(x+\frac{1}{x}\right)^{-10}$ i.e. $x^{10}\,(1+x^2)^{30}$

Now, $(1+x^2)^{30}={}^{30}C_0+{}^{30}C_1(x^2)+{}^{30}C_2(x^2)^2+{}^{30}C_3(x^2)^3+{}^{30}C_4(x^2)^4+\dots+{}^{30}C_{30}(x^2)^{30} \qquad \dots(1)$

$x^{10}(1+x^2)^{30} = x^{10} + {}^{30}C_1x^{12} + {}^{30}C_2x^{14} + {}^{30}C_3x^{16} + {}^{30}C_4x^{18} + {}^{30}C_5x^{20} + \dots + x^{70}$

$\therefore$ Coefficient of x^{20} is ${}^{30}C_5$ or ${}^{30}C_{25}$. $[\because {}^nC_r = {}^nC_{n-r}]$

6. Given, $(1-x^2)^{10}$ and $\left(x-\dfrac{2}{x}\right)^{10}$

To find Ratio of coefficient of x^{10} in $(1-x^2)^{10}$ and the term independent of x in $\left(x-\dfrac{2}{x}\right)^{10}$.

Now, let $(r+1)^{\text{th}}$ term in $(1-x^2)^{10}$ contains x^{10}.

$\therefore T_{r+1} = {}^{10}C_r(-x^2)$

$= (-1)^r\,{}^{10}C_r\,x^{2r}$

To get x^{10}, put $2r = 10$

$\Rightarrow r = 5$

$\therefore T_6 = (-1)^5\,{}^{10}C_5\,x^{10}$

Thus, coefficient of x^{10} is ${}^{-10}C_5$.

Again, let term be independent of x in $\left(x-\dfrac{2}{x}\right)^{10}$.

$\therefore T_{r+1} = {}^{10}C_r(x)^{10-r}\left(-\dfrac{2}{x}\right)^r$

$= (-1)^r\,{}^{10}C_r\,(2)^r\,(x)^{10-2r}$

To get term independent of x, put $10-2r = 0$

$= r = 5$

Thus, the term independent of x in $\left(x-\dfrac{2}{x}\right)^{10}$ is ${}^{-10}C_5(2)^5$.

$\therefore$ Required ratio $= \dfrac{-{}^{10}C_5}{-{}^{10}C_5(2)^5}$

$= \dfrac{1}{(2)^5} = \dfrac{1}{32}$

$= 1:32$

7. Given, $\left(\sqrt[3]{2}+\dfrac{1}{\sqrt[3]{3}}\right)^n$

To find n in $\left(\sqrt[3]{2}+\dfrac{1}{\sqrt[3]{3}}\right)^n$

Now, $T_{6+1} = {}^nC_6\,(2)^{\frac{n-6}{3}}\,\dfrac{1}{(3)^2}$ …(1)

and seventh term from the end $= (n-7+2)^{\text{th}}$ term from beginning $= (n-5)^{\text{th}}$ term from beginning.

$\therefore T_{n-5} = T_{(n-6+1)} = {}^nC_{n-6}\,\{(2)^{1/3}\}^6\left(\dfrac{1}{(3)^{1/3}}\right)^{n-6}$

$= {}^nC_{n-6}\,(2)^2\;3^{\frac{n-6}{3}}$ …(2)

On dividing Eq. (1) by Eq. (2), we get

$$\therefore \text{Required ratio} = \frac{{}^nC_6\,\dfrac{(2)^{\frac{n-6}{3}}}{(3)^2}}{{}^nC_{n-6}\,\dfrac{(2)^2}{(3)^{\frac{n-6}{3}}}} = \frac{(2)^{\frac{n-6}{3}}}{(3)^2}\times\frac{(3)^{\frac{n-6}{3}}}{(2)^2}$$

$= (2)^{\frac{n-6}{3}-2}\times(3)^{\frac{n-6}{3}-2}$

$= (2\times3)^{\frac{n-6}{3}-2} = (6)^{\frac{n-12}{3}}$

It is given that the ratio of seventh term from beginning and from end is $\dfrac{1}{6}$.

$\Rightarrow\ (6)^{\frac{n-12}{3}} = \dfrac{1}{6}$

$\Rightarrow\ (6)^{\frac{n-12}{3}} = (6)^{-1}$

$\Rightarrow\ n-12 = -3$

$\therefore\ n = -3+12 = 9$

8. Given, $(y^{1/5}+x^{1/10})^{55}$

To find Number of terms free from radical sign.

Now, let $(r+1)^{\text{th}}$ terms is independent of radical sign,

$\therefore T_{r+1} = {}^{55}C_r\,y^{11-\frac{r}{5}}\,x^{\frac{r}{10}}$ …(1)

Clearly, T_{r+1} will be independent of radical sign, if $r/5$ and $r/10$ are integers, where $0 \le r \le 55$.

$\therefore r = 0, 10, 20, 30, 40, 50$.

Hence, there are 6 terms in the expansion of the given expression, which are independent of radical sign.

9. We have, $\displaystyle\sum_{k=0}^{4}\frac{5^{4-k}}{(4-k)!}\cdot\frac{x^k}{k!} = \frac{8}{3}$

$\Rightarrow \dfrac{5^4}{4!}+\dfrac{5^3x}{3!1!}+\dfrac{5^2x^2}{2!2!}+\dfrac{5x^3}{1!3!}+\dfrac{x^4}{4!} = \dfrac{8}{3}$

$\Rightarrow \dfrac{1}{4!}[{}^4C_05^4+{}^4C_15^3x+{}^4C_25^2x^2+{}^4C_35x^3+{}^4C_4x^4]$

$= \dfrac{8}{3}$

$\Rightarrow (x+5)^4 = \dfrac{8\times4!}{3}$

$\Rightarrow (x+5)^4 = 64$

$\Rightarrow x+5 = \pm2\sqrt{2}$

$\Rightarrow x = -5\pm2\sqrt{2}$

Then, the largest real value of x is $2\sqrt{2}-5$.

10. The coefficient of r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ terms in the expansion of $(1+y)^m$ are in AP.

$\therefore\ 2T_{r+1} = T_r + t_{r+2}$

$\Rightarrow\ 2\,{}^mC_r = {}^mC_{r-1}+{}^mC_{r+1}$

$\Rightarrow\ \dfrac{2m!}{r!(m-r)!} = \dfrac{m!}{(r-1)!(m+1-r)!}+\dfrac{m!}{(r+1)!(m+r-1)!}$

$\Rightarrow\ \dfrac{2}{r(m-r)} = \dfrac{1}{(m+1-r)(m-r)}+\dfrac{1}{r(r+1)}$

$\Rightarrow\ \dfrac{2}{r(m-r)} = \dfrac{r(r+1)+(m+1-r)(m-r)}{r(r+1)(m+1-r)(m-r)}$

$\Rightarrow\ 2(mr-r^2+r+m-r+1) = r^2+r+m^2+r^2-2mr+m-r$

$\Rightarrow\ 4r^2-4mr-m-2+m^2 = 0$

$\Rightarrow\ 4r^2-m(4r+1)+m^2-2 = 0$

$\Rightarrow\ m^2-m(4r+1)+4r^2-2 = 0$

11. $T_{r+1} = {}^{30}C_r(x^{2/3})^{30-r}\left(-\dfrac{1}{\sqrt{x}}\right)^r$

$= {}^{30}C_r(-1)^r\, x^{20-\frac{2r}{3}-\frac{r}{2}} \Rightarrow 20 - \dfrac{7r}{6} = 13 \Rightarrow r = 6$

$\Rightarrow$ Coefficient $= {}^{30}C_6 = 29.7.9.13.25$

12. Coefficient of x^7 in $\left(ax^2 + \dfrac{1}{bx}\right)^{13} = 0$

Coefficient of x^{-6} in $\left(ax + \dfrac{1}{bx^2}\right)^{13} = 0$

So $a, b \in R \qquad b \neq 0.$

13. $N = \dfrac{(1+2+..+10)^2 - (1^2+2^2+...+10^2)}{2} = 1320$

Sum of digits of $N = 6$

Test your concepts-II

1. $N = \left(\sqrt[8]{5} + \sqrt[5]{2}\right)^{100}$

$T_{r+1} = {}^{100}C_r\,(5)^{\frac{100-r}{8}}\,(2)^{\frac{r}{5}}$

For rational terms $r = 5\lambda$

$\dfrac{100-5\lambda}{8}$ must be integer, $\lambda = 8k+4$

$\Rightarrow r = 40k + 20 \leq 100 \qquad \Rightarrow k \leq 2.$

Number of rational terms = 3

Number of irrational terms = 98

$N = (\sqrt{2}+1)^6$

$\therefore I + f = {}^6C_0(\sqrt{2})^6 + {}^6C_1(\sqrt{2})^5 + \ldots + {}^6C_6$

Also,

$f' = (\sqrt{2}-1)^6 = {}^6C_0(\sqrt{2})^6 - {}^6C_1(\sqrt{2})^5 + \ldots + {}^6C_6$

$\Rightarrow I + f + f' = 2\left\{{}^6C_0(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 + {}^6C_4(\sqrt{2})^2 + {}^6C_6\right\}$

$\Rightarrow I + 1 = 2\{8 + 60 + 30 + 1\} \qquad \Rightarrow I = 197$

$\therefore f = 1 - (\sqrt{2}-1)^6$

2. Let $E = 2^{100} = 2\,.2^{99} = 2\,(8)^{33}$

$= 2\,(1+7)^{33}$

So, $E = 2^{100} = 2\,.(1+7)^{33} = 2\,.\left(1 + 33\times 7 + \dfrac{33\times 32}{2}(7)^2 + ...\right)$

$E = 2 + 7\cdot 2\left(33 + \dfrac{33\times 32}{2}\times 7 + ...\right)$

$E = 2 + 7k$

Now, $\dfrac{E}{7} = \dfrac{7k+2}{7} = k + \dfrac{2}{7}$

Hence, remainder is 2.

3. Given, $(7+4\sqrt{3})^n = \alpha + \beta$

Let $(7-4\sqrt{3})^n = G$

Now, $\alpha + \beta + G = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$

$= 2\,[{}^nC_0(7)^n + {}^nC_2(7)^{n-2}(4\sqrt{3})^2 + {}^nC_4(7)^{n-4}(4\sqrt{3})^4 + \ldots]$

$\alpha + \beta + G$ = an even integer

$\Rightarrow \beta + G$ = an even integer $- \alpha$

$\Rightarrow \beta + G$ = an integer $\qquad [\because \alpha$ is an integer$]$

$\Rightarrow \beta + G = 1 \qquad \left[\begin{array}{l}\because 0 \leq \beta < 1, 0 < G < 1\\ \therefore 0 < \beta + G < 2\\ \Rightarrow \beta + G = 1\end{array}\right]$

$\Rightarrow G = (1 - \beta)$

$\therefore (1-\beta)\,(\alpha+\beta) = G\,(\alpha+\beta)$

$= (7-4\sqrt{3})^n\,(7+4\sqrt{3})^n$

$= [(7)^2 - (4\sqrt{3})^2]^n = (49-48)^n = 1$

4. To find last digit of $(3^p + 2)$.

Where, $P = 3^{4n}$

As, 3^{4n} where $n \in N$.

$\Rightarrow (81)^n \Rightarrow (80+1)^n \Rightarrow 80^\lambda + 1.$

$\therefore 3^P + 2 = 3^{80\lambda+1} + 2 = (3^4)^{20\lambda}\cdot 3 + 2$

$= 3\cdot(81)^{20\lambda} + 2 = 3\,[80+1]^{20\lambda} + 2$

$= 3\,[80\,k + 1] + 2$

$= 240\,k + 3 + 2 = 240\,k + 5$

$\therefore$ Last digit of $3^P + 2$, is 5.

5. Let $E = (6)^{83} + (8)^{32}$

$E = (7-1)^{83} + (7+1)^{32}$

$E = {}^{83}C_0(7)^{83} - {}^{83}C_1(7)^{82} + {}^{83}C_2(7)^{81} - \ldots. + {}^{83}C_{82}(7) - {}^{83}C_{83})$
$+ ({}^{32}C_0(7)^{32} + {}^{32}C_1(7)^{31} + {}^{32}C_2(7)^{30} + \ldots + {}^{32}C_{31}(7) + {}^{32}C_{32}$

$E = {}^{83}C_0(7)^{83} - {}^{83}C_1(7)^{82} + {}^{83}C_2(7)^{81} - \ldots. - {}^{83}C_{81}(7)^2 + {}^{83}C_{82}(7)$
$- 1 + {}^{32}C_0(7)^{32} + {}^{32}C_1(7)^{31} + \ldots + {}^{83}C_{82}(7)^2 + {}^{32}C_{31}(7) + 1$

$E = (7)^2\,({}^{83}C_0(7)^{81} - {}^{83}C_1(7)^{80} + {}^{83}C_2(7)^{79} - .. - {}^{83}C_{81}) + 7\times 83 +$
$(7)^2 - ({}^{32}C_0(7)^{30} + {}^{32}C_1(7)^{29} + \ldots + {}^{32}C_{30}) + 32\times 7$

$E = 49\lambda + 7\times 83 + 49\times\mu + 32\times 7$

Where,

$\lambda = {}^{83}C_0(7)^{81} - {}^{83}C_1(7)^{80} + \ldots - {}^{83}C_{81}$

$\mu = {}^{32}C_0(7)^{30} + {}^{32}C_1(7)^{29} + \ldots + {}^{32}C_{30}$

$E = 49(\lambda + u) + 7(83 + 32)$

$E = 49k + 7(7\times 11 + 6 + 7\times 4 + 4)$ where, $k \in 1$.

$E = 49k + 7(7\times 15 + 10)$

$E = 49k + 7(7\times 15 + 7\times 1 + 3)$

$E = 49k + 7(7\times 16 + 3)$

$E = 49k + 49\times 16 + 21$

Now, $\dfrac{E}{49} = \dfrac{49k}{49} + \dfrac{49\times 16}{49} + \dfrac{21}{49} = k + 16 + \dfrac{21}{49}$

Hence, remainder is 21.

6. Let $E = (11)^{27} + (21)^{27}$

$E = (16-5)^{27} + (16+5)^{27}$

$E = 2[{}^{27}C_0(16)^{27} + {}^{27}C_2(16)^{25}(5)^2] + {}^{27}C_4(16)^{23}(5)^4 + \ldots$
$+ {}^{27}C_{26}(16)\,(5)^{26}]$

$[\because (a-b)^n + (a+b)^n = 2[{}^nC_0a^{n-2}b^2 + \ldots]$

$E = 2\times 16\,[{}^{27}C_0(16)^{26} + {}^{27}C_2(16)^{24}(5)^2 + {}^{27}C_4(16)^{22}(5)^4 + \ldots +$
${}^{27}C_{26}(5)^{26}]$

$E = 2\times 16\times k,\ k \in 1$

Now, $\dfrac{E}{16} = \dfrac{2\times 16\times k}{16} = 2k + 0$

Hence, remainder is zero, when $(11)^{27} + (21)^{27}$ is divided by 16.

7. Let $E = [(15)^{23} + (23)^{23}]$

$E = [(19-4)^{23} + (19+4)^{23}]$

$\Rightarrow \quad E = 2[{}^{23}C_0(19)^{23} + {}^{23}C_2(19)^{21}(4)^2 + {}^{23}C_4(19)^{19}(4)^4 + \ldots + {}^{23}C_{22}(19)(4)^{22}]$

$[\because (a-b)^n + (a+b)^n = 2[{}^nC_0a^n + {}^nC_2a^n + {}^nC_2a^{n-2}b^2 + \ldots]$

$\Rightarrow \quad E = 2 \times 19\,[{}^{23}C_0(19)^{22} + {}^{23}C_2(19)^{20}(4)^2 + {}^{23}C_3(19)^{19}(4)^3 + \ldots + {}^{23}C_{22}(4)^{22}]$

$\Rightarrow \quad E = 2 \times 19\,k, k \in 1$

Hence, where $(15)^{23} + (23)^{23}$ is divided by 19 remainder is zero.

8. Given, $(5 + 2\sqrt{6})^n = p + f$

Where, $n \in$ N, $p \in$ N and $0 > f > 1$.

Let $g = (5 - 2\sqrt{6})^n$

Then, $p + f + g = (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n$

$= 2[{}^nC_0(5)^n + {}^nC_2(5)^{n-2}(2\sqrt{6})^2 + \ldots]$

$p + f + g =$ an even integer

$\Rightarrow \quad f + g =$ an integer $\quad [\because p \in 1]$

$\Rightarrow \quad f + g = 1 \quad [\because 0 < f < 1, 0 < g < 1]$

$0 < f + g < 2$

$\Rightarrow \quad f + g = 1)$

Now, $p + f = (5 + 2\sqrt{6})^n$

$p = (5 + 2\sqrt{6})^n - f$

$$\Rightarrow \quad p = \frac{1}{(5 - 2\sqrt{6})^n} - f$$

$$\Rightarrow \quad p = \frac{1}{1 - f} - f$$

$\Rightarrow \quad (1 - f)p = 1 - f(1 - f)$

$\therefore (1 - f)p = 1 - f(1 - f)$

$\therefore f^2 - f + pf - p = -1$

9. To find remainder when

$(8)^{2n} - (62)^{2n+1}$ is divided by 9.

Now, $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (1 - 63)^{2n+1}$

$= (1 + 63)^n + (1 - 63)^{2n+1}$

Let $E = 8^{2n} - (62)^{2n+1} = (1 + 63)^n + (1 - 63)^{2n+1}$

$E = (1 + {}^nC_1 63 + {}^nC_2(63)^2 + \ldots + (63)^n] + (1 - {}^{2n+1}C_1(63) + {}^{2n+1}C_2(63)^2 + \ldots + (-1)\,(63)^{2n+1})$

$E = 2 + 63\,({}^nC_1 + {}^nC_2(63) + \ldots + (63)^n - {}^{2n+1}C_1 + {}^{2n+1}C_2(63) + \ldots - (63)^{2n}]$

$E = 2 + 63\lambda$, where, $\lambda \in 1$.

$$\text{Now, } \frac{E}{9} = \frac{63\lambda}{9} + \frac{2}{9}$$

Hence, remainder is 2.

10. We have, $13^n + 7^n - 3^n = 13^n + 7^n = 10k + 7^n$

$[\because 13^n - 3^n$ is divisible by $13 - 3 = 10]$

$$= 10k + (7^4)^{\left(\frac{n}{4} - \frac{1}{4}\right)}.\,7 = 10k + (2401)^{\frac{n-1}{4}}.\,7$$

$$= 10k + (2400 + 1)^{\frac{n-1}{4}}.\,7 = 10k + 100k + 7$$

The unit digit is 7, if $\dfrac{n-1}{4} = k$

$n = 4k + 1, k \in 1$

11. $(2010)^n - 2(1010)^n - 4(510)^n + 5(10)^n$.

$\Rightarrow \quad (2000 + 10)^n - 2(1000 + 10)^n - 4(500 + 10)^n + 5(10)^n$.

Here, $(2000 + 10)^n = {}^nC_0\,(2000)^n + {}^nC_1\,(2000)^{n-1} + {}^nC_n\,10^n$

= Except last term the remaining terms are divisible by 10^3.

$= N_1 10^3 + 10^n \quad \ldots(1)$

Similarly, $2(1010)^n = 2N_2.10^3 + 2(10)^n \quad \ldots(2)$

$4(510)^n = N_3.10^3 + 4(10)^n \quad \ldots(3)$

From Eqs. (1), (2) and (3), we get

$= (2010)^n - 2\,(1010)^n - 4\,(510)^n + 5\,(10)^n$

$\Rightarrow \quad N_1.10^3 + 10^n - 2N_2.10^3 - 2\,(10)^n - N_3.(10)^3 + 4(10)^n + 5\,(10)^n$

$\Rightarrow \quad 10^3\,(N_1 - 2N_2 - N_3)$ where, $N_1, N_2, N_3 \in$ N.

$\therefore$ The given number is always divisible by 10^3.

12. Consider $(5 + 2)^{100} - (5 - 2)^{100}$

$= 2[{}^{100}C_1\,5^{99} \cdot 2 + {}^{100}C_3\,5^{97}.2^3 + \ldots + {}^{100}C_{99}5.2^{99}]$

$= 2[1000 \cdot 5^{98} + {}^{100}C_3\,5^{94} + \ldots + 1000\,2^{98}]$

$\Rightarrow \quad$ minimum 000 as last three digits.

Test Your Concepts-III

1. Sum of the coefficient in the expansion of $(1 + 2x)^n = 6561$

$\Rightarrow \quad (1 + 2x)^n = 6561$, when $x = 1$

$\Rightarrow \quad 3^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n = 8$

$$\text{Now } m = \frac{n+1}{1 + \frac{\text{first term}}{\text{Last term}}} = \frac{8+1}{1+1} = 4.5 \qquad [\because x = 1/2]$$

Hence, 5th term is the greatest term.

2. $$m = \frac{10+1}{1 + \frac{2}{\frac{3|x|}{8}}} = \frac{33|x|}{3|x| + 16}$$ this m should lie between 3 and 4

$$3 < \frac{33|x|}{3|x| + 16} < 4$$

$$2 \le |\,x\,| \le \frac{64}{21}$$

Hence, value of x be

$$x \in \left[\frac{-64}{21}, -2\right] \cup \left[2, \frac{64}{21}\right]$$

3. We have, $(x + y + z)^{10}$

(i) The total number of terms is ${}^{3+10-1}C_{10} = \dfrac{12!}{10!2!} = 66$

(ii) Coefficient of $x^3y^2z^5$ of the given expansion is $\dfrac{10!}{3!2!5!}$.

(iii) The sum of the radical of $x^2y^3z^4$ is $2 + 3 + 4 = 9$ is not equal to 10. Since, $x^2y^3z^4$ cannot be term of the $(x + y + z)^{10}$.

(iv) $$\because\ (x + y + z)^{10} = \sum_{\alpha+\beta+\gamma=10} \frac{10!}{\alpha!\beta!\gamma!}\, x^{\alpha}y^{\beta}z^{\gamma}$$

Put $x = y = z = 1$ both sides, we have

$$\Rightarrow \sum_{\alpha+\beta+\gamma=10} \frac{10!}{\alpha!\beta!\gamma!} = 3^{10}$$

Hence proved

4. We have, $(ab + bc + ca)^8$

The general term is $\frac{8!}{x!\,y!\,z!}(ab)^x\,(bc)^y\,(ca)^z$

$$\Rightarrow \frac{8!}{x!y!z!} 8^{x+z}\, b^{x+y}\, c^{y+z}$$

Coefficient of $a^3b^4c^7$ in the above expansion is possible if

$x + z = 3,\ x + y = 4,\ y + z = 7$

$\therefore\ x = 0, y = 4, z = 3$

$\therefore$ Coefficient of $(ab + bc + ca)^8$ is

$$\frac{8!}{0!4!3!} = 280$$

5. We know that, the total number of terms in the expansion $(x_1 + x_2 + x_3 + \ldots + x_m)^n$ is ${}^{m+n-1}C_n$

$\therefore$ The number of terms in the expansion of $(a + b + c + d + e + f)^n$ is ${}^{6+n-1}C_n = {}^{n+5}C_n$

6. (i) The sum of coefficients $= (4 – 3)^{15} = 1$ (put $x = 1$)

(ii) sum of binomial coefficients $= {}^{15}C_0 + {}^{15}C_1 + {}^{15}C_3 + \ldots\ldots + {}^{15}C_{15} = 2^{15}$

(iii) $m = \dfrac{n+1}{1+\dfrac{\text{First term}}{\text{Last term}}} = \dfrac{15+1}{1+\dfrac{4}{3}} = \dfrac{48}{7} = 6.9$

Numerically greatest coefficient $T_7 = {}^{15}C_6 2^{18}.3^6$

7. We have, $(1 + 2x + 3x^2 + 4x^3 + \ldots)^{1/2}$

$\Rightarrow [(1 – x)^{-2}]^{1/2} = (1 – x)^{-1}$

$(1 – x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$\therefore$ Coefficient of x^n is 1.

8. We have, $(1 – x)^{-4}$

The $(r + 1)^{\text{th}}$ term is

${}^{r+4-1}C_{4-1}\,x^r \Rightarrow {}^{r+3}C_3\,x^r$

$$\Rightarrow \frac{(r+3)!}{3!(r+3-3)!}x^r = \frac{(r+3)(r+2)(r+1)}{6}x^r$$

9. We have, $\left(\dfrac{1+x}{1-x}\right)^2$

$\Rightarrow (1 + x)^2\,(1 – x)^{-2}$

$\Rightarrow [1 + 2x + x^2]\,[1 + 2x + 3x^2 + \ldots + (n – 1)\,x^{n-2} + nx^{n-1} + (n + 1)\,x^n + \ldots]$

The coefficient of x^n is $(n + 1) + 2(n) + (n – 1) = 4n$

10. We have, $\dfrac{\left(1+\dfrac{3x}{4}\right)^{-4}\left(\sqrt{16-3x}\right)}{(8+x)^{2/3}} = P + Q\,x$

$$\Rightarrow \left(1+\frac{3x}{4}\right)^{-4} \cdot 4\left(1-\frac{3x}{16}\right)^{1/2} \cdot 8^{-2/3}\,(1 + x/8)^{-2/3}$$

$$\Rightarrow \left(1+\frac{3x}{4}\right)^{-4}\left(1-\frac{3x}{16}\right)^{1/2} \cdot (1 + x/8)^{-2/3}$$

$$\Rightarrow \left(1+\frac{3x}{4}\right)^{-4}\left(1-\frac{3x}{16}\right)^{1/2}\left(1+\frac{3x}{8}\right)^{-2/3}$$

$$\Rightarrow (1 – 3x + \ldots)\left(1-\frac{3x}{32}+\ldots\right)\left(1-\frac{x}{12}+\ldots\right)$$

Constant term is 1 and coefficient of x is

$$\left(-3-\frac{3}{32}-\frac{1}{12}\right) = -\frac{305}{96}$$

$\therefore$ P = 1 and Q $= -\dfrac{305}{96}$

11. We have, $(1 + 3x + 6x^2 + 10x^3 + \ldots)^2$

Let $S = 1 + 3x + 6x^2 + \ldots$

$xS = x + 3x^2 + 6x^3 + \ldots$

$\Rightarrow (1 – x)S = (1 – x)^{-2} \Rightarrow S = (1 – x)^{-3}$

Now, $(1 + 3x + 6x^2 + 10x^3 + \ldots)^2 \Rightarrow [(1 – x)^{-3}]^2 = (1 – x)^{-6}$

Coefficient of x^r in the expansion of $(1 – x)^{-6}$ is ${}^{r+6-1}C_{6-1}$.

[$\because$ coefficient of x^r in $(1 – x)^{-n}$ is ${}^{r+n-1}C_{n-1}$]

$= {}^{r+5}C_5$

$$\Rightarrow \frac{(r+5)!}{5!(r+5-5)!} = \frac{(r+5)(r+4)(r+3)(r+2)(r+1)}{5!}$$

12. $(x^2 – 2y + z)^{10} = \sum \dfrac{10!}{r_1!r_2!r_3!} x^{2r_1}(-2y)^{r_2} z^{r_3}$

$r_1 + r_2 + r_3 = 10$, here $r_1 = 5, r_2 = 3, r_3 = 2$

Coefficient of $x^{10}y^3z^2 = \dfrac{10!}{5!\,3!\,2!}(-2)^3$

Test Your Concepts-IV

1. We have, $(2x^2 – 3x + 1)$

Sum of all coefficient for the given expansion, we put $x = 1$

$\therefore\ (2 – 3 + 1)^{11} = 0$

2. We have,

$(2x^2 – 3x + 1)^{11}$

Let $(2x^2 – 3x + 1)^{11} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$ …(1)

Put $x = 1$ in Eq. (1), we get

$0 = a_0 + a_1 + a_2 + a_3 + \ldots$ …(2)

Put $x = – 1$ in Eq. (1), we get

$6^{11} = a_0 – a_1 + a_2 – a_3 + \ldots$ …(3)

For the sum of even power of x, on adding Eqs. (2) and (3), we get

$6^{11} = 2\,(a_0 + a_2 + a_4 + \ldots)$

$$\Rightarrow a_0 + a_2 + a_4 + \ldots = \frac{6 \cdot 6^{10}}{2} = 3 \cdot 6^{10}$$

3. We have $\displaystyle\sum_{r=1}^{10} {}^{10}C_r \cdot r(r-1) = \sum_{r=1}^{10} \frac{10!}{r!(10-r)!} r(r-1)$

$$= 90 \sum_{r=1}^{10} \frac{(8!)r(r-1)}{r(r-1)(r-2)!(10-r)!}$$

$$= 90\sum_{r=1}^{10}\frac{8!}{(r-2)!(10-r)!} = 90\sum_{r=1}^{10}\frac{8!}{(r-2)!(10-r)!}$$

$$= 90[{}^8C_0 + {}^8C_1 + {}^8C_2 + \ldots + {}^8C_8]$$

$$= 90.2^8 = 45.2^9 \Rightarrow k = 45$$

4. We have,

$$\left(1+\frac{{}^nC_1}{{}^nC_0}\right)\left(1+\frac{{}^nC_2}{{}^nC_1}\right)\left(1+\frac{{}^nC_3}{{}^nC_2}\right)\ldots\left(1+\frac{{}^nC_n}{{}^nC_{n-1}}\right)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right)\ldots\left(1+\frac{1}{n}\right)$$

$$= \frac{(1+n)}{1}\frac{(1+n)}{2}\frac{(1+n)}{3}\ldots\frac{(1+n)}{n} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n+1-r}{r}\right]$$

$$= \frac{(n+1)^n}{n!}$$

5. We have, $(\alpha^2x^2 - 2\alpha x + 1)^{51}$

The sum of the coefficient of the expansion vanishes if $\alpha^2 - 2\alpha + 1 = 0$

$\Rightarrow (\alpha - 1)^2 = 0$

$\Rightarrow \alpha = 1$

6. We have,

$$(1 + x - 3x^2)^{2145} = a_0 + a_1x + a_2x^2 + \ldots$$

Put $x = -1$, we get

$$a_0 - a_1 + a_2 - a_3 \ldots = (1-1-3)^{2145} = (-3)^{2145}$$

$$(-3)^{4\times 526}\times(-3) = (81)^{526}\times(-3)$$

$$= (1)^{526}(-3)$$

$$= -(\ldots 3)$$

∴ End digit is 3.

7. We have,

$$\left(x - \frac{{}^{50}C_1}{{}^{50}C_0}\right)\left(x - 2^2\cdot\frac{{}^{50}C_2}{{}^{50}C_1}\right)\left(x - 3^2\cdot\frac{{}^{50}C_3}{{}^{50}C_2}\right)\ldots\left(x - 50^2\cdot\frac{{}^{50}C_{50}}{{}^{50}C_{49}}\right)$$

$$\Rightarrow x^{50} - x^{49}\left\{\frac{{}^{50}C_1}{{}^{50}C_0} + 2^2\frac{{}^{50}C_2}{{}^{50}C_1} + 3^2\frac{{}^{50}C_3}{{}^{50}C_2} + \ldots + 50^2\frac{{}^{50}C_{50}}{{}^{50}C_{49}}\right\} + \ldots$$

∴ Coefficient of x^{49}

$$= -\left\{\frac{{}^{50}C_1}{{}^{50}C_0} + 2^2\frac{{}^{50}C_2}{{}^{50}C_1} + 3^2\frac{{}^{50}C_3}{{}^{50}C_2} + \ldots + 50^2\frac{{}^{50}C_{50}}{{}^{50}C_{49}}\right\}$$

$$= -\left(\sum_{r=0}^{49}(r+1)^2\frac{{}^{50}C_{r+1}}{{}^{50}C_r}\right) = -\left(\sum_{r=0}^{49}(r+1)^2\frac{{}^{50}C_{r+1}}{{}^{50}C_r}\right)$$

$$= -\left[\sum_{r=0}^{49}(r+1)^2\frac{50!}{(r+1)!(50-r-1)}\times\frac{r!(50-r)!}{50!}\right]$$

$$= -\left[\sum_{r=0}^{49}(r+1)^2\frac{(50-r)}{(r+1)}\right] = -\left[\sum_{r=0}^{49}(r+1)(50-r)\right]$$

$$= \left[\sum_{r=0}^{49}(49r + 50 - r^2)\right]$$

$$= \left[49\sum_{r=0}^{49}r + \sum_{r=0}^{49}50 - \sum_{r=0}^{49}r^2\right]$$

$$= -\left[49\times\frac{49\times 50}{2} + 50\times 50 - \frac{49\times 50\times 99}{6}\right]$$

$$= -[49\times 49\times 25 + 2500 - 49\times 25\times 33]$$

$$= -[25\times 49\,(49-33) + 2500]$$

$$= -[25\times 49\times 16 + 2500]$$

$$= -22100$$

8. Given, $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$ ...(1)

Put $x = 1$ in Eq. (1), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

$$\Rightarrow \sum_{r=0}^{n}C_r = 2^n \qquad \ldots(2)$$

We have, $\displaystyle\sum_{r=0}^{n}\sum_{s=0}^{n}{}^nC_r\cdot{}^nC_s = \left(\sum_{r=0}^{n}{}^nC_r\right)\left(\sum_{s=0}^{n}C_s\right)$

$$= \sum_{r=0}^{n}{}^nC_r\cdot 2^n \qquad \text{[using Eq. (2)]}$$

$$= 2^n\,2^n \qquad \text{[using Eq. (2)]}$$

9. Here, $\displaystyle\sum_{r=0}^{s}\sum_{s=1}^{n}{}^nC_s\cdot{}^sC_r$

$$= \sum_{s=1}^{n}{}^nC_s[{}^sC_0 + {}^sC_1 + {}^sC_2 + \ldots + {}^sC_s] = \sum_{s=1}^{n}{}^nC_s\cdot 2^s$$

$$= (1+2)^n - 1 \Rightarrow 3^n - 1.$$

10. Given,

$$1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n = \sum_{k=0}^{n}a_kx^k$$

$$\Rightarrow \frac{[(1+x)^{n+1} - 1]}{1+x-1} = \sum_{k=0}^{n}a_kx^k$$

$$\left[\because 1 + r + r^2 + r^3 + \ldots + r^n = \frac{r^{n+1}-1}{r-1}\right]$$

$$\Rightarrow \frac{[(1+x)^{n+1} - 1]}{x} = \sum_{k=0}^{n}a_kx^k \qquad \ldots(1)$$

Put $x = 1$ in Eq. (1), we get

$\Rightarrow 2^{n+1} - 1 = \displaystyle\sum_{k=0}^{n}a_k$ Hence proved (i)

Also, coefficient of $x^k \Rightarrow a_k = {}^{n+1}C_{k+1}$

$\therefore a_{n-2} = {}^{n+1}C_{n-1} = \dfrac{n(n+1)}{2}$ Hence proved (ii)

11. Here difference of upper and lower index is constant

$${}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \ldots + {}^{950}C_0$$

= Coefficient of x^{950} in$\{(1+x)^{950} + (1+x)^{951} + \ldots + (1+x)^{1000}\}$

$$= \text{Coefficient of } x^{950} \text{ in } (1+x)^{950}\,\frac{\{(1+x)^{51}-1\}}{(1+x)-1}$$

$$= \text{Coefficient of } x^{951} \text{ in } \{(1+x)^{1001}-(1+x)^{950}\} = {}^{1001}\mathrm{C}_{50}$$

12. Given, $\displaystyle\sum_{r=0}^{n}(-1)^r\,{}^n\mathrm{C}_r\,\frac{1+r\log_e 10}{(1+\log 10^n)^r}$

Let $\log_e 10 = x$

$$= \sum_{r=0}^{n}(-1)^r\,{}^n\mathrm{C}_r\,\frac{1+rx}{(1+nx)^r}$$

$$= \sum_{r=0}^{n}(-1)^r\,{}^n\mathrm{C}_r\left(\frac{1}{(1+nx)}\right)^r + \sum_{r=0}^{n}(-1)^r\,\frac{n}{r}\,{}^{n-1}\mathrm{C}_{r-1}\,\frac{rx}{(1+nx)^r}$$

$$\left[\because {}^n\mathrm{C}_r = \frac{n}{r}\,{}^{n-1}\mathrm{C}_{r-1}\right]$$

$$= \sum_{r=0}^{n}(-1)^r\,{}^n\mathrm{C}_r\left(\frac{1}{(1+nx)}\right)^r - \frac{nx}{1+nx}$$

$$\sum_{r=1}^{n}(-1)^{r-1}\,{}^{n-1}\mathrm{C}_{r-1}\left(\frac{1}{1+nx}\right)^{r-1}$$

$$= \left(1-\frac{1}{1+nx}\right)^n - \frac{nx}{1+nx}\left(1-\frac{1}{1+nx}\right)^{n-1}$$

$$= \left(\frac{nx}{1+nx}\right)^n - \left(\frac{nx}{1+nx}\right)^n = 0$$

13. We have,

$(x+1)(x+2)(x+3)\ldots(x+10)+(x+2)(x+3)\ldots(x+11)+\ldots+(x+11)(x+12)\ldots(x+20)$

Coefficient of x^9 in the given polynomial is

$= (1+2+3+\ldots+10)+(2+3+4+\ldots+11)\ldots(11+12+13+\ldots+20)$

$$= 55+65+75+\ldots+155$$

$$= \frac{11}{2}(55+155) = \frac{11}{2}\times 210 = 1155$$

14. We have, $(1+x)^n(1+y)^n(1+z)^n$

$$= \left(\sum_{p=0}^{n}{}^n\mathrm{C}_p x^p\right)\left(\sum_{q=0}^{n}{}^n\mathrm{C}_q y^q\right)\left(\sum_{s=0}^{n}{}^n\mathrm{C}_s z^s\right)$$

$$= \sum_{0\le p,q,s\le n}({}^n\mathrm{C}_p)({}^n\mathrm{C}_q)({}^n\mathrm{C}_s)x^p y^q z^s)$$

For the coefficient of degree r, we must have $p+q+s=r$ where, p, q, s are integers with $p, q, s \ge 0$.

Sum of such coefficient $= \displaystyle\sum_{p,q,s\ge n}({}^n\mathrm{C}_p)({}^n\mathrm{C}_q)({}^n\mathrm{C}_s)$

$$[\because p+q+s=r]$$

15. We have, $\displaystyle\sum_{k=0}^{100}\left(\frac{k}{k+1}\right){}^{100}\mathrm{C}_k = \sum_{k=0}^{100}\left(\frac{k+1-1}{k+1}\right){}^{100}\mathrm{C}_k$

$$= \sum_{k=0}^{100}\left(1-\frac{1}{k+1}\right){}^{100}\mathrm{C}_k$$

$$= \sum_{k=0}^{100}{}^{100}\mathrm{C}_k - \sum_{k=0}^{100}\frac{{}^{100}\mathrm{C}_k}{k+1} = 2^{100} - \frac{2^{101}-1}{101}$$

$$\left[\because \sum_{r=0}^{n}\frac{{}^n\mathrm{C}_r}{r+1} = \frac{2^{n+1}-1}{n+1}\right]$$

$$= \frac{101(2^{100})-2^{101}+1}{101} = \frac{2^{100}(101-2)+1}{101} = \frac{a(2^{100})+b}{c}$$

On comparing, we get

$$a = 99,\ b = 1 \text{ and } c = 101$$

$\therefore a+b+c = 99+1+101 = 201$

$\therefore$ Number of digits is 3.

16. We have, $\displaystyle \mathrm{S} = \sum_{r=0}^{n-1}\frac{1}{(n-r)^2}\left(\frac{{}^n\mathrm{C}_{r+1}}{{}^n\mathrm{C}_r}\right)^2$

$$\mathrm{S} = \sum_{r=0}^{n-1}\frac{1}{(n-r)^2}\left[\frac{n-r}{r+1}\right]^2 \qquad \left[\because \frac{{}^n\mathrm{C}_{r+1}}{{}^n\mathrm{C}_r} = \frac{n-r}{r+1}\right]$$

$$\mathrm{S} = \sum_{r=0}^{n-1}\frac{1}{(r+1)^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$$

17. Given,

$$(1+2x+3x^2)^{10} = a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_{20}x^{20} \qquad \ldots(1)$$

To find a_1 in the expansions.

Now, differentiating Eq. (1) w.r.t. x, we get

$$10(1+2x+3x^2)^9(2+6x)$$

$$= a_1+2a_2x+3a_3x^2+\ldots+20\,a_{20}x^{19} \qquad \ldots(2)$$

Put $x = 0$ in Eq. (2), we get

$$10\times 2 = a_1 \Rightarrow a_1 = 20$$

18. Given, $(1+x-2x^2)^6 = a_0+a_1x+a_2x^2+\ldots+a_{12}x^{12}$ …(1)

Put $x = -1$ in Eq. (1), we get

$$\Rightarrow (-2)^6 = a_0-a_1+a_2-a_3+\ldots+a_{12} \qquad \ldots(2)$$

Again, put $x = 1$ in Eq. (1), we get

$$0 = a_0+a_1+a_2+a_3+\ldots+a_{12} \qquad \ldots(3)$$

Put $x = 0$ in Eq. (1), we get

$$1 = a_0 \qquad \ldots(4)$$

On adding Eqs. (2) and (3), we get

$$2(a_0+a_2+a_4+a_6+\ldots+a_{12}) = 2^6$$

$$\Rightarrow a_0+a_2+a_4+a_6+\ldots+a_{12} = 2^5$$

$$\Rightarrow a_2+a_4+a_6+\ldots+a_{12} = 2^5-1$$

$$\Rightarrow a_2+a_4+a_6+\ldots+a_{12} = 31$$

19. Given, $\mathrm{T}_0, \mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3, \ldots, \mathrm{T}_n$ represents terms in the expansion of $(x+a)^n$, then

$$(x+a)^n = \mathrm{T}_0+\mathrm{T}_1+\mathrm{T}_2+\mathrm{T}_3+\ldots+\mathrm{T}_n \qquad \ldots(1)$$

On replacing a by ia in Eq. (1), we get

$$(x+ai)^n = (\mathrm{T}_0-\mathrm{T}_2+\mathrm{T}_4-\mathrm{T}_6+\ldots)+i(\mathrm{T}_1-\mathrm{T}_3+\mathrm{T}_5\ldots) \qquad \ldots(2)$$

On replacing a by-ai in Eq. (1), we get

$(x-ai)^n = (T_0 - T_2 + T_4 - T_6 + \ldots) - i(T_1 - T_3 + T_5 \ldots)$ …(3)

On multiplying Eqs. (2) and (3), we get

$(x+ai)^n (x-ai)^n = (T_0 - T_2 + T_4 - T_6 + \ldots)^2 + (T_1 - T_3 + T_5 \ldots)^2$

$\Rightarrow (x^2+a^2)^n = (T_0 - T_2 + T_4 - T_6 + \ldots)^2 + (T_1 - T_3 + T_5 \ldots)^2$

Single Correct Choice Type Questions

1. $\frac{12}{3}+\frac{6}{2}+\frac{15}{5}+\frac{4n}{3}=26 \Rightarrow n=12$

Required coefficient $= {}^{26}C_4 \times {}^{22}C_3 \times \left({}^{19}C_3 \times 2^3\right) \times {}^{16}C_{16}$

$= \frac{26!}{108.16!}$

Hence, the correct answer is (B).

2. $\frac{5^{200}}{8} = \frac{\left(5^2\right)^{100}}{8} = \frac{(1+24)^{100}}{8}$

$= \frac{1+{}^{100}C_1.24+{}^{100}C_2(24)^2+\ldots.+{}^{100}C_{100}(24)^{100}}{8}$

$= \frac{1}{8} + \text{integer} \Rightarrow \left\{\frac{5^{200}}{8}\right\} = \frac{1}{8}$

Hence, the correct answer is (B).

3. $\sum_{r=1}^{12} {}^{24}C_r = {}^{24}C_1 + {}^{24}C_2 + \ldots\ldots + {}^{24}C_{12}$

Now

$2^{24} = {}^{24}C_0 + {}^{24}C_1 + \ldots\ldots + {}^{24}C_{11} + {}^{24}C_{12} + {}^{24}C_{13} + \ldots\ldots. + {}^{24}C_{24}$

$2^{24} = 2\times {}^{24}C_0 + 2\left\{{}^{24}C_1 + {}^{24}C_2 + \ldots\ldots. + {}^{24}C_{11} + {}^{24}C_{12}\right\} - {}^{24}C_{12}$

$\therefore\ {}^{24}C_1 + {}^{24}C_2 + \ldots\ldots. + {}^{24}C_{12} = 2^{23} + \frac{1}{2}.{}^{24}C_{12} - 1$

Hence, the correct answer is (C).

4. Let, $S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots.. + 1001\,x^{1000}$

$\frac{x}{1+x}S = x(1+x)^{999} + 2x^2(1+x)^{998} + \ldots\ldots + 1000x^{1000} + \frac{1001x^{1001}}{1+x}$

Subtract above equations,

$\left(1-\frac{x}{1+x}\right)S = (1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots. + x^{1000} - \frac{1001x^{1001}}{1+x}$

$\Rightarrow S = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999} + \ldots\ldots.. + x^{1000}(1+x) - 1001\,x^{1001}$

$= \frac{(1+x)^{1001}\left[\left(\frac{x}{1+x}\right)^{1001} - 1\right]}{\frac{x}{1+x}-1} - 1001\,x^{1001}$

[sum of G.P]

$= (1+x)^{1002} - x^{1002} - 1002\,x^{1001}$

$\therefore$ Coefficient of x^{50} in S = coefficient of x^{50} in $\left[(1+x)^{1002} - x^{1002} - 1002x^{1001}\right] = {}^{1002}C_{50}$

Hence, the correct answer is (C).

5. $P_n = {}^nC_0\,{}^nC_1\,{}^nC_2 \ldots {}^nC_n$

$\frac{P_{n+1}}{P_n} = \frac{{}^{n+1}C_1}{{}^nC_1}, \frac{{}^{n+1}C_2}{{}^nC_2} \ldots \frac{{}^{n+1}C_n}{{}^nC_n}.\,{}^{n+1}C_{n+1}$

$= \frac{n+1}{n}.\frac{n+1}{n-1}.\frac{n+1}{n-2}\ldots\frac{n+1}{1} = \frac{(n+1)^n}{n!}$

$\Rightarrow \frac{(n+1)^n}{n!} = \frac{10^9}{9!}$

$\Rightarrow n = 9$

Hence, the correct answer is (B).

6. $1+99^n = 1+(100-1)^n = 1 + \left\{{}^nC_0 100^n - {}^nC_1.100^{n-1} + \ldots\ldots\ldots - {}^nC_n\right\}$

Because n is odd $= 100\left\{{}^nC_0.100^{n-1} - {}^nC_1.100^{n-2} + \ldots\ldots\ldots - {}^nC_{n-2}.100 + {}^nC_{n-1}\right\}$

$= 100 \times$ integer whose units place is different from 0

$\left[\because {}^nC_{n-1} = n, \text{has odd digit at unit place}\right]$

$\therefore$ There are two zeros at the end of the sum $99^n + 1$

Hence, the correct answer is (C).

7. Rewriting by adding last and first terms , second and second last term and so on

Sum

$= \{C_0 + (C_1 + C_2 + \ldots\ldots + C_{n-1})\} + = \{(C_0 + C_1) + (C_0 + C_1 + \ldots\ldots. + C_{n-2})\} + \{(C_0 + C_1 + C_2) + (C_0 + C_1 + \ldots\ldots + C_{n-3})\} + \ldots\ldots.\text{ to}\left(\frac{n}{2}\right)$

Terms $= (C_0 + C_1 + \ldots\ldots C_n) \times \frac{n}{2} = n.2^{n-1}$

Hence, the correct answer is (B).

8. Let $t_r = \frac{{}^nC_{r-1}}{{}^nC_r + {}^nC_{r-1}} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} = \frac{{}^nC_{r-1}}{\frac{n+1}{r}\,{}^nC_{r-1}} = \frac{r}{n+1}$

$$S = \sum_{r=0}^{n} \frac{r^3}{(n+1)^3} = \frac{1}{(n+1)^3}\sum_{r=0}^{n} r^3$$

$$\Rightarrow S = \frac{1}{(n+1)^3}\left\{\frac{n(n+1)}{2}\right\}^2 \Rightarrow S = \frac{n^2}{4(n+1)}$$

Now, $S = \frac{25}{24}$ (given) which is only possible for 5

Hence, the correct answer is (C).

9. Since ${}^{10}C_0 + {}^{11}C_1 + {}^{12}C_2 + {}^{13}C_3 + \ldots\ldots\ldots + {}^{999}C_{989}$

$= {}^{1000}C_{989} = {}^{1000}C_{11}$

(Since, ${}^{10}\mathrm{C}_0 = {}^{11}\mathrm{C}_0$ and ${}^{n}\mathrm{C}_r + {}^{n}\mathrm{C}_{r-1} = {}^{n+1}\mathrm{C}_r$)

So, ${}^{12}\mathrm{C}_2 + {}^{13}\mathrm{C}_3 + {}^{14}\mathrm{C}_4 + \ldots\ldots + {}^{999}\mathrm{C}_{989} = {}^{1000}\mathrm{C}_{11} - 12$

Hence, the correct answer is (A).

10. $\mathrm{S}_n = {}^{3n}\mathrm{C}_0 - {}^{3n}\mathrm{C}_1 + {}^{3n}\mathrm{C}_2 + \ldots\ldots + (-1)^n . {}^{3n}\mathrm{C}_n$

But ${}^{3n}\mathrm{C}_0 = {}^{3n-1}\mathrm{C}_0$

$$-{}^{3n}\mathrm{C}_1 = -{}^{3n-1}\mathrm{C}_0 - {}^{3n-1}\mathrm{C}_1$$

$${}^{3n}\mathrm{C}_2 = {}^{3n-1}\mathrm{C}_1 + {}^{3n-1}\mathrm{C}_2$$

$$-{}^{3n}\mathrm{C}_3 = -{}^{3n-1}\mathrm{C}_2 - {}^{3n-1}\mathrm{C}_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(-1)^n . {}^{3n}\mathrm{C}_n = (-1)^n . {}^{3n-1}\mathrm{C}_{n-1} + (-1)^n . {}^{3n-1}\mathrm{C}_n$$

On adding we get $\mathrm{S}_n = (-1)^n . {}^{3n-1}\mathrm{C}_n$

Hence, the correct answer is (B).

11. Coefficient x^9 = sum of products of ${}^{21}\mathrm{C}_0, {}^{21}\mathrm{C}_1 \ldots\ldots\ldots\ldots {}^{21}\mathrm{C}_{10}$. Taking '2' at a time.

$$\text{Coefficient of } x^9 = \frac{\left(\sum_{r=0}^{10} {}^{21}C_r\right)^2 - \sum_{r=0}^{10} {}^{21}C_r^2}{2}$$

$$= \frac{\left(\frac{2^{21}}{2}\right)^2 - \left(\frac{{}^{42}C_{21}}{2}\right)}{2} = \frac{2^{40} - \frac{1}{2} \cdot {}^{42}C_{21}}{2} = 2^{39} - \frac{1}{4} \cdot {}^{42}C_{21}$$

Hence, the correct answer is (D).

12. Required value is $\frac{{}^{10}\mathrm{C}_1}{1} - \frac{{}^{10}\mathrm{C}_2}{2} + \frac{{}^{10}\mathrm{C}_3}{3} - - - - - \frac{{}^{10}\mathrm{C}_{10}}{10}$

To find which, consider

$$(1-x)^{10} = {}^{10}C_0 - {}^{10}C_1 x + {}^{10}C_2 x^2 - - - - - - + {}^{10}C_{10} x^{10}$$

$$\Rightarrow \frac{(1-x)^{10} - 1}{x} = -\left[{}^{10}\mathrm{C}_1 - {}^{10}\mathrm{C}_2 x + - - - - - - + {}^{10}\mathrm{C}_{10} x^9 \right]$$

$$\Rightarrow \int_0^1 \frac{1-(1-x)^{10}}{x} dx = \int_0^1 {}^{10}\mathrm{C}_1 - {}^{10}\mathrm{C}_2 x + - - - - - - + {}^{10}\mathrm{C}_{10} x^9 dx$$

$$= {}^{10}\mathrm{C}_1 - \frac{{}^{10}\mathrm{C}_2}{2} + \frac{{}^{10}\mathrm{C}_3}{3} - - - - - - \frac{{}^{10}\mathrm{C}_{10}}{10}$$

To find LHS consider

$$I_n = \int_0^1 \frac{1-(1-x)^n}{x} dx \Rightarrow I_{n+1} - I_n = \int_0^1 (1-x)^n \, dx = \frac{1}{n+1}$$

$$\therefore I_{n+1} = \frac{1}{n+1} + I_n$$

$$\therefore I_{10} = \int_0^1 \frac{1-(1-x)^{10}}{x} dx = \frac{1}{10} + I_9 = \frac{1}{10} + \frac{1}{9} + I_8 \approx - - -$$

$$= \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + - - - - - + 1$$

Hence, the correct answer is (B).

13. $27^{40} = 3^{120}$

$$3^{119} = (4-1)^{119} = {}^{119}\mathrm{C}_0 4^{119} - {}^{119}\mathrm{C}_1 4^{118} + {}^{119}\mathrm{C}_2 4^{117} - {}^{119}\mathrm{C}_3 4^{116} + \ldots + {}^{119}\mathrm{C}_{118} 4 - 1$$

$\therefore$ $3^{119} = 4k - 1$

$\therefore$ $3^{120} = 12k - 3 = 12(k-1) + 9$

$\therefore$ The required remainder is 9

Hence, the correct answer is (C).

14. Let $x - 1 = t$, then

$$\sum_{r=0}^{2n} a_r t^r = \sum_{r=0}^{2n} b_r (t-1)^r$$

$\therefore a_n =$ coefficient of t^n in $\sum_{r=0}^{2n} b_r (t-1)^r$

= coefficient of t^n b in $(b_0 + b_1(t-1) + \ldots + b_n(t-1)^n + b_{n+1}(t-1)^{n+1} + \ldots + b_{2n}(t-1)^{2n})$

$$= b_n \, {}^{n}\mathrm{C}_0 + b_{n+1} \, {}^{n+1}\mathrm{C}_1 (-1)^1 + b_{n+2} \, {}^{n+2}\mathrm{C}_2 (-1)^2 + \ldots + b_{2n} \, {}^{2n}\mathrm{C}_n (-1)^n$$

$$= (-1)^{n-n} . {}^{n}\mathrm{C}_0 + (-1)^{n+1-n+1}, {}^{n+1}\mathrm{C}_1 + \ldots + (-1)^{2n-n+n} \, {}^{2n}\mathrm{C}_n$$

$$= {}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \ldots + {}^{2n}C_n$$

$$= {}^{2n+1}C_n$$

Hence, the correct answer is (C).

15. Using $\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{n-r+1}{r}$

$$\mathrm{S} = \sum_{r=0}^{n} \left(1 - \frac{2r}{n-r+1}\right) \frac{{}^{n}C_r}{2^r} = \sum_{r=0}^{n} \frac{{}^{n}C_r}{2^r} - \sum_{r=0}^{n} \frac{{}^{n}C_{r-1}}{2^{r-1}} = \frac{1}{2^n}$$

Hence, the correct answer is (A).

16. Let $\left(\sqrt{2}+1\right)^6 = 1 + \mathrm{F}$, where I is an integer and $0 < \mathrm{F} < 1$.

Let $f = \left(\sqrt{2}-1\right)^6$. We have

$$\sqrt{2} - 1 = \frac{1}{\sqrt{2}+1} \Rightarrow 0 < \sqrt{2} - 1 < 1 \Rightarrow 0 < f < 1.$$

Also $1 + \mathrm{F} + f = \left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

$= 2[{}^{6}\mathrm{C}_0 . 2^3 + {}^{6}\mathrm{C}_2 . 2^2 + {}^{6}\mathrm{C}_4 . 2 + {}^{6}\mathrm{C}_6]$

$= 2(8 + 60 + 30 + 1) = 198$

Hence $\mathrm{F} + f = 198 - \mathrm{I}$ is an integer. But $0 < \mathrm{F} + f < 2$.

Therefore $\mathrm{F} + f = 1$, and thus, $\mathrm{I} = 197$.

Hence, the correct answer is (D).

17. Coefficient of x^6 is $\frac{1}{\lfloor 5}\frac{1}{\lfloor 1} + \frac{1}{\lfloor 4}\frac{1}{\lfloor 2} + \frac{1}{\lfloor 3}\frac{1}{\lfloor 3} + \frac{1}{\lfloor 2}\frac{1}{\lfloor 4} + \frac{1}{\lfloor 1}\frac{1}{\lfloor 5}$

$$= \frac{1}{\lfloor 6}\left[{}^{6}\mathrm{C}_1 + {}^{6}\mathrm{C}_2 + {}^{6}\mathrm{C}_3 + {}^{6}\mathrm{C}_4 + {}^{6}\mathrm{C}_5 \right]$$

$$= \frac{1}{\lfloor 6}\left(2^6 - 2\right) = \frac{31}{360}$$

Hence, the correct answer is (C).

18. $(1 + x + x^2 + x^3 + x^4)^{1001} (1 - x)^{1002}$
$= (1 - x) (1 - x^5)^{1001}$, so all the powers of x will be of the $5m$ or $5m + 1$ ($m \in I$)

So coffecient of x^{2009} will be 0

Hence, the correct answer is (A).

19. The coefficient of x^6 in the given expression = coefficient of x^6 in

$$(1 + {}^6C_1x^6)(1 + {}^5C_1x^5)(1 + {}^4C_1x^4)(1 + {}^3C_1x^3 + {}^3C_2x^6)(1 + {}^2C_1x^2 + {}^2C_2x^4)(1 + x)$$

$$= (1 + 6x^6 + 5x^5 + 4x^4)(1 + 2x^2 + 3x^3 + x^4 + 6x^5 + 3x^6)(1 + x)$$

$= \text{Coefficient of } x^6 \text{ in } \left(11x^5 + 17x^6\right)\left(1 + x\right) = 28$

Hence, the correct answer is (B).

20. Coefficient of $x^2 = \frac{1}{2}[(1 + 2 + 2^2 ...... + 2^n)^2]$

$$-\frac{1}{2}[1^2 + 2^2 + ...(2^n)^2]$$

$$= \frac{1}{2}\left[\left(\frac{2^{n+1}-1}{1}\right)^2 - \left(\frac{\left(2^2\right)^{n+1}-1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(2^{n+1}-1\right)\left[\left(2^{n+1}-1\right) - \frac{2^{n+1}+1}{3}\right]\right]$$

$$= \frac{\left(2^{n+1}-1\right)\left(2^{n+1}-2\right)}{3}$$

Hence, the correct answer is (B).

21. Suppose $x = (5\sqrt{3} + 8)^4 = [x] + \{x\}$ $0 < 5\sqrt{3} - 8 < 1$

$0 < \left(5\sqrt{3} - 8\right)^4 < 1$

Let $F = \left(5\sqrt{3} - 8\right)^4$ $0 < F < 1$

$$[x] + \{x\} + F = \left(5\sqrt{3} + 8\right)^4 + \left(5\sqrt{3} - 8\right)^4$$

$$= {}^4C_0\left(5\sqrt{3}\right)^4 + {}^4C_1\left(5\sqrt{3}\right)^3(8) + ...$$
$$+ \left({}^4C_0\left(5\sqrt{3}\right)^4 - {}^4C_1\left(5\sqrt{3}\right)^3(8) + ...\right)$$

$$= 2.\left[C_0\left(5\sqrt{3}\right)^4 + C_2(8)^2\left(5\sqrt{3}\right)^2 + C_4(8)^4\right]$$

$$= 2\left[625(9) + (6)64(25)(3) + (64)(64)\right]$$

$$= 2\left[5625 + 28800 + 4096\right] = 77042$$

$\{x\} + F$ must be an integer

Also $0 < \{x\} + F < 2$

$\Rightarrow \{x\} + f = 1$ $\Rightarrow [x] = 77042 - 1 = 77041$

Hence, the correct answer is (C).

22. $$1.2.3 .... 101(1 + x)\left(\frac{1}{2} - x\right)\left(\frac{1}{3} + x\right)......\left(\frac{1}{101} + x\right)$$

$$= \underline{|101}(x + 1)\left(x - \frac{1}{2}\right)\left(x + \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$$
$$......\left(x - \frac{1}{100}\right)\left(x + \frac{1}{101}\right).(-1)^{50}$$

Coefficient of x^{100} is

$$= \underline{|101}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{101}\right).$$

Hence, the correct answer is (B).

23. $x = \frac{1}{9}\left(10^{20} - 1\right), y = \frac{1}{3}\left(10^{10} - 1\right), z = \frac{2}{9}\left(10^{10} - 1\right)$ and

$x - y^2 = z$.

Hence, the correct answer is (A).

24. Coefficient of abc^3de^2 is $\frac{8!}{3!2!} = 3360$

Hence, the correct answer is (C).

25. $$\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Hence, the correct answer is (D).

26. $x^{15} - x^{13} + x^{11} - x^9 + x^7 - x^5 + x^3 - x$

$$= x(x^8 + 1)(x^4 + 1)(x^2 - 1)$$

$$x^{16} - 1 = (x^8 + 1)(x^4 + 1)(x^2 + 1)(x^2 - 1)$$

$$= \frac{7}{x}(x^2 + 1) = 7\left[\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2\right] > 14$$

$\therefore x^{16} > 15$

Hence, the correct answer is (C).

27. $T_{k+1} = (-1)^k .{}^nC_k(a - k)(b - k)$

$$= (-1)^{k.n}C_k\left[ab - k(a + b) + k^2\right]$$

Thus, the sum of series in ...(1)

$$= \sum_{k=0}^{n}(-1)^k.{}^nC_k\left[ab - k(a + b) + k^2\right]$$

$$= ab\sum_{k=0}^{n}(-1)^k.{}^nC_k - (a + b)\sum_{k=0}^{n}(-1)^k \cdot k \cdot {}^nC_k$$
$$+ \sum_{k=0}^{n}(-1)^k \cdot k^2 \cdot {}^nC_k$$

We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n \quad ...(2)$$

Differentiating both sides w.r.t. x, we get

$$n(1 + x)^{n-1} = 1.\,{}^nC_1 + 2.\,{}^nC_2x + 3.\,{}^nC_3x^2 + ... + n.\,{}^nC_nx \quad ...(3)$$

Multiplying it by x we get

$$n.x(1 + x)^{n-1} = 1.\,{}^nC_1x + 2.\,{}^nC_2x^2 + 3.\,{}^nC_3x^3 + ... + n.\,{}^nC_nx_n$$

Differentiating w.r.t. x, we get

$$n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2}$$
$$=1^2.{}^nC_1+2^2.{}^nC_2x+....+n^2.{}^nC_nx^{n-1} \qquad \ldots(4)$$

Putting $x=-1$ in (2), (3) and (4), we get

$$\sum_{k=0}^{n}(-1)^k.{}^nC_k=0$$

$$\sum_{k=0}^{n}(-1)^k k.{}^nC_k=0$$

$$\sum_{k=0}^{n}(-1)^k k.^2\,{}^nC_k=0$$

Thus, the sum of the series is 0

Hence, the correct answer is (D).

28. $\dfrac{1}{\sqrt{4x+1}}\cdot\dfrac{2}{2^7}\Big[{}^7C_1\sqrt{4x+1}+{}^7C_3(4x+1)^{3/2}$
$\qquad +{}^7C_5(4x+1)^{5/2}+{}^7C_7(4x+1)^{7/2}\Big]$

$$=\frac{1}{26}\left[{}^7C_1+{}^7C_3(4x+1)+{}^7C_5(4x+1)^2+{}^7C_7(4x+1)^3\right]$$

It is a cubic

Hence, the correct answer is (D).

29. Given expression $=\dfrac{{}^{(a+p)^{m-1}}\left[\left(\dfrac{a+q}{a+p}\right)^m-1\right]}{\dfrac{a+q}{a+p}-1}$

$$=\frac{(a+q)^m-(a+p)^m}{q-p}$$

Coefficient of $a^t=\dfrac{{}^mC_tq^{m-t}-{}^mC_tp^{m-t}}{q-p}=\dfrac{{}^mC_t\left(p^{m-t}-q^{m-t}\right)}{p-q}$

Hence, the correct answer is (B).

30. To get common terms in both the expansions

$x^{r_1}\left(y^2\right)^{13-r_1}=\left(x^2\right)^{r_2}\left(y\right)^{14-r_2}$
$r_1=2r_2$ and $26-2r_1=14-r_2$
$r_1=8;\ r_2=4$

∴ Only one term is common.

Hence, the correct answer is (C).

31. The coefficients are 3, 10, 28,...

$3,\ 5\times2,\ 7\times2^2,\ \ldots$

It is an A.G.P with general form $(2n+3)2^n$

∴ The given series is

$$\sum_{r=0}^{n}(2r+3)2^r\,{}^nC_r$$

$$=2\sum 2^r.r.\frac{n}{r}.{}^{n-1}C_{r-1}+3\sum 2^r.{}^nC_r$$

$$=4n\sum_{r=1}^{n}2^{r-1}.{}^{n-1}C_{r-1}+3\sum_{r=0}^{n}2^r.{}^nC_r$$

$$=4n(1+2)^{n-1}+3(1+2)^n=4n.3^{n-1}+3\times3^n$$
$$=3^{n-1}(4n+9)$$

Hence, the correct answer is (C).

32. $\displaystyle\sum_{m=1}^{n}{}^nC_m\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m}\frac{m!}{p!(m-p)!}\cdot\frac{p!}{k!(p-k)!}\right)\right)$

$$=\sum_{m=1}^{n}{}^nC_m\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m}{}^{m-k}C_{p-k}\right)\frac{m!}{k!(m-k)!}\right)$$

$$=\sum_{m=1}^{n}{}^nC_m\left(\sum_{k=1}^{m}2^{m-k}.{}^mC_k\right)$$

$$=\sum_{m=1}^{n}{}^nC_m\left((1+2)^m-2^m\right)=\sum_{m=1}^{n}\left({}^nC_m3^m-{}^nC_m2^m\right)$$

$$=(1+3)^n-1-(1+2)^n+1=4^n-3^n$$

Hence, the correct answer is (B).

33. $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+......+{}^nC_nx^n$

Put $x=1,\ \omega,\ \omega^2$ and add

$$\Rightarrow C_2+C_5+C_8+......=\frac{1}{3}\left\{2^n+(-1)^n\left(\omega^{2n+1}+\omega^{n+2}\right)\right\}$$

$$=\frac{1}{3}\left\{2^{2000}+(-1)^{2000}\left(\omega^{4001}+\omega^{2002}\right)\right\}=\frac{2^{2000}-1}{3}$$

Hence, the correct answer is (D).

34. Coefficient of x^{10} in $\left[{}^{20}C_0(3+x)^{20}-{}^{20}C_1(3+x)^{19}+....+{}^{20}C_{10}(3+x)^{10}\right]$

$=x^{10}$ in $(3+x-1)^{20}$
$=x^{10}$ in $(2+x)^{20}$
$=2^{10}\,{}^{20}C_{10}$

Hence, the correct answer is (C).

35. $\displaystyle\sum_{r=1}^{n}{}^nC_r\left((1+2)^r-2^r\right)=\sum_{r=1}^{n}{}^nC_r(1+2)^r-\sum_{r=1}^{n}{}^nC_r2^r$

$$=\left(4^n-1\right)-\left(3^n-1\right)=4^n-3^n$$

Hence, the correct answer is (D).

36. ${}^{50}C_6-{}^5C_1\,{}^{40}C_6+{}^5C_2\,{}^{30}C_6-{}^5C_3\,{}^{20}C_6+{}^5C_4\,{}^{10}C_6$

= coefficient of x^6 in $[{}^5C_0(1+x)^{50}-{}^5C_1(1+x)^{40}+{}^5C_2(1+x)^{30}-{}^5C_3(1+x)^{20}+{}^5C_4(1+x)^{10}-{}^5C_5(1+x)^0]$

= coefficient x^6 in $[(1+x)^{10}-1]^5$

= coefficient of x^6 in $({}^{10}C_1x+{}^{10}C_2x^2+...)^5$

$={}^5C_1({}^{10}C_2)({}^{10}C_1)^4=2250000$

Hence, the correct answer is (D).

37. $T_{r+1}={}^{10}C_r\,2^{\left(\frac{10-r}{2}\right)}.3^{\frac{r}{5}}$

This is rational, if $\dfrac{10-r}{2}$ and $\dfrac{r}{5}$ are integers.

∴ There are only two rational terms

Namely ${}^{10}\mathrm{C}_0\left(\sqrt{2}\right)^{10}\left(3^{\frac{1}{5}}\right)^0$ and ${}^{10}\mathrm{C}_{10}\left(\sqrt{2}\right)^0\left(3^{\frac{1}{5}}\right)^{10}$

$\therefore$ Sum $= 32 + 9 = 41$

Hence, the correct answer is (A).

38. $$\sum_{K=1}^{\infty}\sum_{r=0}^{K}\frac{1}{3^K}\left({}^K\mathrm{C}_r\right)$$

$$\sum_{K=1}^{\infty}\frac{1}{3^K}\sum_{r=0}^{K}\left({}^K\mathrm{C}_r\right)$$

$$\sum_{K=1}^{\infty}\left(\frac{2}{3}\right)^K \Rightarrow \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = \frac{2}{3}\times\frac{3}{1} = 2.$$

Hence, the correct answer is (B).

39. $$\frac{f(x)}{1-x} = b_0 + b_1x + b_2x^2 + \ldots.. + b_nx^n + \ldots$$

$$\Rightarrow \left(a_0 + a_1x + a_2x^2 + \ldots.. + a_nx^n + \ldots..\right)$$

$$= (1-x)\left(b_0 + b_1x + b_2x^2 + \ldots.. + b_nx^n + \ldots..\right)$$

comparing the coefficient of

x^n an both sides

$a_n = b_n - b_{n-1}$

Hence, the correct answer is (B).

40. Co-efficient of x^4

$= (1 + 5x + 9x^2 + 13x^3 + 17x^4 + \ldots.)(1 + 11x^2 + {}^{11}C_2x^4 + \ldots.)$

$= {}^{11}C_2 + 99 + 17$

Hence, the correct answer is (D).

41. $$\frac{a_0 + a_1 + a_3 + a_4 + a_6 + a_7 + \ldots\ldots\ldots..}{a_2 + a_5 + a_8 + \ldots\ldots\ldots} = K$$

$$a_0 + a_1 + a_3 + a_{11} + \ldots\ldots = K\left(a_2 + a_5 + a_8 + \ldots\ldots\ldots.\right)$$

$$a_0 + a_1 + a_2 + a_4 + a_5 + a_6 + \ldots\ldots = (K+1)\left(a_2 + a_5 + a_8 + \ldots\ldots\ldots.\right) \quad \ldots(1)$$

$$a_0 + a_1 + a_2 + a_4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. = f(1)$$

$$\Rightarrow f(1) = \sum_{i=0}^{2n} a_i = 3^n$$

$$\left(1 + x + x^2\right)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots.. + a_{2n}x^{2n}$$

multiplying with

$$x\left(1 + x + x^2\right)^n = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \ldots\ldots\ldots.$$

Put $x = 1,\ x = w,\ x = w^2$ adding we get

$$3^n = 3\left(a_2 + a_5 + a_8 + \ldots\ldots\ldots\ldots\ldots\ldots.\right)$$

$\therefore$ From (1), $3^n = (K+1).3^{n-1}$

$\therefore \quad K = 2$

Hence, the correct answer is (D).

42. Given

$$(1+x)^n = \mathrm{C}_0 + \mathrm{C}_1x + \mathrm{C}_2x^2 + \ldots.. + \mathrm{C}_nx^n$$

Integrating both sides with respect to x, we get

$$\frac{(1+x)^{n+1}}{n+1} = \frac{\mathrm{C}_0x}{1} + \frac{\mathrm{C}_1x^2}{2} + \frac{\mathrm{C}_2x^3}{3} + \ldots.. + \frac{\mathrm{C}_nx^{n+1}}{n+1} + k$$

Putting $x = 0$,

We get $k = \dfrac{1}{n+1}$

$$\therefore \frac{(1+x)^{n+1} - 1}{n+1}$$

$$= \frac{\mathrm{C}_0x}{1} + \frac{\mathrm{C}_1x^2}{2} + \frac{\mathrm{C}_2x^3}{3} + \ldots.. + \frac{\mathrm{C}_nx^{n+1}}{n+1}$$

Multiplying with x both sides

$$\frac{x(1+x)^{n+1} - x}{n+1} = \frac{\mathrm{C}_0x^2}{1} + \frac{\mathrm{C}_1x^3}{2} + \frac{\mathrm{C}_2x^4}{3} + \ldots\ldots + \frac{\mathrm{C}_nx^{n+2}}{n+1}$$

Differentiating with respect to x

$$\frac{(n+1)x(1+x)^n + (1+x)^{n+1} - 1}{n+1}$$

$$= \frac{2\mathrm{C}_0x}{1} + \frac{3\mathrm{C}_1x^2}{2} + \frac{4\mathrm{C}_2x^3}{3} + \ldots + \frac{(n+2)\mathrm{C}_nx^{n+1}}{n+1}$$

Now putting $x = 1$ both sides, we get

$$\frac{2^{n+1} + (n+1)2^n - 1}{n+1}$$

$$= \frac{2\mathrm{C}_0}{1} + \frac{3\mathrm{C}_1}{2} + \frac{4\mathrm{C}_2}{3} + \ldots.. + \frac{(n+2)\mathrm{C}_n}{n+1}$$

$$\frac{2^n(n+3) - 1}{n+1} = \frac{2\mathrm{C}_0}{1} + \frac{3\mathrm{C}_1}{2} + \frac{4\mathrm{C}_2}{3} + \ldots\ldots.. + \frac{(n+2)\mathrm{C}_n}{n+1}$$

Hence, the correct answer is (C).

43. Put $x = \omega,\ \omega^2$ and adding we get

$$2^{2011} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 + \ldots.$$

Hence, the correct answer is (C).

44. Given $t_n = \displaystyle\sum_{r=o}^{n}\frac{r}{\mathrm{C}_r} = \sum_{r=0}^{n}\frac{n-(n-r)}{\mathrm{C}_{n-r}} \qquad \left(\because {}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}\right)$

$$= \sum_{r=0}^{n}\frac{n}{\mathrm{C}_{n-r}} - \sum_{r=0}^{n}\frac{n-r}{\mathrm{C}_{n-r}}$$

$$= n\mathrm{S}_n - \left[\frac{n}{\mathrm{C}_n} + \frac{n-1}{\mathrm{C}_{n-1}} + \ldots. + \frac{1}{\mathrm{C}_1} + 0\right]$$

$$t_n = n\mathrm{S}_n - \sum_{r=0}^{n}\frac{r}{\mathrm{C}_r}$$

$$t_n = n\mathrm{S}_n - t_n \quad \Rightarrow \quad 2t_n = n\mathrm{S}_n$$

$$\therefore \frac{t_n}{\mathrm{S}_n} = \frac{n}{2}$$

Hence, the correct answer is (A).

45. Let $S = 3C_1 + 7C_2 + 11C_3 + \ldots. + (4n-1)C_n$

Let us write

$$T_r = (4r-1)C_r$$

$T_r = 4r\text{C}_r - \text{C}_r = 4r\frac{n}{r}\text{C}_{r-1} - \text{C}_r$ using ${}^{n}C_r = \frac{n}{r}.\,{}^{n-1}C_{r-1}$

$\therefore \sum_{r=1}^{n} T_r = 4n\sum_{r=1}^{n} C_{r-1} - \sum_{r=1}^{n} C_r$

$= 4\,n.\,2^{n-1} - \left(2^n - 1\right) = 2n\,.2^n - 2^n + 1$

$S = 2^n\left(2n-1\right)+1$ By using

$\left(1+x\right)^{n-1} = C_0 + C_1x + C_2x^2 + \ldots.. + C_{n-1}x^{n-1}$

$2^{n-1} = C_0 + C_1 + C_2 + \ldots. + C_{n-1}$

Hence, the correct answer is (C).

46. $(1+x^4+2x^2)({}^{n}\text{C}_0 + {}^{n}\text{C}_1x + {}^{n}\text{C}_2x^2 + \ldots) = a_0 + a_1x + a_2x^2 + \ldots$

By comparing

$a_0 = 1,\ a_1 = n,\ a_2 = 2 + {}^{n}\text{C}_2$

$\Rightarrow 2n = 1 + 2 + \frac{n\left(n-1\right)}{2}$

$\Rightarrow n^2 - 5n + 6 = 0 \quad \Rightarrow \quad n = 2, 3.$

Largest $n = 3$

Hence, the correct answer is (A)

47. $\left(1 + x + \frac{1}{x^2} + \frac{1}{x^3}\right)^{10} = \frac{\left((1+x)(1+x^3)\right)^{10}}{x^{30}}$

coefficient of x^{30} in $(1+x+x^3+x^4)^{10} = (1+x)^{10}(1+x^3)^{10}$

$= {}^{10}\text{C}_{10}\ {}^{10}\text{C}_0 + {}^{10}\text{C}_9\ {}^{10}\text{C}_3 + {}^{10}\text{C}_8\ {}^{10}\text{C}_6 + {}^{10}\text{C}_7\ {}^{10}\text{C}_9$

$= 1 + 1200 + 9450 + 1200 = 11851$

Last digit of $n = 1$

Hence, the correct answer is (A).

48. $2n + 1 = 53 \Rightarrow n = 26$

so exponent of 5 in (26)! is $\left[\frac{26}{5}\right] + \left[\frac{26}{5^2}\right] = 5 + 1 = 6$

so $p = 7$

Hence, the correct answer is (C).

49. Take $y - x - 1 = \text{X} \Rightarrow \sum_{r=0}^{n} a_r\left(1-X\right)^r = \sum_{r=0}^{n} b_rX^r$

Equating coefficient of X^n both sides $\Rightarrow b_n = \left(-1\right)^n a_n$.

Hence, the correct answer is (B).

50. $\sum_{r=0}^{15} {}^{15}\text{C}_r.r^2 - 15\sum_{r=0}^{15} r.{}^{15}\text{C}_r + \frac{225}{4}\times 2^{15}$

$\sum_{r=0}^{15} r^2.\,{}^{15}\text{C}_r = 15\sum_{r=1}^{15}\left[r-1+1\right]\times {}^{14}\text{C}_{r-1}$

$= 15.2^{14} + 15.14.2^{13} = 2^{15}.60$

$\therefore$ Required Sum $= \frac{225}{4}\times 2^{15} + 2^{15}.60 - 225.2^{14}$

Hence, the correct answer is (C).

51. $\text{P} = \frac{1}{2} \Rightarrow$ coefficient of x^{-12} in expansion of $\left(x + \frac{b}{x^3}\right)^{16}$

$\Rightarrow T_{r+1} = {}^{16}\text{C}_r\ x^{16-r}\left(\frac{b}{x^3}\right)^r$

$= {}^{16}\text{C}_r\,x^{16-r}\,b^r x^{-3r} \quad \Rightarrow \quad = {}^{16}\text{C}_r\,b^r\,x^{16-4r}$

$16 - 4r = -12 \quad \Rightarrow 4r = 28$

$\Rightarrow r = 7$

Coefficient of x^{-12} is $\left({}^{16}\text{C}_7\,b^7\right)$.

Hence, the correct answer is (C).

52. $(x + 1) + (2x + 1)\ (2^2x + 1)\ (2^3x + 1) \ldots\ldots (2^{20}x + 1)$

$= 1\cdot 2\cdot 2^2\cdot 2^3\ldots\ldots 2^{20}\left(1+x\right)\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2^2}\right)$
$\left(x+\frac{1}{2^3}\right)\ldots\ldots\left(x+\frac{1}{2^{20}}\right)$

$= 2^{\frac{20\times 21}{2}}\left(x+1\right)\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2^2}\right)\ldots\ldots\left(x+\frac{1}{2^{20}}\right)$

Coefficient of $x^{20} = 2^{210}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{20}}\right)$

$= \left(2^{21}\right)^{10}\left(\frac{1-\frac{1}{2^{21}}}{1-\frac{1}{2}}\right)$

$= (2^{21})^9\ 2(2^{21} - 1) = 2^{211} - 2^{190}$

Hence, the correct answer is (A).

53. $a_1 = {}^{n}\text{C}_r$

$\frac{{}^{n}C_r}{{}^{n}C_r + {}^{n}C_{r+1}},\ \frac{{}^{n}C_{r+1}}{{}^{n}C_{r+1} + {}^{n}C_{r+2}},\ \frac{{}^{n}C_{r+2}}{{}^{n}C_{r+2} + {}^{n}C_{r+3}}$

$= \frac{r+1}{n+1}, \frac{r+2}{n+1}, \frac{r+3}{n+1}$, clearly in A.P.

Hence, the correct answer is (D).

54. Put $x = y = z = 1$ then $2^n = 128,\ n = 7,\ {}^{7}C_3 = \frac{7.6.5}{1.2.3} = 35$

Hence, the correct answer is (B).

55. Geometric mean of all divisors of N is $\sqrt{N}$

${}^{n}\text{C}_1 + 2.{}^{n}\text{C}_2 + 3.{}^{n}\text{C}_3 + \ldots\ldots.n.{}^{n}\text{C}_n = n.2^{n-1}$

G. M of all divisors of 2011.2^{2010} is $\sqrt{2011.2^{2010}}$

$= 2^{1005}\sqrt{2011}$

Hence, the correct answer is (B).

56. $\int_0^2 (1+x)^{10}\,dx = {}^{10}\text{C}_0 2 + \frac{2^2.{}^{10}\text{C}_1}{2} + \ldots\ldots\ldots + \frac{2^{11}.{}^{10}\text{C}_{10}}{11}$

$= \frac{3^{11} - 1}{11}$

Hence, the correct answer is (D).

57. $(1 + px + x^2)^n = \sum_{r=0}^{2n} a_rx^r \qquad \ldots(1)$

Put $x = 1 \qquad (2 + p)^n = \sum_{r=0}^{2n} a_r$

Differentiate (1) and put $x = 1$

$n(1 + px + x^2)^{n-1}(p + 2x) = \sum_{r=0}^{2n} a_rrx^{r-1}$

$$n(2+p)^{n-1}(p+2)=\sum_{r=0}^{2n} ra_r \Rightarrow \sum_{r=0}^{2n} ra_r = n(p+2)^n$$

$$(2\sum_{r=0}^{2n} ra_r+\sum_{r=0}^{2n} a_r = 2n(2+p)^{n-1}(p+2)+(2+p)^n$$

$$=(2n+1)(p+2)^n$$

Hence, the correct answer is (C).

58. Given $= \displaystyle\sum_{r=0}^{n} r\frac{n!}{(n-r)!r!}y^{n-r}x^r$

$$= nx\sum_{r=0}^{n}\frac{(n-1)!}{((n-1)-(r-1))!(r-1)!}y^{(n-1)-(r-1)}x^{r-1}$$

$$= nx(y+x)^{n-1} = nx(1) = nx$$

Hence, the correct answer is (B).

59. $x - x^2 + x\,[x] = x - x(x - [x]) = x(1 - \{x\})$

Now $x + x_1$ = even integer where $x_1 = \left(2-\sqrt{3}\right)^n$ clearly $x_1 \in (0, 1)\ \forall\ n \in N$.

$\therefore \{x\} + x_1$ = integer

$\Rightarrow \{x\} + x_1 = 1$

Hence, the correct answer is (A).

60. Putting $x = 1$ and -1 and adding

$$a_0 + a_2 + \ldots + a_{50} = \frac{3^{25}+1}{2} = \frac{(1+2)^{25}+1}{2}$$

$$= \frac{{}^{25}C_0 + {}^{25}C_1 2 + {}^{25}C_2.2^2 + \ldots + {}^{25}C_{25}.2^{25} + 1}{2}$$

$$= \frac{2\left[1 + {}^{25}C_1 + {}^{25}C_2.2 + \ldots + {}^{25}C_{25}.2^{24}\right]}{2}$$

$= 2\left[13 + {}^{25}C_2 + \ldots + {}^{25}C_{25}.2^{23}\right]$ is an even integer

Hence, the correct answer is (A).

61. $\displaystyle\sum_{t=0}^{r-1} {}^rC_t 5^t = \sum_{t=0}^{r} {}^rC_t 5^t - 5^r = 6^r - 5^r$

$$\therefore \sum_{r=1}^{n}\left(\sum_{t=0}^{r-1} {}^nC_r.{}^rC_t 5^t\right) = \sum_{r=1}^{n} {}^nC_r\left(6^r - 5^r\right) = 7^n - 6^n$$

Required value $= \displaystyle\underset{n\to\infty}{Lt}\frac{7^n - 6^n}{7^n} = 1$

Hence, the correct answer is (C).

62. Coefficient of

x^{1274} is $1^2 - 2^2 + 3^2 - 4^2 + \ldots + 49^2 - 50^2$

$= -\left(1 + 2 + 3 + \ldots.\ 50\right) = -1275$

Hence, the correct answer is (B).

63. $\displaystyle\sum_{0\le i<j\le n}\sum\left(C_i - C_j\right)^2 = n\left(C_0^2 + C_1^2 + \ldots C_n^2\right) - 2\sum_{0\le i<j\le n}\sum C_iC_j$

$$= (n+1).{}^{2n}C_n - 2^{2n}$$

Hence, the correct answer is (A).

64. $\displaystyle\frac{r}{r+1}.{}^nC_r = \left(1 - \frac{1}{r+1}\right).{}^nC_r$

$$= {}^nC_r - \frac{1}{r+1}.\frac{n!}{r!(n-r)} = {}^nC_r - \frac{1}{n+1}{}^{n+1}C_{r+1}$$

$$\sum_{r=1}^{n}(-r)^{r-1}\frac{r}{r+1}.{}^nC_r = {}^nC_1 - {}^nC_2 + {}^nC_3 -$$

$$\ldots. - \frac{1}{n+1}\left[{}^{n+1}C_2 - {}^{n+1}C_3 + {}^{n+1}C_4 - \ldots..\right]$$

$$= {}^nC_0 - \frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1\right]$$

$$= 1 - \frac{1}{n+1}\left[-1 + (n+1)\right] = \frac{1}{n+1}$$

Hence, the correct answer is (A).

65. $(23)^{14} = (529)^7 = (530-1)^7$

$= {}^7C_0(530)^7 - {}^7C_1(530)^6 + \ldots - 7C_5(530)^2 + {}^7C_6 530 - 1$

$= {}^7C_0(530)^7 - {}^7C_1(530)^6 + \ldots + 3710 - 1$

$= 100\,m + 3709$

$\therefore$ Last two digits are 09

Hence, the correct answer is (C).

66. $\displaystyle\sum\sum\left(C_i + C_j\right)^2 = \sum\sum C_i^2 + C_j^2 + 2C_iC_j$

$= n\left(C_0^2 + C_1^2 + \ldots\ldots + C_n^2\right) + 2\displaystyle\sum\sum C_iC_j$ is

$$= n.{}^{2n}C_n + 2\left(\frac{2^{2n} - {}^{2n}C_n}{2}\right)$$

Since $\left(C_0 + C_1 + \ldots + C_n\right)^2$

$$= C_0^2 + C_1^2 + \ldots\ldots + C_n^2 + 2\sum\sum C_iC_j$$

$$2^{2n} = {}^{2n}C_n + 2\sum\sum C_iC_j$$

Hence, the correct answer is (A).

67. Put $x = i$ in the expansion of $(1+x)^n$ we get

$\left(C_0 - C_2 + C_4 - C_6\ldots..\right) + i\left(C_1 - C_3 + C_5\ldots.\right)$

$= 2^{n/2}\left[\cos{}^{n\pi}/_4 + i\sin{}^{n\pi}/_4\right]$ Take modulus both sides and square it

Hence, the correct answer is (B).

68. $\displaystyle t_{r+1} = {}^{21}C_r\left(\frac{a}{b}\right)^{\frac{21-r}{3}}\frac{b^{r/3}}{a^{r/6}} = {}^{21}C_r\ a^{\frac{42-3r}{6}}\ b^{\frac{2r-21}{3}}$

$\therefore 42 - 3r = 4r - 42$ i.e. $r = 12$

$\therefore$ 13th term contains same powers of a and b

Hence, the correct answer is (B).

69. First two term not containing x^{20}

Coefficient of x^{20} in

$(1+x)^{20} + (1+x)^{21} + \ldots + (1+x)^{30}$

$\therefore\ {}^{20}C_{20} + {}^{21}C_{20} + \ldots. + {}^{30}C_{20} = {}^{31}C_{21}$

Hence, the correct answer is (B).

70. Middle term is greatest then $\frac{n}{n+1} < x < \frac{n+1}{n}$

$\Rightarrow [x] = \{0,1\}$.

Hence, the correct answer is (C).

71. $T_{5+1} = {}^{10}C_5 (\sin p \cos p)^5$ independent of x

For greatest P = 45°

$\tan\left(\frac{45^o}{2}\right) = \frac{2}{\sqrt{h^2+k^2-4}}$ then

$x^2 + y^2 = 4\left(4+2\sqrt{2}\right)$.

Hence, the correct answer is (A).

72. $x(1+x)^n = {}^nC_0x + {}^nC_1\, x^2 + {}^nC_2x^3 + {}^nC_3\, x^4 +$ ....(1)

and $(1+x)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2\, x^{n-2} + {}^nC_3x^{n-3} +$ ...(2)

Differentiating eq. (2) w.r.t. x

$n(1+x)^{n-1} = n \cdot {}^nC_0x^{n-1} + (n-1){}^nC_1x^{n-2} + (n-2){}^nC_2x^{n-3}$ ....(3)

Multiply (1) and (3) the required sum is coefficient of x^n in $n[x\,(1+x)^{2n-1}] = n^{2n-1}C_{n-1} = n\ {}^{2n-1}C_n$

Hence, the correct answer is (A).

73. $a_{124} = 111......1$ (124 times)

$= 1 + 10 + 10^2 + ...10^{123}$

The remainder when 1, 10, 10^2, 10^3, 10^4 are divided by 271 are respectively 1, 10, 100, 187, 244.

These are only possible remainders when $10^k\,(k \in W)$ is divided by 271.

Required remainder = Remainder of {24 (1 + 10 + 100 + 187 + 244) + (1 + 10 + 100 + 187)} = 27

Hence, the correct answer is (C).

74. $n = 100^{50} - 43 = 10^{100} - 43 = 1000...0 - 43$

$= 999 .... 9957$

$\Rightarrow$ sum of digit = 98 (9) + 5 + 7 = 894.

Hence, the correct answer is (C).

75. Sum of coefficient of $(q + r)^{20}\ (1 + (p - 2)\ x)^{20}$ is $= (q + r)^{20}\,(p - 1)^{20}$ [put $x = 1$]

Square of the sum of co–efficient of

$(2rqx - (r + q).y)^{10}$

$= (2rq - (r + q))^{20}$ [put $x = y = 1$]

So $(q + r)^{20}\,(p - 1)^{20} = (2rq - (q + r))^{20}$

$\Rightarrow p = \frac{2rq}{r+q}$

$\Rightarrow p =$ H.M. of r and q

$\leq$ A.M. of r and q

$\leq (r + q)/2$.

Hence, the correct answer is (B).

76. $f(2x + 3) + f(2x + 7) = 2$

Replace $2x$ by $2x + 4$

$f(2x + 7) + f(2x + 11) = 2$

Take Difference

$f(2x + 3) = f(2(x + 4) + 3)$

$f(x) = f(x + 4)$

$t = 4 \Rightarrow n = 16;\ T_{r+1} = {}^{16}C_r\, m^{16-4r}.\, b^r$.

$r = 7;\ T_8 = {}^{16}C_7\, m^{-12}\, b^7$

Coefficient of $m^{-12} = {}^{16}C_7\, b^7$

Hence, the correct answer is (A).

77. $$\left.\begin{aligned} p+q+r &= 102 \\ 2q &= p+r \end{aligned}\right\} \Rightarrow \begin{aligned} q &= 34 \\ p+r &= 68 \end{aligned}$$

$$\text{R. C} = \frac{102!}{0!34!68!} + \frac{102!}{1!34!67!} + ... + \frac{102!}{68!34!0!}$$

$$= \frac{102!}{34!}\left(\frac{1}{0!68!} + \frac{1}{1!67!} + ... + \frac{1}{68!0!}\right)$$

$$= {}^{102}C_{34}\left(68_{C_0} + 68_{C_1} + ... + 68_{C_{68}}\right) = {}^{102}C_{34} \cdot 2^{68}$$

Hence, the correct answer is (A).

78. $$\frac{{}^nC_r}{{}^{r+2}C_r} = \frac{n!}{(n-r)!r!} \cdot \frac{r!2!}{(r+2)!}$$

$$= \frac{2}{(n+1)(n+2)} \cdot \frac{(n+2)!}{(r+2)!(n-r)!}$$

$$= \frac{2!}{(n+1)(n+2)} \cdot {}^{n+2}C_{r+2}$$

$$\therefore S = \frac{2}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^{r+2} . {}^{n+2}C_{r+2}$$

$$= \frac{2}{(n+1)(n+2)}\left[\sum (-1)^r\ {}^{n+2}C_r - {}^{n+2}C_0 + {}^{n+2}C_1\right]$$

$$= \frac{2}{(n+1)(n+2)}[0 - 1 + n + 2] = \frac{2}{n+2}$$

Hence, the correct answer is (B).

79. Let $s = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + ... + (1001)x^{1000}$.

The terms of the given sum follow arithmetic geometric progression with first factors of the terms as 1, 2.3, ... and second factors are in GP with common ratio $r = x/(1+x)$. Therefore

$$s = \frac{(1+x)^{1001}\{1 - [x/(1+x)]^{1001}\}}{1 - [x/(1+x)]} - (1001)x^{1001}$$

$$= (1+x)^{1002}\left[1 - \left(\frac{x}{1+x}\right)^{1001}\right] - (1001)x^{1001}$$

$$= (1+x)^{1002} - (1+x)x^{1001} - (1001)x^{1001}$$

$$= (1+x)^{1002} - (1002)x^{1001} - x^{1002}$$

Therefore the coefficient of x^{50} in the expression = coefficient of x^{50} in the expansion of $(1+x)^{1002}$ which is equal to ${}^{(1002)}C_{50}$

Hence, the correct answer is (C).

80. Required term

$$= \sum_{r=2}^{10} r \cdot (r-1)\ {}^{10}C_r = \sum_{r=2}^{10} r(r-1)\frac{10}{r} \cdot \frac{9}{r-1}\ {}^{8}C_{r-2}$$

$$= 90\sum_{r=2}^{10} {}^{8}C_{r-2} = 90\cdot 2^{8} = 45\cdot 2^{9}$$

Hence, the correct answer is (D).

81. $(1+x)^{50} + (1-x)^{50} = 2\left[{}^{50}C_0 + {}^{50}C_2x^2 + {}^{50}C_4\,x^4 + \ldots + {}^{50}C_{50}x^{50}\right]$

Integrating from 0 to 1 then put $x = 1$

$$\frac{1}{51}\left[(2^{51}-1)-(0-1)\right]$$

$$= 2\left[{}^{50}C_0 + \frac{{}^{50}C_2}{3} + \frac{{}^{50}C_4}{5} + \ldots + \frac{{}^{50}C_{50}}{51}\right]$$

Hence, the correct answer is (A).

82. $\sum_{i=0}^{m}\left({}^{10}\mathrm{C}_i\right)\left({}^{20}\mathrm{C}_{m-i}\right) =$ coefficient of x^m in the product $(1+x)^{10}$ $(1+x)^{20}$

since $(1+x)^{10}\,(1+x)^{20}$

$$= ({}^{10}\mathrm{C}_0 + {}^{10}\mathrm{C}_1x + {}^{10}\mathrm{C}_2\,x^2 + \ldots\ldots + {}^{10}\mathrm{C}_m\,x^m + \ldots\ldots + {}^{10}\mathrm{C}_{10}\,x^{10})$$

$$\times({}^{20}\mathrm{C}_0 + {}^{20}\mathrm{C}_1x + \ldots\ldots\ldots + {}^{20}\mathrm{C}_m\,x^m + \ldots\ldots\ldots + {}^{20}\mathrm{C}_{20}\,x^{20})$$

= coefficient of x^m in the expansion of $(1+x)^{30} = {}^{30}\mathrm{C}_m$

Now ${}^{30}\mathrm{C}_m$ is maximum when $m = 15$

Hence, the correct answer is (B).

83. $(1+x)^{2n+1} = \mathrm{C}_0 + \mathrm{C}_1x + \mathrm{C}_2x^2 + \ldots\ldots + \mathrm{C}_{2n-1}x^{2n-1} + \mathrm{C}_{2n}x^{2n} + \mathrm{C}_{2n+1}x^{2n+1}$

$$(x-1)^{2n+1} = \mathrm{C}_0x^{2n+1} - \mathrm{C}_1x^{2n} + \mathrm{C}_2x^{2n-1} - \ldots\ldots + (-1)^{2n+1}\,x^{2n+1}$$

Multiplying the above two and comparing coefficients of

$$\mathrm{C}_0^2 - \mathrm{C}_1^2 + \mathrm{C}_2^2 - \ldots\ldots - \mathrm{C}_{2n-1}^2 + \mathrm{C}_{2n}^2 - \mathrm{C}_{2n+1}^2 = 0$$

since all the terms of $\left(x^2-1\right)^{2n+1}$ are of even powers

$$\therefore \mathrm{C}_0^2 - \mathrm{C}_1^2 + \mathrm{C}_2^2 - \ldots\ldots - \mathrm{C}_{2n-1}^2 = \mathrm{C}_{2n+1}^2 - \mathrm{C}_{2n}^2$$

$$= \left({}^{(2n+1)}\mathrm{C}_{2n+1}\right)^2 - \left({}^{(2n+1)}\mathrm{C}_{2n}\right)^2 = 1-(2n+1)^2$$

Hence, the correct answer is (A).

84. Let $n = 2m$

then the greatest term in $(1+x)^{2m}$ is $\Rightarrow \frac{\mathrm{T}_{m+1}}{\mathrm{T}_m} > 1,\ \frac{\mathrm{T}_{m+1}}{\mathrm{T}_{m+2}}$

$$> 1 \Rightarrow \frac{m}{m+1} < x < \frac{m+1}{m} \Rightarrow \frac{n}{n+2} < x < \frac{n+2}{n}$$

Hence, the correct answer is (A).

85. Put $x = 1, \omega, \omega^2$ in (1) and adding we get

$$3\left[a_0 + a_3 + a_6 + a_9 + \ldots.\right] = 2^n + (1+\omega)^n + \left(1+\omega^2\right)^n$$

$$= 2^n + \left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$$

$$\therefore a_0 + a_3 + a_6 + a_9 + \ldots\ldots = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

1. In expansion of $\left(x^2 + \frac{1}{x}\right)^p$; general term $\mathrm{T}_{r+1} = {}^{p}\mathrm{C}_r x^{2p-3r}$

$$2p - 3r = 10 \Rightarrow r = \frac{2p-10}{3}$$

Hence, the correct answer is (A, C).

2. $\frac{5^k+3}{4} = \frac{5^k-5+8}{4} = \frac{5(5^{k-1}-1)}{5-1} + 2$

$= 5 + 5^2 + 5^3 + \ldots (k-1)$ terms $+ 2$

So if $k-1 =$ even, the last digit is 2

If $k-1 =$ odd, the last digit is $5 + 2 = 7$.

Hence, the correct answer is (A, C).

3. Let $S = \mathrm{C}_0^2 + 3.\mathrm{C}_1^2 + 5.\mathrm{C}_2^2 + \ldots. + (2n+1).\mathrm{C}_n^2$

$$\Rightarrow \mathrm{S} = (2n+1).\mathrm{C}_n^2 + {-}{-}{-}{-}{-}{-}{-}{-}{-} + \mathrm{C}_0^2$$

Adding $2\mathrm{S} = (2n+2).\left(\mathrm{C}_0^2 + \mathrm{C}_1^2 {-}{-}{-}{-}{-} + \mathrm{C}_n^2\right)$

$$\Rightarrow \mathrm{S} = (n+1).{}^{2n}\mathrm{C}_n$$

$$\Rightarrow \mathrm{S} = (n+1).\frac{2n}{n}\,{}^{2n-1}\mathrm{C}_{n-1} = 2(n+1)^{2n-1}\,\mathrm{C}_n$$

$\therefore$ C is correct and A, B are not correct.

D is also correct, because

$${}^{2n-1}\mathrm{C}_n + (2n+1).{}^{2n-1}\mathrm{C}_{n-1} = 2(n+1).{}^{2n-1}\mathrm{C}_n$$

Hence, the correct answer is (C, D).

4. $0 < 3\sqrt{3} - 5 < 1$

$$0 < \left(3\sqrt{3}-5\right)^n < 1;$$

$$\lambda = \left(3\sqrt{3}-5\right)^{2n+1} \quad 0 < \lambda < 1;\ 0 < \beta < 1$$

$$-1 < \beta - \lambda < 1$$

$$\alpha + \beta = \left(3\sqrt{3}+5\right)^{2n+1}$$

$[\alpha + \beta - \lambda = 2$ an integer $] =$ an even integer

$\Rightarrow \beta - \lambda = 0 \qquad \therefore\ \beta = \lambda \qquad \Rightarrow \alpha$ is even,

$$\alpha = 2.\left[{}^{2n+1}\mathrm{C}_1\left(3\sqrt{3}\right)^{2n}.5 + {}^{2n+1}\mathrm{C}_3\left(3\sqrt{3}\right)^{2n-2}.5^3 + \ldots..\right]$$

$= 2.5[$ an integer $]$

α is divisible by 10

$$(\alpha+\beta)^2 = \left(27 + 25 + 2.5\sqrt{3}\right)^{2n+1} = 2^{2n+1}\left(26 + 5\sqrt{3}\right)^{2n+1}$$

Hence, the correct answer is (A, B, D).

5. (A) $(17)^{1995} + 11^{1995} - 7^{1995} = (10+7)^{1995} + (10+1)^{1995} - 7^{1995} = 1 +$ (a multiple of 10) as 7^{1995} and -7^{1995} cancelled with each other. Therefore the digit at the unit place is 1. Therefore (A) is true.

(B) $(106)^{86} - (85)^{106} = (1+105)^{85} - (1+84)^{106}$. When binomially expanded 1, –1 will be cancelled and in the remaining terms, 105 and 84 occur and are divisible by 7. Therefore (B) is true.

(C) It is easy to see that $1 < (1+0.0001)^{1000}$

$$= \left(1 + \frac{1}{10^4}\right)^{1000} < 2$$

Therefore the integer just greater than $[1+(1/10^4)]^{1000}$ is 2. Hence (C) is true

(D) Put $x = 1$ and -1 and add $0+(-4)^{2010} = a_0 + a_2 + a_4 + a_0 + \ldots.. + a_{4020}$

Therefore (D) is true

Hence, the correct answer is (A, B, C, D).

6. The term independent of x in $\left(\sqrt{x}+\dfrac{k}{x^2}\right)^{10}$

is ${}^{10}C_2\left(\sqrt{x}\right)^8.\left(\dfrac{k}{x^2}\right)^2 = 405$

$\Rightarrow 45k^2 = 405$

$\Rightarrow k^2 = 9 \;\; \Rightarrow k = \pm 3$

Hence, the correct answer is (A, B).

7. $T_{r+1} = {}^{20}C_r 2^{\frac{20-r}{5}}.3^{\frac{r}{2}}$

$r = 0, 10, 20$

$T_1 = {}^{20}C_0 2^4 3^0 = 16$

$T_{11} = {}^{20}C_{10} 2^2 3^5$

$T_{21} = {}^{20}C_{20} 2^0 3^{10}$

Hence, the correct answer is (A, B, C, D).

8. Put $x = i$

$(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$

$x = i$ and $x = -i$

$(1+i)^n = C_0 + C_1i - C_2 - C_3i + C_4 + \ldots..$

$(1-i)^n = C_0 - C_1i - C_2 + C_3i + C_4 + \ldots..$

$\Rightarrow C_0 - C_2 + C_4 + \ldots\ldots = 2^{n/2}\cos\left(\dfrac{n\pi}{4}\right)$

$\Rightarrow C_1 - C_3 + C_5 + \ldots\ldots = 2^{n/2}\sin\left(\dfrac{n\pi}{4}\right)$

Hence, the correct answer is (A, D).

9. T_{r+1} of $\left(x-\dfrac{a}{x}\right)^n$ is ${}^nC_r.x^{n-r}\left(\dfrac{-a}{x}\right)^r$

$= {}^nC_r(-1)^r.a^r.x^{n-2r}$

for term independent of x

$n - 2r = 0 \Rightarrow r = \dfrac{n}{2}$

n must be divisible by 2

T_{r+1} of $\left(x+\dfrac{b}{x^2}\right)^n$ is ${}^nC_rx^{n-r}\left(\dfrac{b}{x^2}\right)^r = {}^nC_rb^r.\,x^{n-3r}$

for term independent of x, $n - 3r = 0 \Rightarrow r = \dfrac{n}{3}$

n must be divisible by 3.

Hence, the correct answer is (A, B, C).

10. $x = y = 1 \Rightarrow \left(a\alpha + a^2 + 1\right)^{2009} = 0 \;\Rightarrow a^2 + a\alpha + 1 = 0$ if a is real then $b^2 - 4ac \geq 0 \Rightarrow \alpha^2 - 4 \geq 0 \;\therefore \alpha = -2, -3, 2$

Hence, the correct answer is (A, B, D).

11. $\left(1+2x+3x^2\right)^{10} = a_0 + a_1x + a_2x^2 + \ldots.. + a_{20}x^{20}$.

Differentiate w.r.t x.

$10\left(1+2x+3x^2\right)^9\left(2+6x\right) = a_1 + 2a_2x + 3a_3x^2 + \ldots + 20.a_{20}x^{19}$

Put $x = 0$: we get $a_1 = 20$

Again differentiate w.r.t x

$10\left(1+2x+3x^2\right)^9.6 + 90\left(1+2x+3x^2\right)^8\left(2+6x\right)^2$

$= 2.a_2 + 6.a_3.x + \ldots. + 20.19.a_{20}x^{18}$

Put $x = 0$; we get $2.a_2 = 60 + 360 \Rightarrow a_2 = 210$

Replace x by $\dfrac{1}{x}$ in the original expansion

$$\left(1+\frac{2}{x}+\frac{3}{x^2}\right)^{10} = \frac{\left(x^2+2x+3\right)^{10}}{x^{20}} = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots. + \frac{a_{20}}{x^{20}}$$

$\Rightarrow \left(x^2+2x+3\right)^{10} = a_0.x^{20} + a_1.x^{19} + \ldots. + a_{19}x + a_{20}$

Put $x = 0$, we get $a_{20} = 3^{10}$

Differentiate w.r.t x

$10\left(x^2+2x+3\right)^9\left(2x+2\right) = 20.a_0.x^{19} + \ldots + a_{19}$

Put $x = 0$, we get $a_{19} = 20.3^9$

Hence, the correct answer is (A, B, C).

12. $\left[\dfrac{136}{12}\right] + 1 =$ Number of rational terms is 12, Number of irrational terms is 125

Number of total terms is 137.

Hence, the correct answer is (A, B, C).

13. $r\dfrac{{}^nC_r}{{}^nC_{r-1}} = (n - r + 1)$

$$\sum_{r=1}^{n} r\frac{C_r^2}{C_{r-1}} = \sum_{r=1}^{n}\{(n+1)-r\}c_r = (n+1)\left(2^n-1\right) - n2^{n-1}$$

$$= (n+2)2^{n-1} - (n+1)$$

$$S = C_0^2 + \sum_{r=1}^{n} r.\frac{C_r^2}{C_{r-1}} = (n+2)2^{n-1} - n$$

$S + n = (n+2)2^{n-1}$ which is divisible by $(n + 2)$ and 2^{n-1} also, S is a positive integer.

Hence, the correct answer is (B, C, D).

14. Use $\displaystyle\sum_{r=0}^{n} {}^nC_r\,x^{n-r}\,y^r = \left(x+y\right)^n = 1$ and proceed

(A) $x^n\displaystyle\sum_{r=0}^{n} {}^nC_r\left(\frac{x}{y}\right)^{-r} = x^n\sum_{r=0}^{n} {}^nC_r\left(\frac{y}{x}\right)^r$

$$= x^n\left(1+\frac{y}{x}\right)^n = (x+y)^n = (1)^n = 1$$

(B) $\sum_{r=0}^{n} r \cdot {}^{n}C_r \cdot x^{n-r} y^r = \sum_{r=0}^{n} r \cdot \frac{n}{r}\, {}^{n-1}C_{r-1}\, x^{n-1-(r-1)} y^{r-1} \cdot y$

$$= ny \sum_{r=0}^{n} {}^{n-1}C_{r-1}\, x^{n-r}\, y^{r-1}$$

$$= ny(x+y)^{n-1} = ny$$

(C) $\sum_{r=0}^{n} r^2 \cdot \frac{n}{r}\, {}^{n-1}C_{r-1}\, x^r y^{n-r}$

$$= n \sum_{r=0}^{n} (r-1+1)\, {}^{n-1}C_{r-1}\, x^r\, y^{n-r}$$

$$= n\left[\sum_{r=0}^{n} (r-1) \cdot \left(\frac{n-1}{r-1}\right) {}^{n-2}C_{r-2}\, x^r\, y^{n-r} + {}^{n-1}C_{r-1}\, x^r y^{n-r}\right]$$

$$= n\left[(n-1)x^2(x+y)^{n-2} + x(x+y)^{n-1}\right]$$

$$= n(n-1)x^2 + nx$$

Hence, the correct answer is (A, B, C).

15. $T_{r+1} = {}^{n}C_r \left(\frac{5}{x^2}\right)^{n-r} x^{4r} = {}^{n}C_r\, 5^{n-r}\, x^{6r-2n}$

For independent term

$6r - 2n = 0 \Rightarrow n = 3r$

Hence, the correct answer is (A, B, C, D).

16. $N = 144^{255} + 192^{255} = 144^{255} - 4^{255} + 192^{255} + 4^{255}$

$= 140(p) + 196(q),\ p, q \in I^+$

Hence, the correct answer is (A, C, D).

17. $$\frac{T_4}{T_2} = \frac{{}^{5}C_3 \left(\frac{1}{x}\right)^2 (x\tan x)^3}{{}^{5}C_1 \left(\frac{1}{x}\right)^4 \cdot x\tan x} = 2x^4 \tan^2 x = \frac{2}{27}\pi^4$$

$\Rightarrow x^2 \tan x = \pm \frac{1}{3\sqrt{3}} \pi^2$

$\Rightarrow x = \pm \frac{\pi}{3}$ can only satisfy the equation.

Hence, the correct answer is (B, C).

18. ${}^{n}C_{r-1} : {}^{n}C_r : {}^{n}C_{r+1} = a : b : c = 1 : 7 : 35$

$n = \frac{a(b+c) + (a+b)c}{b^2 - ac} = 23$

Hence, the correct answer is (C, D).

19. (A) $x^2 < 1$. Given expression $((1+2x)^n + (1+3x)^n)(1-x)^{-1}$

$\therefore$ of $x^n = \sum_{r=0}^{n} {}^{n}C_r . 2^r + \sum_{r=0}^{n} {}^{n}C_r . 3^r = 3^n + 4^n$.

(B) $$\sum_{r=0}^{n} (-1)^r\, {}^{n}C_r \left(\frac{1}{2^r} + \left(\frac{3}{4}\right)^2 + \left(\frac{7}{8}\right)^2 + \ldots\right) = \left(1 - \frac{1}{2}\right)^n + \left(1 - \frac{3}{4}\right)^n + \ldots = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{2n} + \ldots$$

$$= \frac{1/2^n}{1 - 1/2^n} = \frac{1}{2^n - 1}$$

(C) $1! + 2! + 3! + 4! = 33$ and $15/n!$ for $n \geq 5$

Required remainder 3.

(D) $\frac{10!}{2!8!} + \frac{10!}{1!9!} = {}^{10}C_2 + {}^{10}C_1$

$= {}^{11}C_2$

Hence, the correct answer is (A, B, C).

20. $T_3 = {}^{10}C_2 . 2^8 . \left(\frac{3x}{8}\right)^2,\ T_4 = {}^{10}C_3 2^7 \left(\frac{3x}{8}\right)^3,\ T_5 = {}^{10}C_4 2^6 \left(\frac{3x}{8}\right)^4$

$\left|\frac{T_4}{T_3}\right| \geq 1\ \&\ \left|\frac{T_5}{T_4}\right| \leq 1 \quad \Rightarrow \left|\frac{x}{2}\right| \geq 1\ \&\ \left|\frac{21x}{64}\right| \leq 1$

$\Rightarrow x \in (-\infty, -2] \cup [2, \infty)\ \&\ x \in \left[\frac{-64}{21}, \frac{64}{21}\right]$

$\Rightarrow x \in \left[\frac{-64}{21}, -2\right] \cup \left[2, \frac{64}{21}\right]$

Hence, the correct answer is (A, B).

21. ${}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \ldots\ldots + {}^{950}C_0 =$ coefficient of x^{950} in $\left\{(1+x)^{950} + (1+x)^{951} + \ldots\ldots + (1+x)^{1000}\right\}$

$=$ coefficient of x^{950} in $(1+x)^{950} \frac{\left\{(1+x)^{51} - 1\right\}}{1+x-1}$

$=$ coefficient of x^{950} in $\frac{(1+x)^{1001} - (1+x)^{950}}{x}$

$= {}^{1001}C_{951} = {}^{1001}C_{50} = {}^{1002}C_{951} - {}^{1001}C_{51} = {}^{1002}C_{51} - {}^{1001}C_{950}$, Since ${}^{n}C_r + {}^{n}C_{r-1} = {}^{(n+1)}C_r$.

Hence, the correct answer is (A, B, C, D).

22. $1^2 + 2^2 + 3^2 + \ldots.. + n^2 - \{a_1 + a_2 + a_3 + \ldots. + a_n\} = \frac{n(n^2-1)}{3}$

$\therefore S_n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n^2-1)}{3} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

$a_n = S_n - S_{n-1} = \frac{n(n+1)}{2} - \frac{(n-1)n}{2} = n.$

Hence, the correct answer is (A, B, D).

23. We have $(4+3x)^{25} = 4^{25}\left(1 + \frac{3}{4}x\right)^{25}$

Let $(r+1)^{th}$ is term is the term having greatest coefficient

$\Rightarrow$ coefficient of $T_{r+1} \geq$ coefficient of T_r

$\Rightarrow 4^{25}\left\{{}^{25}C_r \left(\frac{3}{4}\right)^r\right\} \geq 4^{25}\left\{{}^{25}C_{r-1}\left(\frac{3}{4}\right)^{r-1}\right\}$

$\Rightarrow \frac{{}^{25}C_r}{{}^{25}C_{r-1}}\left(\frac{3}{4}\right) \geq 1 \Rightarrow \frac{25-(r-1)}{r}\frac{3}{4} \geq 1$

$\Rightarrow 75 - 3r + 3 \geq 4r$

$\therefore r \leq \frac{78}{7} \leq 11.142$

But, r is an integer, hence $r = 11$

Hence, the correct answer is (A, C).

24. $R = [R] + f = \left(8 + 3\sqrt{7}\right)^{20}$

$= {}^{20}C_0\, 8^{20} + {}^{20}C_1 8^{19}\left(3\sqrt{7}\right) + \ldots\ldots\ldots,$

Where $0 < f < 1$

Let $F = \left(8 - 3\sqrt{7}\right)^{20} = {}^{20}C_0$

$8^{20} - {}^{20}C_1 8^{19}\left(3\sqrt{7}\right) + \ldots\ldots$ where $0 < F < 1$

$\because [R] + f + F = 2\left[{}^{20}C_0 . 8^{20} + {}^{20}C_2\, 8^{20} + {}^{20}C_2 8^{18}\left(3\sqrt{7}\right)\right]^2$

$+ \ldots\ldots] =$ an even integer

$\therefore [R] =$ an even integer $-1 =$ an odd integer

Also, $R - [R] = f = 1 - F = 1 - \left(8 - 3\sqrt{7}\right)^{20}$

$= 1 - \dfrac{1}{\left(8 + 3\sqrt{7}\right)^{20}}$

Again $RF = 1 \Rightarrow R(1 - f) = 1 \Rightarrow R\{1 - R + [R]\} = 1$

There fore $R + R[R] = 1 + R^2$

Hence, the correct answer is (B, C, D).

25. Since,

${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$ (given)

$\Rightarrow {}^{70}C_{3r} = {}^{70}C_{r^2}$

$\Rightarrow$ either $3r = r^2$ or $r^2 + 3r = 70$

i. e., either $r = 0, 3$ or $r = 7, -10$

But the given equation is not defined for

$R = 0, -10$

Hence, $r = 3$ or 7

Hence, the correct answer is (C, D).

26. The given expression is the coefficient of x^n in $(1+x)^n + (1+x)^{n+1} + \ldots. + (1+x)^{n+k}$

$\Rightarrow$ Coefficient of x^n in $(1+x)^n \left\{\dfrac{(1+x)^{k+1} - 1}{1 + x - 1}\right\}$

$= \dfrac{(1+x)^n (1+x)^{k+1} - (1+x)^n}{x}$

$\Rightarrow$ Coefficient of x^{n+1} in

$(1+x)^{n+k+1} - (1+x)^n$ i.e ${}^{n+k+1}C_{n+1} = {}^{n+k+1}C_k$

Hence, the correct answer is (A, B).

27. $\because (1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{2n} x^{2n}$

Putting $x = i (i = \sqrt{-1})$

Then, we get

$(1 + i + i^2)^n = (a_0 - a_2 + a_4 - a_6 + \ldots) + i(a_1 - a_3 + a_5 - a_7 + \ldots)$

$\Rightarrow i^n = (a_0 - a_2 + a_4 - a_6 + \ldots) + i(a_1 - a_3 + a_5 - a_7 + \ldots)$

If n is odd, then $\mathrm{Re}(i^n) = 0$

$\Rightarrow a_0 - a_2 + a_4 - a_6 + \ldots. = 0$

If n is even, then $\mathrm{Im}(i^n) = 0$

$\Rightarrow a_1 - a_3 + a_5 - a_7 + \ldots. = 0$

Hence, the correct answer is (A, B).

28. $a_0 + a_1 + a_2 + \ldots = 2^{2n}$ and $a_0 + a_2 + a_4 + \ldots = 2^{2n-1}$

$a_n = {}^{2n}C_n =$ the greatest coefficient, being the middle coefficient

$a_{n-3} = {}^{2n}C_{n-3} = {}^{2n}C_{2n-(n-3)} = {}^{2n}C_{n+3} = a_{n+3}$

Hence, the correct answer is (A, B, C).

29. $I + f = \left(4 + \sqrt{15}\right)^n$

Let $\delta = \left(4 - \sqrt{15}\right)^n \quad 0 < \delta < 1$

$I + f = {}^nC_0 4^n + {}^nC_1 4^{n-1}\sqrt{15} + {}^nC_2 4^{n-2}.15\ldots\ldots\ldots\ldots\ldots$

$I + f + f' = [{}^nC_0\, 4^n + {}^nC_2\, 4^{n-2}\, 15 + \ldots] =$ an even integer

$f + f' = 1 \Rightarrow f' = 1 - f$

$I \rightarrow$ odd Integer

$(1 + f)(1 - f) = 1$

Hence, the correct answer is (A, C).

30. Consider the product of the expansions $(1+x)^n . \left(1 + \dfrac{1}{x}\right)^{n+1} = \dfrac{(1+x)^{2n+1}}{x^{n+1}}$. The given expression is sum of the coefficients of negative powers of x in this product.

$\therefore$ It is equal to ${}^{2n+1}C_0 + - - - + {}^{2n+1}C_n = 2^{2n}$

Hence, the correct answer is (A, B).

31. $x = 0 : a_0 = 1$

Differentiating $4(1 + ax + bx^2)^3 (a + 2bx) = a_1 + 2a_2 x + \ldots.$

Let $x = 0 \qquad 4a = a_1$

The value of the determinant is $-(a_0^3 + a_1^3 + a_2^3 - 3a_0 a_1 a_2) = 0$

$\Rightarrow \because a_0 + a_1 + a_2 \neq 0$ the $a_0 = a_1 = a_2 = 4a$

Differentiating again and putting $x = 0$

$$a = \frac{1}{4} : b = \frac{5}{32}$$

Hence, the correct answer is (B, C, D).

32. $\displaystyle\sum_{k=0}^{7}\left(\frac{{}^7C_k}{{}^{14}C_k}\sum_{r=k}^{14} {}^rC_k . {}^{14}C_r\right)$

$\displaystyle = \sum_{k=0}^{7}\left(\frac{{}^7C_k}{\underline{|14}} \times \underline{|k}\, \underline{|14-k} \sum_{r=k}^{14} \frac{\underline{|r}}{\underline{|k}\, \underline{|r-k}} . \frac{\underline{|14}}{\underline{|r}\underline{|14-r}}\right)$

$\displaystyle = \sum_{k=0}^{7}\left({}^7C_k \sum_{r=k}^{14} {}^{14-k}C_{r-k}\right) = \sum_{k=0}^{7} {}^7C_k . 2^{14-k} = 2^{14}\sum_{k=0}^{7} {}^7C_k \left(\frac{1}{2}\right)^k$

$= 2^{14} . \left(1 + \dfrac{1}{2}\right)^7 = 6^7 > 7^6$

Hence, the correct answer is (A, B).

33. Coefficient of x^{324} is

$= 1^2 - 2^2 + 3^2 - 4^2 + \ldots\ 24^2 + 25^2 = 325$

$\displaystyle \because \sum_{k=0}^{8} {}^8C_k = 2^8 = 256, \sum_{s=0}^{4} {}^4C_s = 2^4 = 16 \sum_{r=0}^{2} {}^2C_r = 4$

$\therefore$ Option (B) $= 256 + 4 \times 16 + 4 + 1 = 325.$

Hence, the correct answer is (A, B, C).

Linked Comprehension Type Questions

1. $(x+2)^{n-1}+(x+2)^{n-2}(x+1)+(x+2)^{n-3}(x+1)^2$
$+\ldots+(x+1)^{n-1}$

$$=(x+2)^n-(x+1)^n$$

$$={}^nC_r\cdot 2^{n-r}-{}^nC_r={}^nC_r\,(2^{n-r}-1)$$

Hence, the correct answer is (B).

2. $2^n+2^{n-1}(x+2)+2^{n-2}(x+2)^2+\ldots\ldots.+(x+2)^n$

$$=\frac{(x+2)^{n+1}-2^{n+1}}{x}$$

Coefficient of $x^r={}^{n+1}C_{r+1}\,2^{n-r}$

Hence, the correct answer is (A).

3. $S=(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots\ldots+1001\,x^{1000}$

$$\frac{x}{1+x}S=x(1+x)^{999}+2x^2(1+x)^{998}+\ldots..$$

Take difference

$$\frac{S}{1+x}=(1+x)^{1\,000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots.$$

$$S=(1+x)^{1\,001}+x(1+x)^{1000}+x^2(1+x)^{999}+\ldots.$$

Coefficient of $x^{50}={}^{1001}C_{50}+{}^{1000}C_{49}+\ldots\ldots+{}^{952}C_1+{}^{951}C_0$

$={}^{1002}C_{50}$

Hence, the correct answer is (A).

4. $\displaystyle\sum_{r=0}^{200}a_r\left(\frac{1}{x}\right)^r=\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{100}=\frac{1}{x^{200}}\left(x^2+x+1\right)^{100}$

$$\Rightarrow\sum_{r=0}^{200}a_rx^{200-r}=\left(x^2+x+1\right)^{100}=\sum_{r=0}^{200}a_rx^r=\sum_{r=0}^{200}a_{200-r}x^{200-r}$$

Equating the coefficients of x^{200-r} we get $a_r=a_{200-r}$.

Hence, the correct answer is (B).

5. Put $x=1$

$$a_0+a_1+a_2+\ldots+a_{200}=3^{100}$$

but $a_r=a_{200-r}$

$2(a_0+a_1+\ldots+a_{99})+a_{100}=3^{100}$

$$a_0+a_1+\ldots+a_{99}=\frac{3^{100}-a_{100}}{2}$$

Hence, the correct answer is (C).

6. $\displaystyle 100(1+2x)\left(1+x+x^2\right)^{100}=\left(1+x+x^2\right)\sum_{r=0}^{200}ra_rx^{r-1}$

$$100(1+2x)\sum_{r=0}^{200}a_rx^r=\left(1+x+x^2\right)\sum_{r=0}^{200}ra_rx^{r-1}$$

Equating the coefficient of x^r

$100a_r+200a_{r-1}=(r+1)a_{r+1}+ra_r+(r-1)a_{r-1}$

$(r+1)a_{r+1}=(100-r)a_r+(200-r+1)a_{r-1}$, *put* $r=36$.

Hence, the correct answer is (A).

7. Setting $x=1,-1$ in equation and subtracting we get

$2(a_1+a_3+a_5+\ldots\ldots.)=3^{10}-1$

Hence, the correct answer is (C).

8. Put $x=1,\ \omega,\omega^2$ in turn and adding we get $3(a_0+a_3+a_6+\ldots)=3^{10}+(1+\omega+\omega^2)^{10}+(1+\omega^2+\omega^4)^{10}=3^{10}$

Hence, the correct answer is (A).

9. $\left(1+x+x^2\right)^{10}=a_0+a_1x+a_2x^2-a_3x^3+a_4x^4+a_5x^5+\ldots..$

$$(1-x)^{10}=1-\binom{10}{1}x+\binom{10}{2}x^2-\binom{10}{3}x^3$$

Multiplying these two and taking the coefficient of x^5, we get the given series is the coefficient of x^5 in $(1-x^3)^{10}$ which is zero.

Hence, the correct answer is (A).

10. $S=\dfrac{(1+x)^n\left[1-\left(\dfrac{1+x}{2}\right)^{n+1}\right]}{1-\left(\dfrac{1+x}{2}\right)}=\dfrac{1}{2^n}\,\dfrac{(1+x)^n.[2^{n+1}-(1+x)^{n+1}]}{(1-x)}$

$$=\frac{1}{2^n}[2^{n+1}(1+x)^n-(1+x)^{2n+1}]\ .\ \frac{1}{1-x}$$

$$=\frac{1}{2^n}[2^{n+1}(1+x)^n-(1+x)^{2n+1}]\,.\,(1+x+x^2+\ldots\infty)$$

Hence, the correct answer is (A).

11. The coefficient of x^n in $S=\dfrac{1}{2^n}\left[2^{n+1}\displaystyle\sum_{r=0}^{n}{}^nC_r-\sum_{r=0}^{n}{}^{2n+1}C_r\right]$

$$=\frac{1}{2^n}[2^{n+1}.2^n-\frac{1}{2}2^{2n+1}]$$

$$=\frac{1}{2^n}\left[2.2^{2n}-2^{2n}\right]=\frac{2^{2n}}{2^n}=2^n$$

Hence, the correct answer is (A).

12. $G.E=\displaystyle\int_0^1\left(1-x^2\right)^n dx$ put $x=\sin\theta$, $dx=\cos\theta\,d\theta$

$$\int_0^{\frac{\pi}{2}}\cos^{2n+1}\theta\,d\theta=\frac{2n}{2n+1}.\frac{2n-2}{2n-1}\times\ldots..\frac{2}{3}\times1=\frac{2^{2n}.(n!)^2}{(2n+1)!}$$

Hence, the correct answer is (C).

13. $(1-x)^n=C_0-C_1x+C_2x^2-\ldots.+(-1)^nC_nx^n$

$\Rightarrow x^{n-1}(1-x)^{n+1}$

$=\left(C_0x^{n-1}-C_1x^n+C_2x^{n+1}-\ldots..+(-1)^nC_nx^{2n-1}\right)(1-x)$

$=C_0\left(x^{n-1}-x^n\right)-C_1\left(x^n-x^{n+1}\right)+\ldots.(n+1)\text{terms}$

$$\Rightarrow\int_0^1 x^{n-1}(1-x)^{n+1}dx$$

$$=\frac{C_0}{n(n+1)}-\frac{C_1}{(n+1)(n+2)}+\ldots\ldots.(n+1)\text{terms}$$

Hence, the correct answer is (C).

14. $\sum_{r=1}^{n}(-1)^{r-1}\left(1+x+x^2+.....+x^{r-1}\right).{}^{n}C_r$

$$=\sum_{x=1}^{n}\frac{\left(1-x^r\right)(-1)^{r-1}.{}^{n}C_r}{1-x}$$

$$=\frac{1}{1-x}\sum_{r=1}^{n}\left[(-1)^{r-1}.{}^{n}C_r-(-1)^{r-1}x^r.{}^{n}C_r\right]$$

$$=\frac{1}{1-x}\left[1+(1-x)^n-1\right]=(1-x)^{n-1}$$

$$\Rightarrow\int_0^1\sum_{r=0}^{n}(-1)^{r-1}\left(1+x+x^2+.....+x^{r-1}\right)^{n}C_r.dx=\int_0^1(1-x)^{n-1}dx$$

$$\Rightarrow\sum_{r=1}^{n}(-1)^{r-1}\left(1+\frac{1}{2}+\frac{1}{3}......+\frac{1}{r}\right).{}^{n}C_r=\frac{1}{n}$$

Hence, the correct answer is (D).

15. $C_0+C_1x+C_2x^2+.......+C_nx^n=(1+x)^n$ Differentiate w.r. t. x we get .

$C_1+2.C_2x+3.C_3x^2+.......+n(1+x)^{n+1}$ Put $x=i$ we get

$(C_1-3.C_3+5.C_5-.......)+i(2.C_2-4.C_4+.......)=n(1+i)^{n-1}$

Taking modulus and squaring we get

$(C_1-3.C_3+5.C_5-....)^2+(2.C_2-4.C_4+.......)^2=n^2 2^{n-1}$

Hence, the correct answer is (B).

16. $$\sum_{0\le i<j\le n}\sum\left(\frac{1}{C_i}+\frac{1}{C_j}\right)=n\sum_{r=0}^{n}\frac{1}{C_r}=na$$

Hence, the correct answer is (A).

17. ${}^{20}C_r.{}^{10}C_0+{}^{20}C_{r-1}.{}^{10}C_1+..........+{}^{20}C_0.{}^{10}C_r$

$=$ Coefficient of x^r in $(1+x)^{20}.(1+x)^{10}$

$=$ Coefficient of x^r in $(1+x)^{30}={}^{30}Cr$

${}^{30}Cr$ is minimum if $r=0$

Hence, the correct answer is (A).

18. $S_n={}^{n}C_0.{}^{n}C_1+{}^{n}C_1.{}^{n}C_2+........+{}^{n}C_{n-1}.{}^{n}C_n$

$={}^{n}C_0.{}^{n}C_{n-1}+{}^{n}C_1.{}^{n}C_{n-2}+........+{}^{n}C_{n-1}.{}^{n}C_0$

$=$ Coefficient of x^{n-1} in $(1+x)^n(1+x)^n={}^{2n}C_{n-1}$

$$\therefore\ \frac{S_{n+1}}{S_n}=\frac{{}^{(2n+2)}C_n}{{}^{2n}C_{n-1}}=\frac{15}{4}\Rightarrow n=2,4$$

Hence, the correct answer is (C).

19. Required value

$=C_0C_0-C_1C_1+C_2C_2--------+(-1)^nC_n.C_n$

$=C_0C_n-C_1C_{n-1}+--------+(-1)^nC_n.C_0$

$=$ Coefficient of x^n in $(1+x)^n(1-x)^n$

$=$ Coefficient of x^n in $\left(1-x^2\right)^n=0$ ($\because n$ is odd)

Hence, the correct answer is (B).

20. $$\frac{4+x}{\sqrt{1+4x}}=(4+x)(1-2x+6x^2-20x^3....)$$

$$=4-7x+22x^2-74x^3+.....$$

Hence, the correct answer is (A).

21. $$(1-2x)^{-2/3}=1+\frac{2}{3}(2x)+\frac{2}{3}.\frac{5}{3}\frac{1}{2!}(2x)^2+....$$

$$(r+1)^{\text{th}}\text{ Term}=\frac{2.5.8......(3r-1)}{r!}\left(\frac{2}{3}x\right)^r$$

Hence, the correct answer is (A).

22. $y=2x+3x^2+4x^3+......\infty$

$$xy=2x^2+3x^3+......$$

$$y(1-x)=2x+\frac{x^2}{1-x}$$

$$x^2-2x+\frac{4}{1-y}=0$$

$$x=1-\frac{1}{\sqrt{1+y}};-1<x<1$$

Hence, the correct answer is (B).

23. $$\left(1+\frac{C_0}{C_1}\right)\left(1+\frac{C_1}{C_2}\right)\cdots\left(1+\frac{C_{n-1}}{C_n}\right)=\prod_{r=1}^{n}\left(1+\frac{C_{r-1}}{C_r}\right)$$

$$=\prod_{r=1}^{n}\left(1+\frac{r}{n-r+1}\right)=\prod_{r=1}^{n}\left(\frac{n+1}{n-r+1}\right)$$

$$=\frac{n+1}{n}\cdot\frac{n+1}{n-1}\cdots\frac{n+1}{1}=\frac{(n+1)^n}{n!}$$

Hence, the correct answer is (A).

24. $n\times C_0+(n-1)\times C_1+..........+1\times(C_{n-1})=n.2^{n-1}$.

Hence, the correct answer is (B).

25. Since $a_r=a_{2n-r}$

$$S=\sum_{r=0}^{2n}\frac{r}{a_r}=\sum_{r=0}^{2n}\frac{2n-r}{a_r}$$

$$S=n\sum_{r=0}^{2n}\frac{1}{a_r}$$

Hence, the correct answer is (A).

26. $\left(1+x+x^2\right)^n=\sum_{r=0}^{2n}a_rx^r$. on dividing both sides by x^n,

we get $\sum_{r=0}^{2n}a_rx^{r-n}=\left(1+x+\frac{1}{x}\right)^n=\sum_{r=0}^{n}C_r\left(x+\frac{1}{x}\right)^r$

on comparing the constant term,

$a_n=c_0+c_2\left({}^{2}C_1\right)+C_4\left({}^{4}C_2\right)+C_6\left({}^{6}C_3\right)+....=1+a$

Hence, the correct answer is (B).

27. $(1+x+x^2)^n=a_0+a_1x+a_2x^2+.....+a_{2n}x^{2n}$

Replacing by $-\frac{1}{x}$ we have

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n=a_0-\frac{a_1}{x}+\frac{a_2}{x^2}+......+\frac{a_{2n}}{x^{2n}}$$

Now ${a_0}^2 - {a_2}^2 - {a_3}^2 + \ldots\ldots + {a_{2n}}^2 =$ coefficient of x in

$$(1+x+x^2)\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$$

$$= a_n$$

Hence, the correct answer is (A).

28. Compare the coefficient of a^{2n-2r} in $\left(1+x^2+x^4\right)^n = a_{n-r} = a_{n+r}$.

Hence, the correct answer is (D).

29. $$\left(\frac{r}{r+1}\right)C_r = \left(1-\frac{1}{r+1}\right)C_r$$

$$\sum_{r=1}^{n}(-1)^{r-1}\frac{r}{r+1}C_r = (C_1 - C_2 + C_3 \ldots.)$$

$$-\frac{1}{n+1}({}^{n+1}C_2 - {}^{n+1}C_3 + \ldots)$$

$$= C_0 - \frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1\right] = \frac{1}{n+1}$$

Hence, the correct answer is (D).

30. $$C_0 + C_1x + C_2x^2 + \ldots. + C_nx^n = (1+x)^n$$

Integrating

$$\int_0^1\left(C_0 + C_1x + \ldots. + C_nx^n\right)dx = \int_0^1 (1+x)^n\,dx = \frac{2^{n+1}-1}{n+1}$$

Hence, the correct answer is (C).

31. $$\sum_{r=0}^{n}{}^nC_r = \operatorname{Re}\sum_{r=0}^{n}{}^nC_r e^{i(r+1)x} = \operatorname{Re}\left(1+e^{ix}\right)^n e^{ix}$$

$$= \operatorname{Re}\left(1+\cos x+\sin x\right)^n(\cos x + i\sin x)$$

$$= \operatorname{Re}\left(2^n\cos^n {}^{x}\!/_{2}\left(\cos {}^{nx}\!/_{2} + i\sin {}^{nx}\!/_{2}\right)(\cos x + i\sin x)\right)$$

$$= 2^n\cos^n {}^{x}\!/_{2}\left[\cos\frac{nx}{2}\cos x - \sin\frac{nx}{2}\cos x\right]$$

$$= 2^n\cos^n\frac{x}{2}\cos\left(\frac{n+2}{2}\right)x$$

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

1. A → (s); B → (p); C → (r); D → (q)

(A) $$S = \sum_{0\le i<j\le n}(i+j)(C_i.C_j) = \sum_{o\le i<j\le n}(n-i+n-j)(C_i.C_j)$$

$$2S = 2n\sum_{o\le i<j\le n}\left(C_iC_j\right)$$

$$S = n\sum C_iC_j = \frac{n}{2}[2^{2n} - {}^{2n}C_n]$$

$$\left[\because \sum_{o\le i<j\le n}(C_iC_j) = \frac{1}{2}\left[2^{2n} - {}^{2n}C_n\right]\right]$$

(B) $$\sum\sum_{0\le i<j\le n}\left({}^nC_i + {}^nC_j\right) = \sum\sum_{0\le i<j\le n}{}^nC_i + \sum\sum_{0\le i<j\le n}{}^nC_j$$

$$= n.2^{n-1} + \sum_{j=1}^{n} j.\,{}^nC_j = n.2^{n-1} + n.2^{n-1} = n.2^n$$

(C) $$\sum\sum_{0\le i<j\le n} i\,{}^nC_j = \sum_{j=1}^{n}{}^nC_j\left(0+1+2+\ldots..+j-1\right)$$

$$= \sum_{j=1}^{n}{}^nC_j\frac{j(j-1)}{2} = \frac{1}{2}\sum_{j=1}^{n} j^2\,{}^nC_j - \frac{1}{2}\sum_{j=1}^{n} j\,{}^nC_j$$

$$= \frac{1}{2}(n+1)n.2^{n-2} - \frac{1}{2}n.2^{n-1} = n(n-1).2^{n-3}$$

(D) $$\sum_{r=0}^{n} r.{}^nC_r = n\sum_{r=1}^{n}{}^{n-1}C_{r-1} = n.2^{n-1}$$

2. A → (s); B → (p); C → (q); D → (r)

(A) Sum of the coefficients in the expansion of $(a^2x^2 - 2ax + 1)^{50} = (a-1)^{50} = 0 \Rightarrow a = 1$. Therefore

$$a + 2a + 3a + \ldots + 10a = 1 + 2 + 3 + \ldots. + 10 = \frac{10.11}{2} = 55$$

(B) Let $x = (3+2\sqrt{2})^6 = [x] + f$ and $g = (3-2\sqrt{2})^6$.

Then $[x] + f + g = x + g = 2$ (some integer)

Now $0 < f, g < 1$ and $[x] + f + g$ is an integer. This implies $f + g = 1$. Therefore $x(1-f) = xg = (3+2\sqrt{2})^6(3-2\sqrt{6})^6 = 1$

(C) We have $T_{r+1} = {}^nC_r(x^3)^{n-r}.\left(\frac{-1}{x^2}\right)^r = (-1)^r.{}^nC_rx^{3n-5r}$

Now $3n - 5r = 5 \Rightarrow r = \dfrac{3n-5}{5} = p$ …(1)

$3n - 5r = 10 \Rightarrow \dfrac{3n-10}{5} = q$ …(2)

Therefore from Eqs. (1) and (2)

$p - q = 1$

Hence ${}^nC_p(-1)^p + {}^nC_q(-1)^q = 0$

${}^nC_p(-1)^p + {}^nC_{p-1}(-1)^{p-1} = 0$

${}^nC_p = {}^nC_{p-1}$

$$2\left(\frac{3n-5}{5}\right) = n + 1 \Rightarrow n = 15$$

(D) We have $T_{r+1} = {}^{1024}C_r5^{(1024-r)/2}.7^{r/8}$ is an integer. Therefore r is a multiple of 8. Since $0 \le r \le 1024$, the number of multiples of 8 which lie between 0 and 1024 (both inclusive) is 129.

3. A → (s); B → (r); C → (q); D → (p)

(A) $${}^{(16+3-1)}C_{3-1} = {}^{18}C_2 = \frac{18\times 17}{2} = 153$$

(B) $$\frac{n}{2} + 1 = \frac{6}{2} + 1 = 4$$

(C) $\dfrac{100}{24} = 4 + F$ (T_1, T_{25}, T_{49}, T_{73} are rationals)

Total number of terms – rational terms = Irrational terms $\Rightarrow 101 - 4 = 97$

(D) put $x = y = z = 1 \Rightarrow \left(1+\frac{1}{3}+\frac{2}{3}\right)^{12} = 2^{12}$

4. A → (s); B → (r); C → (p); D → (q)

(A) $a_0^2 - a_1^2 + a_2^2 \ldots\ldots\ldots + a_{2n}^2 =$ coefficient x^{2n} in $(1+x+x^2)^n$ $(x^2-x+1)^n = a_n$

(B) $a_0\, a_2 - a_1\, a_3 + \ldots\ldots + a_{2n-2}\, a_{2n} =$ coefficient x^{-2} in $(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_{n-1}$

(C) $a_r = a_{2n-r} \Rightarrow a_0 + a_1 + a_2 + \ldots + a_n = \frac{1}{2}(3^n + a_n)$

(D) let $f(x) = (1+x+x^2)^n$

$$a_0 + a_2 + a_4 \ldots\ldots + a_{2n} = \frac{f(1)+f(-1)}{2}$$

$a_0 = 1$

5. A → (r, s, t); B → (s, t); C → (q, r, s, t); D → (p, q, r, s, t)

We first observe that $n = 8$

(A) $r = 5;\ m_0 = \dfrac{8+1}{1+\left|\dfrac{1}{2(1/2)}\right|} = 4.5$

T_5 is greatest

(B) ${}^8C_r\,.2^r \geq {}^8C_{r-1}\,2^{r-1} \Leftrightarrow \frac{9-r}{r} \geq \frac{1}{2} \Leftrightarrow r \leq 6$

But for $r = 6$, ${}^8C_6.2^6 = {}^8C_5.2^5$

$\therefore T_6$ and T_7 are the terms whose coefficients have greatest value (equal to 7 (2^8)).

(C) $T_{r+1} = n_{C_r}.2^r\ 3^{r/3}$ is an integer only if $r = 0$, 3 and 6. So the number of integral terms is 3

(D) $x = \frac{1}{\sqrt{2}} \Rightarrow R = (\sqrt{2}+1)^8 = 1 - (\sqrt{2}-1)^8 + (\sqrt{2}+1)^8$

$$+(\sqrt{2}-1)^8 - 1$$

$= I + f$ where $f = 1 - (\sqrt{2}-1)^8$

$\Rightarrow R(1-f) = (\sqrt{2}+1)^8(\sqrt{2}-1)^8 = 1$

6. A → (p); B → (q); C → (t); D → (s)

(A) $({}^mC_1\,{}^nC_m - {}^mC_2\,{}^{2n}C_m + {}^mC_3\,{}^{3n}C_m - \ldots(-1)^{m-1m}C_m\,{}^{mn}C_m)$

$=$ Coefficient of x^m in the expansion of

$$\left({}^mC_1(1+x)^n - {}^mC_2(1+x)^{2n} + {}^mC_3(1+x)^{3n} \ldots\right.$$
$$\left. + (-1)^{m-1}\,{}^mC_m.(1+x)^{mn}\right)$$

$=$ Coefficient of x^m in the expansion of

$$\left({}^mC_0 - \left[{}^mC_0 - {}^mC_1(1+x)^n + {}^mC_2(1+x)^{2n}\right.\right.$$
$$\left.\left. + {}^mC_3(1+x)^{3m} + \ldots + (-1)^m\,{}^mC_m(1+x)^{mn}\right]\right.$$

$=$ Coefficient of x^m in the expansion of $\left(1-\left(1-(1+x)^n\right)^m\right)$

$=$ Coefficient of x^m in the expansion of $\left(1-(1+x)^n\right)^m$

(B) ${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^mC_m$ is the coefficient of x^m in the expansion of

$$(1+x)^n + (1+x)^{n-1}(1+x)^{n-2} + \ldots + (1+x)^m$$

$$= (1+x)^m\left[1 + (1+x) + (1+x)^2 + \ldots + (1+x)^{n-m}\right]$$

$$= (1+x)^m\left(\frac{1-(1+x)^{n-m+1}}{1-(1+x)}\right) = \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

Thus the given expression is equal to the coefficient of x^m in the expansion of $\frac{(1+x)^{n+1}}{x}$

(C) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n$ – A

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n$

Multiplying eq. A and B and equating coefficients of x^n on both the sides.

Coefficient of x^n in the expansion of

$$(1+x)^n(x+x)^n = {}^nC_0\,{}^nC_n + {}^nC_1\,{}^nC_{n-1} + {}^nC_2\,{}^nC_{n-2}$$
$$+ \ldots + {}^nC_n\,{}^nC_0$$

$\therefore$ Coefficient of x^n in the expansion of $(1+x)^{2n} = {}^{2n}C_n$

(D) $2^m\,{}^nC_m =$ Coefficient of x^m in the expansion of $(1+2x)^n$

$2^{m-1}\,{}^{n-1}C_{m-1} =$ Coefficient of x^{m-1} in the expansion of $(1+2x)^{n-1}$

$=$ Coefficient of x^m in the expansion of $x(1+2x)^{n-1}$

$\therefore$ given expression = coefficient of x^m in the expansion of

$${}^nC_0(1+2x)^n - {}^nC_1x(1+2x)^{n-1} + {}^nC_2x^2(1+2x)^{n-2} - \ldots$$

Integer/Numerical Answer Type Questions

1. $1, \frac{{}^nC_1}{2}, \frac{{}^nC_2}{2^2}$ are in A.P $\Rightarrow n = 8$

$T_{r+1} = {}^8C_r\,\frac{1}{2^r}\,x^{\left(\frac{16-3r}{4}\right)}$ So, $r \in \{0, 4, 8\}$

Hence, the correct answer is (3).

2. $ab = 1$

In expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, $T_{r+1} = {}^{11}C_r\left(\frac{1}{bx}\right)^r(ax^2)^{11-r} = {}^{11}C_r\,\frac{a^{11-r}}{b^r}x^{12-3r}$

for coefficient of x^7; $r = 5$

In $\left(ax-\frac{1}{bx^2}\right)^{11}$, $T_{p+1} = {}^{11}C_p\left(-\frac{1}{bx^2}\right)^p(ax)^{11-p}$

$$T_{p+1} = {}^{11}C_p(-1)^p \frac{a^{11-p}}{b^p}.x^{11-3p}$$

for coefficient of x^{-7}, $p = 6$

Now comparing coefficient $ab = 1 \Rightarrow a = 9$

Hence, the correct answer is (9).

3. $${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^r.{}^nC_r$$

$$= {}^{n-1}C_0 - ({}^{n-1}C_0 + {}^{n-1}C_1) + ({}^{n-1}C_1 + {}^{n-1}C_2) - ({}^{n-1}C_2 + {}^{n-1}C_3) + \ldots + (-1)^r.({}^{n-1}C_{r-1} + {}^{n-1}C_r)$$

$$= (-1)^r.{}^{n-1}C_r$$

$(-1)^r.{}^{n-1}C_r = 28 \Rightarrow r$ must be even

$${}^{n-1}C_r = 28 \Rightarrow {}^{n-1}C_r = 7 \times 4 = \frac{7 \times 8}{2} = {}^8C_2 \Rightarrow n-1 = 8 \Rightarrow n = 9$$

Hence, the correct answer is (9).

4. $$\frac{a_0 + a_1 + a_3 + a_4 + \ldots\ldots\ldots\ldots\ldots}{a_2 + a_5 + a_8 + \ldots\ldots\ldots\ldots} = k$$

$$a_0 + a_1 + a_3 + a_4 + \ldots\ldots\ldots\ldots = k(a_2 + a_5 + a_8 + \ldots\ldots\ldots)$$

$$a_0 + a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots = (k+1)(a_2 + a_5 + a_8 + \ldots\ldots\ldots)$$

$$a_0 + a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots\ldots = f(1)$$

$$\sum_{i=0}^{2n} a_i = 3^n$$

$$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots a_{2n}x^{2n}$$

Multiplying with x

$$x(1 + x + x^2)^n = a_0x + a_1x^2 + a_2x^3 + \ldots\ldots\ldots\ldots$$

put $x = 1, w, w^2$ and adding

$$3^n = 3(a_2 + a_5 + a_8 + \ldots\ldots\ldots)$$

$$3^n = (k+1)3^{n-1} \Rightarrow k = 2$$

Hence, the correct answer is (2).

5. $$(1-x)^n = C_0 - C_1x + C_2x^2 - \ldots. + (-1)^n.C_nx^n$$

$$\Rightarrow x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - \ldots. + (-1)^n.C_n.x^{n+1}$$

$$\Rightarrow \int_0^1 x(1-x)^n\,dx = \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \ldots. + (-1)^n.\frac{C_n}{n+2}$$

$$\therefore f(x) = \int_0^1 x(1-x)^n\,dx = \int_0^1 (1-x)x^n dx = \int_0^1 \left(x^n - x^{n+1}\right)dx$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

$$\therefore \underset{n\to\infty}{Lt}\left(3n^2 + 4n + 2\right).f(n) = 3$$

Hence, the correct answer is (3).

6. ${}^nC_1, {}^nC_2, {}^nC_3$ are in A.P. gives

$$2\,{}^nC_2 = {}^nC_1 + {}^nC_3 \Rightarrow n = 7$$

also ${}^7C_2\, a^5b^2 = 84 \Rightarrow a^5b^2 = 4$

now A.M. ≥ G.M.

$$\frac{\log_2 a^5 + \log_2 b^2}{2} \ge \left(\log_2 a^5 \log_2 b^2\right)^{1/2}$$

so, $\left(\log_{2^{1/5}} a^2\right)\left(\log_{2^{1/2}} b^2\right) \le 4$

so, maximum value is 4.

Hence, the correct answer is (4).

7. $$\frac{1}{2}\left[(1+x)^n - (1-x)^n\right] = {}^nC_1x + {}^nC_3x^3 + {}^nC_5x^5 + \ldots$$

differentiating

$$\frac{1}{2}\left[n(1+x)^{n-1} + n(1-x)^{n-1}\right]$$

$$= {}^nC_1 + 3\cdot{}^nC_3x^2 + 5\cdot{}^nC_5x^4 + \ldots$$

multiply by x

$$\frac{n}{2}\left[x\left[(1+x)^{n-1} + (1-x)^{n-1}\right]\right]$$

$$= {}^nC_1x + 3\cdot{}^nC_3x^3 + 5\cdot{}^nC_5x^5 + \ldots$$

differentiate

$$\frac{n}{2}\left[x\left[(n-1)(1+x)^{n-2} - (n-1)(1-x)^{n-2}\right] + \left[(1+x)^{n-1} + (1-x)^{n-1}\right]\right]$$

$$= {}^nC_1 + 3^2\,{}^nC_3x^2 + 5^2\,{}^nC_5x^4 + \ldots$$

$$1^2.{}^nC_1 + 3^2.{}^nC_3 + 5^2.{}^nC_5 + \ldots\ldots = n(n+1)2^{n-3}$$

Hence $\alpha = 3$.

Hence, the correct answer is (3).

8. $$\frac{C_1}{C_0} + 2^2.\frac{C_2}{C_1} + 3^2.\frac{C_3}{C_2} + \ldots.25^2.\frac{C_{25}}{C_{24}} =$$

$$1 \times 25 + 24 \times 2 + 23 \times 3 \ldots.\, 1 \times 25 = \sum_{r=1}^{25} r(26-r)$$

$$= 2925 = 3000 - 75$$

Hence, the correct answer is (3).

9. $$29^{3487} + 31^{2005} + 19^{2005}$$

$$= (30-1)^{3487} + (30+1)^{2005} + (20-1)^{2005}$$

$$= \left(30^{3487} - {}^{3487}C_1.30^{3486} + \ldots + {}^{3487}C_1.30 - 1\right) + \left(30^{2005} + \ldots + {}^{2005}C_1.30 + 1\right) + \left(20^{2005} - {}^{2005}C_1.20^{2004} + \ldots - 1\right)$$

$$= (\text{multiple of }10) - 1 + 1 - 1 = (\text{multiple of }10 - 1)$$

$\therefore$ the digit in the unit place $= 9$.

Hence, the correct answer is (9).

10. $$T_5 + T_6 = 0 \Rightarrow {}^nC_4\,a^{n-4}b^4 - {}^nC_5\,a^{n-5}b^5 = 0$$

$$\Rightarrow a^{n-5}.\,b^4\,\{{}^nC_4.\,a - {}^nC_5\,b\} = 0$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!}\left[a - \frac{(n-4)}{5}.b\right] = 0$$

$$\Rightarrow 5a - nb + 4b = 0$$

$$\Rightarrow \frac{nb - 5a}{b} = 4$$

Hence, the correct answer is (4).

11. $(\sqrt{2}+1)^n-(\sqrt{2}-1)^n=2\left\{{}^nC_1(\sqrt{2})^{n-1}+{}^nC_3(\sqrt{2})^{n-3}\right.$

$$= \text{integer}$$

$\Rightarrow (\sqrt{2}-1)^n$ is the fractional part of $(\sqrt{2}+1)^n$

$\therefore\ [f(x)]=\left[(\sqrt{2}+1)^n\right]=(\sqrt{2}+1)^n-(\sqrt{2}-1)^n$ is satisfied for n $=5$

Hence, the correct answer is (5).

12. The sum of coefficients of terms not containing $y=3^6$

The sum of coefficients of terms not containing both x and $y=2^6$

So the required number $=3^6-2^6=665$.

Hence, the correct answer is (7).

13. We have

$$(1+ax)^n=1+{}^nC_1(ax)+{}^nC_2(ax)^2+....$$

Therefore

$$({}^nC_1)a=8\Rightarrow an=8$$

$$({}^nC_2)a^2=24\Rightarrow n(n-1)a^2=48$$

Now, $48=n(n-1)a^2=(an)(an-a)=8(8-a)$.

Therefore $8-a=6$

$a=2 \qquad a=2\Rightarrow n=4$

Hence, the correct answer is (6).

14. Coefficient of $x^{10}={}^{10}C_0+{}^{10}C_1+..{}^{10}C_2+........+{}^{10}C_{10}$

$$=2^{10}=1024$$

$$\Rightarrow\left[\frac{K}{256}\right]=4$$

Hence, the correct answer is (4).

15. Put $x=a=1$,

$$2^{20}-0=k\cdot 2^9\cdot 2^9$$

$$k=\frac{2^{20}}{2^{18}}=4$$

Hence, the correct answer is (4).

16. By hypothesis ${}^7C_5\left(\sqrt{9^{x-1}+7}\right)^2\left(\frac{1}{3^{x-1}+1}\right)=84$

Therefore $\dfrac{3^{2(x-1)}+7}{3^{x-1}+1}=4$

Substituting $y=3^{x-1}$ we get $y^2+7=4y+4$

$y^2-4y+3=0$

This gives $y=1, 3$ which implies that $3^{x-1}=1$ or $3^{x-1}=3$.

Therefore $x=1, 2$ the sum of the values $=1+2=3$

Hence, the correct answer is (3).

17. $p=\displaystyle\sum_{r=0}^{n}{}^nC_r.\left(\frac{-1}{1+nx}\right)^r+\sum_{r=0}^{n}{}^nC_r\frac{(-1)^r rx}{(1+nx)^r}$

$$=\left(1-\frac{1}{1+nx}\right)^n+n\sum_{r=1}^{n}{}^{n-1}C_{r-1}\frac{(-1)^r x}{(1+nx)^r}$$

$$=\left(\frac{nx}{1+nx}\right)^n-\left(\frac{nx}{1+nx}\right)\left(1-\frac{1}{1+nx}\right)^{n-1}=0$$

$\therefore |\alpha-\beta|=4$

Hence, the correct answer is (4).

18. Coefficient x^n in $\left\{{}^nC_o(1+x)^{4n}-{}^nC_1(1+x)^{4n-3}+\right.$

$$\left.{}^nC_2(1+x)^{4n-6}...+(-1)^n\ {}^nC_n(1+x)^n\right\}$$

$=$ Coefficient x^n in $(1+x)^n\left\{(1+x)^3-1\right\}^n\Rightarrow 3^n$

Hence, the correct answer is (2).

19. Any term of the expansion is of the form $\dfrac{6!}{a!b!c!}\left(5^{1/3}x\right)^a$ $\left(3^{1/2}y\right)^b z^c, z_c, a, b, c$ non-negative integers and $a+b+c=6$.

For rational coefficients 'a' must be multiple of 3 and b must be multiple of 2.

The following are the possibilities

Sum of coefficients $=\dfrac{6!}{6!}+\dfrac{6!}{2!4!}.3+\dfrac{6!}{4!2!}3^2+\dfrac{6!}{6!}3^3+$

$$\frac{6!}{3!3!}5+\frac{6!}{3!2!}.5.3+\frac{6!}{6!}5^2=1233.$$

Hence, the correct answer is (3).

20. ${}^{99}C_0 99^{50}-{}^{99}C_1(99-1)^{50}+{}^{99}C_2(99-2)^{50}-.....$

$$+{}^{99}C_{98}(99-98)^{50}-{}^{99}C_{99}(99-99)^{50}$$

$$=99^{50}\left[{}^{99}C_0-{}^{99}C_1+{}^{99}C_2-....+{}^{99}C_{98}-{}^{99}C_{99}\right]+$$

$$^{50}C_1 99^{49}\left[{}^{99}C_1-2.{}^{99}C_2+3.{}^{99}C_3-...\right]+....=0$$

Hence, the correct answer is (0).

21. Expression $=x.x^2.x^3......x^{20}\left(1-\dfrac{1}{x}\right)\left(1-\dfrac{2}{x^2}\right)\left(1-\dfrac{3}{x^3}\right)$

$$.......\left(1-\frac{20}{x^{20}}\right)=x^{210}.\text{E}$$

where

$$\text{E}=\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)\left(1-\frac{3}{x^3}\right).......\left(1-\frac{20}{x^{20}}\right)=x^{203}$$

in original expression $=$ coefficient of x^{-7} in E

But $\text{E}=1-\left(\dfrac{1}{x}+\dfrac{2}{x^2}+\dfrac{3}{x^3}+.....\right)$

$$+\left(\frac{1}{x}.\frac{6}{x^6}+\frac{2}{x^2}.\frac{5}{x^5}+\frac{3}{x^3}.\frac{4}{x^4}+....\right)$$

Coeff of $x^{-7}=7+6+10+12-8=13$

Hence, the correct answer is (1).

22. $53^{53}-33^3=\underset{\substack{\text{divisible by 50}\\\text{then also it is also}\\\text{divisible by 10}}}{\left(53^{53}-3^{53}\right)}+3^{53}-\left[\underbrace{33^3-3^3}_{\substack{\text{divisible by 30}\\\text{then it is also}\\\text{divisible by 10}}}+3^3\right]$

$$=3^{53}-3^3$$

$$=3^3\left(3^{50}-1\right)=3^3\left\{(10-1)^{25}-1\right\}=3^3\{10\ \ \lambda-2\}$$

$= 270\lambda - 54 = 270\lambda - 60 + 6$

$= 10\mu + 6$ when λ, μ are integers

Remainder 6

Hence, the correct answer is (6).

23. Coefficient of x^{20} in

$$1.2.2^2.2^3....2^{20}(1+x)(x+\frac{1}{2})(x+\frac{1}{2^2})(x+\frac{1}{2^3}).....(x+\frac{1}{2^{20}})$$

$$= 2^{\frac{20\times21}{2}}(x+1)(x+\frac{1}{2})(x+\frac{1}{2^2})......(x+\frac{1}{2^2})$$

$$= 2^{210}(1+\frac{1}{2}+\frac{1}{2^2}+.......+\frac{1}{2^{20}})$$

$$= 2^{210}\left(\frac{1-\frac{1}{2^{21}}}{1-\frac{1}{2}}\right) = 2^{211} - 2^{190} = 2^{190}(2^{21}-1)$$

Hence, the correct answer is (4).

24. $p(x) = \dfrac{1-x^{18}}{1+x}$

coefficient of y^2 in $\left(\dfrac{1-(y-1)^{18}}{y}\right)$ is coefficient of

y^3 in $-(1-y)^{18} = {}^{18}C_3 = k = 816$

$\Rightarrow \left[\dfrac{k}{400}\right] = 2.$

Hence, the correct answer is (2).

25. $N = 1! + 2! + 3! + ...2011!$

The unit digit of N is 3 and the unit digit of 3^{2011} is unit digit of $3^{4(502)+3}$ which is units digit of 3^3 i.e. 7

Hence, the correct answer is (7).

26. $$E = \frac{1}{{}^{57}C_4}\left({}^{47}C_4 + \sum_{J=0}^{3} {}^{50-J}C_3 + \sum_{K=0}^{5} {}^{56-K}C_{53-K}\right)$$

$$E = \frac{1}{{}^{57}C_4}\Big[{}^{47}C_4 + \left({}^{47}C_3 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3\right)$$

$$+ \left({}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3 + {}^{56}C_3\right)\Big]$$

$$= \frac{{}^{57}C_4}{{}^{57}C_4}$$

Hence, the correct answer is (1).

27. $R_n f_n = \left(5\sqrt{5}+11\right)^{2n+1}\left(5\sqrt{5}-11\right)^{2n+1} = 4^{2n+1}$

$R_3 f_3 = 2^{14}$

Hence, the correct answer is (1).

28. $$A.M = \frac{1}{13}\left[\frac{{}^{25}C_0}{1} + \frac{{}^{25}C_2}{3} + \frac{{}^{25}C_4}{5} + ... + \frac{{}^{25}C_{24}}{25}\right]$$

$$= \frac{1}{13\times26}\left[{}^{26}C_1 + {}^{26}C_3 + ... {}^{26}C_{25}\right]$$

$$= \frac{1}{13\times26}\left[2^{25}\right]$$

$n = 26 \Rightarrow \text{sum} = 8$

Hence, the correct answer is (8).

29. Highest power of $x = 1 + 2 + 3 + \_\_\_\_\_ + n = n\dfrac{(n+1)}{2} = \alpha$ we have to find coefficient of $x^{\alpha-7}$

$\Rightarrow$ either we should leave

$(x^7-7), (x-1)(x^6-6), (x^2-2)(x^5-5), (x^3-3)(x^4-4),$

$(x-1)(x^2-2)(x^4-4)$

If we leave $(x^7-7) \Rightarrow$ coefficient $= -7$

If we leave $(x-1)(x^6-6)$ coefficient $= 6$

If we leave $(x^2-2)(x^5-5) \Rightarrow$ coefficient $= 10$

If we leave $(x^3-3)(x^4-4) \Rightarrow$ coefficient $= 12$

If we leave $(x-1)(x^2-2)(x^4-4) \Rightarrow$ coefficient $= -8$

$\Rightarrow$ Required co-efficient $= -7 + 6 + 10 + 12 - 8 = 13$

Hence, the correct answer is (3).

30. Given $\dfrac{1}{6} = \dfrac{T_7}{T_{n-5}} \Rightarrow 6.{}^nC_6(2^{1/3})^{n-6}(3^{-1/3})^6$

$$= {}^nC_{n-6}(2^{1/3})^6.(3^{-1/3})^{n-6}$$

$$\Rightarrow \frac{2^{\frac{n-3}{3}}}{3} = 4.3^{\frac{-(n-6)}{3}} \Rightarrow \frac{n-3}{3} = 2 \text{ and } \frac{-(n-6)}{3} = -1 \Rightarrow n = 9$$

Hence, the correct answer is (9).

31. We know that

$(a-1)^n = {}^nC_0 a^n - {}^nC_1 a^n - 1 + {}^nC_2 a^{n-2} - ...... + (-1)^{n-1} nC_{n-1}$
$a + (-1)^n{}^nC_n$

$$\therefore \frac{(a-1)^n}{a} = {}^nC_0 a^{n-1} - {}^nC_1 a^{n-2} + {}^nC_2 a^{n-3} - .... + (-1)^{n-1}\,{}^nC_{n-1}$$

$$+ \frac{(-1)^n}{a}\,{}^nC_n$$

Hence $f(n) = \dfrac{(a-1)^n - (-1)^n}{a}$

now, $f(2007) + f(2008) = \dfrac{(a-1)^{2007}+1}{a} + \dfrac{(a-1)^{2008}-1}{a}$

$$= \frac{(a-1)^{2007}(1+a-1)}{a} = (a-1)^{2007}$$

$$= \left(3^{\frac{1}{223}}\right)^{2007} = 3^9 = 3^2.3^7 = k = 7$$

Hence, the correct answer is (7).

32. $32^{33} = 2^{165} = 2\times16^{41} = 2\times(17-1)^{41} = 2\times(17k-1)$

$= 34k - 34 + 32$

So the remainder is 32.

Hence, the correct answer is (9).

33. $(2109)^{360} - (1396)^{333}$

$= (301\times7+2)^{360} - (200\times7-4)^{333}$

Remainder of given number is same as remainder of

$$2^{360}+4^{333}=(64)^{60}+(64)^{111}$$
$$=(1+63)^{60}+(1+63)^{111}$$

$\therefore$ Remainder is 2.

Hence, the correct answer is (2).

34. We have

$$\frac{{}^nC_k}{{}^nC_{k-1}}=\frac{n-k+1}{k}$$

$$\therefore \sum_{k=1}^{n}k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2$$

$$=\sum_{k=1}^{n}k^3\left(\frac{n-k+1}{k}\right)^2=\sum_{k=1}^{n}k\{(n+1)-k\}^2$$

$$=\sum_{k=1}^{n}\left[k(n+1)^2-2(n+1)k^2+k^3\right]$$

$$=(n+1)^2\left(\sum_{k=1}^{n}k\right)-2(n+1)\left(\sum_{k=1}^{n}k^2\right)+\left(\sum_{k=1}^{n}k^3\right)$$

$$=(n+1)^2\left\{\frac{n(n+1)}{2}\right\}-2(n+1)\frac{n(n+1)(2n+1)}{6}+\left\{\frac{n(n+1)}{2}\right\}^2$$

$$=\frac{n(n+1)^2}{2}\left\{(n+1)-\frac{2}{3}(2n+1)+\frac{n}{2}\right\} \qquad \text{..... (1)}$$

$$=\frac{n(n+1)^2}{2}\times\frac{6(n+1)-4(2n+1)+3n}{6}$$

$$=\frac{n(n+1)^2(n+2)}{12}$$

Hence, the correct answer is (4).

35. We have

$$aC_0+(a+b)C_1+(a+2b)C_2+....+(a+nb)C_n$$

$$\sum_{r=0}^{n}(a+rb)\,{}^nC_r=a\left(\sum_{r=0}^{n}{}^nC_r\right)+b\left(\sum_{r=0}^{n}r\,{}^nC_r\right)$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+b\left(\sum_{r=1}^{n}r\,\frac{n}{r}\,{}^{n-1}C_{r-1}\right)$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+bn\left(\sum_{r=1}^{n}{}^{n-1}C_{r-1}\right)$$

$$=a,\,2^n+bn\,2^{n-1}\qquad\left[\because\sum_{r=0}^{n}{}^nC_r=2^n,\sum_{r=1}^{n}{}^{n-1}C_{r-1}=2^{n-1}\right]$$

$$=(2a+bn)2^{n-1}$$

Hence, the correct answer is (6).

36. $${}^nC_1\,x(1-x)^{n-1}+2.\,{}^nC_2x^2(1-x)^{n-2}$$
$$+3.{}^nC_3x^3(1-x)^{n-3}+.....+n.\,{}^nC_nx^n$$

$$=\sum_{r=1}^{n}r.\,\frac{n}{r}\,{}^{n-1}C_{r-1}\,x^r(1-x)^{n-r}$$

$$=n\sum_{r=1}^{n}{}^{n-1}C_{r-1}x.\,x^{r-1}(1-x)^{(n-1)-(r-1)}$$

$$=nx\sum_{r=1}^{n}{}^{n-1}C_{r-1}x^{r-1}(1-x)^{(n-1)-(r-1)}$$

$$=nx\left[x+(1-x)\right]^{n-1}=nx$$

Here $n=8$

Hence required expression becomes $8x$

$$a+b=8$$

Hence, the correct answer is (8).

37. We have

$$\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{{}^{r+2}C_r}=\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!r!}\times\frac{2!r!}{(r+2)!}$$

$$=2\sum_{r=0}^{n}(-1)^r\frac{n!}{(n-r)!(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^r\frac{(n+2!)}{\{(n+2)-(r+2)\}!(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^{r+2}\,{}^{n+2}C_{r+2}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{s=2}^{n+2}(-1)^s\,{}^{n+2}C_s$$

$$=\frac{2}{(n+1)(n+2)}\left[\left(\sum_{s=0}^{n+2}(-1)^s\,{}^{n+2}C_s\right)-\left({}^{n+2}C_0-{}^{n+2}C_1\right)\right]$$

$$=\frac{2}{(n+1)(n+2)}\left[0-\{1-(n+2)\}\right]=\frac{2}{n+2}$$

Hence, the correct answer is (2).

38. We have

$$s=a+(a+d)+(a+2d)+......+(a+nd)$$

$$\Rightarrow s=\frac{n+1}{2}(2a+nd)$$

Now,

$$A=a+(a+d)\,{}^nC_1+(a+2d)\,{}^nC_2+.....+(a+nd)^n\,C_n$$

$$=\sum_{r=0}^{n}(a+rd)\,{}^nC_r$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+d\left(\sum_{r=0}^{n}r.\,{}^nC_r\right)$$

$$=a.2^n+d.n\,2^{n-1}\qquad\left[\because\sum_{R=0}^{N}{}^rC_r=2^n,\sum_{r=0}^{n}r\,{}^nC_r=n\,2^{n-1}\right]$$

$$=(2a+nd)2^{n-1}$$

$$=\left\{\frac{n+1}{2}(2a+nd)\right\}\left\{\frac{2}{n+1}\times2^{n-1}\right\}=\frac{s.2^n}{n+1}$$

$\therefore (n+1)\mathrm{A} = 2^n \mathrm{S}$

Hence, the correct answer is (2).

39. $$\sum_{r=0}^{200} \alpha_r x^r = \sum_{r=0}^{200} \beta_r (1+x)^r$$

$$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots\ldots + \alpha_{200} x^{200}$$

$$= \beta_0 + \beta_1 (1+x) + \ldots\ldots + \beta_{200} (1+x)^{200}$$

Equating coefficient of x^{100}, we get $\alpha_{100} = {}^{100}C_{100} + {}^{101}C_{100} + \ldots\ldots\ {}^{200}C_{100} = {}^{201}C_{101}$

Similarly we can't find $\alpha_{100} \ldots\ldots \alpha_{200}$

$$\sum_{r=100}^{200} \alpha_r = {}^{201}C_{101} + {}^{201}C_{102} + \ldots\ldots\ {}^{201}C_{201}$$

$$A = 2^{200} = (2^4)^{50} = (15+1)^{50} = 15K + 1$$

When A is divided by 15 remainder is 1.

Hence, the correct answer is (1).

40. Highest exponent of $x = 1 + 2 + 3 + \ldots.. + 100$

$$= \frac{100(101)}{2} = 5050$$

Hence, the correct answer is (5).

41. Coefficient of x^{103} in $(1 + x + x^2 + x^3 + x^4)^{199} (x-1)^{201}$

Coefficient of x^{103} in $-(1-x)^2 (1-x^5)^{199}$

Coefficient of x^{103} in $-(1 - 2x + x^2)\,(1 - {}^{199}C_1\, x^5 + {}^{199}C_2\, x^{10} - {}^{199}C_3\, x^{15} + \ldots..)$

Coefficient of $x^{103} = 0$

Hence, the correct answer is (0).

42. Series $$= \sum_{r=0}^{2007} (-1)^r (5r+3)\ {}^{2007}C_r$$

$$= \sum_{r=0}^{2007} (-1)^r 5 \cdot r \cdot \frac{2007}{r}\ {}^{2006}C_{r-1} + \sum_{r=0}^{2007} (-1)^r 3 \cdot {}^{2007}C_r$$

$$= -5(2007) \sum_{r=0}^{2007} (-1)^{r-1} \cdot {}^{2006}C_{r-1} + 3 \sum_{r=0}^{2007} (-1)^r\ {}^{2007}C_r = 0 + 0 = 0$$

Hence, the correct answer is (0).

43. If $$\left(1 - x^3\right)^n = \sum_{r=0}^{n} a_r x^r (1-x)^{3n-2r}$$

$$\Rightarrow \left[(1-x)\left(1+x+x^2\right)\right]^n = (1-x)^n \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^n$$

$$= \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^{n-r}$$

$$\Rightarrow \sum_{r=0}^{n} {}^nC_r (3x)^r \left[(1-x)^2\right]^{n-r} = \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^{n-r}$$

Comparing the coefficients of like power of x on both sides, we get $a_r = {}^nC_r \cdot 3^r$.

$\therefore \quad a_1 = 3.\ {}^nC_1,\ a_2 = 9.\ {}^nC_2$ and $a_3 = 27.\ {}^nC_3$

$\therefore \quad a_1, a_2, a_3$ are in A.G.P. if ${}^nC_1,\ {}^nC_2,\ {}^nC_3$ are in A.P.

$\therefore \quad n = 7$

Hence, the correct answer is (7).

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1. Let $(1 + x + x^2 + \ldots + x^{2n})\,(1 - x + x^2 - x^3 + \ldots + x^{2n})$
$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_{4n} x^{4n}$
So,

$$a_0 + a_1 + a_2 + \ldots. + a_{4n} = 2n + 1 \qquad \ldots(1)$$

$$a_0 - a_1 + a_2 - a_2 - a_3 \ldots + a_{4n} = 2n + 1 \qquad \ldots(2)$$

$$\Rightarrow \quad a_0 + a_2 + a_4 + \ldots. + a_{4n} = 2n + 1$$

$$\Rightarrow \quad 2n + 1 = 61 \Rightarrow n = 30$$

Hence, the correct answer is (30).

2. $(1 + x + x^2)^{10} = {}^{10}C_0 + {}^{10}C_1 x(1+x) + {}^{10}C_2 x^2 (1+x)^2 + {}^{10}C_3 x^3 (1+x)^3 + {}^{10}C_4 x^4 (1+x)^4 + \ldots$

Coeff. of $x^4 = {}^{10}C_2 + {}^{10}C_3 \times {}^3C_1 + {}^{10}C_4 = 615$.

Hence, the correct answer is (615.00).

3. $S = 1.{}^{25}C_0 + 5.{}^{25}C_1 + 9.C_2 + \ldots + (101){}^{25}C_{25}$

$S = 101\ {}^{25}C_{25} + 97\ {}^{25}C_{24} + \ldots + 1\ {}^{25}C_0$

$2S = (102)\,(2^{25})$

$S = 51\,(2^{25})$

Alternate:

$$S = \sum_{r=0}^{25} (4r+1)\ {}^{25}C_r$$

$$= 100 \sum_{r=1}^{25} {}^{24}C_{r-1} + \sum_{r=0}^{25} {}^{25}C_r = 100.2^{24} + 2^{25}$$

$S = 102.2^{24} = 51.2^{25}$

Hence, the correct answer is (51).

4. ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 2 : 5 : 12$

Now, $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-r+1} = \frac{2}{5} \Rightarrow 7r = 2n + 2 \qquad \ldots(1)$$

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r} = \frac{5}{12} \Rightarrow 17r = 5n - 12 \qquad \ldots(2)$$

On solving (1) and (2) we get $n = 118$

Hence, the correct answer is (118).

5. Given $(2x^2+3x+4)^{10}=\sum_{r=0}^{20} a_r x^r$ …(1)

Replace x by $\frac{2}{x}$ in above identity:

$$\frac{2^{10}(2x^2+3x+4)^{10}}{x^{20}}=\sum_{r=0}^{20}\frac{a_r 2^r}{x^r}$$

$$\Rightarrow \quad 2^{10}\sum_{r=0}^{20} a_r x^r=\sum_{r=0}^{20} a_r 2^r\, x^{(20-r)} \qquad \text{(from (1))}$$

now, comparing coefficient of x^7 from both sides (take $r=7$ in L.H.S. and $r=13$ in R.H.S.)

$$2^{10}\, a_7=a_{13}\, 2^{13}\Rightarrow \frac{a_7}{a_{13}}=2^3=8$$

Hence, the correct answer is (8).

6. $(1+x+x^2+x^3)^6=\left((1+x)(1+x^2)\right)^6=(1+x)^6\,(1+x^2)^6$

$$=\sum_{r=0}^{6}{}^6C_r x^r\sum_{r=0}^{6}{}^6C_t x^{2t}=\sum_{r=0}^{6}\sum_{t=0}^{6}{}^6C_r\,{}^6C_t x^{r+2t}$$

For coefficient of x^4 then $r+2t=4$

r	T
0	2
2	1
4	0

Coefficient of $x^4={}^6C_0\,{}^6C_2+{}^6C_2\,{}^6C_1+{}^6C_4\,{}^6C_0=120$

Hence, the correct answer is (120.00).

7. $T_{r+1}={}^{22}C_r(x^m)^{22-r}\left(\frac{1}{x^2}\right)^r={}^{22}C_r x^{22m-mr-2r}={}^{22}C_r\, x^1$

$\because$ ${}^{22}C_3={}^{22}C_{19}=1540$ we get

$\therefore$ $r=3$ or 19

$$22m-mr-2r=1\Rightarrow m=\frac{2r+1}{22-r}$$

$$r=3,\ m=\frac{7}{19}\notin N \text{ rejected}$$

$$r=19,\ m=\frac{38+1}{22-19}=\frac{39}{3}=13\Rightarrow m=13 \text{ accepted}$$

Hence, the correct answer is (13).

8. ${}^{10}C_7+{}^9C_6+{}^8C_5+\ldots+{}^4C_1+{}^3C_0$

$$=\underbrace{{}^4C_0+{}^4C_1}_{{}^5C_1}+{}^5C_2+\ldots+{}^{10}C_7={}^{11}C_7=330$$

Hence, the correct answer is (B).

9. $a={}^{19}C_{10},\ b={}^{20}C_{10}$ and $c={}^{21}C_{10}$

$\Rightarrow a={}^{19}C_9,\ b={}^{20}C_{10}$ and $c={}^{21}C_{10}$

$\Rightarrow b=2a$ and $c=\frac{21}{11}b=\frac{42a}{11}$

$\Rightarrow a:b:c=a:2a:\frac{42a}{11}=11:22:42$

Hence, the correct answer is (C).

10. $2[{}^6C_0.x^6+{}^6C_2x^4(x^2-1)+{}^6C_4x^2(x^2-1)^2+{}^6C_6(x^2-1)^3]$

$\alpha=2(-{}^6C_2-2\,{}^6C_4-3)=-96$ and $\beta=2({}^6C_4+{}^6C_6\,3\,)=36$

$\therefore$ $\alpha-\beta=-132$

Hence, the correct answer is (C).

11. $T_{r+1}={}^{16}C_r\left(\frac{x}{\cos\theta}\right)^{16-r}\left(\frac{1}{x\sin\theta}\right)^r$

$$={}^{16}C_r(x)^{16-2r}\frac{1}{(\cos\theta)^{16-r}(\sin\theta)^r}$$

For independent of x : $16-2r=0\Rightarrow r=8$

$$\Rightarrow \quad T_9={}^{16}C_8\frac{1}{\cos^8\theta\sin^8\theta}={}^{16}C_8\frac{2^8}{(\sin 2\theta)^8}$$

For $\theta\in\left[\frac{\pi}{8},\frac{\pi}{4}\right]$ ℓ_1 is least for $\theta_1=\frac{\pi}{4}$

For $\theta\in\left[\frac{\pi}{16},\frac{\pi}{8}\right]$ ℓ_2 is least for $\theta_2=\frac{\pi}{8}$

$$\frac{\ell_2}{\ell_1}=\frac{(\sin 2\theta_1)^8}{(\sin 2\theta_2)^8}=(\sqrt{2}\,)^8=\frac{16}{1}$$

Hence, the correct answer is (C).

12. Let t_{r+1} denotes $r+1^{\text{th}}$ term of $\left(\alpha x^{\frac{1}{9}}+\beta x^{-\frac{1}{6}}\right)^{10}$

$$\mathrm{t_{r+1}}={}^{10}C_r\alpha^{10-r}(x)^{\frac{10-r}{9}}\ \beta^r x^{-\frac{r}{6}}={}^{10}C_r\alpha^{10-r}\ \beta^r(x)^{\frac{10-r}{9}-\frac{r}{6}}$$

If t_{r+1} is independent of x

$$\frac{10-r}{9}-\frac{r}{6}=0\Rightarrow r=4$$

Maximum value of t_5 is $10K$ (given) $\Rightarrow {}^{10}C_4\,\alpha^6\beta^4$ is maximum

By AM $\geq$ GM (for positive numbers)

$$\frac{\frac{\alpha^3}{2}+\frac{\alpha^3}{2}+\frac{\beta^2}{2}+\frac{\beta^2}{2}}{4}\geq\left(\frac{\alpha^6\beta^4}{16}\right)^{\frac{1}{4}}\Rightarrow \alpha^6\beta^4\leq 16$$

So, $10K={}^{10}C_4 16\Rightarrow K=336$

Hence, the correct answer is (B).

13. $\mathrm{T}_{r+1}={}^nC_r(3)^{\frac{n-r}{2}}(5)^{\frac{r}{8}} \qquad (n\geq r)$

Clearly r should be a multiple of 8.

$\because$ there are exactly 33 integral terms

Possible values of r can be

$0, 8, 16, \ldots\ldots, 32\times 8$

$\therefore$ least value of $n=256$.

Hence, the correct answer is (B).

14. Let $n+5=N$

$${}^NC_{r-1}:{}^NC_r:{}^NC_{r+1}=5:10:14$$

$$\frac{{}^NC_r}{{}^NC_{r-1}}=\frac{N+1-r}{r}=\frac{10}{5}=2$$

$$\frac{N_{C_{r-1}}}{N_{C_r}} = \frac{N-r}{r+1} = \frac{7}{5} \Rightarrow r = 4, N = 11$$

Largest coefficient in expansion of $(1 + x)^{11} = {}^{11}C_6 = 462$

Hence, the correct answer is (C).

15. $T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$

For independent of x, $18 - 3r = 0 \Rightarrow r = 6$

$\therefore \quad T_7 = {}^9C_6\left(\frac{3}{2}\right)^3\left(-\frac{1}{3}\right)^6 = \frac{9.8.7}{6}\frac{1}{8.27} = \frac{7}{18} = k$

$\therefore \quad 18k = 7$

Hence, the correct answer is (D).

16. $\sum\limits_{r=0}^{20} {}^{50-r}C_6 = {}^{50}C_6 + {}^{49}C_6 + {}^{48}C_6 + \ldots. + {}^{30}C_6$

$= {}^{50}C_6 + {}^{49}C_6 + \ldots. + {}^{31}C_6 + \left({}^{30}C_6 + {}^{30}C_7\right) - {}^{30}C_7$

using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$= {}^{50}C_6 + {}^{49}C_6 + \ldots. + \left({}^{31}C_6 + {}^{31}C_7\right) - {}^{30}C_7$

$= {}^{50}C_6 + {}^{50}C_7 - {}^{30}C_7 = {}^{51}C_7 - {}^{30}C_7$

Hence, the correct answer is (B).

17. $T_{r+1} = {}^{10}C_r\left(\sqrt{x}\right)^{10-r}\left(\frac{-k}{x^2}\right)^r$

$T_{r+1} = {}^{10}C_r\, x^{\frac{10-r}{2}}(-k)^r x^{-2r} = {}^{10}C_r\, x^{\frac{10-5r}{2}}(-k)^r$

Constant term: $\frac{10-5r}{2} = 0 \Rightarrow r = 2$

$T_3 = {}^{10}C_2.(-k)^2 = 405$

$k^2 = \frac{405}{45} = 9 \Rightarrow k = \pm 3 \Rightarrow |k| = 3$

Hence, the correct answer is (C).

18. $\left\{\frac{3^{200}}{8}\right\} = \left\{\frac{(3^2)^{100}}{8}\right\} = \left\{\frac{(1+8)^{100}}{8}\right\}$

$= \left\{\frac{1 + {}^{100}C_1.8 + {}^{100}C_2.8^2 + \ldots. + {}^{100}C_{100}8^{100}}{8}\right\}$

$= \left\{\frac{1+8m}{8}\right\} = \frac{1}{8}$

Hence, the correct answer is (A).

19. $6 \times {}^{35}C_r = (k^2 - 3)\, {}^{36}C_{r+1}$

$k^2 - 3 > 0 \Rightarrow k^2 > 3$

$k^2 - 3 = \frac{6x\,{}^{35}C_r}{{}^{36}C_{r+1}} = \frac{r+1}{6}$

Possible values of r for integral values of k, are $r = 5, 35$

Number of ordered pairs are 4 (5, 3), (5, –2), (35, 3), (35, –3)

Hence, the correct answer is (A).

20. $\frac{2^{403}}{15} = 2^3.\frac{2^{400}}{15} = \frac{2^3.(2^4)^{100}}{15}$

$= \frac{8[1+15]^{100}}{15} = \frac{8[1+15\lambda]}{15} = \frac{8}{15} + 8\lambda$

the fractional part of the number $\frac{2^{403}}{15} = \left\{\frac{8}{15}\right\} = \frac{8}{15}$

$\therefore \quad k = 8$

Hence, the correct answer is (A).

21. $(1 - t^6)^3 (1 - t)^{-3} = (1 - t^{18} - 3t^6 + 3t^{12}) (1 - t)^{-3}$

$\Rightarrow$ coefficient of t^4 in $(1 - t)^{-3}$ is ${}^{3+4-1}C_4 = {}^6C_2 = 15$

Hence, the correct answer is (B).

22. $\left(x + \sqrt{x^3 - 1}\right)^6 + \left(x - \sqrt{x^3 - 1}\right)^6$

$= 2[{}^6C_0x^6 + {}^6C_2x^4(x^3 - 1) + {}^6C_4x^2 (x^3 - 1)^2 + {}^6C_6(x^3 - 1)^3]$

$= 2[{}^6C_0x^6 + {}^6C_2x^7 - {}^6C_2x^4 + {}^6C_4x^8 + {}^6C_4x^8 + {}^6C_4x^2 - 2{}^6C_4x^5 + (x^9 - 1 - 3x^6 + 3x^3)]$

Sum of coefficient of even powers of $x = 2[1 - 15 + 15 + 15 - 1 - 3] = 24$

Hence, the correct answer is (B).

23. $(1 + x) (1 - x)^{10} (1 + x + x^2)^9 = (1 - x^2) (1 - x^3)^9$

The coefficient of x^{18} in the expansion of $(1 - x^3)^9 - x^2(1 - x^3)^9$ $= {}^9C_6 - 0 = 84$

Hence, the correct answer is (A).

24. $(1 + x)^{20} = {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2x^2 + \ldots + {}^{20}C_{20}x^{20}$ …(1)

Differential equation w.r.t. x

$20(1 + x)^{19} = {}^{20}C_1.1 + 2.{}^{20}C_2x + \ldots + 20\,{}^{20}C_{20}x^{19}$ …(2)

Multiply equation (2) by x

$20x(1 + x)^{19} = {}^{20}C_1x + 2.{}^{20}C_2x^2 + \ldots + 20\,{}^{20}C_{20}x^{20}$ …(3)

Differential equation (3) w.r.t. x

$20[(1 + x)^{19} + 19x(1 + x)^{18}]$

$= 1.{}^{20}C_1 + 2^2.{}^{20}C_2x + \ldots (20^2)\, C_{20}x^{19}$ …(4)

Put $x = 1$ in equation (4)

$20(2^{19} + 19.2^{18}) = 1^2\,{}^{20}C_1 + 2^2\,{}^{20}C_2 + \ldots + (20^2)\,{}^{20}C_{20}$

$= 420 \times 2^{18}$

$A = 420$, $\beta = 18$ Hence correct option is (A).

Hence, the correct answer is (A).

25. $(10 + x)^{50} + (10 - x)^{50} \Rightarrow a_2 = 2,\ {}^{50}C_2\, 10^{48}$ and $a_0 = 2.\ 10^{50}$

$$\frac{a_2}{a_0} = \frac{{}^{50}C_2}{10^2} = 12.25$$

Hence, the correct answer is (A).

26. ${}^{101}C_1 + {}^{101}C_2\, (1 + q) + {}^{101}C_3\, (1 + q + q^2) + \ldots + {}^{101}C_{101}\, (1 + q + \ldots. + q^{100}) = 2\alpha\frac{\left(1 - \left(\frac{1+q}{2}\right)^{101}\right)}{1-q}$

$\Rightarrow {}^{101}C_1\, (1 - q) + {}^{101}C_2\, (1 - q^2) + \ldots + {}^{101}C_{101}\, (1 - q^{101})$

$= 2\alpha\left(1 - \left(\frac{1+q}{2}\right)^{101}\right)$

$\Rightarrow (2^{101} - 1) - ((1 + q)^{101} - 1) = 2\alpha\left(1 - \left(\frac{1+q}{2}\right)^{101}\right)$

$$\Rightarrow 2^{101}\left(1-\left(\frac{1+q}{2}\right)^{101}\right)=2\alpha\left(1-\left(\frac{1+q}{2}\right)^{101}\right)$$

$$2\alpha=2^{101}\Rightarrow\alpha=2^{100}$$

Hence, the correct answer is (C).

27. Given: sum = coefficient of x^r in expansion of $(1+x)^{20}(1+x)^{20}$ which is equal to ${}^{40}C_r$. it is maximum when $r = 20$

Hence, the correct answer is (A).

28. $n = 8$

middle term $=\left(\frac{n}{2}+1\right)^{th}$ $T_5 = {}^{8}C_4\left(\frac{x^3}{3}\right)^4\left(\frac{3}{x}\right)^4 = 5670$

$x=\pm\sqrt{3}$ Sum $= 0$

Hence, the correct answer is (A).

29. $2{}^nC_5 = {}^nC_4 + {}^nC_6$

$$\Rightarrow\frac{2\times n!}{(n-5)!5!}=\frac{n!}{4!(n-4)!}+\frac{n!}{6!(n-6)!}$$

$$\Rightarrow\frac{2}{5(n-5)}=\frac{1}{(n-4)(n-5)}+\frac{1}{30}$$

$$\Rightarrow n^2-21n+98=0$$

$$\Rightarrow n = 14 \text{ and } n = 7$$

Hence, the correct answer is (D).

30. General term $T_{r+1} = {}^{60}C_r . 7^{\frac{60-r}{5}} . 3^{\frac{r}{10}}$

For rational term

$r = 0, 10, 20, 2, 30, 40, 50, 60$

Number of rational terms = 7

Number of irrational terms = 54

Hence, the correct answer is (A).

31. $$\frac{(T_5)_{\text{start}}}{(T_5)_{\text{end}}}=\frac{{}^{10}C_4(2^{1/3})^6\left(\dfrac{1}{2.3^{1/3}}\right)^4}{{}^{10}C_6(2^{1/3})^4\left(\dfrac{1}{2.3^{1/3}}\right)^6}=(36)^{1/3}\ .\ 4:1$$

Hence, the correct answer is (D).

32. $$\sum_{r=0}^{25}{}^{50}C_r.{}^{50-r}C_{25-r}$$

$$=\sum_{r=0}^{25}\frac{50!}{r!(50-r)!}\times\frac{(50-r)!}{25!(25-r)!}$$

$$={}^{50}C_{25}\sum_{r=0}^{25}{}^{25}C_r = 2^{25}.{}^{50}C_{25}$$

$\therefore$ K $= 2^{25}$

Hence, the correct answer is (C).

33. $$\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{21}C_i}\right)^3=\sum_{i=1}^{20}\left(\frac{i}{21}\right)^3=\frac{1}{(21)^3}\left(\frac{20.21}{2}\right)^2$$

$$\frac{100}{21}=\frac{k}{21}\therefore k=100$$

Hence, the correct answer is (A).

34. $T_3 = {}^5C_2\left(x^{\log_2 x}\right)^2 = 2560$

$x^{\log_2 x} = 16 \Rightarrow \log_2 x\ \log_2 x = \log_2 16 = 4$

$\Rightarrow \log_2 x = \pm 2 \Rightarrow x = 2^2 = 4$ or $x = 2^{-2} = \frac{1}{4}$

Hence, the correct answer is (D).

35. T_{r+1} in $\left(2x^2-\frac{3}{x^2}\right)^6$ will be $T_{r+1} = {}^6C_r(2x^2)^{6-r}\left(-\frac{3}{x^2}\right)^r$

$$= {}^6C_r\,2^{6-r}\,(-1)^r\times 3^r\times x^{12-2r-2r}$$

Term independent of x will be

$\frac{1}{60}\times$ term independent of x in $\left(2x^2-\frac{3}{x^2}\right)^6 - \frac{1}{81}\times$ term of x^{-8} in $\left(2x^2-\frac{3}{x^2}\right)^6$

Case-I:

For term independent of x is $12 - 4r = 0 \Rightarrow r = 3$

$$T_4 = -{}^6C_3\times 2^3\times 3^3 = -20\times 2^3\times 3^3$$

Case-II:

For term of x^{-8} $12 - 4r = -8 \Rightarrow 4r = 20 \Rightarrow r = 5$

$$T_6 = -{}^6C_5.\ 2^1.\ (-1).\ 3^5.\ x^{-8}$$

Required ans. $=\frac{1}{60}\times(-20)2^3\times 3^3-\frac{1}{81}\times 6\times 2\times(-1)\times 3^5$

$$= -72 + 36 = -36$$

Hence, the correct answer is (A).

36. $T_{r+1} = {}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{\lambda}{x^2}\right)^r = {}^{10}C_r\lambda^r\,x^{5-\frac{r}{2}-2r}$

$\Rightarrow 5-\frac{r}{2}-2r=0$ $\Rightarrow r = 2$

Coefficient of $x^2 \Rightarrow {}^{10}C_2\,\lambda^2 = 720$ $\Rightarrow \lambda^2 = 16$

$\Rightarrow \lambda = \pm 4$ $\Rightarrow \lambda = 4$

Hence, the correct answer is (A).

37. Given : $\frac{{}^nC_{r-1}}{{}^nC_r}=\frac{2}{15}\Rightarrow\frac{r}{n-r+1}=\frac{2}{15}$

$\Rightarrow 17r = 2n + 2$ …(1)

also given $\frac{{}^nC_r}{{}^nC_{r+1}}=\frac{15}{70}$ $\Rightarrow\frac{r+1}{n-r}=\frac{3}{14}$

$\Rightarrow 17r = 3n - 14$ …(2)

Solving (1) and (2)

$n = 16, r = 2$

Average of coefficient $=\frac{{}^{16}C_1+{}^{16}C_2+{}^{16}C_3}{3}$

$$=\frac{16+120+560}{3}=232$$

Hence, the correct answer is (A).

38. $S = 2.{}^{20}C_0 + 5.{}^{20}C_1 + \ldots\ldots\ 59.{}^{20}C_{19} + 62.{}^{20}C_{20}$ …(1)

$S = 62.{}^{20}C_{20} + 59.{}^{20}C_{19} + \ldots\ldots\ 59.{}^{20}C_1 + 2.{}^{20}C_0$ …(2)

Add equation (1) and (2)

$$2S = 64[{}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \ldots.. + {}^{20}C_{20}]$$

$$S = 2^6 \cdot 2^{19} = 2^{25}$$

Hence, the correct answer is (B).

39. $T_4 = {}^6C_{3(x)}^{-\frac{3}{2}(1+\log_{10}^x)} \cdot \left(x^{\frac{1}{12}}\right)^3 = 200$

$$\Rightarrow \quad x^{-\frac{3}{2}(1+\log_{10}^x)+\frac{1}{4}} = 10$$

take log on both side

$$\left(-\frac{3}{2}(1+t)+\frac{1}{4}\right)\log_{10}^x = 1$$

Let $\log_{10}^x = t$

$$6t^2 + 5t + 4 = 0$$

$$D < 0$$

Roots imaginary (No option is matched)

Hence, the correct answer is (Bonus).

40. $T_4 = 20 \times 8^7$

$$\Rightarrow {}^6C_3\left(\frac{2}{x}\right)^3\left(x^{\log_8 x}\right)^3 = 20 \times 8^7 \Rightarrow \frac{160}{x^3}x^{3\log_8 x} = 20 \times 8^7$$

$$\Rightarrow x^{\log_2 x - 3} = 8^6 = 2^{18} \Rightarrow \log_2(x^{\log_2 x - 3}) = \log_2 2^{18}$$

$$\Rightarrow (\log_2 x - 3)(\log_2 x) = 18$$

Let $\log_2 x = t$

$$\Rightarrow t^2 - 3t - 18 = 0 \Rightarrow t = 6, -3$$

$\log_2 x = 6 \qquad \Rightarrow x = 2^6 = 8^2$

$\log_2 x = -3 \qquad \Rightarrow x = 2^{-3} = 1/8$

Hence, the correct answer is (A).

41. Coefficient of $x^2 = {}^{15}C_2 \times 9 - 3a\,({}^{15}C_1) + b = 0$

$$\Rightarrow {}^{15}C_2 \times 9 - 45a + b = 0 \qquad \ldots(1)$$

Coefficient of $x^3 = -27 \times {}^{15}C_2 \times 9a \times {}^{15}C_2 - 3b \times {}^{15}C_1 = 0$

$$\Rightarrow -273 + 21a - b = 0 \qquad \ldots(2)$$

(1) + (2)

$a = 28$ and $b = 315$

Hence, the correct answer is (B).

42. General term

$$T_{r+1} = {}^nC_r x^{2n-2r}.x^{-3r}$$

$$\because 2n - 5r = 1 \Rightarrow r = \frac{2n-1}{5}$$

$$\therefore \frac{2n-1}{5} = 23 \text{ or } n - \left(\frac{2n-1}{5}\right) = 23$$

$2n - 1 = 115 \qquad n = 38$

$n = 58 \qquad \therefore$ smallest $n = 38$

Hence, the correct answer is (C).

43. Since we know that, $(x+a)^5 + (x-a)^5$

$$= 2\{{}^5C_0x^5 + {}^5C_2\,x^3.a^2 + {}^5C_4x.a^4\}$$

$$\therefore \left(x+\sqrt{x^3-1}\right)^5 + \left(x-\sqrt{x^3-1}\right)^5$$

$$= 2[{}^5C_0x^5 + {}^5C_2\,x^3\,(x^3-1) + {}^5C_4x(x^3-1)^2]$$

$$\Rightarrow 2[x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 5x]$$

$\therefore$ Sum of coefficients of odd degree terms $= 2$

Hence, the correct answer is (C).

44. We have $({}^{21}C_1 + {}^{21}C_2 \ldots {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 \ldots {}^{10}C_{10})$

$$= \frac{1}{2}[({}^{21}C_1 + \ldots + {}^{21}C_{10})] + ({}^{21}C_{11} + \ldots.. {}^{21}C_{20})] - (2^{10} - 1)$$

$$(\therefore {}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_{10} = 2^{10} - 1)$$

$$= \frac{1}{2}[2^{21} - 2] - (2^{10} - 1)$$

$$= (2^{20} - 1) - (2^{10} - 1) = 2^{20} - 2^{10}$$

Hence, the correct answer is (A).

45. Total number of terms $= {}^{n+2}C_2 = 28$

$$(n+2)(n+1) = 56$$

$$n = 6$$

Sum of coefficients $= (1 - 2 + 4)^n = 3^6 = 729$

In this question, it is considered that similar term (having same power of x) are not added in given expansion. So full marks were given to all aspirants.

Hence, the correct answer is (Bonus).

46. $\left(1-2\sqrt{x}\right)^{50} = {}^{50}C_0 - {}^{50}C_1 2\sqrt{x}$

$$+ {}^{50}C_2\left(2\sqrt{x}\right)^2 + \ldots + {}^{50}C_{50}(2\sqrt{x})^{50} \qquad \ldots(1)$$

$$\left(1+2\sqrt{x}\right)^{50} = {}^{50}C_0 + {}^{50}C_1 2\sqrt{x} + {}^{50}C_2\left(2\sqrt{x}\right)^2$$

$$+ \ldots + {}^{50}C_{50}\left(2\sqrt{x}\right)^{50} \qquad \ldots(2)$$

Adding equation (1) and (2), $\left(1-2\sqrt{x}\right)^{50} + \left(1+2\sqrt{x}\right)^{50}$

$$= [2\,{}^{50}C_0 + {}^{50}C_2 2^2 x + {}^{50}C_4 2^4 x^2 + \ldots..]$$

Putting $x = 1$, we get above as $\frac{3^{50}+1}{2}$

Hence, the correct answer is (C).

47. Consider $(1 + ax + bx^2)(1-2x)^{18}$

$= (1 + ax + bx^2)\,[{}^{18}C_0 - {}^{18}C_1(2x) + {}^{18}C_2(2x)^2 - {}^{18}C_3(2x)^3 + {}^{18}C_4\,(2x)^4 - \ldots.]$

Coefficient of $x^3 = {}^{18}C_3\,(-2)^3 + a.\,(-2)^2.\,{}^{18}C_2 + b(-2).\,{}^{18}C_1 = 0$

Coefficient of $x^3 = -{}^{18}C_3\,8 + a \times 4.\,{}^{18}C_2 - 2b \times 18 = 0$

$$\Rightarrow -\frac{18 \times 17 \times 16}{6}.8 + \frac{4a + 18 \times 17}{2} - 36b = 0$$

$$\Rightarrow -51 \times 16 \times 8 + a \times 36 \times 17 - 36b = 0$$

$$\Rightarrow -34 \times 16 + 51a - 3b = 0$$

$$\Rightarrow 51a - 3b = 34 \times 16 = 544$$

$$\Rightarrow 51a - 3b = 544 \qquad \ldots(1)$$

Only option number (b) satisfies the equation number (1)

Hence, the correct answer is (B).

48. Given expression can be written as

$$\left(\left(x^{1/3}+1\right)-\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right)\right)^{10}=\left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}$$

$$= (x^{1/3} - x^{-1/2})^{10}$$

General term $= T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(-x^{-1/2})r$

$$= {}^{10}C_r\, x^{\frac{10-r}{3}}.(-1)^r.x^{-\frac{r}{2}} = {}^{10}C_r(-1)^r.x^{\frac{10-r}{3}-\frac{r}{2}}$$

Term will be independent of when $\frac{10-r}{3} - \frac{r}{2} = 0$

$\Rightarrow r = 4$

So, required term $= T_5 = {}^{10}C_4 = 210$

Hence, the correct answer is (C).

49. $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}+1\right)^{2n}$

$$=\left[\left(\sqrt{3}+1\right)^2\right]^n-\left[\left(\sqrt{3}-1\right)^2\right]^n$$

$$=2^n\left[\left(2+\sqrt{3}\right)^n-\left(2-\sqrt{3}\right)^n\right]$$

$$=2^n\times 2\left[{}^nC_1 2^{n-1}\sqrt{3}+{}^nC_3.2^{n-3}3\sqrt{3}+......\right]$$

$$=2^{n+1}\ \sqrt{3}\left[{}^nC_1.2^{n-1}+{}^nC_3 2^{n-3}.3+.....\right]$$

$$=\sqrt{3}\times \text{Some integer}$$

∴ Irrational number

Hence, the correct answer is (A).

50. $(1-x-x^2+x^3)^6 = [(1-x)-x^2\,(1-x)]^6$

$$= (1-x)^6\,(1-x^2)^6$$

$$= (1-6x+15x^2-20x^3+15x^4-6x^5+x^6)$$

$$\times (1-6x^2+15x^4-20x^6+15x^8-6x^{10}+x^{12})$$

Coefficient of $x^7 = (-6)\,(-20) + (-20)\,(15) + (-6)\,(-6) = -144$

Hence, the correct answer is (B).

51. $S_1=\sum_{j=1}^{10} j(j-1).{}^{10}C_j=\sum_{j=1}^{10} j(j-1).\frac{10}{j}.\frac{9}{j-1}{}^8C_{j-2}$

$$=90\sum_{j=2}^{10}{}^8C_{j-2}=90.2^8$$

$$S_2=\sum_{j=1}^{10} j\,{}^{10}C_j=\sum_{j=1}^{10}10.{}^9C_{j-1}$$

$$=10\left[{}^9C_0+{}^9C_1+{}^9C_2+....+{}^9C_9\right]=10.2^9$$

$$S_3=\sum_{j=1}^{10} j^2.{}^{10}C_j=\sum_{j=1}^{10} j^2.\frac{10}{j}.{}^9C_{j-1}=10\sum_{j=1}^{10}(j-1+1).{}^9C_{j-1}$$

$$=10\left(\sum_{j=1}^{10}(j-1)^9C_{j-1}+\sum_{j=1}^{10}{}^9C_{j-1}\right)=10(9.2^8+2^9)=55.2^9$$

Hence, the correct answer is (D).

52. $(64)^n - (62)^{2n+1} = (63+1)^n - (63-1)^{2n+1}$

$$=\left[{}^nC_0(63)^n+{}^nC_1(63)^{n-1}+{}^nC_2(63)^{n-2}+......+{}^nC_{n-1}(63)\right.$$

$$+{}^nC_n-\left[{}^{2n+1}C_0(63)^{2n+1}-{}^{2n+1}C_1(63)^{2n}\right.$$

$$\left.+{}^{2n+1}C_2(63)^{2n-1}-.....+(-1)^{2n+1}\ {}^{2n+1}C_{2n+1}\right]$$

$$=63\times[{}^nC_0(63)^{n-1n}+{}^nC_1(63)^{n-2}+{}^nC_2(63)^{n-3}+........]+1$$

$$-63\times\left[{}^{2n+1}C_0(63)^{2n}-{}^{2n+1}C_1(63)^{2n-1}+.....\right]+1$$

$\Rightarrow 63 \times$ some integral value + 2

$\Rightarrow 8^{2n} - (62)^{2n+1}$ when divided by 9 leaves 2 as the remainder.

Hence, the correct answer is (A).

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Single Correct Choice Type Questions

1. Coefficient of x^{11} in exp. Of $(1+x^2)^4\,(1+x^3)^7\,(1+x^4)^{12}$

= (Coefficient of x^a) × (Coefficient of x^b) × (Coefficient of x^c)

Such that $a + b + c = 11$

Here $a = 2m,\ b = 3n,\ c = 4p$

∴ $2m + 3n + 4p = 11$

Case I: $m = 0,\ n = 1,\ p = 2$

Case II: $m = 1,\ n = 3,\ p = 0$

Case III: $m = 2,\ n = 1,\ p = 1$

Case IV: $m = 4,\ n = 1,\ p = 0$

∴ Required coefficient

$$= {}^4C_0\times{}^7C_1\times{}^{12}C_2+{}^4C_1\times{}^7C_3\times{}^{12}C_0+{}^4C_2\times{}^7C_1\times{}^{12}C_1+$$
$${}^4C_4\times{}^7C_1\times{}^{12}C_0$$

$$= 462 + 140 + 504 + 7 = 1113$$

Hence, the correct answer is (C).

2. Clearly $A_r = {}^{10}C_r, B_r = {}^{20}C_r, C_r = {}^{30}C_r$

Now $\sum_{r=1}^{10}{}^{10}C_r\left({}^{20}C_{10}{}^{20}C_r-{}^{30}C_{10}{}^{10}C_r\right)$

$$={}^{20}C_{10}\sum_{r=1}^{10}{}^{10}C_r{}^{20}C_r-{}^{30}C_{10}\sum_{r=1}^{10}{}^{10}C_r\times{}^{10}C_r$$

$$={}^{20}C_{10}\left({}^{10}C_1{}^{20}C_1+{}^{10}C_2{}^{20}C_2+...+{}^{10}C_{10}{}^{20}C_{10}\right)$$

$$-{}^{30}C_{10}({}^{10}C_1\times{}^{10}C_1+{}^{10}C_2\times{}^{10}C_2+...+{}^{10}C_{10}{}^{10}C_{10}) \qquad ...(1)$$

Now expanding $(1+x^{10})$ and $(1+x)^{20}$ by binomial theorem and comparing the coefficients of x^{20} in their product, on both sides, we get

$$^{10}C_0\,{}^{20}C_0 + {}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots\ldots + {}^{10}C_{10}\,{}^{20}C_{10}$$

$= \text{coefficient of } x^{20} \text{ in } (1+x)^{30} = {}^{30}C_{20} = {}^{30}C_{10}$

$\therefore\ {}^{10}C_1\,{}^{20}C_1 + {}^{10}C_2\,{}^{20}C_2 + \ldots.. + {}^{10}C_{10}\,{}^{20}C_{10} = {}^{30}C_{10} - 1$

Again expending $(1 + x^{10})$ and $(x + 1)^{10}$ by binomial theorem and comparing the coefficients of x^{10} in their Product on both sides, we get

$$\therefore \left({}^{10}C_0\right)^2\left({}^{10}C_1\right)^2 + \left({}^{10}C_2\right)^2 + \ldots. + \left({}^{10}C_{10}\right)^2$$

$= \text{Coeff of } x^{10} \text{ in } (1+x)^{20} = {}^{20}C_{10}$

$$\therefore \left({}^{10}C_1\right)^2 + \left({}^{10}C_2\right)^2 + \ldots. + \left({}^{10}C_{10}\right)^2 = {}^{20}C_{10} - 1$$

Substituting these values in equation (1), we get

$$= {}^{20}C_{10}\left({}^{30}C_{10} - 1\right) - {}^{30}C_{10}\left({}^{20}C_{10} - 1\right)$$

$$= {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

1. $$f(m, n, p) = \sum_{i=0}^{p} {}^{m}C_i\; {}^{n+i}C_P\; {}^{P+n}C_{P-I}$$

$$= \sum_{i=0}^{p} {}^{m}C_i\; \frac{(n+i)!}{p!(n+i-p)!} \times \frac{(p+n)!}{(p-i)!(n+i)!}$$

$$= \sum_{i=0}^{p} {}^{m}C_i\; \frac{(p+n)!}{p!(n)!} \times \frac{(n)!}{(n+i-p)!(p-i)!}$$

$$= \sum_{i=0}^{P} {}^{m}C_i\; {}^{P+n}C_P\; {}^{n}C_{P-i}$$

$$f(m, n, p) = {}^{p+n}C_p\left(\sum_{i=0}^{p} {}^{m}C_i\,{}^{n}C_{p-i}\right)$$

$$\left(\sum_{i=0}^{p} {}^{m}C_i\,{}^{n}C_{p-i}\right)$$

$$= \left[{}^{m}C_0 \cdot {}^{n}C_P + {}^{m}C_1 \cdot {}^{n}C_{P-1} + {}^{m}C_2 \cdot {}^{n}C_{P-2} + \ldots.. + {}^{m}C_P \cdot {}^{n}C_0\right]$$

Coefficient of x^p in expansion of $(1 + x)^m(1 + x)^n = (1 + x)^{m+n}$ $= {}^{m+n}C_p$

Now $f(m, n, p) = {}^{p+n}C_p \cdot {}^{m+n}C_p$

$$g(m, n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{{}^{p+n}C_p} = \sum_{p=0}^{m+n} {}^{m+n}C_p$$

$= {}^{m+n}C_0 + {}^{m+n}C_1 + \ldots. + {}^{m+n}C_{m+n} = 2^{m+n}$

$g(m, n) = 2^{m+n}$ and $g(n, m) = 2^{n+m}$

$g(m, n + 1) = 2^{m+n+1} = 2^{m+1+n} = g(m + 1, n)$

$g(2m, 2n) = 2^{2m+2n} = 2^{2(m+n)} = (2^{m+n})^2 = (g(m, n))^2$

Hence, the correct answer is (A, B, D).

Integer/Numerical Answer Type Questions

1. $$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^{n}C_k k^2 \\ \sum_{k=0}^{n} {}^{n}C_k k & \sum_{k=0}^{n} {}^{n}C_k 3^k \end{bmatrix} = 0$$

Terms used here are $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

$$\sum_{k=0}^{n} {}^{n}C_k k = \sum_{k=0}^{n} \frac{n}{k}.k\; {}^{n-1}C_{k-1} = n.2^{n-1}$$

$$= \sum_{k=0}^{n} {}^{n}C_k.k^2 = \sum_{k=0}^{n} \frac{n}{k}.k^2\; {}^{n-1}C_{k-1}$$

$$= n\sum_{k=0}^{n} k\; {}^{n-1}C_{k-1} = n.(2)^{n-1} + n(n-1).2^{n-2}$$

$$\Rightarrow \sum_{k=0}^{n} {}^{n}C_k.3^k = (3+1)^n = 4^n$$

$$\begin{vmatrix} \frac{n(n+1)}{2} & n(2)^{n-1} + n(n-1)2^{n-2} \\ n2^{n-1} & 4^n \end{vmatrix} = 0$$

$$\Rightarrow 4^n.\ \frac{n(n+1)}{2} - n.2^{n-1}[n.2^{n-1} + n(n-1)2^{n-2}] = 0$$

$\Rightarrow 2^{n-1}\,.\,2^n(n+1) - 2^{n-1}\,.\,n[2^{n-1} + (n-1).\,2^{n-2}] = 0$

$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow n = 4$

$$\sum_{k=0}^{4} \frac{{}^{4}C_k}{k+1} = \frac{1}{5}\sum_{k=0}^{4} \frac{5}{k+1}.{}^{4}C_k = \frac{1}{5}\sum_{k=0}^{4} {}^{5}C_{k+1} = \frac{2^5-1}{5} = \frac{31}{5}$$

Hence, the correct answer is (6.20).

2. $$\sum_{r=0}^{n} r({}^{n}C_r)^2 = n\sum_{r=0}^{n}\left({}^{n}C_r\,{}^{n-1}C_{r-1}\right)$$

$$= n\sum_{r=1}^{n} {}^{n}C_{n-r}\,{}^{n-1}C_{r-1} = n\,{}^{2n-1}C_{n-1}$$

So, $X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \ldots. + 10({}^{10}C_{10})^2$

$$= \sum_{n=0}^{10} r({}^{10}C_r)^2 = 10\,{}^{19}C_9$$

Hence, $\frac{X}{1430} = \frac{1}{143}\,{}^{19}C_9 = 646$

Hence, the correct answer is (646).

3. $(1 + x)^2 + (1 + x)^3 + \ldots.. + (1 + x)^{49} + (1 + mx)^{50}$

$$= (1+x)^2\left[\frac{(1+x)^{48} - 1}{(1+x) - 1}\right] + (1+mx)^{50}$$

$$= \frac{1}{x}\left[(1+x)^{50}-(1+x)^{2}\right]+(1+mx)^{50}$$

Coeff. of x^2 in the above expansion

$=$ coeff . of x^3 in $(1+x)^{50}$ + Coeff. of x^2 in $(1+mx)^{50}$

$\Rightarrow {}^{50}C_3 + {}^{50}C_2\, m^2$

$\therefore\ (3n+1)\ {}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2\, m^2$

$$\Rightarrow\ (3n+1) = \frac{{}^{50}C_3}{{}^{51}C_3} + \frac{{}^{50}C_2}{{}^{51}C_3} m^2$$

$$\Rightarrow 3n+1 = \frac{16}{17} + \frac{1}{17}m^2 \Rightarrow n = \frac{m^2-1}{51}$$

Least positive integer m for which n is an integer is $m = 16$ and then $n = 5$

Hence, the correct answer is (5).

4. Let the coefficient of three consecutive terms of $(1+x)^{n+5}$ be ${}^{n+5}C_{r-1}$, ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$, then we have

$${}^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$$

$$\frac{{}^{n+5}C_{r-1}}{{}^{n+5}C_r} = \frac{5}{10} \Rightarrow \frac{r}{n+6-r} = \frac{1}{2}$$

Or $n - 3r + 6 = 0$ …(1)

Also $\dfrac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \dfrac{10}{14} \Rightarrow \dfrac{r+1}{n-r+5} = \dfrac{5}{7}$

Or $5n - 12r + 18 = 0$ …(2)

Solving (1) and (2) we get $n = 6$.

Hence, the correct answer is (6).

CHAPTER 6: PERMUTATION AND COMBINATION

Test Your Concepts-I

1. ${}^{m+n}P_2 = 90$ then $(m+n)(m+n-1) = 10\times 9$

   ${}^{m-n}P_2 = 30$ then $(m-n)(m-n-1) = 30 = 6\times 5$

   on solving $m - n = 6$

   $m = 8$; $n = 2$

2. Consider, ${}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

   $$= \frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-1-r)!}\left[1+\frac{r}{n-r}\right]$$

   $$= \frac{(n-1)!}{(n-1-r)!}\left[\frac{n}{(n-r)}\right] = \frac{n!}{(n-r)!} = {}^nP_r$$

3. Since, each digit can be filled by any of the two numbers 1 or 2, i.e. 2 ways.

   ∴ Number of n-digit numbers

   $$= \frac{2\times 2\times 2\times 2\times\ldots\times 2}{n\ \text{time}} - 2 = 2^n - 2$$

   $\Rightarrow\quad 2^n = 512 = 2^9$

   $\therefore\quad n = 9$

4. Let us first find the number of three-digit numbers having none of the digits even.

   Clearly, there are 5 odd digits, viz. 1, 3, 5, 7, 9.

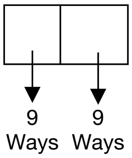
   ∴ Number of three-digit numbers having all digits odd = $5^3 = 125$

   ∴ Required number = 900 – 125 = 775

5. Clearly, there are 1-digit, 2-digits, 3-digits, and 4-digits number from 1 to 1000.

   In 1-digit number no digit is repeated, therefore the number of natural number from 1 to 9, which have none of their digits repeated is 9.

   In 2-digit number, tenth place can be filled with any of the 9 digits 1, 2, 3, … 9. So, there are 9 ways of filling the tenth place. Now, 9 digits are left including 0. So, units place can be filled with any of the remaining 9 ways. Thus, there are 9 × 9 = 81 numbers which have none of their digits repeated.

   In 3-digit number, hundred's place can be filled with any of the 9 digits 1, 2, 3, …, 9. So, there are 9 ways to fill the hundred's place. Now, 9 digits are left including 0, so tenth's place can be filled in 9 ways. Now, the units place can be filled with any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

   Thus, there are 9 × 9 × 8 = 648 number which have none of their digits repeated.

   Hence, the total number of required numbers

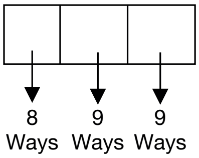
   = 9 + 81 + 648 = 738

6. Clearly, a number between 100 and 1000 has three digits and there are 900 numbers between them.

   Now, let us first find number of numbers of three digit having none of digit as 7.

   Clearly, unit place can be filled by any of digits (0, 1, 2, 3, 4, 5, 6, 8, 9, 10) i.e. in 9 ways.

   and hundred place can be filled by any of digits (1, 2, 3, 4, 5, 6, 8, 9, 10) i.e. in 8 ways.

   ∴ Number of 3-digits numbers having none of the digit as 7

   = 8 × 9 × 9 = 648

   Hence, the number having at least one of their digit is 7

   = 900 – 648 = 252

7. Consider the following box, where green colour is in the middle.

			Green colour			
VII	VI	V	IV	III	II	I

   Clearly, I box can be filled by any of 6 remaining colours i.e. in 6 ways.

   Similarly, II box can be filled in 5 ways.

   III box can be filled in 4 ways.

   V box can be filled in 3 ways.

   VI box can be filled in 2 ways.

   VII box can be filled in 1 ways.

   Hence, the number of ways in which 7 different colours in a rainbow can be arranged.

   Considering green colour in middle = 1 × 2 × 3 × 4 × 5 × 6 = 6!

8. Clearly, a palindrome is a word, which is same while read from backward or forward.

   ∴ 5 lettered palindrome is a word in which first letter is same as last letter and second letter in same as second last letter.

I	II	III	II	I

   Clearly, first place can be filled in 26 ways.

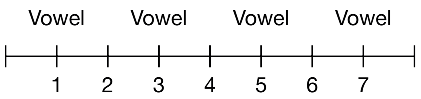
   Second place can be filled in 26 ways.

   and third place can be filled in 26 ways.

   ∴ Required number of 5 letter palindromes = $(26)^3$

9. Clearly, there are three vowels in the word MIRACLE, viz., I, A, E.

   Now, as vowels can occupy only old places.



Therefore, there are 4 places available for 3 vowels and consonants can be arrange on the remaining 4 places.

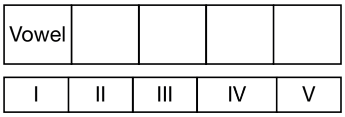
∴ Required number of ways = $^4P_3 \times 4!$

$= 4! \times 4! = 24 \times 24$

$= 576$

10. Clearly, there are 2 vowels in word AKSHI, viz. A and I.

Case I When each word begins with vowel.

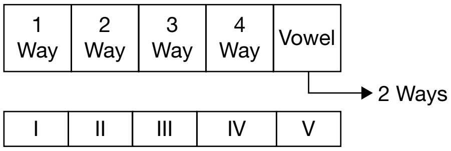


Clearly, first place can be filled by any of vowels (A to I) i.e. in 2 ways, second place can be filled by any of remaining 4 letters, i.e. in 4 ways, third place can be filled by 3 ways, fourth place can be filled by 2 ways and fifth place can be filled by 1 way.

∴ Number of word begins with vowels = $2 \times 4 \times 3 \times 2 \times 1$

$= 48$

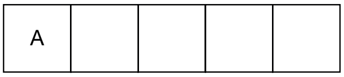
Case II When each word terminates with vowel.



By similar argument as in Case I, we get 48 words that terminates with vowel.

Note that in above two cases, we have some words which begin and terminate with vowel.

Case III When each word begin and terminate with vowel.



Let us consider A in begin and I is last, then second place can be filled by 3 ways, third can be filled by 2 ways and fourth place can be filled by 3 ways, third can be filled by 2 ways and fourth place can be filled by 1 way.

∴ Number of words = $3 \times 2 = 6$

Similarly, if we consider A in last and I in begin, we get same number of words = 6

Hence, required number of words = $48 + 48 - 6 - 6 = 84$

11. Clearly, the first word start with A and we get 24 words starting with A, as rest letters can be arrange in 4! Ways.

Similarly, number of words starting with E are 24, number of words starting with H are 24.

Now, if we start the word with M, then again we get 24 words, i.e. we have 96 words before the words start with S.

Thus, the 86th, word will be start with M.

Now, if we start the word with M, then next letter will be A and remaining 3 letters can be arrange in 3! Ways, i.e. we get 6 words.

Now, if we take letter E after M, then again we get 6 words.

Thus, over all we have = 72 + 6 + 6 = 84 words

Now, next two words are M H A E S and M H A S E.

Hence, the last letter of the 86th word in the list is E.

12. A I L N O S

A/I/L/N/O_ _ _ _ _ _5!.5

SAI _ _ _3!

SALI _ _2!

SALN_ _ _2!

SALOIN 611

SALONI 612th

13. The letters of the word in alphabetical order is A, B, E, F, L

F	A	B	L	E
3	0	0	1	0
4!	3!	2!	1!	0!

Rank of the word = 3.4! + 1.1! + 1 = 74

14. There are nine-digits numbers 223355888 out of which 5 are even number and 4 are odd numbers.

Let us mark out the position to filled up as follows

(1) (2) (3) (5) (6) (7) (8) (9)

Now, the 4 odd number can be placed at any of the four place, marked (2, 4, 6, 8).

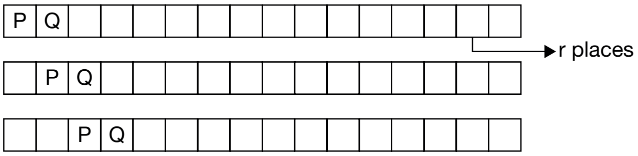
So, the number of ways of arranging the digit (3, 3, 5, 5) out of which 3 and 5 occur twice

$$= \frac{4!}{2!2!} = 6$$

Also, the 5 even numbers (2, 2, 8, 8, 8) out of which 2 occurs twice and 8 occurs thrice may be arranged = $\frac{5!}{2!3!} = 10$

∴ The required number of ways = $6 \times 10 = 60$ ways

15. Clearly, a bundle of 2 specific things (say P and Q) can be put in r places in $(r - 1)$ ways.



Also, 2 things in the bundle can be arranged themselves into 2! ways. Now, number of permutations of $(n - 2)$ things taking $(r - 2)$ things at a time is $^{n-2}P_{r-2}$.

Thus, using the fundamental principle of counting, the required number of permutations will be $2!(r - 1)\ ^{n-2}P_{r-2}$

$$= \frac{(n-2)!}{(n-r)!} \cdot 2 \cdot (r-1)$$

Test Your Concepts-II

1 We have, $\frac{^{20}C_n}{^{25}C_m}$,

Numerator has greatest value, if $n = 10$ and

Denominator has greatest value, if $m = 12$ or 13

$$\therefore \quad \frac{^{20}C_{10}}{^{25}C_{12}} = \frac{20!\times13!\times12!}{10!10!\times25!} = \frac{143}{4025}$$

2. $B_1B_2B_3\ G_1G_2\ldots G_6$

$m = 3!\cdot 7!$

$n = {}^3C_2\ 2!\ 7! = 3!\ 7!$. Hence $k = 1$.

3. There are 6 questions in part A and 7 in part B,

Required no. of ways $= {}^6C_{4.}\ {}^7C_{6.} + {}^6C_5{}^7C_5 + {}^6C_{6.}\ {}^7C_4 = 266$

4. Let each student try r question then there are combination for this is ${}^{20}C_r =$ number of students

Now maximum number of students is binomial coefficient of middle term in expansion of $(1 + x)^{20} = {}^{20}C_{10}$

5. We have, $\{a_1, a_2, a_3, \ldots, a_n\}$

Total number of elements $= n$

∴ Total number of selection 3 elements out of n elements in which a_3 is included is

${}^{n-1}C_{3-1} = {}^{n-1}C_2$

6. Total number of words without any restriction is 6!

Total number of words beginning with G is 5!.

Total number of words ending with S is 5!.

Total number of words beginning with G and ending with S is 4!

Thus, total number of required words $= 6! - 2\times 5! + 4!$.

$= 720 - 240 + 24 = 504$

7. The total number of numbers that can be formed with the digits 1, 3, 5, and 7 taken all at a time is equal to the number of arrangements of 4 digit, taken all at a time

i.e. ${}^4P_4 = 4! = 24$

To find the sum of these 24 numbers, we have to find the sum of the digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers.

If 1 is the digit at unit's place, the remaining three place can be filled in 3! Ways or we can say 1 occurs at unit's place

$3! = (6)$times.

Similarly, each digit occurs 6 times.

So, the total sum of the digits in the unit's place in all these numbers is

$(1 + 3 + 5 + 7) \times 3! = 16 \times 6 = 96$

Similarly, the sum of digit is 96 at ten's, hundred's and thousand places.

Hence, the sum of all the number is

$16 \times 3! \times (10^o + 10^1 + 10^2 + 10^3) = 16 \times 3! \times 1111$

Aliterate Method

Trick sum of all the numbers formed by taking all the given n digits.

= (Sum of all the n digits) × $(n - 1)!$ × (111…n times)

Hence, given digits are 1, 3, 5, and 7.

∴ Required sum $= (1 + 3 + 5 + 7) \times 3! \times (1111) = 16 \times 3! \times 1111$

8. Clearly, first chair can be accommodated by any of the four persons, i.e. in 4 ways. Second chair can be accommodated by any of remaining 3 persons, i.e. in 3 ways and their chain can be accommodated by any of the remaining 2 persons, i.e. in 2 ways.

Thus, number of ways in which four persons can be accommodated in 3 different chairs

$= 4 \times 3 \times 2 = 24$ ways

9. We first arrange the m men. This can be done in $m!$ ways. After m men have taken their seats, we can arrange the n women on $(m + 1)$ seats marked with cross.

$$\times \underset{1\text{st}}{M} \times \underset{2\text{nd}}{M} \times \underset{3\text{rd}}{M} \times \underset{4\text{th}}{M} \times \ldots \times \underset{m\text{th}}{M} \times$$

This can be done in ${}^{m+1}P_n$ ways.

Thus, the required of ways $= m! \times {}^{m+1}P_n = \dfrac{m!(m+1)!}{(m-n+1)!}$

10.

I	II	III	IV	V	VI
4 ways	3 ways	3 ways	3 ways	3 ways	3 ways

Flag

Clearly, first strip can be filled by any of the four colours, i.e. in 4 ways, second strip can be filled by any of the remaining 3 colours, i.e. in 3 ways, third strip can be filled by any of the remaining 3 colours (including colour of first strip) i.e. in 3 ways, fourth strip can be filled by any of the remaining 3 colours (including colour of 2^{nd} strip) i.e. in 3 ways. Similarly, we can fill fifth and sixth strip in 3 ways.

∴ By fundamental principle of counting required number of ways

$= 4 \times 3 \times 3 \times 3 \times 3 \times 3 = 12 \times 81$

11. We know that, a number is divisible by 9, if sum of digits of number is divisible by 9. Now, as we have to form a 9–digit number using distinct digits can be 36 or 45.

Case I Sum of digits is 36.

In this case digits are 0, 2, 3, 4, 5, 6, 7, 8 and required number of 9-digit number

$= {}^9P_9 - {}^8P_8 = 9! - 8!$

Case II Sum of digits is 45.

In this case digits are 1, 2, 3, 4, 5, 6, 7, 8, 9 and required number of 9-digit number $= {}^9P_9 = 9!$

Thus, total number of 9-digit number which are divisible by 9 and having all digits distinct $= 2(9!) - 8!$

⇒ $k(8!) = 2(9!) - 8!$

⇒ $(k + 1)8! = 2(9!) - 8!$

⇒ $k + 1 = 18$

∴ $k = 17$

12. Let us first arrange the 5 lions, This can be done in 5! Ways. After giving the position to 5 lions, tamer can bring 4 tigers at 6 positions marked with P.

$$P\ \underset{1\text{st}}{L}\ P\ \underset{2\text{nd}}{L}\ P\ \underset{3\text{rd}}{L}\ P\ \underset{4\text{th}}{L}\ P\ \underset{5\text{th}}{L}\ P$$

This can be done in ${}^6P_4$ ways.

Thus, the required number of ways $= 5! \times {}^6P_4$

13. We have, given digits 1, 2 and 3 in which digit 2 occur exactly twice.

Digit 1	**Digit 2**	**Digit 3**	**Number of Arrangements**
1	2	4	$\frac{7!}{2!4!} = 105$
2	2	3	$\frac{7!}{2!2!3!} = 210$
3	2	2	$\frac{7!}{3!2!2!} = 210$
4	2	1	$\frac{7!}{4!2!1!} = 105$
0	2	5	$\frac{7!}{2!5!} = 21$
5	2	0	$\frac{7!}{5!2!} = 21$

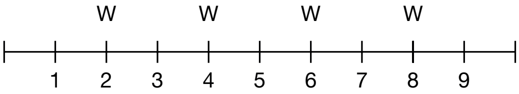
Total number of different seven-digit numbers that can be formed using digits 1, 2 and 3 and the digit 2 occurs exactly twice is 105 + 210 + 210 + 105 + 21 + 21 = 672.

14. Since, each ring has 10 different letters. therefore, total number of attempts are 10 × 10 × 10 = 1000.

Clearly, out of 1000 attempts one will be the code of lock.

∴ Number of unsuccessful attempts = 1000 – 1 = 999

15.



Clearly, for 4 women four places are available and for 5 men five places are available. Now, 2nd place can be filled by any of the 4 women, i.e. in 4 ways, 4th place can be filled by any of the remaining 3 women, i.e. in 3 ways, 6th place can be filled by any of the remaining 2 women, i.e. in 2 ways and the 8th place can be filed by one way.

∴ Number of ways in which 4 women sit at the even places

$= 4 \times 3 \times 2 \times 1 = 4!$

Similarly, number of ways in which 5 men sit at odd places

$= 5 \times 4 \times 3 \times 2 \times 1 = 5!$

Thus, required number of ways = 5! × 4! = 2880

16. The given word 'BANANA' contains 6 letters out of which A occur thrice, N occur twice and rest are distinct.

∴ Total number of arrangements $= \frac{6!}{3!2!} = 60$

When two N's are appear adjacently, treated as one letter. We have to arrange 5 letters out of which A occur thrice and rest are distinct

∴ Required number of arrangement $= \frac{5!}{3!} = 20$

∴ The total number of word two N's do not appear adjacently

= (Total number of words) – (Number of words in which two N's appear adjacently)

= 60 – 20 = 40

Test Your Concepts-III

1. We know that, by joining the two points we can obtain a line, so number of lines joining vertices of octagon $= {}^8C_2 = 28$

∴ Required number of diagonals

= 28 – (Number of sides of octagon)

= 28 – 8 = 20

2. Here, total number of points are $(m + n + k)$ which must give ${}^{m+n+k}C_3$ number of triangles, but m points on l, taking 3 points at a time gives ${}^mC_3$ combinations which produce to triangle, Similarly, ${}^nC_3$ combinations which produce no triangle. Similarly, ${}^nC_3$ and ${}^kC_3$ number of triangles cannot be formed. Therefore, the required number of triangles are

$${}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$$

3. Let n be the number of sides of polygon

∴ Number of diagonals $= {}^nC_2 - n$

$\Rightarrow \frac{n(n-1)}{2} - n = 44 \Rightarrow \frac{n(n-3)}{2} = 44$

$\Rightarrow n^2 - 3n - 88 = 0$

$\Rightarrow n^2 - 11n + 8n - 88 = 0$

$\Rightarrow n(n - 11) + 8(n - 11) = 0$

$\Rightarrow (n - 11)(n + 8) = 0$

∴ $n = 11$ [∵ n connot be negative]

4. Line and line $= {}^8C_2 \cdot 1 = 28$

Circle and circle $= {}^7C_2 \cdot 2 = 43$

Triangle and triangle $= {}^6C_2 \cdot 6 = 90$

Line and circle $= {}^8C_1 \cdot {}^7C_1 \cdot 2 = 112$

Line and triangle $= {}^8C_1 {}^6C_2 \cdot 2 = 96$

Circle and triangle $= {}^7C_1 {}^6C_1 \cdot 6 = 252$

Total point of intersection = 28 + 42 + 90 + 112 + 96 + 252 = 620

5. Number of ways is equal to number of ways of selection of 4 points out of given 18 points such that no two them are consecutive.

Assume that all points P1,p2,.....P18 are in a line, we select 4 point which are not consecutive by ${}^{18-4+1}C_4$ ways, but in these ways we select P1 and P18 and in a circle P1 and P18 are consecutive. So we need to subtract cases when p1 and p18 already selected . this can be done by ${}^{13}C_2$

Number of ways $= {}^{15}C_4 - {}^{13}C_2 = 1287$

6. Number of points required for the fixed circle is 3.

So, first select any three points from the 10 points in ${}^{10}C_3$ ways.

In these ways, circle with four concyclic points is selected in ${}^4C_3$ ways. But it should be taken once.

∴ Total number of circles $= ({}^{10}C_3 - {}^4C_3) + 1$

$= \left(\frac{10 \times 9 \times 8}{3 \times 2 \times 1} - 4\right) + 1$

$= (120 - 4) + 1 = 117$

7. We know that, two circles intersect at two distinct points, two straight lines intersect at one point, one circle and one line intersect at two distinct points, therefore total number of points of intersection are as follows.

Number of Ways of Selection	Point of Intersection
Two straight lines: $^4C_2$	$^4C_2 \times 1 = 6$
Two circles: $^4C_2$	$^4C_2 \times 2 = 12$
One line and one circle: $^4C_1 \times {}^4C_1$	$^4C_1 \times {}^4C_1 \times 2 = 32$

Thus, total number of point of intersection = 32 + 12 + 6 = 50

8. Clearly, there are three horizontal lines and four vertical lines. As, we know that a parallelogram can be formed by two vertical lines and two horizontal lines.

$\therefore$ Required number of parallelogram = $^3C_2 \times {}^4C_2$

$$= 3 \times \frac{4 \times 3}{2 \times 1} = 18$$

9. Let the given box is of size 4 × 3.

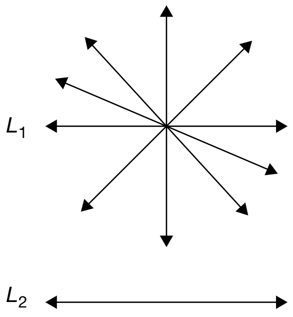
Now, on considering block of size 1 × 1, we get 4 × 3 = 12 squares (number of ways of selecting a unit length horizontally × number of ways of selecting a unit length vertically).

On considering block of size 2 × 2, we get 3 × 2 = 6 squares (number of ways of selecting a length of unit 2 horizontally × number of ways of selecting a length of unit 2 vertically)

On considering block of size 3 × 3, we get 2 × 1 squares

$\therefore$ Total number of squares = 12 + 6 + 2 = 20

10. Given, n concurrent lines and another line parallel to one of them.



As, we have seen in figure if $L_1 \parallel L_2$, then any other two lines from congruent lines will intersect L_2 at two points and that form a triangle.

$\therefore$ Required number of triangle = number of ways of selecting two lines from $(n - 1)$ concurrent lines

$$= {}^{n-1}C_2 = \frac{(n-1)(n-2)}{2}$$

11. Total number of points are 18 which must give $^{18}C_3$ number of triangles, but on each side of triangles 7 points are lying.

Now, on taking 3 points at a time from any one side gives $^7C_3$ combinations which produce no triangle.

$\therefore$ Required number of triangle = $^{18}C_3 - {}^7C_3 - {}^7C_3 - {}^7C_3$

$$= \frac{18 \times 17 \times 16}{3 \times 2 \times 1} - 3 \times \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 816 - 105 = 711$$

12. We have to select 5 squares, taking at least one from each row and this can be done in following ways

$R_1(4)$	$R_2(2)$	$R_3(2)$	Number of Selections
1	2	2	$^4C_1 \times {}^2C_2 \times {}^2C_2 = 4$
2	1	2	$^4C_2 \times {}^2C_1 \times {}^2C_2$ $= 6 \times 2 = 12$
2	2	1	$^4C_2 \times {}^2C_2 \times {}^2C_1$ $= 6 \times 1 \times 2 = 12$
3	1	1	$^4C_3 \times {}^2C_1 \times {}^2C_1$ $= 4 \times 2 \times 2 = 16$

$\therefore$ Total number of ways = 4 + 12 + 12 + 16 = 44

13. We know that, in a rectangle of $n \times p$ $(n < p)$

Number of rectangles of any size in $\dfrac{np}{4}(n + 1)(p + 1)$

and number of square of any size is $\displaystyle\sum_{r=1}^{n} (n + 1 - r) \times (p + 1 - r)$.

$\therefore$ Number of rectangles excluding square

$$= \frac{np}{4}(n + 1)(p + 1) - \sum_{r=1}^{n} (n + 1 - r)(r + 1 - r).$$

Put $n = 4$, $p = 6$ in above, we get

$$= \frac{6 \times 4}{4}(4 + 1)(6 + 1) - \sum_{r=1}^{4} (5 - r)(7 - r)$$

$$= 6 \times 7 \times 5 - \sum_{r=1}^{4} (35 - 12r + r^2)$$

$$= 210 - \left(35 \times 4 - 12 \times \frac{4 \times 5}{2} + \frac{4 \times 5 \times 9}{6}\right)$$

$$= 210 - (140 - 120 + 30) = 210 - 140 + 120 - 30 = 160$$

Test Your Concepts-IV

1. Let P1 and P2 do not want to be together

Required no. of ways = Total – when P1 and P2 are together

$= 5! - 4!.2! = 72$

2. Let G1 and G2 want to be together then remaining 10 guests have no choices so lets divide them in group of 6 and 4 = $\dfrac{10!}{6!4!}$ Now 4 person will join G1 and G2 and they select side for 2 ways then their arrangement keeping G1 and G2 together

No required no. of ways are = $\dfrac{10!}{6!4!}$ .2!.6!.5!.2! =20.10!

3. (i) The total number of person = 4 + 4 = 8. These 8 person can be seated at the round table = (8 – 1)! = 7! = 5040 ways.

(ii) since, these is no restriction on men, therefore first of all we arrange the 4 men.

Now, 4 men can be seated a round table = 3! = 6 ways

Now, there a 4 places for 4 women, therefore 4 women can be seated in 4! Way = 24 ways

∴ Required number of ways = 6 × 24 = 144 ways

(iii) consider two particular women as one person, therefore total number of person = 7. These 7 person can be seated at the round table = (7 – 1)!2! = 6!2! = 720 × 2 = 1440 ways

(iv) Two particular women not sit together = (Total number of arrangement) – (Number of arrangement of particular women sit together)

= 5040 – 1440 = 3600

(v) Consider 4 women as one person, therefore total number of person = 5. These 5 person can be seated at round table = (5 – 1)! = 4! = 24 ways.

Now, also 4 women can be arrange = 4! = 24 ways

∴ Required number of ways = 24 × 24 = 576 ways

(vi) Let P_1, P_2 be two particular women and H be the person sit between two particular women.

∴ arrangement of 2 particular women = 2! = 2 ways

Now, one person out of six is between P_1, $P_2 = {}^6C_1 = 6$

Thus, we have 5 persons and 1 group (P_1, P_2, and H) that can be arranged in = 5! Ways

∴ Total number of ways = 2 × 6 × 5! = 1440

4. (i) The given word 'DAUGHTER' contains 8 different letters when the vowels A, U, E are always together, treated as one letters.

We have to arrange 6 letters i.e. 6! = 720 ways, but corresponding to each of these arrangement, the vowels in the group (A, U, E) can be arrange themselves in 3! = 6 ways.

∴ The total number of arrangement when vowels always together = 720 × 6 = 4320

(ii) Clearly, the total number of words formed by using all the 8 letters of the given word is 8! = 40320

∴ The number of words in which vowels are never together = (total number of words) – (number of words in which vowels are always together)

= (40320 – 4320) = 36000

5. There are 18 beads out of which 8 being blue, 5 red, 4 green and 1 yellow. In this arrangement anti-clockwise or clockwise are not different.

∴ Total number necklace formed = $\dfrac{(18-1)!}{2!8!5!4!}$

$= \dfrac{17!}{2!\cdot 8!\cdot 5!\cdot 4!}$

6. (i) He can invite 2 friends three times each

Let us select first those 2 friends in ${}^3C_2$ ways

Now these two friends each three times can be invited on 6 days in $\dfrac{6!}{3!3!}$

(ii) Another possibility is that he invites all three friends two times each. Then

Number of ways = $\dfrac{6!}{2!2!2!}$

(iii) One more possibility is that he invites one friend three times, one two times, and one three times. This process can be done by ${}^3C_1{}^2C_1$

Number of ways = $\dfrac{6!\times 6}{3!2!}$

Hence, total number of ways N = ${}^3C_2 \times \dfrac{6!}{3!3!} + \dfrac{6!\times 6}{3!2!} + \dfrac{6!}{2!2!2!} = 510$

7. According to the question, if all white roses come together, then we consider 3 white roses as one group.

Now, we have to arrange 6 + 1 = 7 thing in a circle.

Then, number of arrangements = $\dfrac{(7-1)!}{2} = \dfrac{6!}{2}$

Now, 3 white roses can be arranged themselves in 3! Ways.

∴ Required number of arrangements = $\dfrac{6!}{2} \times 3!$

$= \dfrac{6!}{2} \times 6 = 3 \times 6! = 3 \times 720 = 2160$

8. Let the seat occupied by A be numbered as 1 and the remaining 5 seats be number as 2, 3, 4, 5, 6 in anticlockwise direction. Here, two cases arises.

Case I B is on right of A i.e. at number 2. Then, seat number 3 can be occupied by C or D in ${}^2C_1$ ways and remaining 3 person can have remaining three seats in 3! ways.

Hence, the number of arrangements in this case is 2 × 6 = 12.

Case II C is at the right of A i.e. at number 2. Then, B can occupy any seat from number 3 or 4 or 5. Then, D must be on the right of B. So, we are left with 2 persons and 2 seats, which can be occupied in 2! ways.

Hence, the number of arrangement in this case is ${}^3C_1 \times 2! = 6$.

These cases are exclusive. So, by the same rule

Total number of arrangement is 12 + 6 = 18

9. First group can be given 8 different things out of 15 in ${}^{15}C_8$ ways second group can be given 4 different thing out remaining 7 in ${}^7C_4$ and 3 different things out of 3 remaining 3 things in ${}^3C_3$ ways.

Hence, the required number of ways (here, order of groups are not important) = ${}^{15}C_8 \times {}^7C_4 \times {}^3C_3$

$= \dfrac{15!}{8!7!} \times \dfrac{7!}{4!3!} \times \dfrac{3!}{3!} = \dfrac{15!}{8!4!3!}$

10. Here, 200 different things are to be divided in 100 equal groups.

∴ the total numbers of ways = $\dfrac{200!}{(2!)^{100} \times (100)!}$

$= \dfrac{200!}{(100)!2^{100}}$

11. There are 50 identical things distributed among 8 persons. In which 3 of them get 8 things each, two of them get 7 things each and remaining 3 get 4 thing each

8 + 8 + 8 + 7 + 7 + 4 + 4 + 4 = 50

3(8) + 2(7) + 3(4) = 50

Now, number of ways to distribute (3 + 2 + 3) items among 8 persons in the groups 3, 2, and 3 items

$$= \frac{(3+2+3)!}{3!2!3!} = \frac{8!}{(3!)^2(2!)}$$

Test Your Concepts-V

1. Consider the product $(x^0 + x^1 + x^2 + \ldots + x^9)\,(x^0 + x^1 + x^2 + \ldots + x^9)\ldots 6$ factors

The number of ways in which the sum of the digits will be equal to 12 is equal to the coefficient of x^{12} in the above product.

So, required number of ways

$= \text{Coefficient of } x^{12} \text{ in } (x^0 + x^1 + x^2 + \ldots + x^9)^6$

$= \text{Coefficient of } x^{12} \text{ in} \left(\dfrac{1-x^{10}}{1-x}\right)^6$

$= \text{Coefficient of } x^{12} \text{ in } (1 - x^{10})^6\,(1 - x)^{-6}$

$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6}\,(1 - {}^6C_1x^{10} + \ldots)$

$= \text{Coefficient of } x^{12} \text{ in } (1 - x)^{-6} - {}^{2+6-1}C_{6-1}$

$= {}^{17}C_5 - 6 \times {}^7C_5 = 6062$

2. Required number of ways is the number of positive integral solution of $x + y + z = 11$,

where $1 \le x, y, z \le 6$.

$\therefore$ Required number of ways = Coefficient of x^{11} in

$(x + x^2 + x^3 + \ldots + x^6)^3$

$= \text{Coefficient of } x^8 \text{ in } (1 + x + x^2 + \ldots + x^5)^3$

$= \text{Coefficient of } x^8 \text{ in } \left(\dfrac{1-x^6}{1-x}\right)^3$

$= \text{Coefficient of } x^8 \text{ in } (1 - x^6)^3\,(1 - x)^{-3}$

$= \text{Coefficient of } x^8 \text{ in } (1 - x)^{-3} - {}^3C_1.$ Coefficient of x^2 in $(1 - x)^{-3}$

$= {}^{8+3-1}C_{3-1} - {}^3C_1 \cdot {}^{2+3-1}C_{3-1}$

$= {}^{10}C_2 - 3 \times {}^4C_2 = 27$

3. Four letters can be selected in the following ways

(i) All different i.e. C, O, R, G ${}^4C_4 = 1$

(ii) 2 like and 2 different $1 \cdot {}^3C_2 = 3$

(iii) 3 like and 1 different $1 \times {}^3C_1 = 3$

Therefore, required number of ways = 1 + 3 + 3= 7

4. In 'MORADABAD' we have 6 different types of letters 3A's, 2D's and rest four different. We have, to form words of 4 letters.

Now, four letters word can be formed in following ways

(i) All different number of ways ${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360$

(ii) Two different two alike number of ways ${}^2C_1 \times {}^5C_2 \times \dfrac{4!}{2!} = 240$

(iii) 3 alike 1 different number of ways ${}^1C_1 \times {}^5C_1 \times \dfrac{4!}{3!} = 20$

(iv) 2 alike of one type and 2 alike of other type, number of ways ${}^2C_2 \times \dfrac{4!}{2!\times 2!} = 6$

$\therefore$ Total numbers of words = 360 + 240 + 20 + 6 = 626

5. In 'COMBINATION' we have 8 different types of letters 2Is is, 2 Os, 2 Ns and rest 5 different.

We have to form words by taking two same and two different letters and this can be done in ${}^3C_1 \times {}^7C_2 \times \dfrac{4!}{2!}$ ways $= 3 \times 21 \times 12 = 756$

6. $xyz = 3 \times 2^3 \times 5^3$

Now the power of '3' is 1, then 3 can be distributed among three variables in '3' ways. Similarly the power of '2' is 3, hence 3 can be distribute among x, y and z in '10' ways and same is the case of 5^3.

Hence total ways = $3 \times 10 \times 10 = 300$

7. Consider, $30 = 2 \times 3 \times 5 = 10 \times 3 \times 1 = 2 \times 15 \times 1$

$= 6 \times 5 \times 1 = 1 \times 1 \times 30$

Hence, the total number of positive integral solution of abc = 30 is

$3! + 3! + 3! + 3! + \dfrac{3!}{2!} = 27$

8. Let $a = x_1 + x_2 + x_3$ and $b = y_1 + y_2$

$\therefore$ We have, $a, b > 0$ and $ab = 77$

Now, consider $77 = 7 \times 11 = 1 \times 77$

$\therefore$ Number of possible positive integral solution of ab = 77 is 2! + 2! = 4.

But $a \ne 1$ and $b \ne 1$ as $x_1, x_2, x_3, y_1, y_2 \in N$.

$\therefore$ $a = 7, b = 11$ or $a = 11; b = 7$

Case I: When $a = 7, b = 11$

We have, $x_1 + x_2 + x_3 = 7$ and $y_1 + y_2 = 11$

Clearly, the number of positive integral solution of

$x_1 + x_2 + x_3 = 7$ is ${}^{7-1}C_{3-1} = {}^6C_2 = \dfrac{6\times 5}{2\times 1} = 15$

and the number of positive integral solution of $y_1 + y_2 = 11$ is

${}^{11-1}C_{2-1} = {}^{10}C_1 = 10$

Case II: When $a = 11, b = 7$

We have $x_1 + x_2 + x_3 = 11$ and $y_1 + y_2 = 7$

the number of positive integral solution $= {}^{11-1}C_{3-1}\,{}^{7-1}C_{2-1} = {}^{10}C_2\,{}^6C_1 = 270$

$\therefore$ Total number of solutions = 150 + 270 = 420

9. We have, $x_1\,x_2\,x_3\,x_4\,x_5 = 240$, where x_1, x_2, x_3, x_4, x_5 are natural numbers.

Now, by using prime factorisation of 240, we can write the given equation as

$x_1x_2x_3x_4x_5 = 2^4 \times 3 \times 5$

Now, we can assign 3 and 5 to any of 5 unknowns in 5 × 5 ways.

We can assign 2^4 to x_1, x_2, x_3, x_4, x_5 in

$a + b + c + d + e = 4$

$\Rightarrow$ ${}^{4+5-1}C_{5-1} = {}^{8}C_4$, ways

$\therefore$ Required number of ways $= {}^{8}C_4 \times 25 = 1750$

10. Consider, $24 = 2^3 \times 3$. Now, consider three boxes x, y, z. 3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in three boxes in ${}^{3+3-1}C_{3-1} = {}^{5}C_2$ ways. Thus, total number of positive integral solutions is equal to the number of distribution which is given by $3 \times {}^{5}C_2 = 30$.

Now, to find the integral solution, where negative integers are also allowed, any two of the factors in each factorisation may be negative. Therefore, the number of ways to associate negative sign in each case is ${}^{3}C_2 = 3$.

Hence, total number of integral solution, is $30 + 3 \times 30 = 120$

Note if any box remains empty, say x, then $x = 1$

11. Let the number of green balls be x. Then, the number of red balls is $2x$. Let the number of blue balls be y.

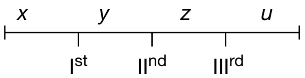
Then, $x + 2x + y = 10$

$\Rightarrow$ $3x + y = 10$

$\Rightarrow$ $y = 10 - 3x$

Clearly, x can take value 0, 1, 2, 3. The corresponding values of y are 10, 7, 4 and 1. Thus, the possibilities are (0, 10, 0), (2, 7, 1), (4, 4, 2) and (6, 1, 3), where (r, b, g) denotes the number of red, blue and green balls.

12. Let x be the number of books to the left of the first book chosen, y be the number of books between the first and second, z be the number of books between the second and third and u be the number of books to the right of the third book.



Then, $x, u \geq 0$, $y, z \geq 1$ and $x + y + z + u = 97$

Let $y_1 = y - 1$ and $z_1 = z - 1$. Then, $y_1 \geq 0$, $z_1 \geq 0$ such that

$$x + y_1 + z_1 + u = 95 \qquad \ldots(1)$$

$\therefore$ Total number of ways in which 3 books can be selected = Number of non-negative integral solution of (1)

$= {}^{95+4-1}C_{4-1} = 98C_3$

$\therefore$ $k = 3$

13. Given $n_1 = x_1x_2x_3$

and $n_2 = y_1y_2y_3$

n_1 and n_2 can be subtracted without borrowing at any stage, if $x_i \leq y_i$

Value of y_3	Value of x_3
9	0, 1, 2, … 9
8	0, 1, 2, … 8
7	0, 1, 2, … 7
6	0, 1, 2, … 6
5	0, 1, 2, … 5
4	0, 1, 2, … 4
3	0, 1, 2, 3
2	0, 1, 2
1	0, 1
0	0

Similarly for y_2 and x_2 but for y_1 and x_1 unit digit can not be zero.

So total pairs of n_1 and $n_2 = (1 + 2 + 3 + \ldots + 10)^2(1 + 2 + 3 + \ldots + 9) = 45.55^2$

14. Put $x = a - 1$, $y = b - 1$, $z = c - 1$, $t = d - 1$, then $a, b, c, d \geq 0$ and $(a - 1) + (b - 1) + (c - 1) + (d - 1) = 20$

$\Rightarrow$ $a + b + c + d = 24$

which is equivalent to the number of ways of distributing 24 identical objects into 4 different boxes, if empty boxes are allowed which is

${}^{24+4-1}C_{4-1} = {}^{27}C_3$

15. Let $x = p - 5$, $y = q - 5$ and $z = r - 5$, then $p, q, r \geq 0$ and the given equation reduces to

$$p + q + r = 15 \qquad \ldots(1)$$

Now, we have to find non-negative integral solution of Eq. (1). The total number of such solution is

$${}^{15+3-1}C_{3-1} = {}^{17}C_2 = \frac{17 \times 16}{2} = 136$$

16. Here $x \geq 1$, $y \geq 1$, $z \geq 1$

$x - 1 \geq 0$; $y - 1 \geq 0$; $z - 1 \geq t_3$

Let $x - 1 = t_1 : y - 1 = t_2$, $z - 1 = t_3$

Now, $1 + t_1 + 1 + t_2 + 1 + t_3 = 6$

$t_1 + t_2 + t_3 = 3$ where $t_1, t_2, t_3 \geq 0$

No. of non-negative integral solutions $= {}^{3+3-1}C_{3-1} = 10$

Test Your Concepts-VI

1. Let D_r denote the number of ways in which r things goes to r corresponding places wrongly.

$$\therefore \quad D_r = r!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^r \frac{1}{r!}\right]$$

Case I: Exactly two letters are placed in wrong envelopes.

$\therefore$ Required number of ways $= {}^{5}C_2 \cdot D_2$

Case II: Exactly three letters are placed in wrong envelopes.

$\therefore$ Required number of ways $= {}^{5}C_2 \cdot D_3$

Case III: Exactly four letters are placed in wrong envelopes.

$\therefore$ Required number of ways $= {}^{5}C_4 \cdot D_4$

Case IV: All five letters are placed in wrong envelopes.

$\therefore$ Required number of ways $= D_5$

So, total number of required ways

$$= {}^5C_4 \cdot D_2 + {}^5C_3 \cdot D_3 + {}^5C_4 \cdot D_4 + D_5$$

$$= 10 \cdot 1 + 10 \cdot 2 + 5 \cdot 9 + 44$$

$$= 10 + 20 + 45 + 44 = 119$$

Alternate Method

Required number of ways = Number of arrangement of 5 letters in 5 envelopes – Number of ways of placing all letters in right envelope

$$= {}^5P_5 - 1 = 5! - 1 = 120 - 1 = 119$$

2. We know that, if there are n letters and n corresponding envelopes, then the number of ways in which letters can be placed in the evelopes, so that no letter is placed in the correct envelope is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Here, $n = 10$

$\therefore$ Required number of ways $= 10!\left[\frac{1}{2!} - \ldots + \frac{1}{10!}\right]$

3. $E_2(10!) = \left[\frac{10}{2}\right] + \left[\frac{10}{2^2}\right] + \left[\frac{10}{2^3}\right] = 8 = p$

$$E_3(10!) = \left[\frac{10}{3}\right] + \left[\frac{10}{3^2}\right] = 4 = q$$

$$E_5(10!) = \left[\frac{10}{5}\right] = 2 = r$$

$$E_7(10!) = \left[\frac{10}{7}\right] = 1 = s$$

4. $x = 20 \times 19 \times 18 \times 17 \times \ldots \times 11$

$$= \frac{\{20 \times 19 \times 18 \times \ldots \times 11\} \cdot (10!)}{(10!)}$$

$$E_2(20!) = \left[\frac{20}{2}\right] + \left[\frac{20}{2^2}\right] + \left[\frac{20}{2^3}\right] + \left[\frac{20}{2^4}\right]$$

$$= 10 + 5 + 2 + 1 = 18$$

$$\therefore \quad E_2\left\{\frac{(20)!}{(10)!}\right\} = 18 - 8 = 10$$

5. $E_2(60!) = \left[\frac{60}{2}\right] + \left[\frac{60}{2^2}\right] + \left[\frac{60}{2^3}\right] + \left[\frac{60}{2^4}\right] + \left[\frac{60}{2^5}\right]$

$$= 30 + 15 + 7 + 3 + 1 = 56$$

$$E_2(60!) = \left[\frac{60}{5}\right] + \left[\frac{60}{5^2}\right]$$

$$= 12 + 2 = 14$$

$$\therefore \quad E_{10}(60!) = 14$$

6. Since, perfect square of every natural number ends with {0, 1, 5, 6, 9}, but square of any natural number cannot have unit's digit {2, 3, 7, 8} …(1)

Also, $1! + 2! + 3! + 4! + 5! + \ldots + x!$, when $x \geq 4$ units place ends with 3.

Thus, $x \in \{1, 2, 3\}$

Now, if $x = 1 \Rightarrow 1! = 1$ [perfect square]

if $x = 2 \Rightarrow 1! = 2! = 3$ [cannot be perfect square]

if $x = 3 \Rightarrow 1! + 2! + 3! = 9$ [perfect square]

Thus, x can take value1 and 3.

$\therefore$ Integral solution for x is two.

7. $${}^{2002}C_{1001} = \frac{(2002)!}{(1001)!(1001)!}$$

Number of zero's in (2002)! are = 400 + 80 + 16 + 3 = 499 = exponent of 5 in 2002!

Number of zero's in $(1001!)^2 = 2(200 + 40 + 8 + 1) = 498$ = 2(exponent of 5 in 1001!)

Hence, number of zero's in $\dfrac{(2002)!}{(1001!)^2} = 1$

Single Correct Choice Type Questions

1. There are 9 balls in the box out of which three balls can be drawn in ${}^9C_3$ ways. But these ways also include those ways in which 3 black balls are drawn. The total number of such ways is ${}^6C_3$.

Hence, the number of ways in which at least one black ball is included

= Total number of ways of drawing 3 balls from 9 balls

Number of ways of drawing 3 balls from 9 balls

$$= {}^9C_3 - {}^6C_3 = 84 - 20 = 64$$

Hence, the correct answer is (B).

2. Use 1, 2, 3, 4, 5 then we get 5! Use 0, 1, 2, 4, 5 then we get 4 × 4! number of ways = 216

Hence, the correct answer is (A)

3. Using 1, 4, 9 total three digits numbers are $3 \cdot 3 \cdot 3 = 27$. So each digit can appear 9 times at unit, 10's and 100's place. Sum = 9 (14) 111 = 126 × 111 = 13986

Hence, the correct answer is (C).

4. Product will be divisible by 3 if atleast one digit is 0, 3, 6, 9 hence total 4 digit no. $= 9.10^3$, number of 4 digit no. without $0, 3, 6, 9 = 6^4$. number of no. $= 9000 - 1296 = 7704$

Hence, the correct answer is (D).

5. Suppose the arrangement is $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$. If we select five consentive positions both x_4 and x_5 will always be selected, and either of them should contain 5.

Hence $2 \times 7!$

Hence, the correct answer is (B).

6. Given $m \leq 30$, $n \leq 30 \Rightarrow$ total cases $= 30 \times 30 = 900$

required condition : $2n \geq m$, $2m \geq n$

Let us find ordered pairs (m, n) such that $2n < m$, $2m < n$. By symmetry will get same answer for both conditions hence let's evaluate only one $2m < n$

value of n	no. of points (m, n)
1, 2	0
3, 4	1
5, 6	2
...... 29, 30	14

Required number = $900 - 2 \times \sum 14 = 480$

Hence, the correct answer is (B).

7. $1440 = 2^5 \times 3^2 \times 5^1$

Number of unordered points (x, y) such that $xy = 1400$ is $\frac{6 \times 3 \times 2}{2} = 18$

product of divisors of $1440 = 1440^{18}$

[Because all divisors can be clubbed in 18 pairs each giving product 1440]

$= 2^{90} \times 3^{36} \times 5^{18}$

$= (2^3 \cdot 3)^{30}\, 3^6 \cdot 5^{18}$

$= (24)^{30}\, 3^6 \cdot 5^{18}$

there $x = 30$

Hence, the correct answer is (B).

8. Both 'a' and 'b' are factors of $2^3 5^7 11^{13}$ and so, $a = 2^x 5^y 11^z$ and $b = 2^p 5^q 11^r$ for some non-negative integers (x, y, z, p, q, r). Since, lcm of a and b is $2^3 5^7 11^{13}$. So, max $(x, p) = 3$, $\max(y, q) = 7$ and $\max(z, r) = 13$. So, (x, p) can be equal to (0, 3) (1, 3) (2, 3) (3, 3), (3, 2) (3, 1) or (3, 0). So, there are 7 choices. Similarly for (y, q) and (z, r) there are 15 and 27 choices, so, net ordered pairs $= 7 \times 15 \times 27 = 2835$

Hence, the correct answer is (A).

9. Let the square has dimensions $n \times n$. If n is even, total no. of blue (diagonal) tiles $= n + n = 2n$

If n is odd, then total no. of blue tiles $= n + n - 1 = 2n - 1$

(The middle tile is be counted twice). Hence, $2n - 1 = 121$

$\Rightarrow n = 61$

So, net red tiles = $61 \times 61 - 121 = 3600$

Hence, the correct answer is (C).

10. Choose any three elements from set A such that $f(x) = y_1$ by ${}^7C_3$ ways, and rest 4 elements of set A and 4 elements of set B can be arranged by $3^4 - 3(2^4) + 3$ ways. So total number of functions are ${}^7C_3\left(3^4 - 3.2^4 + 3\right) = 1260$.

Hence, the correct answer is (D).

11. Let d be the common difference of the A.P., then

$4 = abc = (b - d)\, b(b + d) = b(b^2 - d^2)$

$\Rightarrow b^3 = 4 + bd^2 \geq 4, \left[ab > 0, d^2 \geq 0\right] \Rightarrow b \geq 2^{2/3}$

Hence, the correct answer is (B).

12. $x_1, x_2, ...x_{10}, a, a, a$ (sticks) have to be arranged so that no two a's are together and a is not at the ends.

R . N. is $= 10! \times {}^9C_3 = 10! \times 84$

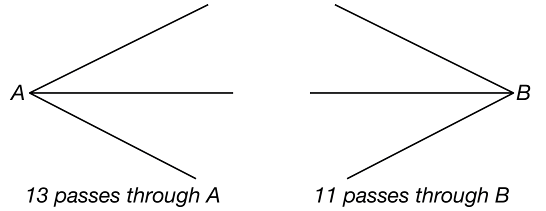
Hence, the correct answer is (C).

13. Fill 10 places each by the digits 0 to 4 by 5^{10} ways. For positive number, we exclude zero.

$\therefore$ Required number $= 5^{10} - 1$

Hence, the correct answer is (C).

14.



Number of intersection points

$= {}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ $\quad$ ($\therefore$ Two points A and B)

$= 535.$

Hence, the correct answer is (B).

15.

	A	B
C	5	4
D	4	5

Required number of ways of selecting 8 items out of '18' so that selected ones belongs equally to A, B, C, D.

$$= \left({}^5C_2\right)^2 . \left({}^4C_2\right)^2 + \left({}^5C_1\right)^2 . \left({}^4C_3\right)^2 + \left({}^5C_3\right)^2 . \left({}^4C_1\right)^2 + \left({}^5C_4\right)^2 + \left({}^4C_4\right)^2$$

$= 100 \times 36 + 25 \times 16 + 100 \times 16 + 25 + 1$

$= 3600 + 400 + 1600 + 26 = 5626$

Hence, the correct answer is (B).

16. It is a relative order problem.

Required ways $= {}^6C_2 \cdot 1 \cdot {}^4C_2 \cdot 1 = 90$ ways

Hence, the correct answer is (D).

17. The boys can be seated in 5! ways.

If the girls g_1 and g_2 do not want to sit by the side of A_1(*says*).

The two gaps.

$A_6 - A_1$ and $A_1 - A_2$ must be filled by two of the remaining and in ${}^4P_2$ ways.

The other four gaps can be filled in 4! ways

Hence the number of ways $= 5! \times {}^4P_2 \times 4! = 5! \times \frac{4!}{2!} \times 4!$

$= 2.6!4!$

Hence, the correct answer is (C).

18. Wonder beginning with A - - - - = 4! = 24

G - - - - = $\frac{4!}{2!} = 12$

I A - - - 3! = 6

IG - - - $\frac{3!}{2!} = 3$

I N A A G 1

I N A G A 1

Hence, the correct answer is (D).

19. Given

$$^{n-2}C_{10} = 3 \cdot {}^{n-3}C_9 \Rightarrow \frac{(n-2)!}{(n-12)!10!}$$

$$= 3\frac{(n-3)!}{(n-12)!9!} \Rightarrow n-2 = 30 \Rightarrow n = 32$$

Hence, the correct answer is (A).

20. No. of function from A to B = $7^4 = 2^{401}$ No. of injection from A to B = $^7P_4 = 840$

Not injections = Total functions – No. of Injective function

$= 2401 - 840 = 156$

Hence, the correct answer is (B).

21. Total number of ways of selecting 3 out of 12 is $^{12}C_3 = 220$. All three are together i.e., (123, 234, 345, …, 212) = 12 ways

$$\left.\begin{array}{ll} \text{2 together and 1 away} & \\ 12 & 45\ldots 11(8) \\ 23 & 56\ldots 12(8) \end{array}\right\} \to 12 \times 8 = 96 \text{ ways}$$

Required number of arrangements = 220 – 12 – 96 = 112

Hence, the correct answer is (C).

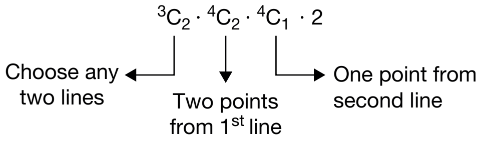
22. 2520 $= 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1$. Hence sum of all divisions is $\frac{2^4-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot \frac{5^2-1}{5-1} \cdot \frac{7^2-1}{7-1} = 15 \times 13 \times 6 \times 8 = 9360$ other than 1 and 2520 is 9360 – 2521 = 6839

Hence, the correct answer is (A).

23. **Case I:** When we choose one point from each line so number of ways = $^4C_1 \cdot {}^4C_1 \cdot {}^4C_1$

Case II: When we choose two points from any one line and one point from other line.

This can be done by



$^3C_2 \cdot {}^4C_2 \cdot {}^4C_1 \cdot 2$
Choose any two lines
Two points from 1st line
One point from second line

2 is multiplied because we can also choose one point from 1st line and two points from second line.

So total number of ways

$$\left({}^4C_1\right)^3 + {}^3C_2 \cdot {}^4C_2 \cdot {}^4C_1 \cdot 2 = 208$$

Hence, the correct answer is (C).

24. Let $x_1 = 2a - 1$, $x_2 = 2b - 1$, $x_3 = 2c - 1$

$2a - 1 + 2b - 1 + 2c - 1 = 51$

$\Rightarrow a + b + c = 27$, $a, b, c \geq 1$

coeff x^{24} in $(1 - x)^{-3} = {}^{26}C_2 = 325$

Hence, the correct answer is (C).

25. Any three numbers x, y, z from {1, 2, 3, ...10} can be chosen to get triplet (x, y, z) w.r.t coordinate $x < y < z$ in $^{10}C_3$ ways any two numbers from the form (x, y, z) can be chosen in $^{10}C_2$ ways total no. of triplets $= {}^{10}C_2 + {}^{10}C_3 = 165$

Hence, the correct answer is (C).

26. Number of ways of making 30 in 7 shots is equal

coeff. of x^{30} in $(x^0 + x^2 + x^3 + x^4 + x^5)^7$

$$=((1 + x + x^2 + x^3) + x^4 (x + 1))^7$$

$$= x^{28} (x + 1)^7 + {}^7C_1 \,.\, x^{24} (x + 1)^6 (1 + x^2 + x^3) + {}^7C_2 \; x^{20} (x + 1)^5 (x^3 + x^2 + 1)^2$$

$$= {}^7C_5 + {}^7C_1 ({}^6C_3 + {}^6C_2 + {}^6C_0) + {}^7C_2 ({}^5C_1 + 2)$$

$$= 420$$

Hence, the correct answer is (C).

27. $36 = 2^2.3^2$

No. of integer [2, 2000] divisible by 2 = 1000, divisible by 3 = 666

Divisible by 2 and 3 = $\left[\frac{2000}{6}\right] = 333$

No. of integers neither divisible by 2 nor divisible by 3 = 1999 – (1000 + 666 – 333) = 666

Hence, the correct answer is (A).

28. N = 321546

Number of numbers beginning with 1	= 120	1					
Number of numbers beginning with 2	= 120	2					
Starting with 31	= 24	3	1				
Starting with 3214	= 2	3	2	1	4		
Finally	= 1	3	2	1	5	4	6

Hence N has 267*th* position

Hence, the correct answer is (A).

29. No. of students gave wrong answers to atleast i questions = 2^{n-i} No. of students gave wrong answers to exactly n questions = No. of students gave wrong answers to at least n questions = $2^0 = 1$

No. of students gave wrong answers to exactly i questions = $2^{n-i} - 2^{n-(i+1)}$

$$\text{No. of wrong answers} = \sum_{i=1}^{n-1} i\left(2^{n-i} - 2^{n-(i+1)}\right) + (n)$$

$$= 2047$$

$$1 + 2 + 2^2 + ...2^{n-1} = 2047$$

$$n = 11$$

Hence, the correct answer is (B).

30. Required no. ways $= {}^{10}C_8 \left[{}^8C_1 \frac{10!}{3!} + {}^8C_2 \frac{10!}{2!2!} \right]$

$$= {}^{10}C_8 . \frac{10!.25}{3}$$

Hence, the correct answer is (A).

31. $^8P_{8-6} = {}^8P_2 = 56$

Hence, the correct answer is (D).

32. Coefficient of x^{30}in$\left(x^2+x^3+x^4+......x^{16}\right)^8$

Coefficient of x^{14} in $(1+x+x^2+\ldots+x^{14})^8$
Coefficient of x^{14} in $(1-x^{15})^8\,(1-x)^{-8}$
$= {}^{21}C_{14} = {}^{21}C_7$

Hence, the correct answer is (A).

33. $(2k_1+1)+(2k_2+1)+(2k_3+1)+(2k_4+1)=20$
$\rightarrow k_1+k_2+k_3+k_4=8$
$k_i \geq 0$

Required number $= {}^{4+8-1}C_8 = {}^{11}C_3 = 165$

Hence, the correct answer is (B).

34. Single digits = 9
Two digits = 9 × 10 = 90 → positions = 180
Three digits start with 1 = 100 → 100 × 3 = 300
Three digits start with 2 → 300
Three digits start with 3 → 300
Three digits start with 4 → 300
Three digits start with 5 → 300
Three digits start with 6 → 300
Three digits start with 700 → 3
701 → 3
702 → 3
703 → 3
704 → 3
705 → 3

Total positions 2007
2007th digit is '5'

Hence, the correct answer is (B).

35. Coefficient x^{20} in $(1+x+x^2+.......+x^{10})^4$
Coefficient x^{20} in $(1-x^{11})^4\,(1-x)^{-4}$
= Coefficient x^{20} in $(1-4x^{11}+\ldots)(1-x)^{-4}$
$= {}^{4+20-1}C_{20} - 4\,{}^{4+9-1}C_9$
$= {}^{23}C_3 - 4\left({}^{12}C_3\right) = 891$

Hence, the correct answer is (A).

36. $n\,(D_2)=500$, $n\,(D_5)=200$,
$n(D_2 \cap D_5) = 100 = n(D_{10})$ [∵ L.C.M of (2, 5) = 10
$n(D_2 \cup D_5) = nD_2 + nD_5 - n(D_2 \cap D_5)$
$\Rightarrow n(D_2 \cup D_5) = 600$
Out of these, 250 are in D_4

Hence, the correct answer is (A).

37. Number of arrangements of 4 letters all of which are different = 4!
Number of arrangements of 4 letters consisting of 2 identical and 2 different = ${}^2C_1 \times {}^3C_2 \times \frac{4!}{2!}$
Number of arrangements of 4 letters consisting of 2 identical of one kind and 2 identical of another kind = $\frac{4!}{(2!)^2}$
Sum = 24 + 72 + 6 = 102

Hence, the correct answer is (C).

38. $10800 = 2^6\,3^3\,5^2$
$4m+2 = 2(2m+1)$, 2 must occur exactly once
$1 \times (3+1)(2+1) = 12$

Hence, the correct answer is (B).

39. $x_1+x_2+x_3+x_4=15, \quad x_i \geq 2$
$y_1+y_2+y_3+y_4=7; \quad y_i = x_i - 2 \geq 0$
${}^{7+4-1}C_{4-1} = {}^{10}C_3$

Hence, the correct answer is (A).

40. $120 = 2^3\,3^1\,5^1$
Let factor $a = 2^x\,3^y\,5^z = x_1x_2x_3$
Number of positive integral solution

$$=\left(\sum_{x=0}^{3} {}^{x+2}C_2\right)\left(\sum_{y=0}^{1} {}^{y+2}C_2\right)\left(\sum_{z=0}^{1} {}^{z+2}C_2\right)$$

$$=\left({}^2C_2+{}^3C_2+{}^4C_2+{}^5C_2\right)\left({}^2C_2+{}^3C_2\right)$$

$$\left({}^2C_2+{}^3C_2\right)=320$$

Hence, the correct answer is (B).

41. Arrange the remaining letters in 5! ways

| P | E | M | T | C

In the 6 gaps arrange the 3 S s and 3 I s

in $\frac{6!}{3!3!}$ ways = 20

The answer is $5! \times 20 = 2400$

Hence, the correct answer is (C).

42. $$\sum \frac{r}{{}^nC_r} = \sum \frac{n-r}{{}^nC_r} = \frac{n}{2}\sum \frac{1}{{}^nC_r}$$

$$\because \sum_{r=0}^{n} \frac{n^2-3n+2}{2\,{}^nC_r} = \sum_{r=0}^{n} \frac{r}{{}^nC_r}$$

$$\Rightarrow \frac{n^2-3n+2}{2}\sum_{r=0}^{n}\frac{1}{{}^nC_r} = \frac{n}{2}\sum_{r=0}^{n}\frac{1}{{}^nC_r} \Rightarrow n^2-3n+2=n$$

$$\Rightarrow n^2-4n+3=0 \;\Rightarrow\; n=3(\because n>1)$$

Area of equilateral Δ with side $\sqrt{3}$ is $\frac{3\sqrt{3}}{4}$

Hence, the correct answer is (B).

43. B, s_1, s_2, s_3, s_4, s_5, D
For a particular class the total number of different tickets from first intermediate station is 5
No. of different tickets from second intermediate station is 4
So the total no. of different tickets is 5 + 4 + 3 + 2 + 1 = 15.
for the remaining classes the total no. of different tickets is = 3 × 15 = 45
Required number of ways = ${}^{45}C_{10}$

Hence, the correct answer is (C).

44. Required number of way

$$= \underbrace{{}^6C_1}_{\substack{\text{Selection of}\\ \text{one couple}}} \cdot \underbrace{{}^5C_4 \cdot 2^4}_{\substack{\text{Choose 4 couple and select}\\ \text{either husband or wife from each}\\ \text{pair}}} = 480$$

Hence, the correct answer is (B).

45. Arrange two rings on one particular finger in ${}^6P_2$ ways. Now the remaining 4 rings can be arranged in remaining 3 fingers. The answer is ${}^6P_2 \times 3^4$

Hence, the correct answer is (A).

46. First arrange 12 persons $A_4, A_5, ...A_{15}$ in ${}^{15}P_{12}$ ways. There remains 3 Places. Keep A_1 in the first place and arrange A_2, A_3 in the remaining two places in 2! ways

Req no of arrangements = ${}^{15}P_{12}.1.2!$

Hence, the correct answer is (A).

47. The other 5 letters in 11 places can be arranged in ${}^{11}P_5$ ways Then there remains 6 places. Choose two of them in ${}^6C_2$ ways, in these places we can put G, R only (i) one way. Similarly others.

$$\text{Req} = {}^{11}P_5 \ {}^6C_2 \cdot {}^4C_2 \cdot {}^2C_2 = \frac{\lfloor\underline{11}}{8}.$$

Hence, the correct answer is (A).

48. Four letter words in which all the letters are different $= \lfloor\underline{4} = 24$

Four letter words in which 3 letters are alike and one different $= \dfrac{\lfloor\underline{4}}{\lfloor\underline{3}} \times 3 = 12$

Four letter words in which two letters are alike and rest are different $= \dfrac{4!}{2!} \times 3 = 36$

$$\text{Re}q = 24 + 12 + 36 = 72$$

Hence, the correct answer is (C).

49. If a, b, c are in A.P. the $2b = a + c \Rightarrow a + c$ is even

both a and c are either even or odd

From 1 to 21 there are 10 even and 11 Odd numbers

Req no = ${}^{10}C_2 + {}^{11}C_2 = 100$

Hence, the correct answer is (C).

50. Let $15 = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^f$

$$E_2[\lfloor\underline{15}] = \left[\frac{15}{2}\right] + \left[\frac{15}{2^2}\right] + \left[\frac{15}{2^3}\right] = 7 + 3 + 1 = 11$$

$$E_5(15!) = \left[\frac{15}{5}\right] + \left[\frac{15}{5^2}\right] = 3$$

$$E_3 = [\lfloor\underline{15}] = 6,\ E_7[\lfloor\underline{15}] = 2,\ E_{11}[\lfloor\underline{15}] = E_{13}[\lfloor\underline{15}] = 1$$

The no of proper division = 12.74.3.2.2-2 = 4030

Hence, the correct answer is (B).

51. Seven digits fill up by the digits 1,2 and 3

Case: 1

$1\,1\,1\,1\,1\ \underline{3}\ \ \underline{2} = \dfrac{7!}{5!} = 42$

Case: 2

$1\,1\,1\,1\ \underline{2}\ \ \underline{2}\ \ \underline{2} = \dfrac{7!}{4!3!} = 35$

Hence, the correct answer is (C).

52. $x + y + z + t = 20,$

$$x \ge -1, \quad y \ge -1, \quad z \ge -1, \quad t \ge -1$$

$$x + 1 \ge 0,\ y + 1 \ge 0,\ z + 1 \ge 0,\ t + 1 \ge 0$$

$$k + l + m + n = 24,$$

where $k = x + 1, \quad l = y + 1, \quad m = z + 1, \quad n = t + 1$

$$k \ge 0, \qquad l \ge 0, \qquad m \ge 0, \qquad n \ge 0$$

number of integral solutions is

${}^{24+4-1}C_{4-1} = {}^{27}C_3$ (or) ${}^{27}C_{24}$

Hence, the correct answer is (D).

53. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2007}$

$\Rightarrow \quad (x + y)\,2007 = xy$

$\Rightarrow \quad xy - 2007x - 2007y = 0$

$$(x - 2007)(y - 2007) = 2007^2 = 3^4 \times 223^2$$

The number of pairs is equal to the number of divisors of 2007^2 that is $(4+1)\times(2+1) = 15$

Since $x < y$ so required number of pairs = 7

Hence, the correct answer is (C).

54. **Case 1:** All three digits are repeated {111, 222, …, 999} = 9

Case 2: One digit is repeated without using zero {114, 117, 225, 228, 336, 339, 441, 447, 552, 663, 669, 771, 774, 882, 885, 993, 996} $= 18 \times \dfrac{3!}{2!} = 54$

Using zeroes {330, 660, 990} = 3 × 2! = 6

and {300, 600, 900} = 3

So required numbers = 9 + 54 + 6 + 3 = 72

Hence, the correct answer is (D).

55.

1	1	x	y
1	x	1	y
1	x	y	1

$3 \times {}^9C_2 \times 2 = 108 \times 2 = 216$

1	x	x	y
1	x	y	x
1	y	x	x

$3 \times {}^9C_2 \times 2 = 216$

Hence, the correct answer is (C).

56. From 1 to 9 there are $9 \times 1 = 9$ digits.

From 10 to 99 there are $90 \times 2 = 180$ digits

From 100 to 599 there are $500 \times 3 = 1500$ digits.

From 600 to 659 there are $60 \times 3 = 180$ digits.

From 660 to 666 there are $7 \times 3 = 21$ digits.

Hence, the correct answer is (B).

57. If we put minimum number of balls required in each box, then remaining 120 balls can be put in $^{135}C_{15}$ ways without any restriction.

Hence, the correct answer is (B).

58. $2^a + 3^b + 5^c = 2^a + (4-1)^b + (4+1)^c$

$= 2^a + 4\lambda_1 + (-1)^b + 4\lambda_2 + (1)^c \quad (\lambda_1, \lambda_2$ are integers)

If $a = 1$, b should be even and c can be any number. On the other hand if $a \neq 1$, b should be odd and c can be any number.

$\Rightarrow$ Total number of ordered pairs $= 5 \times 10 + 9 \times 5 \times 10 = 500$.

Hence, the correct answer is (B).

59. Assume that the number of stairs to be climbed two-steps at a time is $2r$ (r varies from 0 to n). The number of stairs to be climbed one-step at a time, is, thus, $2n - 2r$. Total number of steps required to be taken $= 2n - 2r + r = 2n - r$.

Number of ways of selecting r steps out of

$2n - r = {}^{2n-r}C_r$.

Hence, total number of ways of climbing up $= \sum_{r=0}^{n} {}^{2n-r}C_r$

Hence, the correct answer is (A).

60. Excluding 7 remaining 3 digit can be selected in $^9C_3$ ways and arranged in 4! Ways.

Number of ways $= {}^9C_3\, 4! -$ (when '0' and '7' always included and '0' occupies the first place)

$= {}^9C_3 \cdot 4! - {}^8C_2 \cdot 3! = 84 \cdot 4 \cdot 3! - 28 \cdot 3! = 7 \cdot 3![48 - 4]$

$= 6 \cdot 7 \cdot 44 = 1848$

Hence, the correct answer is (C).

61.

Identical	Distinct
0	10
1	9
2	8
.	.
.	.
.	.
.	.
10	0

$\because$ No. of ways

$= {}^{20}C_{10} + {}^{20}C_9 + {}^{20}C_8 + ... + {}^{20}C_0 = \dfrac{2^{20} + {}^{20}C_{10}}{2}$

Hence, the correct answer is (D).

62. No. of ways = No. of non-negative integral solutions to equation $x_1 + x_2 + x_3 = 10,\ 0 \leq x_1, x_2 x_3 \leq 6$

$\Rightarrow$ Coefficient of x^{10} in $(1 + x + x^2 + ...x^6)3$

$\Rightarrow$ Coefficient of x^{10} in $(1 - x^7)^3\,(1 - x)^{-3}$

$= {}^{12}C_{10} - 3 \times {}^5C_3 = 36$

Hence, the correct answer is (D).

63. $n(A)$: number of words starting with A

$n(B)$: number of words starting with E

$n(C)$: number of words starting with middle term I

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$

$- n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

$= 6! + 6! + 6! - 5! - 5! - 5! + 4! = 1824$

Hence, the correct answer is (B).

64. If all 5 girls sit together $= {}^5C_5 \times 5! \times 2 \times {}^{11}P_9$

If 4 girls sit together and no other person with them $= {}^5C_4 \times 4! \times 4 \times {}^{11}P_{10}$

If 4 girls and one by sit together $= {}^5C_4 \times 4! \times 4 \times {}^9C_1 \times {}^{11}P_9$

Total ways

$= {}^{11}P_9 \times 5! \times 2 + 5! \times 4 \times {}^{11}P_{10} + 5! \times 4 \times {}^9C_1 \times {}^{11}P_9$

Hence, the correct answer is (A).

65. Number of cubes obtained from one face which are painted on only one side $= (n - 2)^2$

No. of cubes which are unpainted $= (n - 2)^3$

$(n - 2)^2 \times 6 = (n - 2)^3$

$\Rightarrow n - 2 = 6 \qquad \Rightarrow n = 8$

Hence, the correct answer is (C).

66. The letters other than vowels are SHSHNK which can be arranged in $\dfrac{6!}{2!2!}$ ways

Now in its each case, let the first A be placed in the r^{th} gap then the number of ways to place the 2nd A will be $(7 - r - 1)$.

So, the total number of ways $= \dfrac{6!}{2!2!} \sum_{r=1}^{5} (6 - r)$

$= \dfrac{6!}{2!2!} \times (5 + 4 + 3 + 2 + 1) = 2700.$

Hence, the correct answer is (A).

67. Given A(1, 2) and B(5, 5). Difference of x-coordinates $= 5 - 1 = 4$

$\therefore$ Exactly 4 rightward steps are needed.

Difference of y-coordinates $= 5 - 2 = 3$.

$\therefore$ Exactly 3 upward steps are needed.

Note: Order of the steps is immaterial.

Denote each rightward step by R and each upward step by U.

$\therefore$ The problem is arranging the letters RRRRUUU

No. of arrangements $= \dfrac{7!}{3! \times 4!} = 35$

Hence, the correct answer is (A).

68. – – – – – (5 blanks)

1^{st} blank can be filled in 9 ways

2^{nd} blank can be filled in 9 ways ________ since repetition is allowed.

3^{rd} blank can be filled in 9 ways

4^{th} blank can be filled in 9 ways

Now, we have to fill the 5^{th} blank carefully such that the number is divisible by 3. Add the 4 numbers in the first 4 blanks.

If their sum is in the form $3n$, then fill the last blank by 3, 6 or 9 so that the sum of all digits is divisible by 3.

If their sum is in the form $3n+1$, than fill the last blank by 2, 5 or 8.

If their sum is in the form $3n+2$, than fill the last blank by 1, 4 or 7.

Therefore, in any case, the last blank can be filled in 3 ways only.

$= 9 \times 9 \times 9 \times 9 \times 3 = 3^9$

Hence, the correct answer is (A).

69. Least digit used = 4

$\therefore$ We can use 4, 5, 6, 7, 8, 9. But remember that at least one 4 must be used.

– – – – (4 blanks)

1^{st} blank can be filled in 6 ways.

2^{nd} blank can be filled in 6 ways.

3^{rd} blank can be filled in 6 ways.

4^{th} blank can be filled in 6 ways.

$\therefore$ 4 blanks can be filled in 6^4 ways. But out of these, some may contain no 4 at all. Let us find them.

– – – – (4 blanks)

Each blank can be filled in 5 ways (by 5, 6, 7, 8, or 9)

$\therefore$ 5^4 ways (no 4 at all)

$= 6^4 - 5^4$ (at least one 4)

$= 671$.

Hence, the correct answer is (B).

70. $p = 1, q = 1, 2, 3, 4, 5, 6 \Rightarrow 6$

$p = 2, q = 1, 3, 4, 5, 6 \Rightarrow 3$

$p = 3, q = 1, 2, 4, 5, 6 \Rightarrow 4$

$p = 4, q = 1, 3, 5 \Rightarrow 3$

$p = 5, q = 1, 2, 3, 4, 6 \Rightarrow 5$

$p = 6, q = 1, 5 \Rightarrow 2$

Hence, the correct answer is (A).

71. There are I – 4, S – 4, P – 2, M – 1

Required number of words

$$\overset{\text{Gap selection}}{{}^8C_4} \times \overset{\text{Arrangement of 4I, 2P, M}}{\left(\frac{7!}{2!4!}\right)}$$

$= {}^8C_4 \times 7 \times {}^6C_4$

Hence, the correct answer is (B).

72. A_1 is always above A_2 so fix the position by ${}^{10}C_2$ ways and arrange rest persons by 8! ways.

Total arrangement $= {}^{10}C_2 \cdot 8! = \dfrac{10!}{2}$

Hence, the correct answer is (B).

73. Number of required ways $= {}^5C_4\, {}^8C_6 + {}^5C_5\, {}^8C_5 = 196$

Hence, the correct answer is (B).

74. Any interior intersection point corresponds to 4 of the fixed points, namely the 4 end points of the intersecting segments. Conversely, any 4 labeled points determine a quadrilateral, the diagonals of which intersect once within the circle.

Number of interior intersection points $= {}^9C_4 = 126$

Hence, the correct answer is (A).

75. Let the number of green balls be x. Then the number of red balls is $2x$. Let the number of blue balls be y. Then, $x + 2x + y = 10 \Rightarrow 3x + y = 10 \Rightarrow y = 10 - 3x$.

Clearly, x can take values 0, 1, 2, 3. The corresponding values of y are 10, 7, 4 and 1. Thus, the possibilities are (0, 10, 0), (2, 7, 1), (4, 4, 2) and (6, 1, 3), where (r, b, g) denotes the number of red, blue, green balls.

Hence, the correct answer is (B).

76. $= 5 \times 9^4 \times 1 + {}^5C_2 \times 9^3 \times 2 + {}^5C_3 \times 9^2 \times 3 + {}^5C_4 \times 9 \times 4 + {}^5C_5 \times 5$

$= 5 \times 10^4$

Hence, the correct answer is (A).

77. Total selections $= {}^{10}C_3 = 120$

Number of selections in which 3 stations are adjacent = 10

Number of selections in which 2 stations are adjacent = 6

But there are 10 such pairs.

$\Rightarrow$ Total invalid selections $= 10 + 6 \times 10 = 70$

Hence, the correct answer is (A).

78. A flight of stairs has 10 steps. A person can go up the steps one at a time, two at a time or any combination of 1s and 2s.

Let $x + 2y = 10$

where x is the number of times he takes single steps and y is the number of times he takes two steps.

	Cases	Total no. of ways
I :	When $x = 0$ and $y = 5$	$\dfrac{5!}{5!} = 1(2\,2\,2\,2\,2)$
II :	When $x = 2$ and $y = 4$	$\dfrac{6!}{2!\,4!} = 15(1\,1\,2\,2\,2\,2)$
III :	When $x = 4$ and $y = 3$	$\dfrac{7!}{4!\,3!} = 35(1\,1\,1\,1\,2\,2\,2)$
IV :	When $x = 6$ and $y = 2$	$\dfrac{8!}{2!\,6!} = 28(1\,1\,1\,1\,2\,2)$
V :	When $x = 8$ and $y = 1$	${}^9C_1 = 9\ (1\,1\,1\,1\,1\,1\,1\,1\,2)$
VI :	When $x = 10$ and $y = 0$	1 (1 1 1 1 1 1 1 1 1 1)

Hence, total no. of ways

$= 1 + 15 + 35 + 28 + 9 + 1 = 89$.

Hence, the correct answer is (D).

79. $(2^3 - 1)(2^4 - 1)(2^2 - 1) = 315$

Hence, the correct answer is (B).

80. $6!\left[1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6}\right]$

Hence, the correct answer is (B).

81. Let the number of balls in the 3 boxes be x, y, z respectively, then $x + y + z = 8, x, y, z \geq 1$ $\therefore$ Required number $= {}^{5+3-1}C_5 = {}^{7}C_5 = 21$

Hence, the correct answer is (B).

82. $x + y + z + 12 = 0, x, y, z$ are negative integers.

Let $x = -a, y = -b, z = -c, a, b, c$ are positive integers.

Then required number of points (x, y, z)

$=$ number of positive integral solutions of $a + b + c = 12$

$=$ coefficient of x^{12} in $(x + x^2 + \ldots)^3$

$=$ coefficient of x^9 in $(1 - x)^{-3}$

$= {}^{11}C_2 = \frac{11 \times 10}{2} = 55$

Hence, the correct answer is (A).

83. A voter can vote atleast one candidate and maximum four candidates so total number of ways $= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 = 385$

Hence, the correct answer is (B).

84. No. of matches in the first round $= {}^6C_2 + {}^6C_2$

No. of matches in the second round $= {}^6C_2$

No. of matches in the semifinal round $= {}^4C_2$

Total $= 15 + 15 + 15 + 6 + 1 = 52$

Hence, the correct answer is (C).

85. From the given condition, $\left\{\frac{x}{3}\right\} + \left\{\frac{3x}{3}\right\} + \left\{\frac{y}{2}\right\} + \left\{\frac{3y}{4}\right\} = 0$

$\Rightarrow \frac{x}{3}, \frac{3x}{2}, \frac{y}{2}, \frac{3y}{4}$ must be integers

$\therefore x = 6, 12, 18, 24$

$y = 4, 8, 12, 16, 20, 24, 28$

$\because$ No. of order pairs $= 4 \times 7 = 28$

Hence, the correct answer is (D).

86. $\sum_{i=1}^{n} \frac{1}{x_i} = \frac{93}{50}$

$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \frac{93}{50}$

$\Rightarrow \frac{93}{n} = \frac{93}{50}$

$\Rightarrow n = 50 = 2 \cdot 5^2$

Total no. of divisors $= k = (1 + 1)(2 + 1)$

$\Rightarrow k = 6$

Hence, the correct answer is (C).

87. $n! + (n + 1)! + (n + 2)! = n!$

$\Rightarrow$ Either 7 divides $(n + 2)$ or 49 divides $n!$

$\Rightarrow n = 5, 12, 14, 15, 16$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. The five digits numbers starting with 1 are

${}^8C_4 = 70$ $\begin{array}{|c|c|c|c|c|}\hline 1 & & & & \\ \hline\end{array}$

The five digits numbers starting with 2 are

${}^7C_4 = 35$ $\begin{array}{|c|c|c|c|c|}\hline 2 & & & & \\ \hline\end{array}$

Total $= 70 + 35 = 105$

$\therefore (105)^{th}$ five digits no. is 26789

Hence, (A), (B), (C) and (D) are correct.

2. (A) we have to make two groups of 7 persons and 4 persons $= 11!/4!7!$

No of ways of arranging them in circular tables $= (11!/4!7!).6!.3!$

(B) for first table of capacity of 7 person(5 girls + 2 boys) and 4 boys for second table $= {}^6C_2.6!.3! = \frac{(6!)^2}{8}$

(C) for first table of capacity of 7 person(1 girl + 6 boys) and 4 girls for second table $= {}^5C_1.6!.3! = \frac{5!6!}{4}$

(D) its addition of option C and D $= \frac{(6!)^2}{8} + \frac{5!6!}{4} = 6!(120)$

Hence, (A), (B), (C) and (D) are correct.

3. (A) $\left({}^6C_3 \times {}^4C_2\right)5!5! = (5!)^3$

$$\begin{array}{cccccccccc}\hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

(B) ${}^6C_1 \times 9!$

(C) $7 \times (4! \times 6!) = 4!7!$

(D) $\frac{10!}{6!} = {}^{10}P_4$

Hence, (A), (B), (C) and (D) are correct.

4. $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline\end{array}$

Required ways

$$= {}^6C_1 \times {}^5C_1 \times {}^4C_1 \times \frac{4!}{3!} + {}^6C_1 \times {}^5C_1 \times {}^4C_2 \times \frac{4!}{2!2!} = 1560$$

Hence, (B) and (D) are correct.

5. (A) $3!\,(1\ 1\ 1\ 1)\,(1 + 3 + 5 + 7)$

(B) $(1\ 1\ 1\ 1)\,(1 + 3 + 5 + 7).4^3$

(C) ${}^3P_2\,(1\ 1\ 1)\,(1 + 3 + 5 + 7) \neq 5328$

(D) $4^3\,(1 + 3 + 5 + 7) = 432$

Hence, the correct answer is (A).

6. No. of squares are

$S = 1^2 + 2^2 + 3^2 + \ldots\ldots.. + 8^2 = \frac{8(9)(17)}{6} = 204$

No. of rectangles $r = {}^9C_2 \cdot {}^9C_2 = 1296$

No. of non-congruent rectangles $= \sum_{r=1}^{8} r = 36$

No. of non-congruent squares $= 8$

Hence, (A), (B) and (C) are correct.

7. $$=1+\sum_{r=1}^{n} r\underline{|r}$$

$$=1+\sum_{r=1}^{n}(r+1-1)\underline{|r}$$

$$=1+\sum_{r=1}^{n}\left(\underline{|r+1}-\underline{|r}\right)=1+\underline{|n+1}-1=\underline{|n+1}$$

Hence, (A), (B), (C) and (D) are correct.

8. $n=2^{10}\,3^{7}\,5^{11}$

Total divisors = (10 +1)(7 + 1)(11 + 1) = 1056

(A) For perfect square we can choose from $(2^0, 2^2, 2^4, 2^6, 2^8, 2^{10}, 3^0, 3^2, 3^4, 3^6, 5^0, \ldots, 5^{10})$

$$=6\times 4\times 6=144$$

(B) Proper divisors = 1056 – 2 = 1054

(C) Odd divisors = (7 + 1)(11 + 1) = 96

(D) Even divisors = 1056 – 96 = 960

Hence, (A), (B) and (C) are correct.

9. There are 4! = 24 numbers which have digit 1 as their leading (the ten thousands') digit. Out of these, delete those which have digit 5 in their unit's place.

They are 3! = 6 in all.

Hence 24 – 6 = 18 numbers in set S have 1 as their leading digit.

The same number if 2 is the leading digit.

Now, let 3 be the leading digit.

There are 4 numbers in S which start with 31;

4 numbers in S which start with 32.

The next number is S is 3 4 1 5 2.

Hence the rank of 3 4 1 5 2 = (18 + 18 + 4 + 4 + 1)th

= 45th (A) is false

The cardinality of S is 5! – 4! = 96 (B) is false

Give an example to show that (C) is false

The greatest element of S is 54321 and the least is 12345.

Hence the difference in them = 41967

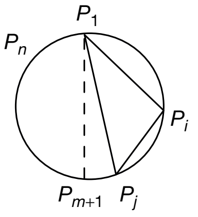
Hence, the correct answer is (D).

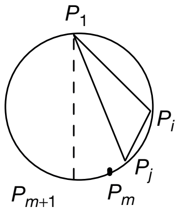
10. $x+y+z=10$

$x\geq 1, y\geq 1, z\geq 1$

${}^{10-1}C_{3-1}={}^{9}C_{2}$

Hence, (A) and (D) are correct.

11. 



n = even = $2m$ (say) n = odd = $2m+1$ (say)

If n is even, then P_{m+1} is diametrically opp. to P_1. We count the number of obtuse triangles $P_1P_jP_k$ with $\angle P_1$ acute. Then both P_j and P_k should lie on the same side of the line P_1P_{m+1}. Hence there are $2\times{}^{m-1}C_2$ obtuse triangles, since, there are $(m-1)$ vertices an each side of line P_1P_{m+1}. For all n vertices, there are $n\times 2\times{}^{m-1}C_2$ obtuse triangles. But there is over counting here, for any obtuse triangles, there are 2 acute angles, so, the net obtuse triangles are $\frac{1}{2}\times n\times 2\times{}^{m-1}C_2$

Similarly for odd n ; $\frac{1}{2}\times n\times 2\times{}^{m-1}C_2$

Hence, (A) and (C) are correct.

12. Consider the six subsets of possible combinations.

Let

$$A_1=\{(x,17,24)\,|\,1\leq x\leq 40\}$$
$$A_2=\{(x,24,17)\,|\,1\leq x\leq 40\}$$
$$A_3=\{(17,x,24)\,|\,1\leq x\leq 40\}$$
$$A_4=\{(24,x,17)\,|\,1\leq x\leq 40\}$$
$$A_5=\{(17,24,x)\,|\,1\leq x\leq 40\}$$
$$A_6=\{(24,17,x)\,|\,1\leq x\leq 40\}$$

In each set, there are 40 elements but some combinations have been counted twice, like (17, 17, 24) belongs to both A_1 and A_3. And in total there are 6 such elements.

So, Net possibilities = 240 – 6 = 234 hence time taken is 39 min.

Hence, the correct answer is (B).

13. The maximum numbers of trials required.

$$=(n)+(n-1)+(n-2)+\ldots\ldots\ldots+2+1$$

$$=\frac{n(n+1)}{2}={}^{n+1}C_2=\sum_{k=1}^{n}k$$

Hence, (C) and (D) are correct.

14. The required number of selections of 4 letters = coeff. of x^4 in the expansion of

$$\left(x^0+x^1\right)\left(x^0+x^1\right)\left(x^0+x^1+x^2\right)\left(x^0+x^1+x^2\right)\left(x^0+x^1\right)$$

$$=\text{coeff. of } x^4 \text{ in } (1+x)^3\left(1+x+x^2\right)^2=18$$

Hence, (A) and (C) are correct.

15. $p={}^5C_4\times{}^2C_1=10;\ q={}^5C_2\times({}^2C_1)^3=80;$

$r={}^5C_0\times({}^2C_1)^5=32$

$2q=5r$, $8p=q$, and $2(p+r)>q$

Hence, (A), (B) and (D) are correct.

16. (A) ${}^{21-1}C_{5-1}=4845$

(B) coefficient of x^{21} in $(x^2+x^3+\ldots..+x^{13})^5={}^{15}C_{11}=1365$

(C) $^{24}C_4 = 10626$

(D) $^{16}C_1 \times {}^7C_2 = 336$

Hence, (A), (B) and (D) are correct.

17. Let us consider a dummy ball numbered 5.

Case I:

When it goes to box no. 5 then the required ways is same as derangement of 4 which is $4!\sum_{r=0}^{4}\frac{(-1)^r}{r!} = 9$

Case II:

When it does not go the box no. 5 then the required ways =

$$5!\sum_{r=0}^{5}\frac{(-1)^r}{r!} = 44$$

So total no. of ways = 9 + 44 = 53

Hence, the correct answer is (C).

18. $144 = 2^4 \cdot 3^2$

(a) no. of divisors = $(4 + 1) \cdot (2 + 1) = 15$

(b) Sum of divisors = $(1 + 2 + 2^2 + 2^3 + 2^4)\ (1 + 3 + 3^2) = 403$

(c) Product of divisors = $(144)^{\frac{15}{2}} = (12)^{15}$

(d) Sum of reciprocals of divisors

$$= \frac{\text{sum of divisiors}}{144} = \frac{403}{144}$$

Hence, (A), (B), (C) and (D) are correct.

19. (A) $(2!)\ \frac{5!}{2!}$

(B) $\frac{6!}{2!}$

(C) $\frac{4!}{2!}\ (2!)$

(D) 360 – 120

Hence, (A), (B), (C) and (D) are correct.

20. **Case: I**

When all the selected coupons bear the same letter.

One letter can be selected from three letters in $^3C_1$ ways

⇒ Total number of ways of choosing 10 coupons bearing the same letter is $^3C_1 \times 1 = 3$

Case: II

When selected coupons bear two letters only

The number of ways of selecting two letters from 3 is $^3C_2$

⇒ The number of ways in which selected coupons bear two letters only = $^3C_2(2^{10} - 2)$

Hence required number of ways = $3 + {}^3C_2(2^{10} - 2) = 3(2^{10} - 1)$

Hence, the correct answer is (A).

21. In 1st round all the integers, which leaves the remainder 1 when divided by 15, will be marked last number of this category is 991

Next number to be marked is (991 + 15 – 1000) = 6 again, second round of integers which leaves the remainder '6' when divided by 15 will be marked.

Last number of this category is 996

Next number to be marked is (996 + 15 – 1000) = 11

Thus third round of integers which leaves the remainder 11 when divided by 15, will be marked last number of this category is 986

Next number to be marked is 986 + 15 – 1000 = 1 which is already been marked.

In first round 67, second round 67, and in third round 66 are marked.

∴ Marked number = 200

Unmarked numbers = 800.

Hence, the correct answer is (D).

22. If $Z = n + 1$ then $x \,\&\, y \in \{1, 2, ...n\}$

number of ways of $x \,\&\, y = n^2$

If $Z = n$ then $x, y \in \{1, 2, 3, ...n - 1\}$

number of ways of choosing $x \,\&\, y = (n - 1)^2$ and so on

Number of ways $= 1^2 + 2^2 + ... + n^2 = \sum n^2$

Hence, (A), (B) and (C) are correct.

23. (A) The number of ways = $^{10}C_3 \times 3! = 720$

(B) Each lady and her child can be arranged in a fixed order only.

∴ The total number of ways in which interview can be held

$$= \frac{6!}{2!2!2!} = 90$$

(C) **Case I:**

No box empty.

Then the number of ways = 3! = 6

Case II:

If one of the boxes is empty, then number of ways = $^3C_1(2^3 - 2) = 18$.

∴ total number of ways = 6 + 18= 24

(D) Total – (All red) – (All white)

$^{11}C_4 - {}^5C_4 - {}^6C_4 = 330 - 5 - 15 = 310$

Hence, (A) and (B) are correct.

Linked Comprehension Type Questions

1. Required number of ways

$$= {}^nC_1 \cdot (3)^{n-1}$$

one element is common

each element has three choices either S_1, S_2 or A

Hence, the correct answer is (B).

2. Each element ∈ $S_1 \cup S_2$ in 3 ways

Hence, the correct answer is (A).

3. As a, b, c are in A.P. $b = \frac{a + c}{2}$

$$(a + c) + \frac{a + c}{2}$$

$\Rightarrow a + c = 14$

⇒ number of solution = 13

Hence, the correct answer is (C).

4. $x_1 + x_2 + x_3 + x_4 = 22$

   $x_i = 2n_i + 1$

   $x_i \in \mathrm{I}^+ \cup \{0\}$

   $i = 1,2,3,4$

   $n_1 + n_2 + n_3 + n_4 = 9$

   number of solutions = $^{12}C_9$.

   Hence, the correct answer is (B).

5. The no. of ways of arranging 2`s is $^{15}C_4$. Fill the first empty position left after arranging the 2`s with a 0(1 way) and pick the remaining five places the position the remaining five zeros ® $^{10}C_5$ ways.

   $\therefore\ ^{15}C_4 \times 1 \times {}^{10}C_5$

   Hence, the correct answer is (A).

6. Put *a* in the first position, (1 way). Pick five other positions for the remaining O`s ($^{14}c_5$ ways), put *a* 1 in the first of the remaining positions (1 way), then arrange the remaining four 1`s ($^{8}C_4$ ways)

   $\therefore\ ^{14}C_5 \times {}^{8}C_4$

   Hence, the correct answer is (B).

7. AAELLLPR

   A --------- $\frac{7!}{3!}$

   E --------- $\frac{7!}{3!2!}$

   L --------- $\frac{7!}{2!2!}$

   PAA --------- $\frac{5!}{3!}$

   PAE --------- $\frac{5!}{3!}$

   PAL --------- $\frac{5!}{2!}$

   PARAE --------- $\frac{3!}{3!}$

   PARALE --------- $\frac{2!}{2!}$

   PARALLEL --------- 1, rank is 2623

   Hence, the correct answer is (D).

8. E E I I J T

   E --------- $\frac{5!}{2!} = 60$

   I --------- $\frac{5!}{2!} = 60$

   JE --------- $\frac{4!}{2!} = 12$

   JIE --------- $3! = 6$

   JIIE --------- $2! = 2$

   JIITEE --------- 1!. It is 141th word.

   Hence, the correct answer is (B).

9. One digit number = 6

   2 digit number = $6 \times 7 = 42$

   3 digit number = $6 \times 7 \times 7 = 294$

   4 digit number = $6 \times 7 \times 7 \times 7 = 2058$

   5 digit number starting with 1 -------- = $7^4 = 2401$

   5 digit number starting with 2 -------- = $7^4 = 2401$

   5 digit number starting with 30 -------- = $7^3 = 343$

   5 digit number starting with 31 -------- = $7^3 = 343$

   5 digit number starting with 320 -------- = $7^2 = 49$

   5 digit number starting with 321 -------- = $7^2 = 49$

   5 digit number starting with 3220 -------- = 7 = 7

   5 digit number starting with 32216 -------- = 7 = 7

   8000

   Hence, the correct answer is (C).

10. Req selections = $6 \times 5 \times 4 - 1 = 119$

    Hence, the correct answer is (A).

11. Req. selections = $4 \times 5 \times 4 = 80$

    Hence, the correct answer is (A).

12. Req. selections = $5 \times 4 \times 3 = 60$

    Hence, the correct answer is (D).

13. If *a*, *b*, *c* are in AP $\Rightarrow 2b = a + c$

    $\therefore$ Both *a*, *c* are odd (or) both are even

    $\Rightarrow$ number of ways = $2(^9C_2) = 72$

    Hence, the correct answer is (C).

14.

R_1	1	4	7	10	13	16
R_2	2	5	8	11	14	17
R_3	3	6	9	12	15	18

$x^3 + y^3$ is divisible by '3'

$\Rightarrow$ we can select both *x*, *y* from R_3 (or) one element from R_1 and another element from R_2 and it can be done is $^6C_2 + {}^6C_1$ $^6C_{1.} = 15 + 36 = 51$

Hence, the correct answer is (D).

15. Number of lines formed by joining the points

    $$= {}^{18}C_2 - \left({}^3C_2 + {}^4C_2 + {}^5C_2 \right) + 3 = 137$$

    Hence, the correct answer is (B).

16. Number of triangles formed

    $$= {}^{18}C_3 - \left({}^3C_3 + {}^4C_3 + {}^5C_3 \right) = 801$$

    Hence, the correct answer is (A).

17. Number of quadrilaterals

    $$= {}^4C_2 \times {}^5C_2 + {}^3C_2 \times {}^5C_2 + {}^3C_2 \times {}^4C_2 + {}^3C_1 \times {}^4C_2 + {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^5C_2 + {}^3C_2 \times {}^4C_1 \times {}^5C_1 = 378$$

    Number of Pentagons

    $$= {}^3C_1 \times {}^4C_2 \times {}^5C_2 + {}^3C_2 \times {}^4C_1 \times {}^5C_2 + {}^3C_2 \times {}^4C_2 \times {}^5C_1 = 390$$

Number of hexagons = ${}^3C_2 \cdot {}^4C_2 \cdot {}^5C_2$
$= 3 \times 6 \times 10 = 180$
Total number of n – gons = 948
Hence, the correct answer is (C).

18. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \rightarrow {}^9C_6$ ways

Hence, the correct answer is (A).

19. $x_1 < x_2 < x_3 = x_4 < x_5 < x_6 \rightarrow {}^9C_4$ ways

Hence, the correct answer is (B).

20. $x_1 < x_2 < x_3 \le x_4 < x_5 < x_6$

$$\left[\begin{array}{l}\text{Case I: } x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \rightarrow {}^9C_6 \\ \text{Case II: } x_1 < x_2 < x_3 = x_4 < x_5 < x_6 \rightarrow {}^9C_5\end{array}\right.$$

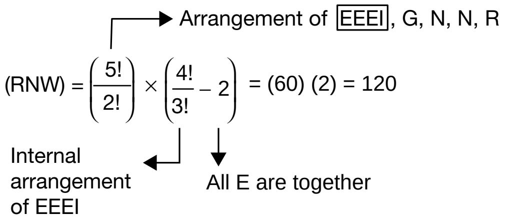
$= {}^9C_3 + {}^9C_4 = {}^{10}C_4$

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

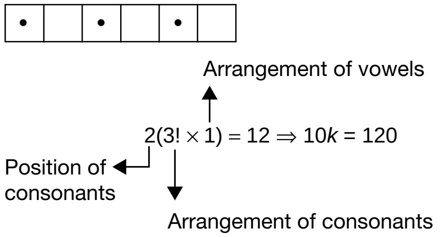
1. A → (s); B → (s); C → (p, q); D → (r)

(A) $\underbrace{(\text{EEE I})}\text{GNNR}$



$$(\text{RNW}) = \left(\frac{5!}{2!}\right) \times \left(\frac{4!}{3!} - 2\right) = (60)\,(2) = 120$$

(B)

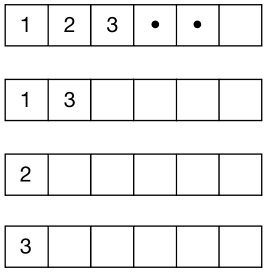


$2(3! \times 1) = 12 \Rightarrow 10k = 120$

(C) $5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 60 - 20 + 5 - 1 = 44$

$\Rightarrow$ divisible by 22, 44

(D)



$= \frac{3!}{2!} = 3$

$= \frac{4!}{2!2!} = 6$

$= \frac{5!}{3!} = 20$

$= \frac{5!}{2!2!} = 30$

Required = 59

2. A → (p, q, r); B → (p, q, r); C → (p, q, s); D → (p, q, s)

(A) $xyz = 24 = 2^3 \times 3^1$ No. of the integral solutions

$\lambda = {}^{3+3-1}C_{3-1}\ {}^{1+3-1}C_{3-1} = {}^5C_2 \times {}^3C_2 = 2 \times 3 \times 5 = 30$

(B) $m = \frac{120}{x_4}$ (or) $x_1x_2x_3x_4 = 120 = 2^3.3^1.5^1$

(C) No. of solutions $\lambda_1 = {}^{10-1}C_{4-1} = {}^9C_3 = \frac{3.4.7}{1} = 84$

(D) $x_1 + x_2 + x_3 + x_4 + x_5 = 20,\ X_i = x_i - 2, i = 1,2,3,4,5$

$x_1 + x_2 + x_3 + x_4 + x_5 = 10,\quad x_i \ge 1$

No. of the integral solutions ${}^{10-1}C_{5-1} = {}^9C_4 = \frac{9.7.2}{1} = 126.$

3. A → (s); B → (r); C → (q); D → (p)

A: The letters are BEMNRU

Starting with	B, 5!	= 120 words
Starting with	E, 5!	= 120 words
Starting with	M, 5!	= 120 words
Starting with	NB, 4!	= 24 words
Starting with	NE, 4!	= 24 words
Starting with	NM, 4!	= 24 words
Starting with	NR, 4!	= 24 words
Starting with	NUB, 3!	= 6 words
Starting with	NUE, 3!	= 6 words
	NUMBER	= 1 word

∴ The desired rank is $120 \times 3 + 24 \times 4 + 2 \times 6 + 1 = 469$

B: Primes: 2, 3, 5

aaaa	—	${}^3C_1 = 3$
aaab	—	${}^3C_2 \times 2 = 6$
aabb	—	${}^3C_2 = 3$
aabc	—	${}^3C_3 \times 3 = 3$

Total = 15

C: Number should be divisible by 10^{2n}, so here we have to determine number of zeroes in the value of 100!, each zero comes from the combination of 2 and 5

Power of 5 $= \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 24$

∴ No. is divisible by 10^{24} or 100^{12}

D: So, in 4 digit number of either all digits are same, digits are from two distinct digits.

All digits are same the number of ways = 9

Two different digits are used (excluding zero) $= {}^9C_2 \times (2^4 - 2)$
$= 504$

Two different digits are used (including zero) $= {}^9C_1 \times (2^3 - 1)$
$= 63$

∴ Total ways = 9 + 504 + 63 = 576

4. A → (p, q, r, s); B → (q); C → (p, q, r, s, t); D → (p, q, r, s)

(A) Possible division of books is (4, 4, 4)

So $\quad n = \frac{12!}{4!4!4!(3)!} \times 3!$

$$\Rightarrow \frac{(4!)^4 n}{12!} = \frac{(4!)^4}{12!}\,\frac{12!}{4!4!4!3!}\times 3!$$

$$= 24$$

(B) 12 different things is distributed in three groups by, $n = \dfrac{12!}{4!4!4!3!}$ ways.

So, $\dfrac{(4!)^3(3!)^2}{12!}n = 3! = 6$

(C) $n = \dfrac{12!}{2!2!2!3!3!3!2!} = \dfrac{12!}{(2!)^3 3!(3!)^2 2!}$

$$\Rightarrow \frac{(3!)^3(2!)^4 5!}{12!}n = \frac{(3!)^3(2!)^4 5!}{12!}\,\frac{12!}{(2!)^4(3!)^3} = 120$$

(D) $x_1x_2x_3x_4x_5 = 2\times 3\times 5^2\times 7$

$\lambda = (1+1)(1+1)(2+1)(1+1) = 24$

5. A → (q); B → (p); C → (s); D → (r)

A: Coefficient of x^{18} in $(1 + x + \ldots + x^{12})^3$

$= (1 - x^{13})^3\,(1 - x)^{-3}$

$= (1 - 3x^{13})\,(1 + {}^3C_1\,x + ({}^4C_2))\,x^2 + \ldots$

Which is ${}^{20}C_{18} - 3\cdot{}^7C_5 = {}^{20}C_2 - 3\cdot{}^7C_2 = 190 - 63 = 127$

B: A team of 5 can be prepared in having 3G and 2B and 4G and 1B

Total ways $= {}^5C_3\times{}^5C_2 + {}^5C_4\times{}^5C_1 = 125$

C: We first fill the small boxes with 2 of the three bundles and the remaining boxes with the remaining bundles.

The desired number is ${}^3P_2\cdot 4! = 6\cdot 24 = 144$

D: We choose any colour and take 2 balls or take any number from 1–6 and choose 2 colours.

The desired number is

${}^5C_1\,{}^6C_2 + {}^6C_1\,{}^5C_2 = 75 + 60 = 135$

6. A → (r); B → (r); C → (q); D → (p)

(A) One Indian wife and one American wife can be selected in ${}^2C_1\times{}^2C_1$ ways and keeping an unmarried person in between these two wives the total number of linear arrangements are ${}^2C_1\times{}^2C_1\times\lfloor\underline{7}\times\lfloor\underline{2} = 40320$

(B) Required number of ways $= \lfloor\underline{8} = 40320$

(C) Required number of ways

$\lfloor\underline{(7-1)}\times\lfloor\underline{2}\times{}^2C_1\times{}^2C_1 = 5760$

(D) Number of ways in which interviews can be arranged

$= 9\times{}^8C_2\times{}^6C_2\times{}^4C_2\times{}^2C_2 = 22680$

7. A → (r); B → (s); C → (p); D → (p)

AA, SSS, TT, EE, RR, M, B, L

(A) Take RAAT as one unit. Therefore 10 + 1 = 11 units can be arranged in $\dfrac{11}{3!\times 2!}$ ways.

(B) After fixing S in first position and R in last position the remaining 12 letters can be arranged in remaining 12 positions in $\dfrac{12!}{(2!)^4}$ ways

(C) First 7 positions can be filled with A's, E's, M, B, L in $\dfrac{7!}{(2!)^2}$ ways. The remaining 7 positions can be filled with S's, T's, R's in $\dfrac{7!}{3!\times(2!)^2}$ ways.

(D) 7 even positions can be filled with A's, S's, R's in $\dfrac{7!}{(2!)^2\times 3!}$ ways. 7 odd positions can be filled with T's, E's, M, B, L in $\dfrac{7!}{(2!)^2}$

8. A → (q, s); B → (q, s); C → (p); D → (r)

(A) Non-negative integral solutions of

R + B + G + Y = 8 is ${}^{8+4-1}C_{4-1} = {}^{11}C_3$

(B) $D_1 \neq 0\ D_1 + D_2 + D_3 + D_4 = 9$

$D_1 - 1 + D_2 + D_3 + D_4 = 8$

${}^{8+4-1}C_{4-1} = {}^{11}C_3$

(C) Select '3' places out of 21 places $= {}^{21}C_3$

(D) $\left[\dfrac{17}{3}\right] + \left[\dfrac{17}{3^2}\right] = 5 + 1 = 6$

9. A → (p, q); B → (q, t); C → (p, q, r, s); D → (q, r)

(A) i) When number is of three digit

5 or 6 only		

First place can be filled in 2 ways, second place can be filled in 4 ways third place can be filled in 3 ways.

∴ Number of ways = 2 × 4 × 3 = 24

ii) When number is of four digit

2 or 3 only		

First place can be filled in 2 ways, second place can be filled in 4 ways and third place can be filled in 3 ways and fourth place can be filled in 2 ways.

∴ Number of ways = 2 · 4 · 3 · 2 = 48

∴ Total number of ways = 24 + 48 = 72

∴ $n = 72 = 2\times 2\times 2\times 3\times 3$ (P, Q)

(B) **Case I:**

When number of three digits

a) The three digit number with '0' at unit place, first place can be filled in 3 ways and second place can be filled in 3 ways

		O

∴ Number of ways = 3 × 3 = 9

b) The three digit number with 2 or 4 at unit place. First place can be filled in 2 ways and second place can be filled in 3 ways

		2 Or 4

∴ Number of ways = 2 × 3× 2 = 12

Case II:

When number of four digits

(a) The four digit number with '0' at unit place

			0

First place can be filled in 2 ways (1 or 2), second place can be filled in 3 ways and third place can be filled in 2 ways.

∴ Number of ways = 2 × 3 × 2 = 12

b) The four digit number with '2' at unit place

			2

First place can be filled in 1 way, second place can be filled in 3 ways and third place can be filled in 2 ways.

∴ Number of ways = 1 × 3 × 2 = 6

c) The four digit number with '4' at unit place

			4

First place can be filled in 2 ways (1 or 2), second place can be filled in 3 ways and third place can be filled in 2 ways

∴ Number of ways = 2 × 3 × 2 = 12

∴ Total number of ways = 9 + 12 + 12 + 6 + 12 = 51

∴ $n = 51 = 3 \times 17$ (Q , T)

C) ROORKEE

Number of ways $\dfrac{7!}{2!2!2!} = 630$

Number of words begin with R = $\dfrac{6!}{2!2!} = 180$

Number of words end with E = $\dfrac{6!}{2!2!} = 180$

and number of words begin with

R and end with E = $\dfrac{5!}{2!} = 60$

∴ Required number of words

= 630 – 180 – 180 + 60 = 330

∴ $n = 330 = 2 \times 3 \times 5 \times 11$ (P, Q, R, S)

D) Total number of games = ${}^{k-2}C_2 + 6 = 84 \Rightarrow {}^{k-2}C_2 = 78$ $k = 15$

Integer/Numerical Answer Type Questions

1.

b	No. of ways of a, c, d
2	1.2^2
3	2.3^2
4	3.4^2
	.
9	8.9^2

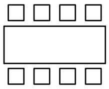
$$\text{RNW} = \sum n.(n+1)^2 = \sum_{n=1}^{8}(n+1)^3 - \sum_{n=1}^{8}(n+1)^2$$

$$= (2^3 + \ldots\ldots.. + 9^3) - (2^2 + \ldots.. + 9^2)$$

$$= (45)^2 - \frac{9 \times 10 \times 19}{6} = 1740$$

Tens place = 4

Hence, the correct answer is (4).

2. 

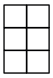
$${}^2C_1 \cdot 4!4!\left(\frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 2(24)(12 - 4 + 1)$$

$$= 432$$

Sum of digits = 9

Hence, the correct answer is (9).

3.  No. of ways

$$\overset{\text{P and C of 3 seats}}{{}^6P_3} - \underset{\text{Choose two rows}}{{}^3C_2} \times \underset{\text{P and C of 3 seats out of 4 seats}}{{}^4P_3}$$

$$= 120 - 72 = 48$$

Hence, the correct answer is (3).

4. $(1+a)(1+b) = 1 + a + b + ab$

$(1+a)(1+b)(1+c) = 1 + a + b + c + ab + bc + ca$

$(1+a)(1+b)(1+c)(1+d) =$

$$1 + \sum a + \sum ab + \sum abc + abcd$$

$$\therefore \text{ R.N.W} = \{(1+1)(1+2)\ldots\ldots..(1+10)\} - 1 - \sum_{i=1}^{10} i$$

$\underline{|11} - 1 - 55 \Rightarrow \underline{|11} - 56$

$$\therefore \left[\frac{56}{11}\right] = 5$$

Hence, the correct answer is (5).

5. $${}^{2002}C_{1001} = \frac{(2002)!}{(1001)!(1001)!}$$

Number of zeros in (2002)! are

$$= \left[\frac{2002}{5}\right] + \left[\frac{2002}{5^2}\right] + \left[\frac{2002}{5^3}\right] + \left[\frac{2002}{5^4}\right]$$

= 400 + 80 + 16 + 3 = 499

no. of zeroes in $(1001\,!)^2 = 2(200 + 40 + 8 + 1) = 498$

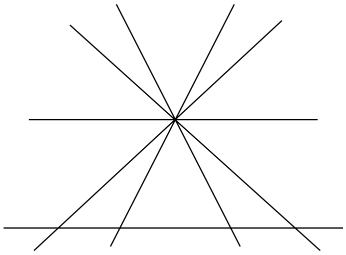
Hence no. of zeroes is $\dfrac{(2002)!}{(1001!)^2} = 1$

Hence, the correct answer is (1).

6. $3^n \geq 900 \quad \Rightarrow \quad n \geq 7$

Hence, the correct answer is (7).

7. Choose any two non-parallel sides = ${}^4C_2 = 6$



Hence, the correct answer is (6).

8. $1+\sum_{n=1}^{6} n = 1+21 = k+15 \Rightarrow k = 7$

Hence, the correct answer is (7).

9. There are 3 choices for the first letter and two choices for each subsequent letter. Hence using fundamental principle number of good words $= 3.2^{n-1} = 384$
$2^{n-1} = 128 = 2^7$
$n = 8$
Hence, the correct answer is (8).

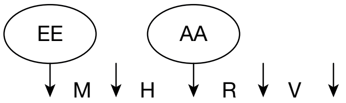
10. Sum of 7 digits to be a multiple of '9' is possible if left two numbers are divisible by '9' they are (1, 8), (2, 7) (3, 6), (4, 5). Hence required number $= 4 \times 7!$
Since sum of the numbers is divisible by 9
Hence, the correct answer is (4).

11. Last two digits are 25, 75 remaining places can be filled in ${}^5P_2$ ways. Hence for digit numbers are $= 40 = 8m \Rightarrow m = 5$
Hence, the correct answer is (5).

12. $4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=9$ (number of derangements)

Hence, the correct answer is (9).

13.



M, H, R, V can be arranged in 4! ways.
Now, there are 5 places where the 2 packages of 2 vowels each can be put.

Numbers of ways $= {}^5C_2$

Case – I

when EE and AA are together

No. of ways ${}^5C_2 \times 2! \times 4!$

(EE and AA can interchange their places)

Case – II

When EA and EA are together

No. of ways ${}^5C_2 \times 4! \times 2! \times 2!$

∴ Total number of ways =

${}^5C_2 \times 4! \times 2!(1+2)$

$= \frac{5!}{2!3!} \times 4! \times 2! \times 3 = \frac{5! \times 4!}{2!}$

$= 120 \times 12 = 1440$

Hence, the correct answer is (8).

14. AAAAA | BBBBBB
M
middle did it must be A
so that even number of A's and B's are available
take AABBB on one side of A (6^{th} place) and then their angle about M in a unique way

Number of ways $= \frac{5!}{2!.3!} = 10$

Hence, the correct answer is (0).

15. Each one of four people calls the fifth person and give their information. In turn the fifth person having all bits of information calls every one to tell everything.
No. of calls required is $4 + 4 = 8$.
Hence, the correct answer is (8).

16. Exponent of 5 in 25! is 6
then exponent of 5 in 30! is 7
25!, 26!, 27!, 28!, 29!
Hence, the correct answer is (5).

17. Let the vertices of req. pentagon be V_1, V_2, V_3, V_4, V_5.
Let the no. of vertices of the polygon b/w $V_1.V_2$; V_2, V_3; V_3, V_4 ; V_4V_5 and V_5V_1 are x_1, x_2, x_3, x_4, x_5 respectively.

∴ req. no. of ways is the $\frac{n}{5}.{}^{n-6}C_4$ (number of the positive integral solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = n - 5$)

$\therefore \frac{n}{5}.{}^{n-6}C_4 = 36$

$(n(n-6)(n-7)(n-8)(n-9) = 12.6.5.4.3)$

$\Rightarrow n = 12 \Rightarrow n - 3 = 9$

Hence, the correct answer is (9).

18. The 8 vertices of the block gives $8_{C_3} = 56$ triangles, each of which is either acute or right angled. Each vertex of the block serves as the vertex of the right angle in three triangles whose hypotenuse are face diagonals of the block, and three triangles whose hypotenuse are space diagonals of the block. Hence there are $8(3+3) = 48$ right triangles and so $56 - 48 = 8$ acute triangles.
Hence, the correct answer is (8).

19. $N = 2^2.503.2011$

no. of solutions = no. of divisors of $N^2 = 45$

for $x < y$ no. of solutions of $\frac{45-1}{2} = 22$

Hence, the correct answer is (4).

20. $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 = {}^9C_7$

$x_1 = x_2 < x_3 < x_4 < x_5 < x_6 < x_7 = {}^9C_6$

$x_1 = x_2 < x_3 < x_4 = x_5 < x_6 < x_7 = {}^9C_5$

$x_1 < x_2 < x_3 < x_4 = x_5 < x_6 < x_7 = {}^9C_6$

Required number of ways $= {}^9C_7 + {}^9C_6 + {}^9C_5 + {}^9C_6$

$= {}^{10}C_7 + {}^{10}C_6 = {}^{11}C_7 = {}^NC_k$

$N - K = 11 - 7 = 4$

Hence, the correct answer is (4).

21. Let x, y, z be the number of sticks received by the children A, B and C. Then the line segments formed by them form a triangle if $1 \le x \le 5, 1 \le y \le 5, 1 \le z \le 5$

$\therefore$ n is the number of positive integral solutions of the equation $x + y + z = 12$ Where $x, y, z \le 5$.

$\Rightarrow$ $n =$ Coefficient of x^{12} in $\left(x + \ldots. + x^5\right)^3$ = coefficient of x^9 in $(1 + x + x^2 + x^3 + x^4)^3$

= coefficient of x^9 in $(1 + x^5)^3(1 - x)^{-3}$

= coefficient of x^9 in $(1 - 3x^5 + 3x^{10} - x^{15})\,(1 - x)^{-3}$

$= 1\times {}^{11}C_2 - 3\times {}^{6}C_2$

$= 55 - 45 = 10$

Hence, the correct answer is (9).

22. $\left(\dfrac{n+1}{4}\right)(2^{n+1} - n - 2) = \displaystyle\sum_{k=1}^{n} k.2^{n+1-k} = \text{S}$ where

$\text{S} = 1.2^n + 2.2^{n-1} + 3.2^{n-2} + \ldots + n.2$

$\dfrac{1}{2}\text{S} = 2^{n-1} + 2.2^{n-2} + \ldots. + (n - 1).2 + n$

Subtracting, $\dfrac{1}{2}\text{S} = (2^n + 2^{n-1} + \ldots. + 2) - n = 2(2^n - 1) - n$

$\therefore\ \text{S} = 2(2^{n+1} - 2) - 2n = 2(2^{n+1} - n - 2)$

Hence, $\dfrac{n+1}{4} = 2$(i.e) $n = 7$

Hence, the correct answer is (7).

23. The number of ways of distributing blankets is equal to the number of solutions of the equation

$3a + 2b + c = 15$

$a, b, c \ge 1$, which is equal to the coefficient of t^{15} in

$(t^3 + t^6 + t^9 + t^{12}\ \ldots\ldots)(t^2 + t^4 + \ldots..)(t + t^2 + \ldots.)$

= coefficient of t^9 in $(1 + t^2 + t^2 + t^4 + t^5 + 2t^6 + t^7 + 2t^8 + 2t^9)(1 + t + t^2 + \ldots. + t^9)$

$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 + 2 = 12$

Hence, the correct answer is (6).

24. $(2p + 1) + (2q + 1) + (2r + 1) + (2s + 1) = 20$

$\Rightarrow p + q + r + s = 8$

The no. of non negative integral solutions is ${}^{8+4-1}C_3 = 165 \Rightarrow \left[\dfrac{n}{20}\right] = 8$

Hence, the correct answer is (8).

25. $x_1 + x_2 + x_3 = 15$

$0 \le x_1 \le 5,\ 0 \le x_2 \le 10,\ 0 \le x_3 \le 15$

$n =$ co-efficient of $x^{15}(1 - x^6)\,(1 - x^{11})\,(1 - x^{16})\,(1 - x)^{-3}$

$n = 66$ then $\dfrac{n}{22} = 3$

Hence, the correct answer is (3).

26. Number of zeroes $= x = \displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right]$

$$x < \sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right] \text{ since } \left[\frac{n}{5^i}\right] < \frac{n}{5^i}$$

$$x < \frac{n}{5}\sum_{i=1}^{\infty}\frac{1}{5^i}$$

$$\Rightarrow x < \frac{n}{5}\times\frac{5}{4}$$

$\Rightarrow n > 4x \Rightarrow n > 80$

If $n = 80,\ x = \left[\dfrac{80}{5}\right] + \left[\dfrac{80}{25}\right] = 16 + 3 = 19$

$n = 85, x = \left[\dfrac{85}{5}\right] + \left[\dfrac{85}{25}\right] = 20$

$n = 90, x = \left[\dfrac{90}{5}\right] + \left[\dfrac{90}{25}\right] = 21$

$\Rightarrow n \in \{85, 86, 87, 88, 89\}$

Hence, the correct answer is (5).

27. Case I: Let two identical digits are both one.

Number of natural number = 3.9.8 = 216

Case II: Let two identical digits are not one.

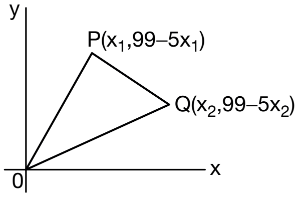
Number of natural number = 3.9.8 = 216

N = 432

Sum of digits = 9.

Hence, the correct answer is (9).

28. Area $= \dfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x_1 & 99 - 5x_1 & 1 \\ x_2 & 99 - 5x_2 & 1 \end{vmatrix}$



$= \dfrac{1}{2}\,[x_1(99 - 5x_2) - x_2(99 - 5x_1)]$

Area $= \dfrac{1}{2}\ [(x_1 - x_2)99]$

Area is an integer then both x_1 and x_2 are simultaneously either even or both odd.

Now x_1 varies from 0 to 19 $\left\langle\begin{array}{l}\text{10 even}\\ \text{10 odd}\end{array}\right.$

hence ${}^{10}C_2 + {}^{10}C_2 = 2\cdot{}^{10}C_2 = 90$.

Hence, the correct answer is (9).

29. $$\overbrace{{}^8C_3}^{\text{Choose any 3 vertices}} - \underbrace{4 \cdot {}^8C_1}_{\text{Choose one side of octagon}} - \underbrace{8}_{\text{Choose two sides of octagon}} = 9K$$

Hence, the correct answer is (2).

30. $a + b + c = 21 \Rightarrow b + c > a \Rightarrow a + b + c > 2a \Rightarrow 2a < 21 \Rightarrow a \le 10$.

So $1 \le a, b, c \le 10$.

The cases when $a > b > c$ are (10, 9, 2), (10, 8, 3), (10, 7, 4), (10, 6, 5), (9, 8, 4), (9, 7, 5) and (8, 7, 6). So, number of cases when a, b, c are all distinct is 7 × 3! = 42.

The cases when $a = b > c$ or $a > b = c$ are (10, 10, 1), (9, 9, 3), (8, 8, 5) and (9, 6, 6). So number of cases when two same and 1 different is $4 \times 3!/2! = 12$.

The cases when $a = b = c$ is (7, 7, 7). The total number of ordered triplets = 42 + 12 + 1 = 55.

Hence, the correct answer is (5).

31. $x_1 + x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 1$

Number of ways ${}^{5-1}C_{3-1} = {}^{4}C_2 = 6$

Hence, the correct answer is (6).

32. Number of ways

$$\underbrace{(3!)}_{\text{Circular arrangement of 5 Boys where 2 Boys are together}}\ \overbrace{(2!)}^{\text{Internal arrangement of 2 Boys}}\ \underbrace{(4!)}_{\text{Girls arrangement in 4 gaps}} = 288$$

Hence, the correct answer is (4).

33. Number of such quadrilaterals = 10 (when four consecutive points are taken)

Hence, the correct answer is (7).

34. $$\text{No. of such triangles} = \underbrace{10}_{\text{Choose a common side}}\ \overbrace{{}^{6}C_1}^{\text{Choose a third vertex}} = 60$$

Hence, the correct answer is (6).

35. A = {7, 14, 21, ..., 98}

B = {1, 8, 15, ..., 99}

C = {2, 9, 16, ..., 100}

D = {3, 10, 17, ..., 94}

E = {4, 11, 18, ..., 95}

F = {5, 12, 19, ..., 96}

G = {6, 13, 20, ..., 97}

y can contain one element of A, (all from B, none from G), (all from C, None from F) = 1 + 15 + 15 + 14 = 45 $\Rightarrow \lambda = 5$

Hence, the correct answer is (5).

36. $$\underbrace{1 \times 8^3}_{\text{5 at first place}} + \underbrace{3 \times 7 \times 8^2}_{\text{5 at any of 2nd | 3rd | 4th place}} = 1856$$

1st	2nd	3rd	4th

$\Rightarrow$ B = 8

Hence, the correct answer is (8).

37. $\frac{1}{4} \cdot {}^{n}C_3 = {}^{n-1}C_3 \Rightarrow n = 12 \Rightarrow \frac{n}{3} = 4$

Hence, the correct answer is (4).

38. The no. of non negative integral solutions of $x_1 + x_2 + \ldots + x_8 = 7$ is ${}^{n+p-1}C_{p-1} = {}^{7+8-1}C_{8-1} = {}^{14}C_7$

Hence, the correct answer is (7).

39. $x > y > z \quad \Rightarrow {}^{10}C_3 = 120$

$x = y > z \quad \Rightarrow {}^{10}C_2 = 45$

$\Rightarrow \quad k = 165 \quad \Rightarrow \frac{k}{20} = 8.25 \qquad \Rightarrow \left[\frac{k}{20}\right] = 8$

Hence, the correct answer is (8).

40. $$\sum_{r=3}^{5} {}^{5}P_r\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + (-1)^r \frac{1}{r!}\right]$$

$= 20 + 45 + 44 = 109$

Hence, the correct answer is (9).

41. End digit of 7^{4n+1} is 7, of 7^{4n+2} is 9 of $7^{4n+3} = 3$ and that the of $7^{4n} = 1$ for $n \in N$

Now, 754 = 4.188 + 2

Thus end digit of $(2137)^{754}$ will be same as that of 7^{4n+2} is 9.

Hence, the correct answer is (9).

42. Total no. of ways = ${}^{7}P_5$

If all five persons occupy rooms leaving out either first or second floor then no. of ways = 2 × 5!

$\therefore$ Required no. of ways = ${}^{7}P_5 - 2 \times 5! = 2280$

$$\sqrt{\frac{2280}{10} - 164} = 8$$

Hence, the correct answer is (8).

43. Images of 1, 2, 3, 4, 5 are respectively.

7, 7, 7, 8, 9; 7, 7, 8, 8, 9; 7, 7, 8, 9, 9

7, 8, 8, 8, 9; 7, 8, 8, 9, 9; 7, 8, 9, 9, 9

$\therefore$ there are six functions.

Hence, the correct answer is (6).

44. No. of 4 digit numbers = 3 × 5 × 4 × 3 = 180

No. of 5 digit numbers = 5 × 5 × 4 × 3 × 2 = 600

No. of 6 digit numbers = 5 × 5 × 4 × 3 × 2 = 600

$n = 1380$

$\Rightarrow \frac{n}{230} = 6$

Hence, the correct answer is (6).

45. There are exactly $(n - 1)$ sub sets of S containing two elements having 1 as least element; exactly $(n - 2)$ subsets of S having 2 as least element and so on.

Thus $\sum_{A \in X} \min(A) = (1)(n-1) + 2(n-2) + \_\_\_\_ + (n-1)(1)$

$$= \sum_{r=1}^{n-1} r(n-r) = {}^{n+1}C_3$$

Hence, the correct answer is (3).

46. No. of n digit numbers in which no two consecutive digits are same = 9^n

$\Rightarrow$ no. of numbers from 1 to 10^6 in which no two consecutive digits are same

$$= \sum_{n=1}^{6} 9^n = 597870$$

Required number = $10^6 - 597870 = 402130 = 402128 + 2$

$K = 2$

Hence, the correct answer is (2).

47. Let $P(x) = x^3 + ax^2 + bx + c$

$x^2 + 1 = (x + i)(x - i)$

$P(i) = 0$, $P(-i) = 0$ we get
$b = 1, a = c$
No. of ways of choosing $a, b, c = 10 = 10 \times 1$
$\therefore$ K = 1
Hence, the correct answer is (1).

48. There are three major ways $\alpha\alpha\beta\beta\alpha$, $\alpha\beta\beta\alpha\alpha$ and $\alpha\beta\alpha\beta\alpha$
Each major way has six empty spaces. The number of ways of putting letters at these empty spaces must be non-negative integer function of $x_1 + x_2 + \ldots + x_6 = 6$
$= {}^{6+6-1}C_{6-1} = {}^{11}C_5$
No.of arrangements is $= 3 \times {}^{11}C_5 \times 6! \Rightarrow$ K = 3
Hence, the correct answer is (3).

49. $x < y < z$, these are 1 2 7, 3 6 1, 1 4 5, 2 3 5
total $= 3! \times 4 = 24 = 20 + 4$
$\therefore$ K = 4
Hence, the correct answer is (4).

50. The number of ways in which all the letters are in wrong envelopes

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 4!\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 12 - 4 + 1 = 9$$

Hence, the correct answer is (9).

51. $${}^{100}C_{50} = \frac{100!}{50!50!}$$

Exponent of 7 in 100 ! $= \left[\frac{100}{7}\right] + \left[\frac{110}{7^2}\right] + 0 = 16$

Exponent of 7 in 50! $= \left[\frac{50}{7}\right] + \left[\frac{50}{7^2}\right] = 8$

Exponent of 7 in $(50!)^2 = 16$

$\therefore$ Exponent of 7 in ${}^{100}C_{50} = 16 - 16 = 0$

Hence, the correct answer is (0).

52. For $n = 6$
$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729 < 900$
For $n = 7$
$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 2187 > 900$
Number of n-digits formed > 900
Since the least n is required.
Hence, the correct answer is (7).

53. 240 is divisible by $4n + 2$ or 120 is divisible by $2n + 1$.
Number of the form $(2n + 1)$, $n \in$ I are all odd natural numbers.
Thus we have to find all odd numbers dividing 120.
These numbers are 1, 3, 5, 15.
Hence, number of divisors = 4.
Hence, the correct answer is (3).

ARCHIVE: JEE MAIN

1. The equation does not mention that whether same coloured marbles are distinct or identical, So assuming they are distinct our required answer $= {}^{12}C_4 - {}^{5}C_4 = 490$
And. If same coloured marbles are identical then required answer $= 2 + 3 + 4 + 4 = 13$
Hence, the correct answer is (490).

2. N → 2, A → 2, I → 2, E, X, M, T, O → 1

Category	Selection	Arrangement
s2 alike of one kind and 2 alike of other kind	${}^3C_2 = 3$	$3 \times \frac{4!}{2!2!} = 18$
2 alike and 2 different	${}^3C_1 \times {}^7C_2$	${}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$
All 4 different	${}^8C_4$	${}^8C_4 \times 4! = 1680$

Total = 2454
Hence, the correct answer is (2454).

3. MOTHER
Alphabetical order of letters E H M O R T
Position of word MOTHER in dictionary
$= 2 \times 5! + 2 \times 4! + 3 \times 3! + 2! + 1$
$= 240 + 48 + 18 + 2 + 1 = 309$
Hence, the correct answer is (309).

4. Ways $= {}^6C_4 \cdot 1^4 \cdot 3^2$
$= 15 \times 9 = 135$
Hence, the correct answer is (135).

5. Let three digit number is xyz
$x + y + z = 10; x \geq 1, y \geq 0\ z \geq 0$ ...(1)
Let $a = x - 1 \Rightarrow x = a + 1$ where $a \geq 0$
Put in (1)
$a + y + z = 9; 0 \leq a \leq 8, 0 \leq y, z \leq 9$
No. of non-negative integral solution$= {}^{9+3-1}C_{3-1} - 1$
$= 55 - 1 = 54$
But in these solutions (10, 0, 0) is one of the solutions which is eliminated
Hence, the correct answer is (54).

6. SS LL YABU
Required ways

$$= \underbrace{{}^2C_1}_{\substack{\text{selection of}\\ \text{two alike}\\ \text{letters}}} \times \underbrace{{}^5C_2}_{\substack{\text{selection of}\\ \text{of two}\\ \text{distinct letters}}} \times \underset{\text{arrangement}}{\underline{\frac{4!}{2!}}} = 240$$

Hence, the correct answer is (240).

7. LETTER
vowels = EE, consonant = LTTR
_ L _ T _ T _ R _
Required number of ways $= \left(\frac{4!}{2!}\right){}^5C_2 \times 1 = 120$
Hence, the correct answer is (120).

8. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 is ${}^5C_1 \times \frac{6!}{2!}$

Hence, the correct answer is (A).

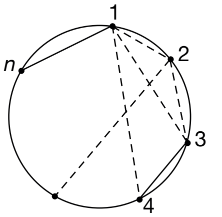
9. _ _ _ 2 _

No. of five digits numbers = No. of ways of filling remaining 4 places = 8 × 8 × 7 × 6

$$\therefore \quad k = \frac{8 \times 8 \times 7 \times 6}{336} = 8$$

Hence, the correct answer is (A).

10. Number of blue lines = Number of sides = n



Number of red lines = number of diagonals

$$= {}^nC_2 - n$$

$${}^nC_2 - n = 99\,n \quad \Rightarrow \quad \frac{n(n-1)}{2} - n = 99\,n$$

$$\frac{n-1}{2} - 1 = 99 \quad \Rightarrow \quad n = 201$$

Hence, the correct answer is (C).

11. $S = (2.{}^1p_0 - 3.{}^2p_1 + 4.{}^3p_2$ …. Upto 51 terms) + (1! – 2! + 3! ….. upto 51 terms) $[\because {}^np_{n-1} = n!]$

$\therefore S = (2 \times 1! - 3 \times 2! + 4 \times 3! \ldots. + 52.51!) + (1! - 2! + 3! \ldots.. + (51)!)$

$= (2! - 3! + 4! \ldots.. + 52!) + (1! - 2! + 3! - 4! + \ldots.. + (51)!)$

$= 1! + 52!$

Hence, the correct answer is (C).

12.

A	B	C
1	2	2 (3 possibilities)
1	1	3 (3 possibilities)

Total number of selections

$$= ({}^5C_1\,{}^5C_2\,{}^5C_2) \cdot 3 + ({}^5C_1\,{}^5C_1\,{}^5C_3) \cdot 3$$

$$= 5 \cdot 10 \cdot 10 \cdot 3 + 5 \cdot 5 \cdot 10 \cdot 3 = 2250$$

Hence, the correct answer is (D).

13.

Family 1	Family 2	Family 3
3	3	4

Favourable cases

$$= \underset{\text{Arrangment of 3 Families}}{3!} \cdot \underset{\text{Internal Arragment of family members}}{3! \times 3! \times 4!}$$

$= (3!)^3.(4!)$

$$\text{Required ways} = \frac{4!}{2!} \times {}^5C_2 \times \frac{2!}{2!} = 12 \times 10 = 120$$

Hence, the correct answer is (B).

14. Let m – men, 2 women

${}^mC_2 \times 2 = {}^mC_1 . {}^2C_1 . 2 + 84$

$m^2 - 5m - 84 = 0$

$(m - 12)(m + 7) = 0$

$m = 12$

Hence, the correct answer is (A).

15. There are 9 numbers so 4 even places are available for 3 odd number.

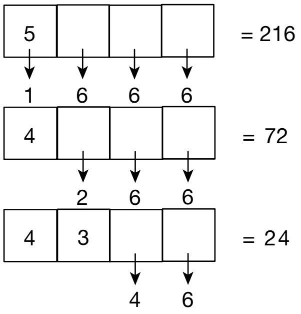
Total number of possible numbers

$$= \left({}^4C_3 \times \frac{3!}{2!}\right)\left(\frac{6!}{4!\,2!}\right) = 180$$

Hence, the correct answer is (A).

16. Given digits are 0, 1, 2, 3, 4, 5

requires four number greater than 4321



total case = 22

{subtract two case 4320 and 4321}

total = 216 + 72 + 22

= 310

Hence, the correct answer is (C).

17. Given: (8 males, 5 females)

Committee to be selected = 11 members

m = no. of ways the committee is formed with at least 6 males.

⇒ (6M, 5F) or (7M, 4F) or (8M, 3F)

$= {}^8C_6 \times {}^5C_5 + {}^8C_7 \times {}^5C_4 + {}^8C_8 \times {}^5C_3 = 78$

n = no. of ways the committee is formed with atleast 3 female

⇒ (8M, 3F) or (7M, 4F) or (6M, 5F)

$\Rightarrow \; {}^8C_8 \times {}^5C_3 + {}^8C_7 \times {}^5C_4 + {}^8C_6 \times {}^5C_5$

$= 10 + 40 + 28 = 78$

$\Rightarrow \; m = n = 78$

Hence, the correct answer is (B).

18. Let the six digit number be abcdef for this number to be divisible by 11, $|(a + c + e) - (b + d + f)|$ must be multiple of 11

$\therefore$ possibility is $a + c + e = b + d + f = 12$

Case: 1 $\{a, c, e\} = \{7, 5, 0\}$
and $\{b, d, f\} = \{9, 2, 1\}$

So, number of numbers = 2 × 2! × 3! = 24

Case: 2 $\{a, c, e\} = \{9, 2, 1\}$
and $\{b, d, f\} = \{7, 5, 0\}$

So, number of numbers = 3! × 3! = 36

total ⇒ 24 + 36 = 60

Hence, the correct answer is (B).

19. $\square \quad \square \quad \square \quad \square$

There are 3 ways to fill 1 position (0, 1, 3)

For remaining digits there are 5 ways

= 3, 5, 5, 5 – 1 (0000 not included)

= 375 – 1 = 374

Hence, the correct answer is (D).

20. 2 girls 3 boys

$$\left.\begin{array}{llll} \text{Case(i)} & A & B & {}^5C_2 \times {}^5C_2 = 10 \times 10 \\ & \checkmark & \times & \\ \text{Case(ii)} & A & B & {}^5C_2 \times {}^5C_2 = 10 \times 10 \\ & \times & \checkmark & \\ \text{Case(iii)} & A & B & \\ & \times & \times & {}^5C_3 \times {}^5C_2 = 10 \times 10 \end{array}\right\}$$

Total no. of ways = 300

Hence, the correct answer is (A).

21. First select any one beam by ${}^{20}C_1$ ways. Now we need to select another beam which is not adjacent so it can be done by ${}^{17}C_1$. But in this selection each case is considered twice so

required no. = $\dfrac{{}^{20}C_1 \times {}^{17}C_1}{2} = 170$

Hence, the correct answer is (B).

22. Total 31 $\Rightarrow \underbrace{21 \text{distinct} \;\&\; 10 \text{identical}}_{10}$

$$S = {}^{21}C_{10} + {}^{21}C_9 + {}^{21}C_8 + {}^{21}C_7 + {}^{21}C_6 + \ldots\ldots\ {}^{21}C_0$$

$$S = {}^{21}C_{11} + {}^{21}C_{12} + {}^{21}C_{13} \ldots\ldots\ {}^{21}C_{21}$$

Adding, $2S = 2^{21}$

$$S = 2^{20}$$

Hence, the correct answer is (B).

23. Given 5 boys and n girls

total ways of forming team of 3 member under given condition

$= {}^5C_1.{}^nC_2 + {}^5C_2.{}^nC_1$

$\Rightarrow {}^5C_1.{}^nC_2 + {}^5C_2.{}^nC_1 = 1750$

$\Rightarrow \dfrac{5n(n-1)}{2} + 10n = 1750$

$\Rightarrow n^2 + 3n = 700$

$\Rightarrow n^2 + 3n - 700 = 0$

$\Rightarrow n = 25$

Hence, the correct answer is (B).

24. Total cases

(1 + 2 + ….+ 8) + (1 + 2 + …+ 7) + (1 + 2 + … 6) + (1 + 2 +… + 5) + (1 + 2 +… + 4) + (1 + 2 + 3) + (1 + 2) + 1

= 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1

= 120

Hence, the correct answer is (D).

25. $k = \{4, 8, 12, 16, 20\}$

$f(k)$ can take value from set {3,6,9,12,15,18} this can be done is ${}^6C_5 \times \lfloor\underline{5} = \lfloor\underline{6}$ ways for reaming elements $\lfloor\underline{15}$ so total number of onto functions = $\lfloor\underline{6} \times \lfloor\underline{15}$

Hence, the correct answer is (C).

26. $\therefore$ Required number of ways = ${}^6C_4 \times {}^3C_1 \times 4!$

= 15 × 3 × 24 = 1080

Hence, the correct answer is (D).

27. Required Number of ways = ${}^4C_3.\ {}^4C_3.+ ({}^4C_2.{}^3C_1)^2 + ({}^4C_1.{}^3C_2)^2 + ({}^3C_3)^2 = 485$

Hence, the correct answer is (B).

28. ALLMS

No. of words starting with

A : A _ _ _ _ _ $\dfrac{4!}{2!} = 12$

L : L _ _ _ _ _ $4! = 24$

M : M _ _ _ _ _ $\dfrac{4!}{2!} = 12$

S : SA _ _ _ _ _ $\dfrac{3!}{2!} = 3$

: SL _ _ _ _ _ $3! = 6$

SMALL → 58th word

Hence, the correct answer is (A).

29. Below the line $x + y = 41$ and in first quadrant

As $x = 1$, $y = 1, 2, \ldots\ldots, 39$ … 39 points

$x = 2$, $y = 1, 2, \ldots\ldots., 38$ … 38 points

.

.

.

$x = 39, y = 1$ 1 point

Total points are = 1 + 2 + 3 +……+ 39 = 39.40/2 = 780

Hence, the correct answer is (B).

30. Four digits number can be arranged in 3 × 4! ways.

Five digits number can be arranged in 5! ways.

Number of integers = 3 × 4! + 5! = 192.

Hence, the correct answer is (D).

31. $n(A \times B) = 8$

Number of required subset = ${}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$
$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 = 219$

Hence, the correct answer is (C).

32. We know,

$$T_n = {}^nC_3,\ T_{n+1} = {}^{n+1}C_3$$

$$T_{n+1} - T_n = {}^{n+1}C_3 - {}^nC_3 = 10$$

$\Rightarrow \quad {}^nC_2 = 10$

$\Rightarrow \quad n = 5.$

Hence, the correct answer is (B).

33. Number of white balls = 10

Number of green balls = 9 and Number of black balls = 7

$\therefore$ Required probability = (10 + 1)(9 + 1)(7 + 1) – 1 = 11.10.8 – 1 = 879

[∵ The total number of ways of selecting one or more items from p identical items of one kind, q identical items of second kind; r identical items of third kind is $(p+1)(q+1)(r+1)-1$]

Hence, the correct answer is (A).

34. Number of required triangles

$$= {}^{10}C_3 - {}^{6}C_3$$

$$= \frac{10\times9\times8}{6} - \frac{6\times5\times4}{6} = 120 - 20 = 100$$

Hence, the correct answer is (B).

35. The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box empty is same as the number of ways of selecting $(r-1)$ places out of $(n-1)$ different places, that is ${}^{n-1}C_{r-1}$.

Hence required number of ways $= {}^{10-1}C_{4-1} = {}^{9}C_3$

∴ Both statements are correct and second statement is a correct explanation of statement – 1.

Hence, the correct answer is (C).

36. Total number of ways $= {}^{3}C_2 \times {}^{9}C_2$

$$= 3 \times \frac{9\times8}{2} = 3\times36 = 108$$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. $N_1 = {}^5C_1 \times {}^4C_4 = 5$

$N_2 = {}^5C_2 \times {}^4C_3 = 40$

$N_3 = {}^5C_3 \times {}^4C_2 = 60$

$N_4 = {}^5C_4 \times {}^4C_1 = 20$

$N_5 = {}^5C_5 = 1$

$\therefore N_1 + N_2 + N_3 + N_4 + N_5 = 126$

Hence, the correct answer is (D).

2. Either one boy will be selected or no boy will be selected. Also out of four members one captain is to be selected.

∴ Required number of ways $= ({}^4C_1 \times {}^6C_3 + {}^6C_4) \times {}^4C_1$

$= (80 + 15) \times 4 = 380$

Hence, the correct answer is (A).

3. ∵ Card numbered 1 is always placed in envelope numbered 2, we can consider two cases.

Case I: Card numbered 2 is placed in envelope numbered 1. Then it is dearrangement of 4 objects, which can be done in $4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 9$ ways

Case II: Card numbered 2 is not placed in envelope numbered 1.

Then it is dearrangement of 5 objects, which can be done in $5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = 44$ ways

∴ Total ways $= 44 + 9 = 53$

Hence, the correct answer is (C).

4. ∵ Each person gets at least one ball.

∴ 3 Persons can have 5 balls in the following systems

Person	I	II	III
No. of balls	1	1	3

or

Person	I	II	III
No. of balls	1	2	2

The number of ways to distribute the balls in first system

$= {}^5C_1 \times {}^4C_1 \times {}^3C_3$

Also 3, persons having 1, 1 and 3 balls can be arranged in $\frac{3!}{2!}$ ways.

∴ No. of ways to distribute 1, 1, 3 balls to the three persons

$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 60$

Similarly the total no. of ways to distribute 1, 2, 2 balls to the three persons $= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!} = 90$

∴ The required number of ways $= 60 + 90 = 150$

Hence, the correct answer is (B).

5. We have to form 7 digit numbers, using the digits 1, 2 and 3 only, such that the sum of the digits in a number = 10.

This can be done by taking 2, 2, 2, 1, 1, 1, 1 or by taking 2, 3, 1, 1, 1, 1, 1

∴ Number of ways $= \frac{7!}{3!4!} + \frac{7!}{5!} = 77.$

Hence, the correct answer is (C).

6. The letter of word COCHIN in alphabetic order are C, C, H, I, N, O.

Fixing first letter C and keeping C at second place, rest 4 can be arranged in 4! ways.

Similarly the words starting with CH, Cl, CN are 4! In each case.

Then fixing first two letters are CO next four places when filled in alphabetic order give the word COCHIN.

∴ Numbers of words coming before COCHIN are

$4 \times 4! = 4 \times 24 = 96$

Hence, the correct answer is (C).

Comprehension Type Questions

1. ∵ a_n = number of all n digit +ve integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0.

and b_n = number of such n digit integers ending with 1

c_n = number of such n digit integers ending with 0

Clearly, $a_n = b_n + c_n$ (∵ a_n can end with 0 or 1)

Also $b_n = a_{n-1}$

and $c_n = a_{n-2}$ [$\because$ if last digit is 0, second last has to be 1]

$\therefore$ We get $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$

Also $a_1 = 1$, $a_2 = 2$,

By this recurring formula

$$a_3 = a_2 + a_1 = 3$$

$$a_4 = a_3 + a_2 = 3 + 2 = 5$$

$$a_5 = a_4 + a_3 = 5 + 3 = 8$$

Also $b_6 = a_5 = 8$

Hence, the correct answer is (B).

2. By recurring formula, $a_{17} = a_{16} + a_{15}$ is correct

Also $c_{17} \neq c_{16} + c_{15}$

$\Rightarrow \quad a_{15} \neq a_{14} + a_{13} \quad (\because c_n = a_{n-2})$

$\therefore$ Incorrect

Similarly, other parts are also incorrect.

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions:

1. A → (p); B → (s); C → (q); D → (q)

(A) For the permutations containing the word ENDEA we consider 'ENDEA' as single letter. Then we have total ENDEA, N, O, E, L i.e. 5 letters which can be arranged in 5! Ways.

$\therefore$ (A) → (p)

(B) If E occupies the first and last position, the middle 7 positions can be filled by N, D, E, A, N, O, L.

These can be arranged in $\frac{7!}{2!} = 21 \times 5!$ Ways.

$\therefore$ (B) → (s)

(C) If none of the letters D, L, N occur in the last five positions then we should arrange N, D, L, N at first four positions and rest five *i.e.* E, E, E, A, O at last five positions. This can be done in

$\frac{4!}{2!} \times \frac{5!}{3!}$ ways. (C) → (q)

(D) As per question A, E, E, E, O can be arranged at 1st, 3rd, 5th, 7th and 9th positions and rest N, D, N, L at rest 4 positions. This can be done in $\frac{5!}{3!} \times \frac{4!}{2!}$ ways = 2 × 5! Ways (D) → (q)

Integer/Numerical Answer Type Questions

1. Non-visiting days = 11

$$\times D_1 \times D_2 \times D_3 \times \ldots\ldots \times D_{10} \times D_{11} \times$$

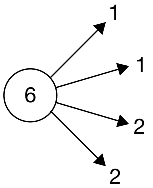
gaps → 12

required ways $= {}^{12}C_4 = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{8!\,4!} = \frac{12\cdot 11\cdot 10\cdot 9}{24}$

$= 99 \times 5 = 495$

Hence, the correct answer is (495).

2. Required ways = (Group formation)(Room allocation)



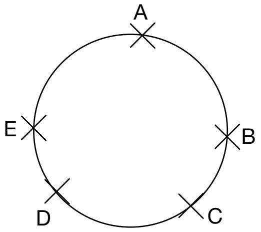
$$= \left[\frac{6}{1!\,1!\,2!\,2!} \times \frac{1}{2!} \times \frac{1}{2!}\right] 4!$$

$$= \frac{6 \times 120}{2 \times 2 \times 2 \times 2} \times 24 = 1080$$

Hence, the correct answer is (1080).

3. We can maximum choose 2 hats of same colour because if we choose 3 hats of same colour then two consecutive person will set same colour hat

$\therefore$ select two colours to be repeated in ${}^3C_2$ say we choose RR BB G.



Select one person give green hat in ${}^5C_1$ ways say we give green hat to A now two reds can be given to either EC and BD i.e. two ways and reaming two blue can only be given in one ways

$\therefore$ Total no. of ways $= {}^3C_2\,{}^5C_1\,2 = 30$ ways

Hence, the correct answer is (30).

4. The last 2 digits, in 5-digits number divisible by 4, can be 12, 24, 32, 44 or 52.

Also each of the first three digits can be by of {1, 2, 3, 4, 5}

Hence 5 options for each of the first three digits and total 5 options for last 2-digits

$\therefore$ Required number of 5 digit numbers are

= 5 × 5 × 5× 5 = 625

Hence, the correct answer is (625).

5. $x = 10!$ And $y = {}^{10}C_1\,{}^9C_8\,\frac{10!}{2!} = 50 \times 9!$

$$= 10 \cdot 9 \cdot 5 \cdot 9! = 45 \cdot 10!$$

$$\therefore \frac{y}{9x} = \frac{45 \cdot 10!}{9(10!)} = 5$$

Hence, the correct answer is (5).

6. $n = 5! \times 6!$

For second arrangement,

5 boys can be made to stand in a row in 5! Ways, creating 6 alternate space for girls. A group of 4 girls can be selected in ${}^5C_4$ ways. A group of 4 and single girl can be arranged at 2 places out of 6 in ${}^6P_2$ ways. Also 4 girls can arrange themselves in 4! ways.

$\therefore\ m = 5! \times {}^6P_2 \times {}^5C_4 \times 4!$

$$\frac{m}{n} = \frac{5! \times 6 \times 5 \times 5 \times 4!}{5! \times 6!} = 5$$

Hence, the correct answer is (5).

7. Number of adjacent lines $= n$

Number of non-adjacent lines $= {}^nC_2 - n$

$\therefore\ {}^nC_2 - n = n \quad \Rightarrow \dfrac{n(n-1)}{2} - 2n = 0$

$\Rightarrow n^2 - 5n = 0 \quad \Rightarrow n = 0$ or 5

But $n \geq 2 \quad \Rightarrow n = 5$

Hence, the correct answer is (5).

8. n_1, n_2, n_3, n_4 and n_5 are positive integers such that $n_1 < n_2 < n_3 < n_4 < n_5$

Then for $n_1 + n_2 + n_3 + n_4 + n_5 = 20$

If n_1, n_2, n_3, n_4 take minimum values 1, 2, 3, 4 respectively then n_5 will be maximum 10.

$\therefore$ Corresponding to $n_5 = 10$, there is only one solution

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4.$

Corresponding to $n_5 = 9$, we can have, only solution

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 5$ i.e., one solution

Corresponding to $n_5 = 8$, we can have,

$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 6.$

or $n_1 = 1, n_2 = 2, n_3 = 4, n_4 = 4.$

i.e., 2 solution

For $n_5 = 7$, we can have

$n_1 = 1, n_2 = 1, n_3 = 4, n_4 = 6$

or $n_1 = 1, n_2 = 3, n_3 = 4, n_4 = 5$ i.e. 2 solutions

For $n_5 = 6$, we can have

$n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 5$

i.e., one solution

Thus there can be 7 solutions.

Hence, the correct answer is (7).

CHAPTER 7: PROBABILITY

Test Your Concepts-I

1. Let S be the sample space of the experiment.

$$\therefore S = \begin{array}{l}\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\\vdots\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{array}$$

Clearly, $n(S) = 6 \times 6 = 36$

Now, A = {(2, 1) (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

B = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}

C = {(1, 1)}

(i) False, as A contains 18 sample point.

(ii) True, as C contains only single sample point

(iii) True, as $A \cap B = \phi$

(iv) True, as A' = Event of getting an odd number on the first die = B and B' = event of getting an even number on the first die A, also $A' \cap B' = \phi$

(v) True [using part (iv)]

(vi) True, as $A \cup B = S$

2. Total number of ways of placing 3 letters in 3 envelopes

$$= {}^3P_3 = 3! = 6$$

Number of ways of placing 3 letters in 3 envelopes such that only one letter goes in right envelope

$$= {}^3C_1 \times 1 = 3$$

$$\therefore \text{Required probability} = \frac{3}{6} = \frac{1}{2}$$

3. Total number of ways of arranging n persons around a table i.e. $n(S) = (n - 1)$

Let E be the event of arranging n persons around a table, so that two specified individuals sitting next to each other, i.e.

$$n(E) = (n - 2)! \times 2!$$

$$\therefore \quad P(E) = \frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{(n-1)}$$

$$= \frac{(n-1)-2}{(n-1)} = \frac{n-3}{(n-1)} = \frac{(n-3)}{(n-1)}$$

$$\therefore \quad \text{Odd against of an event } E = \frac{P(\bar{E})}{P(E)} = \frac{\frac{(n-3)}{(n-1)}}{\frac{2}{(n-1)}} = \frac{(n-3)}{2}$$

$$= (n - 3): 2$$

4. Here, in 52 cards there are 12 face cards and four suits, in each suits have 3 face cards.

$$\therefore \text{Required probability} = \left(\frac{3}{52} \times \frac{2}{51} \times \frac{1}{50}\right) \times 4 = \frac{1}{5525}$$

5. Die marked with 1, 2, 2, 3, 3, 3 is thrown 3 times,

$$P(1) = \frac{1}{6}, P(2) = \frac{2}{6}, P(3) = \frac{3}{6}$$

P(4 or 6) = P[(1, 1, 2) (3 times) or (1, 2, 3) (6 case) or (2, 2, 2)]

$$= 3 \times \frac{1}{6} \times \frac{1}{6} \times \frac{2}{6} + 6 \times \frac{1}{6} \times \frac{2}{6} \times \frac{3}{6} + \frac{2}{6} \times \frac{2}{6} \times \frac{2}{6}$$

$$= \frac{6 + 36 + 8}{216} = \frac{50}{216}$$

6. ∵ Total number of two-digit numbers that can be formed by digits 1, 2, 3, 4, 5 = $5^2 = 25$

Now, as we know that a number is divisible by 4, if last two-digits of number divisible by 4.

So, the possible two-digit numbers divisible by 4 and formed by digits 1, 2, 3, 4, 5 are 12, 24, 32, 44, 52.

$$\therefore \text{Required probability} = \frac{5}{25} = \frac{1}{5}.$$

7. The total number of ways in which 8 persons can speak is ${}^8P_8 = 8!$. The number of ways in which A, B and C can be arranged in the specified speaking order is ${}^8C_3$. There are 5! ways in which the other five can speak. So, favourable number of ways is ${}^8C_3 \times 5!$.

$$\therefore \text{Required probability} = \frac{{}^8C_3 \times 5!}{8!} = \frac{8!}{3! \times 5!} \times 5! \times \frac{1}{8!}$$

$$= \frac{1}{3!} = \frac{1}{6}$$

8. Number of ways of choosing two numbers from numbers 1, 2, …, 40 = ${}^{40}C_2$.

As, we know that sum of two numbers is odd, if one is even and other is odd.

$$\therefore \text{Number of favourable ways} = {}^{20}C_1 \times {}^{20}C_1$$

$$= 20 \times 20 = 400$$

$$\therefore \text{Required probability} = \frac{400}{{}^{40}C_2} = \frac{400 \times 2}{40 \times 39} = \frac{20}{39}$$

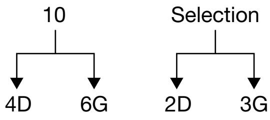
9. All the three persons has three options to apply a house.

$\therefore$ Total number of cases = $(3)^3$

Now, favourable cases = 3(with all has applied for has applied for house 1 or 2 or 3)

$$\therefore \text{Required probability} = \frac{3}{3^3} = \frac{1}{9}$$

10.



$$\therefore \text{Required probability} = \frac{{}^4C_2 \cdot {}^6C_3}{{}^{10}C_5} = \frac{10}{21}$$

Now, $3m - n = 9$

11. Total arrangement of 2 red, 3 green and 4 blue counters in a row

$$= \frac{(2+3+4)}{2!3!4!} = \frac{9!}{2!3!4!} = 1260$$

First, we arrange 2 red counter and 3 green counters.

$$\text{Number of ways} = \frac{5!}{3!2!} = \frac{120}{6 \times 2} = 10$$

we have, R R G_1 G_1 G_1

Now, there are 6 places at we place 4 blue counters in ${}^6C_4$ ways.

$\therefore$ Required probability $= \dfrac{15 \times 10}{1260}$

$= \dfrac{15}{126} = \dfrac{5}{42}$

But, $\dfrac{m}{n} = \dfrac{5}{42}$

$\therefore\ |8m - n| = |5 \times 8 - 42| = 2$

12. Number of ways of selecting three integers from 20 integers $= {}^{20}C_3$

As we know that, product of three integers is even, if atleast one of the number is even.

Now, let us first find the number of ways of selecting 3 odd integers $= {}^{10}C_3$

$\therefore$ Number of favourable ways $= {}^{20}C_3 - {}^{10}C_3$

$\therefore$ Required probability $= \dfrac{{}^{20}C_3 - {}^{10}C_3}{{}^{20}C_3}$

$= 1 - \dfrac{10 \times 9 \times 8}{20 \times 19 \times 18} = 1 - \dfrac{2}{19} = \dfrac{17}{19}$

13. Total number of ways of selecting 4 people, $n(S) = {}^{12}C_4$

Let E be the event of getting exactly one married couple in selecting of 4 people. Clearly, 4 people may consist 3 men and 1 women (one of the wives of selected men) or 2 men and 2 women (it include the wife of exactly one of the selected men) or 1 man and 3 women (include the wife of selected man.)

$\therefore n(E) = {}^6C_3 \times {}^3C_1 + {}^6C_2 \times {}^2C_1 \times {}^4C_1 + {}^6C_1 \times {}^1C_1 \times {}^5C_2$

$= 60 + 120 + 60$

$= 240$

$\therefore P(E) = \dfrac{n(E)}{n(S)} = \dfrac{240}{{}^{12}C_4}$

$= \dfrac{240 \times 4 \times 3 \times 2 \times 1}{12 \times 11 \times 10 \times 9}$

$= \dfrac{24 \times 2}{11 \times 9} = \dfrac{16}{33}$

14. We have, S = {1, 2, 3, 4, … 100}

Since, three distinct numbers are to be selected from 100 natural numbers $= {}^{100}C_3$

A = All the three of them are divisible by 2 and 3

$\Rightarrow$ divisible by 6 i.e. {6, 12, 18, …, 96}

$n(A) = {}^{16}C_3$

Required probability $= \dfrac{{}^{16}C_3}{{}^{100}C_3} = \dfrac{4}{1155}$

Test Your Concepts-II

1. We have, A and B are mutually exclusive events.

$\therefore P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$

(i) $\because P(A) + P(B) \le 1 \Rightarrow P(A) \le 1 - P(B) \Rightarrow P(A) \le P(\bar{B})$ (True)

(ii) $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) = P(\bar{A}) - P(B)$ (given relations is False)

(iii) $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1$ (given relations is False)

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = P(B)$ (given relations is False)

2. We have, S = {1, 2, 3, 4, 5, 6}

A(number greater than 3) {4, 5, 6}

B{number less than 5} = {1, 2, 3, 4}

$A \cap B = \{4\}$

$P(A) = \dfrac{3}{6},\ P(B) = \dfrac{4}{6},\ P(A \cap B) = \dfrac{1}{6}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \dfrac{3}{6} + \dfrac{4}{6} - \dfrac{1}{6} = 1$

3. We have, 4 red, 3 pink, 5 pink, 5 yellow and 8 white roses in a lawn.

Let A be the event that a blind man touch a red rose and B be the event that a blind man touch a white rose.

$\therefore n(A) = 4$

and $n(B) = 8$

$\therefore P(A \cup B) = P(A) + P(B)$ $[\because A \cap B = \phi]$

$= \dfrac{4}{20} + \dfrac{8}{20} = \dfrac{12}{20} = \dfrac{3}{5}$

4. Total number of ways of selecting 5 people out of 9 $= {}^9C_5$

Let A be the event that selection of 5 people consists a certain married couple and B be the event that selection of 5 people never consists a certain married couple.

$\therefore n(A) = {}^7C_3,\ n(B) = {}^7C_5$ and $n(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$

$= \dfrac{{}^7C_3 + {}^7C_5}{{}^9C_5} = \dfrac{{}^8C_3}{{}^9C_5} = \dfrac{8 \times 7 \times 6 \times 5 \times 4}{9 \times 8 \times 7 \times 6 \times 5} = \dfrac{4}{9}$

5. Since, $P(A) = P(B) = 2P(C)$

Now, $P(A) + P(B) + P(C) = 1$

$\therefore\quad P(C) = \dfrac{1}{5}$

$P(A) = P(B) = \dfrac{2}{5},\ P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$

Probability of B or C wins $= \dfrac{2}{5} + \dfrac{1}{5} = \dfrac{3}{5}$

6. We Have, $n(S) = 100$

$n(P \cap C') = 25$

$n(P' \cap C) = 20$

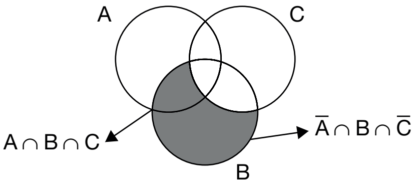
$n(P \cap C) = 15$

$\therefore$ Required probability,

$P(P \cap C') + P(P' \cap C) = \dfrac{25}{100} + \dfrac{20}{100}$

$= \dfrac{45}{100} = \dfrac{9}{20}$

7. From Venn diagram, we can see that



$$P(B \cap C) = P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$= \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{12}$$

8. $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/10}{1/4} = \frac{2}{5}$$

9. We have, let event A appeared sum is 8.

$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$

Let event B be the second die always exhibits 4.

$B = \{(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4)\}$

$A \cap B = \{(4, 4)\}$

$P(A) = 5/36, P(B) = 6/36, P(A \cap B) = 1/36$

To find $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{6/36} = \frac{1}{6}$$

10. Total number of balls in a bag = (3 + 6 + 7) = 16

Let A be the event first balls is white and B be the event second ball is blue.

$$\therefore P(A) = \frac{6}{16}, P(B/A) = \frac{7}{15}$$

$$P(A \cap B) = P(A) \times P(B/A) = \frac{6 \times 7}{16 \times 15} = \frac{7}{40}$$

11. We have, $S = \{1, 2, 3, \ldots, 10\}$

A = (Minimum number is 4)

$= \{4, 5, 6, 7, 8, 9, 10\}$

B = {Maximum number is 8}

$= \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cap B = \{4, 5, 6, 7, 8\}$

$$P(A) = \frac{{}^6C_2}{{}^{10}C_3}, P(B) = \frac{{}^7C_2}{{}^{10}C_3}, P(A \cap B) = \frac{{}^3C_1}{{}^{10}C_3}$$

$$\therefore P(A \cup B) = \frac{{}^6C_2}{{}^{10}C_3} + \frac{{}^7C_2}{{}^{10}C_3} - \frac{{}^3C_1}{{}^{10}C_3}$$

$$= \frac{15}{120} + \frac{21}{120} - \frac{3}{120} = \frac{33}{120} = \frac{11}{40}$$

12. $$\text{As, } P\left(\frac{C}{D}\right) = \frac{P(C \cap D)}{P(D)} = \frac{P(C)}{P(D)} \qquad \left[\begin{array}{l} \because \quad C \subset D \\ \therefore \quad P(C \cap D) = P(C) \end{array}\right] \qquad \ldots(1)$$

$$\left.\begin{array}{ll} \text{As,} & P(D) \le 1 \\ \therefore & \dfrac{1}{P(D)} \ge 1 \\ \text{and} & \dfrac{P(C)}{P(D)} \ge P(C) \end{array}\right\} \qquad \ldots(2)$$

From Eqs. (1) and (2), we have

$$P\left(\frac{C}{D}\right) = \frac{P(C)}{P(D)} \ge P(C)$$

13. We have, let A be the event of face one and B be the event of either face one or face two has turn up.

$P(A) = 0.10, P(B) = 0.10 + 0.32 = 0.42$

Required probability.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.10}{0.42} = \frac{5}{21} \qquad [\because A \cup B = A]$$

14. Let A be the event automobile will be stolen and found with in one week.

$\therefore P(A \cap B) = 0.0006$

and B be the event automobile will be stolen, then

$P(B) = 0.0015 \Rightarrow P(A \cap B) = 0.0006$

$$\text{Required probability} = P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0006}{0.0015} = 0.4$$

15. Let A be the event the person will get an electric contract and B be the event the person will get plumbing contract we have,

$$P(A) = \frac{2}{5}, P(\bar{B}) = \frac{4}{7}, P(B) = 3/7$$

$$P(A \cup B) = \frac{2}{3}$$

To find $P(A \cap B)$

$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$= \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{42 + 45 - 70}{105} = \frac{17}{105}$$

Test Your Concepts-III

1. Given, $$P(A) = \frac{1}{5}, P(B) = \frac{1}{6}$$

$$P(A \cap C) = \frac{1}{20}$$

$$P(B \cup C) = \frac{3}{8}$$

Now, since A and C are independent.

$$\text{So, } P(A \cap C) = P(A) \cdot P(C) = \frac{1}{20}$$

$$\Rightarrow \frac{1}{5}, P(C) = \frac{1}{20}$$

$$\Rightarrow P(C) = \frac{1}{4}$$

$$P(B \cup C) = P(B) + P(C) - P(B \cap C)$$

$$\Rightarrow \frac{3}{8} = \frac{1}{6} + \frac{1}{4} - P(B \cap C)$$

$$\Rightarrow P(B \cup C) = \frac{5}{12} - \frac{3}{8} = \frac{1}{24}$$

$$\text{Now, } P(B) \cdot P(C) = \frac{1}{6} + \frac{1}{4} = \frac{1}{24} = P(B \cap C)$$

$\Rightarrow$ B and C are independent.

2. Let A be the event of getting first two balls are red and B be the event of getting second two balls are blue.

$$\therefore P(AB) = P(A) \cdot P(B) = \frac{{}^4C_2}{{}^7C_2} \times \frac{{}^3C_2}{{}^7C_2}$$

$$= \frac{4 \times 3}{7 \times 6} \times \frac{3 \times 2}{7 \times 6} = \frac{2}{49}$$

3. Let A, B and C be three events such that,

$P(A) = p_1 = P(B) = p_2, P(C) = p_3$

Now, $P(\bar{A}) = 1 - P(A) = (1 - p_1)$

$$P(\bar{B}) = 1 - P(B) = (1 - p_2)$$
$$P(\bar{C}) = 1 - P(C) = (1 - p_3)$$

Probability of that not happening of A, B and C.

$P(\bar{A}\cap\bar{B}\cap\bar{C}) = P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C})$

$= (1 - p_1)(1 - p_2)(1 - p_3)$

∴ Probability of happening of atleast one of them

= 1 – (Probability of not happening of all of them)

$= 1 - (1 - p_1)(1 - p_2)(1 - p_3)$

4. Let A, B, C and D are the person and their respective probabilities to solve the problem is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}$.

i. e. $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$, $P(C) = \frac{1}{4}$ and $P(D) = \frac{1}{8}$

Probability that A fails to solve the problem,

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$$

Similarly, B, C and D fail to solve the problem is

$$P(\bar{B}) = 1 - P(B) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(\bar{C}) = 1 - P(C) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(\bar{D}) = 1 - P(D) = 1 - \frac{1}{8} = \frac{7}{8}$$

Now, probability that all of them are fail to solve the problem,

$$P(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}) = P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C})\cdot P(\bar{D})$$
$$= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{8} = \frac{21}{256}$$

Probability that, the problem is solved correctly by atleast one of them

= 1 – (Problem is not solved by all of them)

$$= 1 - \frac{21}{256} = \frac{235}{256}$$

5. Probability that bird is killed in a shot $= \frac{1}{3}$

Probability that bird is not killed in a shot $= 1 - \frac{1}{3} = \frac{2}{3}$

Now, probability that the bird is not killed in three shots

$$= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3} = \frac{8}{27}$$

Thus, probability that bird is killed $= 1 - \frac{8}{27} = \frac{19}{27}$

6. Given, odds against solving a problem by A are 4: 3.

∴ $P(\bar{A}) = \frac{4}{4+3} = \frac{4}{7}$

and $P(A) = \frac{3}{4+3} = \frac{3}{7}$

and odds in favour of B are 7 : 5.

∴ $P(B) = \frac{7}{7+5} = \frac{7}{12}$

and $P(\bar{B}) = 1 - \frac{7}{12} = \frac{5}{12}$

P(Only one will solve the problem) $= P(A)\cdot P(\bar{B}) + P(\bar{A})\cdot P(B)$

$$= \frac{3}{7}\times\frac{5}{12} + \frac{4}{7}\times\frac{7}{12} = \frac{5}{28} + \frac{1}{3}$$

$$= \frac{15+28}{84} = \frac{43}{84}$$

7. Probability (ending on R in the second step)

= Probability (landing on B, then move to R) + Probability (landing on W, then move to R)

$$= \frac{1}{6}\cdot(1) + \frac{1}{6}\cdot\left(\frac{1}{2}\right) = \frac{1}{4}$$

8. Set A have 4 elements and set B have 5 elements.

∴ Total number of functions from A to B $= (5)^4 = 625$

Number of monotonic function (either increasing or decreasing)

$$= 2 \times {}^5C_4 = 2 \times 5 = 10$$

Required probability $= \frac{10}{625} = \frac{2}{125}$

9. We have,

$$x + \frac{100}{x} > 50 \quad \text{or} \quad x^2 + 100 > 50x$$

$\Rightarrow x^2 - 50x + 100 > 0$

$\Rightarrow x^2 - 50x + (25)^2 + 100 - (25)^2 > 0$

$\Rightarrow (x - 25)^2 - 525 > 0$

$\Rightarrow (x - 25)^2 > 525$

$\Rightarrow (x - 25) < -\sqrt{525} \quad \text{or} \quad x - 25 > \sqrt{525}$

$\Rightarrow x < 25 - \sqrt{525} \quad \text{or} \quad x > 25 + \sqrt{525}$

As, x is a positive integer and $\sqrt{525} = 22.91$

We must have, $x \le 2$ or $x \ge 48$

Thus, the favourable number of cases is (2 + 53) = 55

∴ Required probability $= \frac{55}{100} = \frac{11}{20} = \frac{m}{n}$

$\Rightarrow m = 11, n = 20$

Now, $|m - n| = |11 - 20| = 9$

10. We have, $n(S) = (5)^5 = 3125$

For computing favourable outcomes, 2 boxes which are to remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in of 221 or 311 in

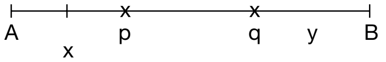
$$3!\left(\frac{5!}{2!2!2!} + \frac{5!}{3!2!}\right) = 150 \text{ ways}$$

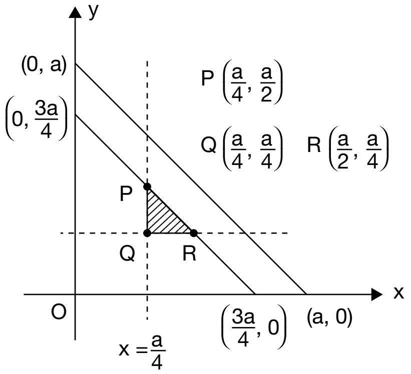
Now, $n(E) = {}^5C_2 \times 150$

Required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{{}^5C_2\times 150}{3125} = \frac{12}{25} = \frac{m}{n}$

∴ $|2m - n| = |2\times 12 - 25| = 1$

11.





Total conditions

$x+y<a$ and $0<x<a$, $0<y<a$

Favourable conditions

$x \geq \frac{a}{4}, y \geq \frac{a}{4}, a-(x+y) \geq \frac{a}{4}$

$x \geq \frac{a}{4}, y \geq \frac{a}{4}, x+y \leq \frac{3a}{4}$

Required probability $= \dfrac{\frac{1}{2}\cdot\frac{a}{4}\cdot\frac{a}{4}}{\frac{1}{2}\cdot a\cdot a} = \frac{1}{16}$

12. Here, 4 red and 4 blue balls and four balls are drawn one-by-one.

There are two ways in which balls are alternate in colour RBRB BRBR.

∴ Required probability = P(RBRB) + P(BRBR)

$$= \frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}+\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$$

$$= \frac{2\times4\times4\times3\times3}{8\times7\times6\times5} = \frac{6}{35}$$

13.

3^a ends in →	1	3	7	9
7^b ends in ↓				
1			8	
3				
7	8			
9				8

Out of 16 cases, 3 are favourable.

$\Rightarrow P = \frac{3}{16}$

14. Let A be the event of getting 10 on first throw, B be the event of getting 11 on second throw and C be the event of getting 12 on third throw.

$\therefore$ P(A) $= \frac{3}{36}$, P(B) $= \frac{2}{36}$, P(C) $= \frac{1}{36}$

Since, the trials are independent, therefore

$$P(A \cap B \cap C) = P(A)\cdot P(B)\cdot P(C)$$

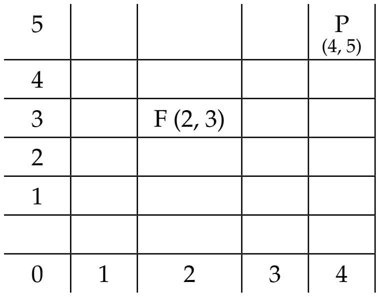
$$= \frac{3}{36}\times\frac{2}{36}\times\frac{1}{36} = \frac{1}{36}\times\frac{1}{36}\times\frac{1}{6} = \frac{1}{7776}$$

15. Here, $n(S) = \frac{9!}{4!5!} = 126$

$n(A)$ = 0 to F and F to P

$$= \frac{5!}{2\times3!}\times\frac{4!}{2!\times2!} = 60$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{60}{126} = \frac{10}{21}$$



Test Your Concepts-IV

1. Consider, the events ε_1, ε_2 and A.

ε_1 = selection of 1st purse

ε_2 = selection of 2nd purse

A = drawn coin is copper

Now, $P(\varepsilon_1) = \frac{1}{2}, P(\varepsilon_2) = \frac{1}{2}$

$$P(A/\varepsilon_1) = \frac{4}{7};\quad P(A/\varepsilon_2) = \frac{6}{8}$$

∴ Required probability $P(A) = P(\varepsilon_1)\times P(A/\varepsilon_1)+P(\varepsilon_2)\times P(A/\varepsilon_2)$

$$= \frac{1}{2}\times\frac{4}{7}+\frac{1}{2}\times\frac{6}{8} = \frac{37}{56}$$

2. First cube has five red faces and 1 blue face.

$\therefore P(R) = \frac{5}{6}, P(B) = \frac{1}{6}$

Let second cube has n red faces shown same colour $= \frac{1}{2}$

$$\Rightarrow \frac{5}{6}\times\frac{n}{6}+\frac{1}{6}\times\frac{6-n}{6} = \frac{1}{2} \quad\Rightarrow\quad \frac{5n}{36}+\frac{6-n}{36} = \frac{1}{2}$$

$$\Rightarrow 4n+6 = \frac{36}{2} = 18 \quad\Rightarrow\quad 4n+6 = 18$$

$$\Rightarrow 4n = 12 \quad\therefore\quad n = 3$$

3. Given, families have one, two or three children.

A family have four grand children is possible, if they have two or three children.

Now, two children have two possible cases (2, 2), (1, 3) and (3, 1) and three children have three possible cases (1, 1, 2), (1, 2, 1) and (2, 1, 1).

Required probability

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}$$
$$\times\frac{1}{2}+\frac{1}{4}\times\frac{1}{4}\times\frac{1}{2}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}$$

$$= \frac{1}{8} + \frac{1}{32} + \frac{1}{32} + \frac{1}{128} + \frac{1}{128} + \frac{1}{128}$$

$$= \frac{16+4+4+1+1+1}{128} = \frac{27}{128}$$

4. If S_1 and S_2 are in the same pair, then exactly one wins. The number of ways 16 players are divided into 8 pairs is $n(S) = \dfrac{16!}{(2!)^8 \times 8!}$

The number of ways in which S_1 and S_2 are in same pairs,

$$n(E) = \frac{14!}{(2!)^7 \times 7!}$$

$\therefore$ The probability of S_1 and S_2 in the same pairs $= \dfrac{\dfrac{14!}{(2!)^7 \times 7!}}{\dfrac{16!}{(2!)^8 \times 8}}$

$$= \frac{1}{15}$$

The probability of anyone winning in the pair of S_1 S_2 is P (certain event) = 1

Hence, the probability that the pair of S_1, S_2 being in two pairs separately and anyone of S_1, S_2 wins is given by the probability S_1, S_2 being in two pairs separately and S_1 wins, S_2 loses + the probability of S_1, S_2 being in two pairs separately and S_1 loses, S_2 wins, it is given by

$$\left(1 - \frac{1}{15}\right) \times \frac{1}{2} \times \frac{1}{2} + \left(1 - \frac{1}{15}\right) \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{15}$$

Required probability $= \dfrac{1}{15} + \dfrac{7}{15} = \dfrac{8}{15}$

5. We have,

$$\sum_{i=1}^{6} P_i = \sum_{i=1}^{6} \lambda i$$

[$\because P_i$ = probability of number shows on a face of die]

$\Rightarrow 1 = \lambda(1 + 2 + 3 + 4 + 5 + 6)$ [$\because \Sigma P_i = 1$]

$$\lambda = \frac{1}{21}$$

$\therefore$ Required probability $= 3\lambda = 3 \times \dfrac{1}{21} = \dfrac{1}{7}$

6. We have, 3 red and 3 green balls.

There are four possible, if three balls are drawn.

ε_1 = all are red balls

ε_2 = all are green balls

ε_3 = 1 green and red 2

ε_4 = 2 red and 1 green

$\therefore$ Required probability to find all different colour,

$$P(A) = P(\varepsilon_3) \times P(A/\varepsilon_3) + P(\varepsilon_4) \times P(A/\varepsilon_4)$$

$$= \frac{{}^3C_1 \times {}^3C_2}{{}^6C_3} \times \frac{{}^2C_1 \times {}^1C_1 \times {}^3C_1}{{}^6C_3} + \frac{{}^3C_2 \times {}^3C_1}{{}^6C_3} \times \frac{{}^1C_1 \times {}^2C_1 \times {}^3C_1}{{}^6C_3}$$

$$= \frac{3 \times 3 \times 2 \times 3}{20 \times 20} + \frac{3 \times 3 \times 2 \times 3}{20 \times 20}$$

$$= \frac{2 \times 3 \times 3 \times 2 \times 3}{20 \times 20} = \frac{27}{100}$$

7. We have, 3 goods and 2 defective articles in A and 4 goods and 1 defective articles in B.

A new to C is formed by taking 3 articles from A and 2 from B.

Good	defective	good	defective	good	defective
3	2	4	1	3 + 2	0 + 0
3	2	4	1	2 + 2	1 + 0
3	2	4	1	1 + 2	2 + 0
3	2	4	1	3 + 1	0 + 1
3	2	4	1	2 + 1	1 + 1
3	2	4	1	1 + 1	2 + 1

Required probability

$$= \frac{{}^3C_3 \times {}^4C_2 \times 0}{{}^5C_3 \times {}^5C_2} + \frac{{}^3C_2 \times {}^2C_1 \times {}^4C_2}{{}^5C_3 \times {}^5C_2} \times \frac{1}{5} + \frac{{}^3C_1 \times {}^2C_2 \times {}^4C_2}{{}^5C_3 \times {}^5C_2}$$

$$\times \frac{2}{5} + \frac{{}^3C_3 \times {}^4C_1 \times {}^2C_0 \times {}^1C_1}{{}^5C_3 \times {}^5C_2} \times \frac{1}{5} + \frac{{}^3C_2 \times {}^4C_1 \times {}^2C_1 \times {}^1C_1}{{}^5C_3 \times {}^5C_2} \times \frac{2}{5}$$

$$+ \frac{{}^3C_1 \times {}^2C_2 \times {}^1C_1 \times {}^4C_1}{{}^5C_3 \times {}^5C_2} \times \frac{3}{5}$$

$$= \frac{36}{500} + \frac{36}{500} + \frac{4}{500} + \frac{48}{500} + \frac{36}{500} = \frac{8}{25}$$

8. Consider, the event ε_1, ε_2 and A.

ε_1 = specific home is unlocked, ε_2 = specific home is locked,

A = master key open houses

$$P(\varepsilon_1) = 40\% = \frac{40}{100}, P(\varepsilon_2) = 60\% = \frac{60}{100}$$

$$P(A/\varepsilon_1) = 1, P(A/\varepsilon_2) = 3/8$$

$\therefore$ Required probability.

$$P(A) = P(\varepsilon_1) \times P(A/\varepsilon_2) + P(\varepsilon_2) \times P(A/\varepsilon_2)$$

$$= \frac{40}{100} \times 1 + \frac{60}{100} \times \frac{3}{8} = \frac{320+180}{800} = \frac{500}{800} = \frac{5}{8}$$

9. Consider, the events, ε_1, ε_2 and A.

ε_1 = red marbles in first draw

ε_2 = green marbles in first draw

A = red marble in second draw without replacement

$$P(\varepsilon_1) = \frac{6}{9}, P(\varepsilon_2) = \frac{3}{9}$$

$$P(A/\varepsilon_1) = \frac{5}{8}, P(A/\varepsilon_2) = \frac{6}{8}$$

$\therefore$ Required probability

$$P(A) = P(\varepsilon_1) \times P(A/\varepsilon_1) + P(\varepsilon_2) \times P(A/\varepsilon_2)$$

$$= \frac{6}{9} \times \frac{5}{8} + \frac{3}{9} \times \frac{6}{8} = \frac{30+18}{72} = \frac{48}{72} = \frac{2}{3}$$

10. Let E_1 be the event that chosen bag is A,
E_2 be the event that chosen that chosen bag is B
and A be the event that chosen ball is red.

Now, $P(E_1) = P(E_2) = \frac{1}{2}$

$$P\left(\frac{A}{E_1}\right) = \frac{3}{5},\ P\left(\frac{A}{E_2}\right) = \frac{5}{9},$$

To find $P\left(\frac{E_1}{A}\right)$

By Bayes' theorem, we have

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{3}{5}}{\frac{1}{2}\times\frac{3}{5}+\frac{1}{2}\times\frac{5}{9}} = \frac{\frac{3}{5}}{\frac{3}{5}+\frac{5}{9}} = \frac{\frac{3}{5}}{\frac{52}{45}} = \frac{3}{5}\times\frac{45}{52} = \frac{27}{52}$$

$$\therefore P\left(\frac{E_1}{A}\right) = \frac{27}{52}$$

11. Let E_1 = coin shown head, E_2 = coin shows tail
A = Drawn ball is blue

$$P(E_1) = \frac{1}{2} = P(E_2)$$

$P(A/E_1)$ = Probability of drawing a blue ball from box l = $\frac{2}{7}$

$P\left(\frac{E_2}{A}\right)$ = Probability of drawing a blue ball from box II = $\frac{6}{8}$

To find, $P\left(\frac{E_1}{A}\right)$

By Bayes' theorem, we have

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{7}}{\frac{1}{2}\times\frac{2}{7}+\frac{1}{2}\times\frac{6}{8}} = \frac{\frac{2}{7}}{\frac{2}{7}+\frac{6}{8}} = \frac{2}{7}\times\frac{56}{58} = \frac{8}{29}$$

12. Let E_1 be the event that the letter came from 'DHARMAPURI' and E_2 be the event that the letter came from 'KRISHNAGIRI'.
Let A be the event that two consecutive letters RI are visible as post mark.
Since, letters have come either from 'KRISHNAGIRI' or 'DHARMAPURI'.

$$\therefore P(E_1) = P(E_2) = \frac{1}{2}$$

If E_1 has occurred, then it means that letters came from 'DHARMPURI'.
In the word 'DHARMAPURI' there are 10 letters in which RI occurs in the end.
Considering RI as one letter there are 9 letters out of which one can be in 9 ways.

$$\therefore P\left(\frac{A}{E_1}\right) = \frac{1}{9}$$

If E_2 has occurred, then the letters came from 'KRISHNAGIRI'.
In the word 'KRISHNAGIRI' there are 11 letters in which RI occurs twice. Considering one of the two RI as one letter there as 10 letters.

$$\therefore P\left(\frac{A}{E_2}\right) = \frac{2}{10}$$

By Bayes' theorem, we have

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{2}{10}}{\frac{1}{2}\times\frac{1}{9}+\frac{1}{2}\times\frac{2}{10}} = \frac{\frac{2}{10}}{\frac{1}{9}+\frac{2}{10}} = \frac{2}{10}\times\frac{90}{28} = \frac{9}{14}$$

13. Let $E_1, E_2, E_3. \ldots, E_{100}$ and A be the events defined as follow:

E_1: There are one coin of 1 rupee rest are different $P(E_1) = \frac{1}{100}$

E_2: There are two coin of 1 rupee rest are different $P(E_2) = \frac{1}{100}$

E_{100}: All coins are 1 rupee coin $P(E_{100}) = \frac{1}{100}$

A: Drawing a one rupee coin from the purse

$$P\left({}^{A}\!/_{\varepsilon_1}\right) = \frac{1}{100};\ P\left({}^{A}\!/_{\varepsilon_2}\right) = \frac{2}{100};\ldots P\left({}^{A}\!/_{\varepsilon_{100}}\right) = \frac{100}{100}$$

∴ Required probability

$$P\left({}^{\varepsilon_1}\!/_{A}\right) = \frac{\frac{1}{100}\times\frac{1}{100}}{\frac{1}{100}\times\frac{1}{100}+\frac{1}{100}\times\frac{2}{100}+\frac{1}{100}\times\frac{3}{100}+\ldots..+\frac{1}{100}\times\frac{100}{100}}$$

$$= \frac{\frac{1}{100}}{\frac{1}{100}+\frac{2}{100}+\frac{3}{100}+\frac{100}{100}} = \frac{\frac{1}{100}}{\frac{1}{100}(1+2+3+4+\ldots..+100)}$$

$$= \frac{1}{\frac{100\times101}{2}} = \frac{1}{50\times101} = \frac{1}{5050}$$

14. R: they obtained the same result.

$B_1: A \cap \bar{B}; P(B_1) = \frac{1}{8}\cdot\frac{11}{12}$

$B_2: \bar{A} \cap B; P(B_2) = \frac{7}{8}\cdot\frac{1}{12}$

$B_3: A \cap B; P(B_3) = \frac{1}{8}\cdot\frac{1}{12}$

$B_4: \bar{A} \cap \bar{B}; P(B_4) = \frac{7}{8}\cdot\frac{11}{12}$

Now, $P(R/B_1) = 0$, $P(P/B_2) = 0$, $P(R/B_3) = 1$

$$P(R/B_4) = \frac{7}{8}\cdot\frac{11}{12}\cdot\frac{1}{1001}$$

$$P(B_3/R) = \frac{\frac{1}{8}\cdot\frac{1}{12}}{\frac{1}{8}\cdot\frac{1}{12}+\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{1}{1001}} = \frac{13}{14}$$

Test Your Concepts-V

1. Let p = Probability of getting head on tossing a coin

n = Number of times coin tossed = 10

x = Number of times getting head

Clearly, $X - B\left(10, \frac{1}{2}\right)$

$$\therefore P(X = x) = {}^{10}C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}, x = 0, 1, 2, \ldots, 10$$

$$\Rightarrow P(X = x) = {}^{10}C_x\left(\frac{1}{2}\right)^{10}, x = 0, 1, 2, \ldots, 10$$

(i) P(exactly six heads) = P(X = 6)

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} = {}^{10}C_4\left(\frac{1}{2}\right)^{10}$$

$$= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\times\frac{1}{2^{10}}$$

$$= \frac{105}{2^9} = \frac{105}{512}$$

(ii) P(atleast six heads) = P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10}({}^{10}C_4 + {}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0)$$

$$= \frac{1}{2^{10}}\times(210 + 120 + 45 + 10 + 1)$$

$$= \frac{386}{2^{10}} = \frac{193}{512}$$

(iii) P (atmost six heads)

$$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^{10}C_0\left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2^{10}}\right)^{10}({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_6)$$

$$= \frac{1}{2^{10}}[2^{10} - ({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10})]$$

$$= \frac{1}{2^{10}}[2^{10} - ({}^{10}C_3 + {}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0)]$$

$$= \frac{1}{2^{10}}[2^{10} - (120 + 45 + 10 + 1)]$$

$$= 1 - \frac{176}{2^{10}} = 1 - \frac{2^4\times 11}{2^{10}}$$

$$= 1 - \frac{11}{64} = \frac{53}{64}$$

2. Let p = Probability of winning a match by India = $\frac{1}{2}$

n = Number of matches = 5

x = Number of matches win by India

Clearly, $X \sim B\left(5, \frac{1}{2}\right)$

$$\therefore P(X = x) = {}^5C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}, x = 0, 1, 2, \ldots, 5$$

$$= {}^5C_x\left(\frac{1}{2}\right)^5, x = 0, 1, 2, \ldots, 5$$

∴ P(that India wins atleast three matches)

$$= P(X = 3) + P(X = 4) + P(X = 5)$$

$$= {}^5C_3\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_5\left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5({}^5C_3 + {}^5C_4 + {}^5C_5)$$

$$= \left(\frac{1}{2}\right)^5(10 + 5 + 1) = \frac{2^4}{2^5} = \frac{1}{2}$$

3. Let P = Probability of getting head on tossing a coin = $\frac{1}{2}$

n = Number of times coin tossed = 7

x = Number of times man get head

Clearly, $X \sim B\left(7, \frac{1}{2}\right)$

$$\therefore P(X = x) = {}^7C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{7-x}, x = 0, 1, 2, \ldots 7 = {}^7C_x\left(\frac{1}{2}\right)^7$$

P (that man wins the toss on more than three occassions)

= 1 – P (that man wins the toss atmost three occassion)

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - \left[{}^7C_0\left(\frac{1}{2}\right)^7 + {}^7C_1\left(\frac{1}{2}\right)^7 + {}^7C_2\left(\frac{1}{2}\right)^7 + {}^7C_3\left(\frac{1}{2}\right)^7 +\right]$$

$$= 1 - \left(\frac{1}{2}\right)^7({}^7C_0 + {}^7C_1 + {}^7C_2 + {}^7C_3)$$

$$= 1 - \left(\frac{1}{2}\right)^7(1 + 7 + 21 + 35)$$

$$= 1 - \frac{64}{2^7} = 1 - \frac{1}{2} = \frac{1}{2}$$

4. Let p be the probability of getting 5 on a die = $\frac{1}{6}$

n = Number of times die thrown = 7

x = Number of times getting 5

Clearly, $X \sim B\left(7, \frac{1}{6}\right)$

$$\therefore P(X = x) = {}^7C_x\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{7-x}, x = 0, 1, 2, \ldots, 7$$

Required probability = P(X = 4) = ${}^7C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^3$

5. Let p = Probability of getting head on tossing a coin

n = Number of times coin tossed = 10

x = Number of times getting head

Clearly, $X \sim B\left(10, \frac{1}{2}\right)$

$$\therefore P(X = x) = {}^{10}C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}, x = 0, 1, 2, \ldots, 10$$

$$= {}^{10}C_x\left(\frac{1}{2}\right)^{10}, x = x = 0, 1, 2, \ldots, 10$$

Now, P (getting an odd number of heads)

$= P(X = 1) + P(X = 3) + P(X = 5) + P(X = 7) + P(X = 9)$

$$= {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_9\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10}\left({}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_3 + {}^{10}C_1\right)$$

$$= \frac{1}{2^{10}}\left(2.{}^{10}C_1 + 2.{}^{10}C_3 + {}^{10}C_5\right)$$

$$= \frac{1}{2^{10}}(20 + 240 + 252) = \frac{512}{2^{10}} = \frac{1}{2}$$

6. Let p = Probability of getting a score of 9 on throwing a pair of dice

$$= \frac{4}{36} = \frac{1}{9}$$

n = Number of times a pair of dice thrown = 3

x = Number of times getting a score of 9

Clearly, $X \sim B\left(3, \frac{1}{9}\right)$

$$\therefore P(X = x) = {}^3C_x\left(\frac{1}{9}\right)^x\left(\frac{8}{9}\right)^{3-x}, x = 0, 1, 2, 3$$

Required probability

$$= P(X = 2) = {}^3C_2\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)^1 = 3 \times \frac{1}{81} \times \frac{8}{9} = \frac{8}{243}$$

Single Correct Choice Type Questions

1. $n(s) = 6^3$

$n(E)$ = Available 6 different numbers or 3 places in ${}^6P_3$ Hence, $P(E) = 5/9$

Hence, the correct answer is (A).

2. P(atleast one white ball is selected) = 1 – P(no white ball is selected)

$$P(A) = 1 - \left[\frac{{}^7C_2}{{}^{11}C_2}\right] = \frac{34}{55}$$

Hence, the correct answer is (D).

3. E →First boys can be arranged in $\lfloor 4$ ways, then there are 5-gaps between boys in 5-gaps, 5-girls can be arranged in $\lfloor 5$ ways

$$P(E) = \frac{\lfloor 5 \lfloor 4}{\lfloor 9} = \frac{5 \times 4 \times 3 \times 2}{5 \times 6 \times 7 \times 8 \times 9} = \frac{1}{126}$$

Hence, the correct answer is (B).

4. Let 3-stopping stations be S_1, S_2, S_3 then are $m - 3$ stations remaining. Between these $m - 3$ stations there are $m - 2$ places select any 3 for S_1, S_2, S_3 then there are no two stopping stations are adjacent

$$P(E) = \frac{{}^{m-2}C_3}{{}^mC_3}$$

Hence, the correct answer is (C).

5. Among every 6-consecutive integers one divisible by 6 and other integers leaves remainders 1, 2, 3, 4, 5 when divided by 6

The numbers which leave the remainder 1 and 5 are relatively prime to 6

Required probability $\frac{2}{6} = \frac{1}{3}$

Hence, the correct answer is (B).

6. $P(A \cap B) = P(A).P(B)$

$P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$

$0.\,8 = 0.3 + P(B)\,(1 - 0.3)$

$0.\,5 = P(B)(0.7) \Rightarrow P(B) = \frac{5}{7}$

Hence, the correct answer is (C).

7. Out of 20 shoes 4 be taken in ${}^{20}P_4$.

Ways of getting no. pair = 20 × 18 × 16 × 14

Probability of no. pair $= \frac{224}{323}$

Probability of at least one pair $= 1 - \frac{224}{323} = \frac{99}{323}$

Hence, the correct answer is (D).

8. Out of 10 persons 3 can be selected in ${}^{10}P_3$ ways = 10.9.8 = 720. First person in 10 ways. Other 2 in ${}^7C_2$ ways in which 2 are together in 6 ways.

Hence,

$$P(E) = \frac{10 \times ({}^7C_2 - 6) \times 2}{720} = \frac{5}{12}.$$

Hence, the correct answer is (D).

9. Each coupon can be selected in 15 ways. The total number of ways of choosing 7 copouns is 15^7. If largest number is 9, then the selected numbers have to be from 1 to 9 excluding those consisting of only 1 to 8.

Probability desired is $\frac{9^7 - 8^7}{15^7}$

$$= \left(\frac{3}{5}\right)^7 - \left(\frac{8}{15}\right)^7$$

Hence, the correct answer is (D).

10. The total numbers of arrangements is $\frac{11!}{2!2!2!} = \frac{11!}{8}$

The number of arrangements in which C, E, H, I, S appear in that order

$$= \left({}^{11}C_5\right)\frac{6!}{2!2!2!} = \frac{11!}{8.5!}$$

Probability $= \dfrac{11!}{8.5!} \div \dfrac{11!}{8} = \dfrac{1}{5!} = \dfrac{1}{120}$

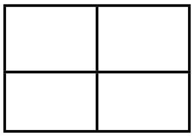
Hence, the correct answer is (C).

11. $n(s) = {}^{64}C_3$

Let 'E' be the event of selecting three squares which form the letter 'L'

The number of ways selecting squares consisting of 4 unit squares is $7 \times 7 = 49$.

Each square with 4 unit squares form 4L shapes consisting of 3 unit squares



$n(\mathrm{E}) = 7 \times 7 \times 4 = 196$ $\dfrac{n(\mathrm{E})}{n(\mathrm{S})} = \dfrac{196}{{}^{64}C_3}$

Hence, the correct answer is (A).

12. $np = 6$, $npq = 2$

$\Rightarrow q = \dfrac{1}{2}, p = \dfrac{1}{2}, n = 8$

$\mathrm{P}(X = 2) = {}^{8}C_2 \left(\dfrac{1}{2}\right)^2 \left(\dfrac{1}{6}\right)^6 = \dfrac{8 \times 7}{2} \times \left(\dfrac{1}{2}\right)^8 = \dfrac{28}{256}$

Hence, the correct answer is (B).

13. $np = 4$, $npq = 3$

$\Rightarrow q = \dfrac{3}{4}, p = \dfrac{1}{4}, n = 16$

$\mathrm{P}(X = r) = {}^{16}C_r \left(\dfrac{1}{4}\right)^r \left(\dfrac{3}{4}\right)^{16-r}$

Hence, the correct answer is (A).

14. $a, b \in \{7m, 7m+1, 7m+2, 7m+3, 7m+4, 7m+5, 7m+6\}$

$a^2, b^2 \in \{7m_1, 7m_1+1, 7m_1+4, 7m_1+2, 7m_1+2, 7m_1+4, 7m_1+1\}$

$\therefore a^2, b^2$ must be of the form $7m$.

Probability $= \dfrac{1}{49}$

Hence, the correct answer is (D).

15. For ranked 1 and 2 players to be winners and runners up res., they should not be paired with each other in any rounded. Rank 1 always wins. Therefore, the required probability $\dfrac{30}{31} \times \dfrac{14}{15} \times \dfrac{6}{7} \times \dfrac{2}{3} = \dfrac{16}{31}$

Hence, the correct answer is (A).

16. Let H denote the head, T the tail.

* Any of the head or tail

$\mathrm{P(H)} = \dfrac{1}{2}, \mathrm{P(T)} = \dfrac{1}{2}$ $\mathrm{P}(*) = 1$

HHHH*** $= \left(\dfrac{1}{2}\right)^4 \times 1 = \dfrac{1}{16}$

THHHH** $= \left(\dfrac{1}{2}\right)^5 \times 1 = \dfrac{1}{32}$

THHHH $= \left(\dfrac{1}{2}\right)^5 \times 1 = \dfrac{1}{32}$

**THHHH $= \left(\dfrac{1}{2}\right)^5 \times 1 = \dfrac{1}{32}$

Required probability $= \dfrac{5}{32}$

Hence, the correct answer is (B).

17. $\mathrm{P(no\ heads)} = {}^{5}C_0 \left(\dfrac{1}{2}\right)^5$

$\mathrm{P(1\ head)} = {}^{5}C_1 \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2}\right)^4$

$\mathrm{P(2\ heads\ not\ consecutive)} = {}^{4}C_2 \left(\dfrac{1}{2}\right)^2 \left(\dfrac{1}{2}\right)^3 \quad {}_X T_X T_X T_X$

$\mathrm{P(3\ heads\ not\ consecutive)} = {}^{3}C_3 \left(\dfrac{1}{2}\right)^3 \left(\dfrac{1}{2}\right)^2 \quad {}_X T_X T_X$

Required probability $= \dfrac{1}{32} + \dfrac{5}{32} + \dfrac{6}{32} + \dfrac{1}{32} = \dfrac{13}{32}$

Hence, the correct answer is (C).

18. P(get a number bigger than 3) $= \dfrac{1}{2}$

P(get 5 in throw) $= \dfrac{1}{6}$

E → get 5 in last throw when he gets a number bigger than 3

$\mathrm{P(E)} = \dfrac{1}{6} + \dfrac{1}{2}.\dfrac{1}{6} + \dfrac{1}{2}.\dfrac{1}{2}.\dfrac{1}{6} + \ldots.\ \infty$

$= \dfrac{1}{6} \times \dfrac{1}{1 - \dfrac{1}{2}} = \dfrac{1}{3}$

Hence, the correct answer is (A).

19. We observe that $7^1, 7^2, 7^3$ and 7^4 ends in 7, 9, 3 and 1 respectively. Thus 7^l ends in 7, 9, 3 or 1 according as l is of the form $4k + 1$, $4k + 2$, $4k - 1$ or $4k$ respectively. If S is the sample space, then $n(\mathrm{S}) = (100)^2$. $7^m + 7^n$ is divisible by 5 if (i) m is of the form $4k + 1$ and n is of the form $4k - 1$ or (ii) m is of the form $4k + 2$ and n is of the form $4k$ or (iii) m is of the form $4k - 1$ and n is of the form $4k + 1$ or (iv) m is of the form $4k$ and n is of the form $4k + 1$.

Thus number of favourable ordered pairs $(m, n) = 4 \times 25 \times 25$.

Hence, required probability is $\dfrac{1}{4}$

Hence, the correct answer is (A).

20. $\because$ LHS is an integer

⇒R.H.S must be an integer for 'x' is a multiple of 30.

i. e., $x = 30, 60, \ldots 990$

Required probability $= \dfrac{33}{1000}$

Hence, the correct answer is (C).

21. Required probability

$$= \frac{{}^{m}C_1 \times {}^{mn-2}C_{n-2} \times \frac{(mn-n)!}{(n!)^{m-1}}}{\left(\frac{(mn)!}{(n!)^m}\frac{1}{m!}\right)m!} = \frac{(n-1)}{(mn-1)}$$

Hence, the correct answer is (B).

22. $n_1 + n_2 = 100$

$$n_1, n_2 > 1600$$
$$n_1(100 - n_1) > 1600$$
$$n_1^2 - 100n_1 + 1600 < 0$$
$$(n_1 - 80)(n_1 - 20) < 0 \Rightarrow n_1 \in (20, 80)$$
$$n_1 = 21, 22, \ldots\ldots\ldots 79$$

$\therefore$ Required probability $= \frac{59}{99}$

Hence, the correct answer is (D).

23. E_1: 2 white, 2 non-white balls
E_2: 3 white, 1 non-white ball
E_3: 4 white balls
E: Two drawn balls are white

$$P(E) = \frac{\frac{1}{3}\cdot\frac{{}^4C_2}{{}^4C_2}}{\frac{1}{3}\left\{\frac{{}^2C_2}{{}^4C_2} + \frac{{}^3C_2}{{}^4C_2} + \frac{{}^4C_2}{{}^4C_2}\right\}}$$

$$= \frac{1}{\frac{1+3+6}{6}} = \frac{6}{10} = \frac{3}{5}$$

Hence, the correct answer is (C).

24. $x + y = 2n$ and $x, y \in N$
$\Rightarrow$ sample space (S) $= 2n - 1$
and out of this sample space only one point (n, n) lies on the line $y = x$
So, required probability,

$$P = \frac{(2n-1)-1}{2n-1} = \frac{2n-2}{2n-1}$$

Hence, the correct answer is (C).

25. E_1: Coin having both side head $\Rightarrow P(E_1) = \frac{n}{2n+1}$

E_2: Coin is unbiased $\Rightarrow P(E_2) = \frac{n+1}{2n+1}$
E: Getting head

$$P(E) = P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right)$$

$$= \frac{n}{2n+1} \times 1 + \frac{n+1}{2n+1}\cdot\frac{1}{2} = \frac{37}{50} \Rightarrow n = 12$$

Hence, the correct answer is (C).

26. Let A be the event that sum of digits is 9
Let B be the event that number is a perfect square
(B) ≡ 01, 04, 09, 16, 25, 36, 49, 64, 81

$(A \cap B) \equiv 09, 36, 81$

$$p\left(\frac{A}{B}\right) = \frac{p(A \cap B)}{p(B)} = \frac{3}{9} = \frac{1}{3}$$

Hence, the correct answer is (A).

27. (1, 2, 3 to 6) , (1, 3, 4 to 6), (1, 4, 5 to 6), (1, 5, 6) (2, 3 to 4 to 6), (2, 4, 5 to 6), (2, 5, 6) (3, 4, 5 to 6), (3, 5, 6), (4, 5, 6)

$$P(E) = \frac{20}{216}$$

Hence, the correct answer is (C).

28. $n(s) = {}^{2n}C_2 \cdot {}^{2n-2}C_2 \ldots\ldots {}^{2}C_2$

$n(E) = n^2(n-1)^2(n-2)^2 \ldots. 1 = (\lfloor n)^2$

Hence, the correct answer is (C).

29. $16x^2 + 8(a + 5)x - 7a - 5 = 0$ has imaginary root
$\Rightarrow 64(a + 5)^2 + 64(7a + 5) < 0 \Rightarrow -15 < a < -2$

Hence, the required probability $= \frac{\int_{-15}^{-2} da}{\int_{0}^{20} da} = \frac{[a]_{-15}^{-2}}{[a]_0^{20}} = \frac{13}{20}$

Hence, the correct answer is (A).

30. $\log_a b = \frac{\log b}{\log a}$

(let $b = 2^m$ and $a = 2^n$ where m and n denotes the exponents on the bose 2 in the given set $= \frac{m}{n}$)

Therefore, $\log_a b$ is an integer only if n divides m.

Now, total no. of ways m and n can be chosen $= 25 \times 24 = 600$
For favourable cases

Let $n = 1$	So m can take values 1, 2, 3, 4, 5, 6, ..., 24	= 24
If $n = 2$	m = 4, 6, 8, 10, 12, 14, 16, 12, 14, 16, 18, 20, 22, 24	= 11
$n = 3$	m = 6, 9, 12, 15, 18, 21	= 7
$n = 4$	m = 8, 12, 16, 20, 24	= 5
$n = 5$	m = 10, 15, 20, 25	= 4
$n = 6$	m = 12, 18, 24	= 3
$n = 7$	m = 14, 21	= 2
$n = 8$	m = 16, 24	= 2
n = 9, 10, 11, 12	one value of m, for each	= 4
		= 62

Hence, required probability $= \frac{62}{600} = \frac{31}{300}$

Hence, the correct answer is (B).

31. The no. of ways in which 'm' passengers can alight at 'n' floors $= n^m$.
The no. of ways in which all of them can alight at different floors is $= {}^nP_m$

The probability of the required event $= \frac{{}^nP_m}{n^m}$

Hence, the correct answer is (D).

32. Let E_1 be the event of both getting the correct answer and E_2 be the even of both getting wrong answer . Let E be the even of both obtaining the same answer.

$$P(E_1) = \frac{1}{8}\frac{1}{12} = \frac{1}{96}$$

$$P(E_2) = \left(1-\frac{1}{8}\right)\left(1-\frac{1}{12}\right) = \frac{77}{96}$$

$$P\left(\frac{E}{E_1}\right) = 1, P\left(\frac{E}{E_2}\right) = \frac{1}{1001}$$

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right).P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)} = \frac{13}{14}$$

Hence, the correct answer is (B).

33. $f'(x) = 3x^2 + 2ax + b$, $f(x)$ is increasing $f'(x) \geq 0\ \forall\ x$ and for $f'(x) = 0$ should not form an interval

$\therefore a^2 - 3b \leq 0$

This is true for exactly 16 ordered pairs (a, b) $1 \leq a, b \leq 6$ (1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6) (3, 3) (3, 4) (3, 5) (3, 6) (4, 6)

$\therefore P(E) = \frac{16}{36} = \frac{4}{9}$

Hence, the correct answer is (A).

34. A's chance of winning in a throw $= \frac{5}{36}$

A's chance of losing in a throw $= \frac{31}{36}$

B's chance of winning in a throw $= \frac{1}{6}$

B's chance of losing in a throw $= \frac{5}{6}$

$$\text{A can win the game} = \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots\ldots\ldots.$$

$$= \frac{5}{36}\left[1 + \frac{155}{216} + \left(\frac{155}{216}\right)^2 + \cdots\cdots\right] = \frac{30}{61}$$

Hence, the correct answer is (A).

35. In pool A, pairing may be done in following ways:

Way number 1: A_1 A_2 and A_3A_4

Way number 2: A_1 A_3 and A_2A_4

Way number 3: A_1 A_4 and A_2A_3

Outcomes of different ways are

W – 1 A_1A_3 A_1A_4

W – 2 A_1A_2 A_1A_4 A_2A_3 A_3A_4

W – 3 A_1A_2 A_1A_3 A_2A_4 A_3A_4

(i) The probability that A_1A_2 reach the semifinal

$$= \frac{1}{3}\left(0 + \frac{1}{4} + \frac{1}{4}\right) = \frac{1}{6}$$

(ii) The probability that A_1A_2, B_1B_2 reach the semifinal

$$= \frac{1}{6} \times \frac{1}{12} = \frac{1}{72}$$

(since probability of B_2 B_3 is same as that of A_2A_3 which is $\frac{1}{12}$)

Hence, the correct answer is (A).

36. For expression $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ to be an integer, x should lie between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and y should lie between $[0, \pi]$

$x = 1$, and $y = 1, 2, 3$

Required probability $= \frac{3}{16}$

Hence, the correct answer is (B).

37. G = Original signal green. A = A receive correct signal, B = B receive signal correct. E is signal received by B is green.

$$P\left(\frac{G}{E}\right) = \frac{P(G \cap E)}{P(E)}$$

$$P(E) = P(GAB) + P\left(G\bar{A}\bar{B}\right) + \left(\bar{G}A\bar{B}\right) + P\left(\bar{G}\bar{A}B\right)$$

$$= \frac{4}{5}.\frac{3}{4}.\frac{3}{4} + \frac{4}{5}.\frac{1}{4}.\frac{1}{4} + \frac{1}{5}.\frac{3}{4}.\frac{1}{4} + \frac{1}{5}.\frac{1}{4}.\frac{3}{4} = \frac{46}{80}$$

$$P(G \cap E) = P(GAB) + P\left(G\bar{A}\bar{B}\right) = \frac{40}{80}$$

Hence, the correct answer is (C).

38. r_1, r_2, r_3 must be of the four $3n$, $3n + 1$, $3n + 2$

$$P(E) = \frac{3!.{}^2C_1.{}^2C_1.{}^2C_1}{6^3} = \frac{2}{9}$$

Hence, the correct answer is (C).

39. Let the numbers be $a, b, \frac{1}{ab}$. We are given

$$a + b + \frac{1}{ab} > \frac{1}{a} + \frac{1}{b} + ab$$

$$\text{Now, } (a-1)(b-1)\left(\frac{1}{ab} - 1\right)$$

$$= 1 + \left(a + b + \frac{1}{ab}\right) - \left(ab + \frac{1}{a} + \frac{1}{b}\right) - 1 > 0$$

$\Rightarrow$ either all three of $a - 1$, $b - 1$, $\frac{1}{ab} - 1$ are positive or exactly one of them is positive

But all three of $a - 1$, $b - 1$, $-1 + \frac{1}{ab}$ cannot be positive

Hence, the correct answer is (A).

40. $P(A \cap B') = P(A) - P(A \cap B) = 0.7 - 0.3 = 0.4$

$$P(B/A \cup B') = \frac{P(B \cap (A \cup B'))}{P(A \cup B')} = \frac{P(A \cap B)}{P(A) + P(B') - P(A \cap B')}$$

$$= \frac{0.3}{0.7 + 0.5 - 0.4} = \frac{3}{8}$$

Hence, the correct answer is (A).

41. E_i = Bag has exactly i black balls $3 \leq i \leq 10$ $P(E_i) = \frac{1}{8}$

E = Three balls drawn are black

$$P(E_9 / E) = \frac{P(E_9)\,P(E/E_9)}{\sum\limits_{i=3}^{9} P(E_i)\,P(E/E_i)}$$

$$= \frac{\frac{1}{8}\ \frac{{}^{9}C_3}{{}^{10}C_3}}{\frac{1}{8}\left[\frac{{}^{3}C_3}{{}^{10}C_3} + \frac{{}^{4}C_3}{{}^{10}C_3} + \ldots\ldots + \frac{{}^{9}C_3}{{}^{10}C_3} + \frac{{}^{10}C_3}{{}^{10}C_3}\right]}$$

$$= \frac{{}^{9}C_3}{{}^{3}C_3 + {}^{4}C_3 + \ldots\ldots + {}^{9}C_3 + {}^{10}C_3} = \frac{{}^{9}C_3}{{}^{11}C_4} = \frac{14}{55}$$

Hence, the correct answer is (A).

42. Total ways = 50!

Favourable ways = ${}^{50}C_5 \times 45!$

$\therefore$ Probability $= \frac{{}^{50}C_5 \,.\, 45!}{501} = \frac{1}{120}$

Hence, the correct answer is (D).

43. $\underset{x \to 0}{Lt}\left(\frac{a^x + b^x}{2}\right)^{2/x} = 6 \;\; a \in \{1 \text{ to } 6\}\; b \in \{1 \text{ to } 6\}$

as section 36 may = $n(s)$

$\Rightarrow \quad e^{\lim\limits_{x\to 0}\left(\frac{a^x+b^x}{2}-1\right)\frac{2}{x}} \quad \Rightarrow \quad e^{\lim\limits_{x\to 0}\left(\frac{a^x-1+b^x-1}{x}\right)} = 6$

$\Rightarrow \quad e^{\ln a + \ln b} = 6$

$\Rightarrow \quad ab = 6$

$(1, 6)(2, 3)(3, 2)(6, 1)$

Required ways = 4 = n(A)

$P(A) = \frac{4}{36} = \frac{1}{9}.$

Hence, the correct answer is (C).

44. 1, 6, 11.......................96

2, 7, 12 97

3, 8, 13,......................98

4, 9, 14.......................99

5, 10, 15 100

$x^4 - y^4 = (x - y)\,(x + y)\,(x^2 + y^2)$

$n(E) = 5\,({}^{20}C_2) + 6({}^{20}C_1)({}^{20}C_1)$

$= 3350$

Hence, the correct answer is (A).

45. P(no two adjacent persons are selected)

$= \frac{n(n-4)(n-5)/6}{n(n-1)(n-2)/6} = \frac{(n-4)(n-5)}{(n-1)(n-2)} = \frac{2}{7}$

$\Rightarrow 5n^2 - 57n + 136 = 0 \Rightarrow n = 8$

Hence, the correct answer is (C).

46. E_1: Selected coin is fair = $\frac{m}{n}$; $P\left(A/E_1\right) = \frac{1}{2} \times \frac{1}{2}$

E_2: Selected coin is biased = $\frac{n-m}{n}$; $P\left(A/E_2\right) = \frac{1}{3} \times \frac{1}{3}$

A = Getting head on first toss and tail in second toss

$$P\left(\frac{E_1}{A}\right) = \frac{\frac{m}{n}.\frac{1}{2}.\frac{1}{2}}{\frac{m}{n}.\frac{1}{2}.\frac{1}{2} + \frac{n-m}{n}\frac{2}{3} \times \frac{1}{3}} = \frac{\frac{m}{4n}}{\frac{m}{4n} + \frac{2n-2m}{9n}} = \frac{9m}{m+8n}$$

Hence, the correct answer is (C).

47. Unit place digits in 33^n are 3, 9, 7, 1

$n(s) = 4$, $p(E) = 1/4$

Hence, the correct answer is (C).

48. Here $n(S) = {}^{12}C_4$ and $n(E) = \overbrace{2(8)}^{\text{Choosing 3 from right or left end}} + \underbrace{8(7)}_{\text{Choosing 3 consecutive from 2nd to 11th place}} = 72$

$\therefore P(E) = \frac{72}{{}^{12}C_4} = \frac{8}{55}$

Hence, the correct answer is (B).

49. The number of triplets with 7 as the least = ${}^{13}C_2$

$n(s) = {}^{20}C_3$. Therefore $p = \frac{{}^{13}C_2}{{}^{20}C_3} = \frac{13}{190}$

Hence, the correct answer is (A).

50. $P(B^C) > P(A^C)$ Given

$\Rightarrow P(B) < P(A)$

$P(A \cap B^C) + P(A \cap B) = P(A) > P(B) = P(A^C \cap B) + P(A \cap B)$

$\Rightarrow P(A \cap B^C) > P(A^C \cap B).$

$\therefore$ (A) is false and so is (B)

$P(B \mid A) = \frac{P(A \cap B)}{P(A)} < \frac{P(A \cap B)}{P(B)} = P(A \mid B) \qquad \because$ (C) is true.

Hence, the correct answer is (C).

51. The total number of ways to put four identical oranges and six distinct apples into five distinct boxes is $\left({}^{5+4-1}C_4\right).\,5^6 = 70 \times 5^6$

To satisfy the criteria that each box contains two objects we make three cases (based on number of oranges to go into a box)

1. Two oranges in each of the two boxes and no oranges in the other three boxes. Number of ways = ${}^{5}C_2 \times \frac{\underline{|6}}{\underline{|2}\,\underline{|2}\,\underline{|2}} =$ 900
2. Two oranges in one box, one orange in each of the two other boxes

   $(5) \times \left({}^{4}C_2\right) \times \frac{\underline{|6}}{\underline{|2}\,\underline{|2}\,\underline{|1}\,\underline{|1}} = 5.6.180$

   $= 5400$
3. One orange in each of the four boxes

   $= 5 \,.\, \frac{\underline{|6}}{\underline{|2}\,\underline{|1}\,\underline{|1}\,\underline{|1}} = 5 \times 360 = 1800$

   The total number of ways = 900 + 5400 + 1800 + 8100

$$\text{Probability} = \frac{8100}{70\times 5^6} = \frac{162}{21875}$$

Hence, the correct answer is (B).

52. $\frac{x^2-60x+800}{x-30} < 0 \quad \Rightarrow \frac{(x-20)(x-40)}{x-30} < 0$

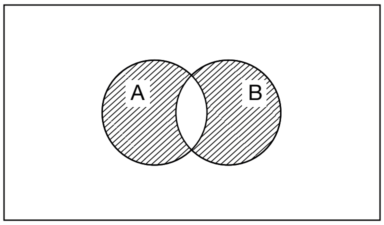
$\therefore x \in \{1, 2, \ldots\ldots, 19\} \cup \{31, 32 \ldots.., 39\}\ [x \in N]$

$\therefore$ number of favorable cases = 19 + 9 = 28. total no., of cases = 100

$\therefore$ required probability $= \frac{28}{100} = \frac{7}{25}$

Hence, the correct answer is (C).

53.



P (exactly one event out of A and B occurs)

$= P[(A \cap B') \cup (A' \cap B)]$

$= P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$

$\therefore P(A) + P(B) - 2P(A \cap B) = 1 - a$ .... (1)

similarly, $P(B) + P(C) - 2P(B \cap C) = 1 - 2a$ .... (2)

$P(C) + P(A) - 2P(C \cap A) = 1 - a$ .... (3)

$P(A \cap B \cap C) = a^2$ .... (4)

Now $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$= \frac{1}{2}[P(A) + P(B) - 2P(A \cap B) + P(B) + P(C) - 2P(B \cap C) + P(C) + P(A) - 2P(C \cap A)] + P(A \cap B \cap C)$

$= \frac{1}{2}[1 - a + 1 - 2a + 1 - a] + a^2$ [using (1), (2), (3) and (4)]

$= \frac{3}{2} - 2a + a^2 = \frac{1}{2} + (a-1)^2 > \frac{1}{2}$

Hence, the correct answer is (B).

54. Total number of ways = 36 × 36 = 1296

Let a_i = the number of ways in which a sum of i ($2 \le i \le 12$) is obtained on two dice. We have

$a_i = i - 1$ for $2 \le i \le 7$ and $a_i = 13 - i$ for $8 \le i \le 12$.

The number of ways in which their throws can be equal

$= a_2^2 + a_3^2 + \ldots\ldots\ldots + a_{12}^2 = 1^2 + 2^2 + \ldots\ldots. + 5^2 + 6^2 + 5^2 \ldots\ldots.. + 1^2$

$= 2(1^2 + 2^2 + \ldots.. + 5^2) + 6^2 = 2 \times \frac{5\times 6\times 11}{6} + 36 = 146$

$\therefore$ the number of ways in which the throw are unequal = 1296 – 146 = 1150

$\therefore$ the required probability $= \frac{1150}{1296} = \frac{575}{648}$.

Hence, the correct answer is (D).

55. 3 ball for 1st Box $= {}^{12}C_3 = {}^{12}C_9$

Choose any three balls for 1st box

Favourable $= {}^{12}C_9 \times 2^9$

Rest balls have 2 choices

Total $= 3^{12}$

Required Probability $= \frac{{}^{12}C_9 \times 2^9}{9^6}$

Hence, the correct answer is (B).

56. 1) $ad = 1, bc = 4 \Rightarrow (a = 1, d = 1, b = 2, c = 2)$

2) $ad = 1, bc = 2 \Rightarrow (a = 1, d = 1, b = 1, c = 2), (a = 1, d = 1, b = 2, c = 1)$

3) $ad = 2, bc = 1 \Rightarrow (a = 1, d = 2, b = 1, c = 1), (a = 2, d = 1, b = 1, c = 1)$

4) $ad = 2, bc = 4 \Rightarrow (a = 1, d = 2, b = 2, c = 2), (a = 2, d = 1, b = 2, c = 2)$

5) $ad = 4, bc = 1 \Rightarrow (a = 2, d = 2, b = 1, c = 1)$

6) $ad = 4, bc = 2 \Rightarrow (a = 2, d = 2, b = 1, c = 2), (a = 2, d = 2, b = 2, c = 1)$

$\Rightarrow$ required probability $= \frac{10}{16} = \frac{5}{8}$

Hence, the correct answer is (D).

57. $E_1 = (A \cap \bar{B}) \cup (\bar{A} \cap B)$, $E_2 = (B \cap \bar{C}) \cup (\bar{B} \cap C)$, $E_3 = (C \cap \bar{A}) \cup (\bar{C} \cap A)$,

$E_4 = A \cap B \cap C$, $E_5 = A \cup B \cup C$.

$\because \quad P(A \cup B \cup C) = \frac{1}{2}[P(A) + P(B) - 2P(A \cap B)] + P(B) + P(C) - 2P(B \cap C) + P(A) + P(C) - 2P(A \cap C)] + P(A \cap B \cap C)]$

$P(A \cup B \cup C) = \frac{1}{2}[P(E_1) + P(E_2) + P(E_3)] + P(E_4)$

$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right] + \frac{1}{9} = \frac{11}{8}$

Hence, the correct answer is (D).

58.

Box I	Box II	Box III
3 W	2 W	1 W
1 B	2 B	3 B

W W B + W B W + B W W

$\text{Req. Prob.} = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{2}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4}$

$= \frac{18+6+2}{4\times 16} = \frac{26}{4\times 16} = \frac{13}{32}$

Hence, the correct answer is (A).

59. Player A can win if A throws (1, 6) or (6, 1) and B throws ((1, 1,), (2, 2), (3, 3), (4, 4), (5, 5) or (6, 6)). Thus the number of ways is 12.

similarly the number of ways in which B can win is 12.

total number of ways in which either A wins or B wins = 24

Thus the number of ways in which none of the two wins = $6^4 - 24$.

$\therefore$ the required probability $= \frac{6^4 - 24}{6^4} = \frac{53}{54}$.

Hence, the correct answer is (C).

60. L and W can be filled at 14 places in 2^{14} ways.

$\therefore n(S) = 2^{14}$.

Now 13 L's and 1W can be arranged at 14 places in 14 ways.

Hence $n(A) = 14$

$\therefore \quad p = \dfrac{14}{2^{14}} = \dfrac{7}{2^{13}}$

Hence, the correct answer is (A).

61. P(a) = 0.3; P(b) = 0.5; P(c) = 0.2 $\Rightarrow a, b, c$ are exhaustive

P(same horse wins all the three races) = P(aaa or bbb or ccc)

$= (0.3)^3 + (0.5)^3 + (0.2)^3 = \dfrac{27+125+8}{1000} = \dfrac{160}{1000} = \dfrac{4}{25}$

P(each horse wins exactly one race)

= P(*abc or acb or bca or bac or cab or cba*)

$= 0.3 \times 0.5 \times 0.2 \times 6 = 0.18 = \dfrac{9}{50}$

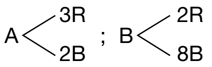
Hence, the correct answer is (A).

62. E: $B_1\, B_2\, B_3 \times \times \times$ where × means B or G

$P(E) = \dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{2} = \dfrac{1}{8} = \dfrac{8}{64}$

Hence, the correct answer is (C).

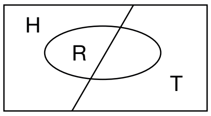
63.



R = event that a red marble is drawn

$P(R) = P(R \cap H) + P(R \cap T)$

$= P(H)\, P(R/H) + P(T) \cdot P(R/T)$



$= \dfrac{1}{2}\left[\dfrac{3}{5} + \dfrac{2}{10}\right] = \dfrac{8}{10}\cdot\dfrac{1}{2} = \dfrac{2}{5}$

Hence, the correct answer is (B).

64. $P\left(\overline{\overline{E_1}/\overline{E_2}}\right) = 1 - P\left(\overline{E_1}/\overline{E_2}\right) = 1 - P\dfrac{\left(\overline{E_1} \cap \overline{E_2}\right)}{P(\overline{E_2})}$

$= 1 - \left(\dfrac{1 - P(E_1 \cup E_2)}{1 - P(E_2)}\right) = 1 - \dfrac{\left(1 - \dfrac{3}{10} - \dfrac{1}{4} + \dfrac{1}{5}\right)}{1 - \dfrac{1}{4}}$

$= 1 - \dfrac{13}{15} = \dfrac{2}{15}$

Hence, the correct answer is (A).

65. $n(E) = {}^{52}C_2 + {}^{52}C_4 + {}^{52}C_6 + \ldots\ldots + {}^{52}C_{52}$

$= \dfrac{1}{2}\left(2^{52}\right) - 1 \Rightarrow 2^{51} - 1$

$m(E) = {}^{26}C_1 . {}^{26}C_1 + {}^{26}C_2 . {}^{26}C_2 + \ldots\ldots. + {}^{26}C_{26} . {}^{26}C_{26}$

$= {}^{52}C_{26} - 1$

$p(E) = \dfrac{m}{n} = \dfrac{{}^{52}C_{26} - 1}{2^{51} - 1}$

Hence, the correct answer is (A).

66. Let the orthocentre be (x, y)

Image of orthocentre with respect to any sides lies on circumcircle of triangle

$\therefore (x-2)^2 + (y-3)^2 = (5-2)^2 + (5-3)^2$

$\Rightarrow x^2 + y^2 - 4x - 6y = 0$

$\Rightarrow x^2 - 4x + (y-3)^2 - 9 = 0$

$\Rightarrow x = 2 \pm \sqrt{13 - (y-3)^2}$

y can take the values 1, 2, 3, 4, 5, 6

Probability $= \dfrac{3}{5}$

Hence, the correct answer is (C).

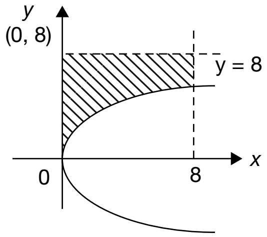
67. $n(S) = 8 \times 8 = 64 \because x, y \in [0, 8],\ n(E) = \displaystyle\int_0^8 \left(8 - \sqrt{2}\sqrt{x}\right) dx$

$= \left(8x - \dfrac{\sqrt{2}x^{3/2}}{3/2}\right)_0^8$

$= 64 - \dfrac{2\sqrt{2}}{3} 8 \times 2\sqrt{2}$

$= 64\left(1 - \dfrac{1}{3}\right) = 64 \times \dfrac{2}{3}$

= Required probability $= \dfrac{2}{3}$



Hence, the correct answer is (D).

68. First three digit number is 000

So 000, 121, … 990 are divisible by 11

$n(E) = (91)^3$

$n(S) = (10)^9$

The number of multiples of 11 from 000 to 999 is 91

Required probability $= \left(\dfrac{91}{1000}\right)^3$

Hence, the correct answer is (D).

69. In order that all balls are red after 3 replacements, two of the three balls selected must have been green. There could be three cases

I : Red, Green, Green

II : Green, Red, Green

III : Green, Green, Red (Since they are now all red)

The probability is

Case (I) is $\dfrac{1}{2}.\dfrac{1}{2}.\dfrac{1}{4} = \dfrac{1}{16}$

Case (II) is $\dfrac{1}{2}.\dfrac{3}{4}.\dfrac{1}{4} = \dfrac{3}{32}$

Case (III) is $\dfrac{1}{2}.\dfrac{1}{4}.1 = \dfrac{1}{8}$

The required probability $= \dfrac{1}{16} + \dfrac{3}{32} + \dfrac{1}{8} = \dfrac{9}{32}$

Hence, the correct answer is (D).

70. If the tallest girls are A and B, A can be assigned to group I and then (n – 1) girls can be chosen for this group out of a total of

($2n$ – 2) girls.

This can be done in ${}^{(2n-2)}C_{n-1}$ ways.

$$p=\frac{{}^{(2n-2)}C_{n-1}}{{}^{2n}C_n}=\frac{n}{2(2n-1)}$$

Hence, the correct answer is (D).

71. E_1: getting head

E_2: getting tail

A: score is 3.

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{1}{6}}{\frac{1}{2}\times\frac{{}^5C_2}{32}+\frac{1}{2}\times\frac{1}{6}}=\frac{8}{23}$$

Hence, the correct answer is (C).

72. Seven coupons are selected at random with replacement, the probability that the largest number appearing an a selected coupon atmost 9 is $\left(\frac{9}{15}\right)^7=\left(\frac{3}{5}\right)^7$ and at most 8 is $\left(\frac{8}{15}\right)^7$

Required probability is $\left(\frac{3}{5}\right)^7-\left(\frac{8}{15}\right)^7$

Hence, the correct answer is (D).

73. P(at least one correct pair) = 1 – P(No correct pair)

$$=1-\frac{{}^{10}C_6.2^6}{{}^{20}C_{10}}=\frac{211}{323}$$

Hence, the correct answer is (B).

74. If $a\neq 0$ the numbers with 0 & a are

a000, a00a, a0a0, aa00, aaa0, aa0a, a0aa

These are 9.7 = 63

Now favourable cases are

i) *aaab, aaba, abaa, baaa* are 4.9.8 = 288

ii) *aabb, abab, abba* are 3.9.8 = 216

iii) *aaaa* are 9

$$\text{So probability}=\frac{288+216+9}{63+288+216+9}=\frac{513}{576}=\frac{57}{64}$$

Hence, the correct answer is (D).

75. E_1 : Bag 1 selected, $P(E_1)=\frac{1}{3}$

E_2 : Bag 2 selected, $P(E_2)=\frac{1}{3}$

E_3 : Bag 3 selected, $P(E_3)=\frac{1}{3}$

A : Ball is red, $P\left(A/E_1\right)=\frac{2}{a^2-4a+10}$,

$$P\left(A/E_2\right)=\frac{1}{a^2-4a+9}$$

$$P\left(A/E_3\right)=\frac{3}{a^2-4a+7}$$

$$\text{Required probability}=\frac{1}{3}\left(\frac{6}{a^2-4a+10}\right)$$

$$(P(A))_{\max}=\frac{1}{3}\times\frac{6}{6}=\frac{1}{3}$$

Hence, the correct answer is (A).

76. Suppose there exist three rational points or more on the circle $x^2+y^2+2gx+2fy+c=0$

If (x_1, y_1) (x_2, y_2) (x_3, y_3) be those three points $S_{11}=0$, $S_{22}=0$, $S_{33}=0$

On solving we get g, f, c as rational

Possible values of p are 1, 2

q are 1, 2

(p, q) be chosen as 4 ways

(p, q) can be chosen without restriction in 6 × 6 = 36

$$\text{Probability}=1-\frac{4}{36}=1-\frac{1}{9}=\frac{8}{9}$$

Hence, the correct answer is (D).

77. Let P(A) = x and P(B) = y, then $x>1/2$, $y>1/2$,

$$P(A-B)=x-xy=3/25 \qquad \ldots(1)$$

$$\text{and } P(B-A)=y-xy=8/25 \qquad \ldots(2)$$

Solving (1) and (2) we get $x=\frac{3}{5}$ and $y=\frac{4}{5}$

$$\Rightarrow \quad xy=\frac{12}{25}$$

Hence, the correct answer is (C).

78. Let E be the event that B throws a number more than 8. Then $P(E)=1-P(\bar{E})$

$|\bar{E}|$ = Number of positive integral solutions of $x+y+z\leq 8$

$\therefore$ $|\bar{E}|={}^8C_3=56$ and $|S|=216$

$$\therefore \quad P|\bar{E}|=\frac{56}{216}=\frac{7}{27}$$

Hence, the correct answer is (D).

79. P(GBB) + P(BGB) + P(BBG) =

$$\frac{3}{4}\times\frac{2}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{2}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{2}{4}\times\frac{1}{4}=\frac{13}{32}$$

Hence, the correct answer is (B).

80. Let S be the sample space of the composite experiment of drawing 6 in the first draw and then four in second draw then $|S|={}^{10}C_6\times{}^{10}C_4$

$$\therefore \text{Required Probability}=\frac{{}^{10}C_6\times{}^6C_3\times{}^4C_1}{{}^{10}C_6\times{}^6C_4}$$

$$=\frac{80\times 24}{10\times 9\times 8\times 7}=\frac{8}{21}$$

Hence, the correct answer is (A).

81. $\dfrac{^4C_2}{^7C_2}\times\dfrac{^3C_2}{^5C_2}=\dfrac{4\times3}{7\times6}\times\dfrac{3}{10}=\dfrac{3}{35}$

Hence, the correct answer is (B).

82. Let p be the probability that the team wins a game. Let $q = 1 - p$. Then the random variable "number of wins" follows the binomial distribution $P(X = K) = {}^4C_k\, q^{4-k}\, p^k$, $k = 0, 1, 2, 3, 4$.

Required probability $= {}^4C_3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3+{}^4C_4\left(\frac{2}{3}\right)^4=\frac{16}{27}$

Hence, the correct answer is (C).

83. E_1 : Bag having 2 red balls

E_2 : Bag having 3 red balls

E_3 : Bag having 4 red balls

E_4 : Bag having 5 red balls

A : Two red balls are chosen, $P\left(A/E_1\right)=\frac{2}{10}$,

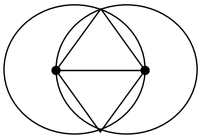
$$P\left(A/E_2\right)=\frac{3}{10},\ P\left(A/E_3\right)=\frac{6}{10},\ P\left(A/E_4\right)=\frac{10}{10},$$

From Bayes' theorem, the required probability

$$=\frac{1}{\frac{1}{10}+\frac{3}{10}+\frac{6}{10}+\frac{10}{10}}=\frac{1}{2}$$

Hence, the correct answer is (C).

84.



$n(A) = 2$ (area of sector - area of trianlge) $= \dfrac{8\pi-6\sqrt{3}}{3}$

$$n(S) = 8\pi-\left(\frac{8\pi-6\sqrt{3}}{3}\right)$$

$$p(A) = \frac{4\pi-3\sqrt{3}}{8\pi+3\sqrt{3}}$$

Hence, the correct answer is (D).

85. Product of four numbers with last digit 1, 3, 7 and 9 is possible when we choose each number from 1, 3, 7 and 9

Probability $=\frac{4}{10}\cdot\frac{4}{10}\cdot\frac{4}{10}\cdot\frac{4}{10}=\frac{16}{625}$ (every digit these four numbers out of 10 numbers is $\frac{4}{10}$).

Hence, the correct answer is (B).

86. Let p = probability of even faces 2, 4, 6 is $=\frac{1}{2}$

q = probability of odd faces 1, 3, 5 is $=\frac{1}{2}$

Required probability =

$$2n+1C_1(p)^1(q)^{2n}+2n+1C_2(p)^3(q)^{2n-2}+$$

$$\ldots\ldots\ldots+2n+1C_{2n+1}(p)^{2n+1}(q)^0$$

$$\left(\frac{1}{2}\right)^{2n+1}\left(2n+1C_1+2n+1C_3+\ldots\ldots\ldots+2n+1C_{2n+1}\right)=\frac{1}{2}$$

Hence, the correct answer is (D).

87. No. of trials $n = 21$

Success is solving the problem

$$\therefore p=P(A\cup B\cup C)=1-P(\overline{A})P(\overline{B})P(\overline{C})$$

$$=1-\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}=\frac{1}{2}$$

$$q=\frac{1}{2}$$

now,

$$P(X\geq 11)={}^{21}C_{11}\left(\frac{1}{2}\right)^{21}+{}^{21}C_{12}\left(\frac{1}{2}\right)^{21}+\ldots+{}^{21}C_{21}\left(\frac{1}{2}\right)^{21}$$

$$=\frac{2^{20}}{2^{21}}=\frac{1}{2}$$

Hence, the correct answer is (B).

88. $\underset{\text{II times}}{\underline{HHH....H}}\ \underset{\text{9 times}}{\underline{\qquad\qquad}}\rightarrow\left(\frac{1}{2}\right)^{11}$

$$\underset{\text{II times}}{\underline{THH....H}}\ \underset{\text{8 times}}{\underline{\qquad\qquad}}\rightarrow\left(\frac{1}{2}\right)^{12}$$

Blank spaces mean anything heads or tails can come

$$-\underset{\text{II times}}{\underline{TH....H}}\ \underset{\text{7 times}}{\underline{\qquad\qquad}}\rightarrow\left(\frac{1}{2}\right)^{12}$$

$$\underset{\text{7 times}}{\underline{\qquad\qquad}}\ \underset{\text{II times}}{\underline{THH....H}}\rightarrow\left(\frac{1}{12}\right)^{2}$$

Required. Prob $=\left(\frac{1}{2}\right)^{11}+9\times\left(\frac{1}{2}\right)^{12}=\frac{11}{2^{12}}$

Hence, the correct answer is (A).

89. Let the number of marble be $2n$ (where n is large)

Required probability $=\lim\limits_{n\to\infty}\dfrac{n\times{}^nC_4}{^{2n}C_5}\times\dfrac{^nC_3\times{}^nC_2}{^{2n}C_5}$

$$=\lim_{n\to\infty}\frac{n\times n(n-1)(n-2)(n-3)}{4!}\times\frac{n(n-1)(n-2)}{3!}$$

$$\times\frac{n(n-1)}{2!}\times\frac{(5!)^2((2n-5)!)^2}{(2n!)^2}$$

$$=\lim_{n\to\infty}\frac{n^4(n-1)^3(n-2)^2(n-3)((2n-5)!)^2\times5\times5\times4\times3!}{3!2!(2n!)^2}$$

$$=\lim_{n\to\infty}\frac{50.n^4(n-1)^3(n-2)^2(n-3)}{\left(2n(2n-1)(2n-2)(2n-3)(2n-4)\right)^2}=\frac{50}{1024}=\frac{25}{512}.$$

Hence, the correct answer is (A).

90. $n(S) = {}^{50}C_5$ = Total number of ways

$n(E) = {}^{30}C_2\times{}^{19}C_2$ = Number of favourable ways

$$P(E)=\frac{^{30}C_2\times{}^{19}C_2}{^{50}C_5}.$$

Hence, the correct answer is (B).

91. $P_n={}^nC_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1}+{}^nC_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}+\ldots$

$$P_n = \frac{1}{2}\left[\left(\frac{5}{6}+\frac{1}{6}\right)^n + \left(\frac{5}{6}-\frac{1}{6}\right)^n\right]$$

$$P_n = \frac{1}{2}\left[1+\left(\frac{2}{3}\right)^n\right]$$

$$P_{n-1} = \frac{1}{2}\left[1+\left(\frac{2}{3}\right)^{n-1}\right]$$

$$2P_{n-1} - 1 = \left(\frac{2}{3}\right)^n \cdot \frac{3}{2}$$

$$\left(\frac{2}{3}\right)^n = \left[\frac{4P_{n-1}-2}{3}\right]$$

$$\Rightarrow P_n = \frac{1}{2}\left[1+\frac{4P_{n-1}-2}{3}\right]$$

$$6P_n = 4P_{n-1} + 1 \Rightarrow 6P_n - 4P_{n-1} = 1$$

Hence, the correct answer is (A).

92. Let A be the event that the bulb will last for 1000 hours or more $P(A) = 0.7$ and B be the event that the bulb will last for at most 1500 hours $P(B) = 0.8$ Probability that it lasts for 1000 to 1500 hours

$$= P(A \cap B)$$

$$= P(A) + P(B) - P(A \cup B)$$

$$= 0.7 + 0.8 - 1 = 0.5$$

Hence, the correct answer is (B).

93. A and B gets 0 or (1) or (2) or (3) heads.

$$\text{Prob.} = \frac{1}{8}\cdot\frac{1}{8} + \frac{3}{8}\cdot\frac{3}{8} + \frac{3}{8}\cdot\frac{3}{8} + \frac{1}{8}\cdot\frac{1}{8} = \frac{20}{64}$$

Hence, the correct answer is (C).

94. ‘9’ diagonals intersect at 15 interior points and 6 vertices

$^6C_2$ = diagonals + sides

∴ diagonals = $^6C_2 - 6 = 9$

Total ways = $^9C_2$

Selection of any 4 vertices gives 1 diagonal which intersect inside

$$\text{Probability} = \frac{^6C_4}{^9C_2} = \frac{15}{36} = \frac{5}{12}$$

Hence, the correct answer is (A).

95. 1st 2nd 3rd 4th

Case 1: WWRR

$$P_1 = \frac{8}{10}\times\frac{8}{10}\times\frac{2}{10}\times\frac{1}{10}$$

Case 2: WRWR

$$P_2 = \frac{8}{10}\times\frac{2}{10}\times\frac{9}{10}\times\frac{1}{10}$$

Case 3: RWWR

$$P_3 = \frac{2}{10}\times\frac{9}{10}\times\frac{9}{10}\times\frac{1}{10}$$

Probability = $P_1 + P_2 + P_3 = 0.0434$

Hence, the correct answer is (B).

96. P(non occurrence of (A_i)) $= 1 - \dfrac{1}{i+1} = i/(i+1)$.

P(non occurrence of any of events)

$$= \left(\frac{1}{2}\right)\cdot\left(\frac{2}{3}\right)\ldots\left\{\frac{n}{(n+1)}\right\} = \frac{1}{(n+1)}$$

Hence, the correct answer is (C).

97. Let E_1 be the event that one of first n urns is chosen, E_2 be the event that $(n+1)^{\text{th}}$ urn is chosen.

A be the event that two balls drawn are black.

$$P(E_1) = \frac{n}{n+1};\ P(E_2) = \frac{1}{n+1}$$

$$P\left(\frac{A}{E_1}\right) = \frac{^6C_2}{^{10}C_2} = \frac{1}{3};\ P\left(\frac{A}{E_2}\right) = \frac{^5C_2}{^{10}C_2} = \frac{2}{9}$$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2).P\left(\frac{A}{E_2}\right)} = \frac{1}{16}$$

$$\Rightarrow \frac{\frac{1}{n+1}\cdot\frac{2}{9}}{\frac{n}{n+1}\cdot\frac{1}{3} + \frac{1}{n+1}\cdot\frac{2}{9}} = \frac{1}{16} \Rightarrow n = 10$$

Hence, the correct answer is (A).

98. $^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \le 100$

$$(n-4)n(n+1) \le 600$$

If $n = 9$ then $(n-4)n(n+1) = 450$

If $n = 10$ then $(n-4)n(n+1) = 660 \Rightarrow n \le 9$

Hence, $2 \le n \le 9 \Rightarrow n \in \{2, 3, 4, 5, 6, 7, 8, 9\}$

$$\text{Required probability} = \frac{8}{100} = \frac{2}{25}$$

Hence, the correct answer is (C).

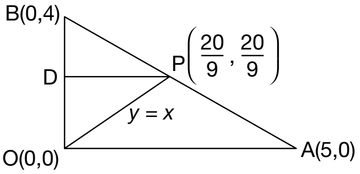
99. No. of dearrangement of four objects

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 12 - 4 + 1$$

$$P = \frac{9}{4!} = \frac{3}{8}$$

Hence, the correct answer is (A).

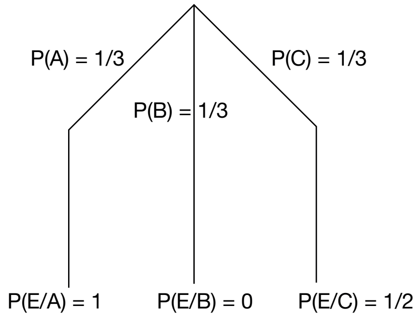
100.



$$P(E) = \frac{\text{Area of } \Delta OPB}{\text{Area of } \Delta OAB} = \frac{4}{9}$$

Hence, the correct answer is (C).

101.



$$P(E)=\frac{1}{3}\times1+\frac{1}{3}\times0+\frac{1}{3}\times\frac{1}{2}=\frac{1}{2}$$

Required probability =

$$P(A/E)=\frac{P(E/A)P(A)}{P(E)}=\frac{1/3}{1/2}=\frac{2}{3}$$

Hence, the correct answer is (C).

102. $n(s)$ = ${}^{8}C_3 = 8\times7$; n(A) = 8 (Each vertex gives us only one equilateral triangle (or) consider one diagonal of a cube, each end gives us 4 equilateral triangles.

Hence, the correct answer is (C).

103. $20P^2-13P+2\le0$

$\Rightarrow(4P-1)(5P-2)\le0$

$\Rightarrow\frac{1}{4}\le P\le\frac{2}{5}$

$P=1-(1-0.2)^n=1-(0.8)^n$

$0.25\le P\le0.40$

$\Rightarrow\quad 0.6\le(0.8)^n\le0.75$

$n=2$

Hence, the correct answer is (D).

104. $50p + 50(3P) = 1 \Rightarrow P = 0.005$

Now there are seven perfect square not exceeding not exceeding 50 and three greater than 50 but not exceeding 100. So the probability of choosing a perfect square is 7P + 3(3P) = 0.08.

Hence, the correct answer is (C).

105. When 4 points are selected we get one inside intersecting point. So probability is $\dfrac{{}^{n}C_4}{{}^{\left({}^{n}C_2-n\right)}C_2}$

Hence, the correct answer is (C).

106. With 10 dices, the number on each face being odd, we can never obtain an odd number, since the sum of 10 odd number can be odd. Hence required probability = 0.

Hence, the correct answer is (D).

107. There are 15 ways { (1, 1, 2), (2, 1, 3), (3, 1, 4), (4,1,5), (5, 1, 6), (1, 2, 3), (2, 2, 4), (3, 2, 5), (4, 2, 6), (1, 3, 4), (2, 3, 5), (3, 3, 6), (1, 4, 5), (2, 4, 6), (1, 5, 6)} in which the third outcome is the sum of the first two.

Atleast on '2' appears in exactly 8.

Hence, the correct answer is (D).

108. $x + y = 2a$, let $z = xy = x(2a - x) = 2ax - x^2 = a^2 - (x - a)^2$

z will be max if $x = a$. In this case $y = a$

Thus the $z_{\max} = a^2$

Now $P\left(xy\nless\frac{3}{4}a^2\right)=P\left(xy\ge\frac{3}{4}a^2\right)$

$$P\left(x(2a-x)\ge\frac{3}{4}a^2\right)=P\left(4x^2-8ax+3a^2\le0\right)$$

$$=P\left((2x-3a)(2x-a)\le0\right)$$

$$=P\left(\frac{a}{2}\le x\le\frac{3a}{2}\right)$$

Length of fav. Interval = a

Required probability = $\frac{a}{2a}=\frac{1}{2}$.

Hence, the correct answer is (A).

109. Put $\tan 2x = t, -\infty < t < \infty$

$$\left[t^2\right]=t+a,\ a\in N$$

$\therefore$ t should be integer

$t^2 - t - a = 0$

$$t=\frac{1\pm\sqrt{1+4a}}{2}$$

Here 1 + 4a is odd number and which is perfect square 1 + 4a = 9, 25, ..., $(63)^2$

$\therefore$ Number of a takes the value is 31

$$P=\frac{31}{1000}$$

Hence, the correct answer is (C).

110. If one box has exactly a white ball and the other box has exactly b white balls, then $\frac{a}{20}\frac{b}{20}=0.21\Rightarrow ab=84$ as a, b are integers.

$\Rightarrow$ {a, b} is either {6, 14} or {7, 12} since total numbers of black ball is different from white balls $a+b\ne20$.

Thus probability that both balls are black is

$$\frac{(20-a)(20-b)}{400}=\frac{(20-7)(20-12)}{400}$$

$$=\frac{13\times8}{400}=0.26$$

Hence, the correct answer is (D).

111. Probability that matrix is symmetric = $\frac{7^6}{7^9}=\frac{1}{7^3}$.

Again that matrix is skew symmetric = $\frac{7^3}{7^9}=\frac{1}{7^6}$

One matrix containing all elements = 0; is common in both type of matrices

Required probability = $\frac{1}{7^3}+\frac{1}{7^6}-\frac{1}{7^9}$.

Hence, the correct answer is (D).

112. Let n(A) = total numbers divisible by 2, n(B) = total numbers divisible by 3, n(C) = total numbers divisible by 6 and n(D) = total numbers divisible by 8.

Hence $n(A\cup B\cup C\cup D)=n(A\cup B)$

$$=n(A)+n(B)-n(C)=\left[\frac{199}{2}\right]+\left[\frac{199}{3}\right]-\left[\frac{199}{6}\right]$$

$$=99+66-33=132.$$

Required probability $=\dfrac{132}{198}=\dfrac{2}{3}$

Hence, the correct answer is (D).

113. Let the probability of choosing integer k

$p(k)=\dfrac{\alpha}{k^4}$, α is constant of proportionality

$$\sum_{k=1}^{2m}p(k)=1\Rightarrow\alpha=\left(\sum_{k=1}^{2m}\frac{1}{k^4}\right)^{-1}$$

$a=p$ (choosing an odd number)

$$a=\sum_{k=1}^{2m}p(2k-1)=\alpha\cdot\sum_{k=1}^{m}\frac{1}{(2k-1)^4}$$

$$1-a=\alpha\cdot\sum_{k=1}^{m}\left(\frac{1}{2k}\right)^4<\alpha\cdot\sum_{k=1}^{m}\frac{1}{(2k-1)^4}=a$$

$$\therefore a>\frac{1}{2}.$$

Hence, the correct answer is (B).

114. Favourable cases for bijection is ${}^{100}C_9\times 9!$

For total cases

'B' is set of one element, no. of function = 1

'B' is set of two element, no. of function $=2^9$

'B' is set of three element, no. of function $=3^9$

......

......

'B' = Y No. of function -101^9

Total cases $=1+{}^{100}C_1.2^9+{}^{100}C_2.\,3^9+\ldots+{}^{100}C_{100}.\,101^9$

$$=\sum_{r=1}^{101}r^9.\,{}^{100}C_{r-1}$$

Required probability $=\dfrac{{}^{100}C_9.9!}{\sum_{r=1}^{101}r^9.\,{}^{100}C_{r-1}}$

Hence, the correct answer is (D).

115. The number of favourable cases is

${}^nC_0\times 2^0+{}^nC_1.\,2^1+\ldots+{}^nC_n2^n=3^n$.

Total number of cases is $2^n\times 2^n=4^n$. Thus required probability is $\left(\dfrac{3}{4}\right)^n$

Hence, the correct answer is (D).

116. No. of ways of selecting 6 points on circle is 5!. No. of different arrangement such that ABC is disjoint from DEF is 3! × 3!

$\therefore$ Probability $\dfrac{3!\times 3!}{5!}=\dfrac{3}{10}$

Hence, the correct answer is (B).

117. No. of determinants formed $=n(s)=2^4=16$

The determinants $\begin{vmatrix}1&0\\0&1\end{vmatrix}\begin{vmatrix}1&0\\1&1\end{vmatrix}\begin{vmatrix}1&1\\0&1\end{vmatrix}$ are positive

$\Rightarrow\ n(E)=3$

$\therefore\ P(E)=\dfrac{3}{16}$

Hence, the correct answer is (B).

118.
$$\begin{array}{lllll} G_1=1 & 4 & 7 & ---- & 3n-2\\ G_2=2 & 5 & 8 & ---- & 3n-1\\ G_3=3 & 6 & 9 & ---- & 3n \end{array}$$

We have $n(s)={}^{3n}C_2$ for $n(E)$ both numbers can be selected from same group or one number from G_2 and one from G_3

$\therefore\ n(E)=3\cdot{}^nC_2+{}^nC_1\times{}^nC_1$

$\therefore\ P(E)=\dfrac{3.{}^nC_2+n^2}{{}^{3n}C_2}=\dfrac{5n-3}{3(3n-1)}$

Hence, the correct answer is (A).

119. $n(S)={}^{52}C_3$, $n(E)={}^4C_3\times{}^{13}C_1\times{}^{13}C_1\times{}^{13}C_1$

$$P=\frac{n(E)}{n(S)}=\frac{4\times 13\times 13\times 13\times 6}{52\times 51\times 50}=\frac{169}{425}$$

Hence, the correct answer is (A).

120.
$$\left.\begin{aligned} n(S)&=2^7-2\\ n(E)&=\underset{\text{For women}}{2}\underbrace{(2^3-1)}_{\text{Men can choose any of two}} \end{aligned}\right\}$$

$$P(E)=\frac{2^4-2}{2^7-2}=\frac{14}{126}=\frac{7}{63}=\frac{1}{9}$$

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. We know that $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\therefore$ Here, $P(A\cap B)=A(A)P(B)$

$\Rightarrow$A, B are independent events.

$\Rightarrow A,\bar{B};\ \bar{A},B$ and $\bar{A},\bar{B}$ are also pairwise independent events.

$$P(\bar{A}\cup\bar{B})=P(\overline{A\cap B})=1-P(A\cap B)$$
$$=1-P(A)P(B)$$
$$=P(\bar{A})+P(A)P(\bar{B})$$

$\therefore\ P\left(\overline{A\cup B}\right)=P\left(\bar{A}\cap\bar{B}\right)=P\left(\bar{A}\right)P\left(\bar{B}\right)$

and $P(A/B)=\dfrac{P(A\cap B)}{P(B)}=P(A)$

Hence, the correct answer is (C, D).

2. Total no. of words $=\dfrac{6!}{3!}=120$

Number of words containing CAN $=\dfrac{4!}{2!}=12$

Number of words containing ADA $=4!=24$

Number of words containing both $=2!=2$

CAN and ADA

Probability that neither CAN or ADA occurs

$$= \frac{120-(12+24-2)}{120} = \frac{43}{60}$$

If vowels and consonants occur alternately then the pattern will be of type CANADA or ACANAD whose number of ways = 2 × 3! = 12

$$\text{Probability} = \frac{12}{120} = \frac{1}{10}$$

Number of words in which all vowels and all consonants occur together = 2! × 3! = 12

$$\text{Probability} = \frac{12}{120} = \frac{1}{10}$$

Hence, the correct answer is (B, C, D).

3. Let us make a sample space of 52 dimensions (i.e.,) each point of sample space is an ordered point of the form $\{x_1, x_2, \ldots, x_{52}\}$. Total number of sample points are 52! Number of sample point on which x_{13} is spade is $^{13}C_1 \times 51!$

$$\therefore \text{ Probability of getting spade on } x_{13} = \frac{^{13}C_1 \times 51!}{52!} = \frac{1}{4}.$$

Similarly number of sample points on $^x52$ is spade $= {}^{13}C_1 \times 51!$

$$\therefore \text{ Probability of getting spade on last trail} = \frac{1}{4}$$

Number of sample point having king at x_5 and queen x_{10} will be $^4C_1 \times {}^4C_1 \times 50!$

$$\text{Required Probability} = \frac{16 \times 50!}{52!} = \frac{4}{663}$$

Hence, the correct answer is (B, C, D).

4. (A) Let A be the event of getting exactly 3 defectives in the examination of 8 record players and B be the event 9th record player is defective

$$P(A) = \frac{^4C_3 \times {}^{11}C_5}{^{15}C_8}$$

(B) Probability of 9th one examined is the last defective

$$= \frac{^4C_3 \times {}^{11}C_5}{^{15}C_8} \times \frac{1}{7} = \frac{8}{195}$$

(C) $P\left(\frac{B}{A}\right) = \frac{1}{7}$

Hence, the correct answer is (A, B, C).

5. $n(S) = {}^9C_1 + {}^9C_2 + \ldots + {}^9C_9 = 2^9 - 1 = 511$

(A) Choose any 7-numbers

$$n(E) = \underset{\text{Order is fixed}}{\underset{\downarrow}{{}^9C_7 \times 1}} = 36$$

(B) One digit is 3 so other digit choosing from rest 8 digits

$$n(E) = {}^8C_0 + {}^8C_1 + \ldots + {}^8C_8 = 2^8 = 256$$

(C) $n(E) = {}^6C_1 + {}^6C_2 + \ldots + {}^6C_6 = 2^6 - 1 = 63$

(D) $n(E) = {}^5C_0 + {}^5C_1 + \ldots + {}^5C_5 = 2^5 = 32$

Hence, the correct answer is (A, B, C, D).

6. $P(1) = \frac{2}{4} = \frac{1}{2}$, $P(2) = \frac{2}{4} = \frac{1}{2}$, $P(3) = \frac{2}{4} = \frac{1}{2}$

$P(1 \cap 2) = \frac{1}{4} = P(1)\,P(2)$, similarly for $P(2 \cap 3)$ & $P(1 \cap 3)$

Hence, the correct answer is (A, B).

7. Let the no. of blue marbles is a and no. of green marbles is b.

$$\therefore \frac{ab}{^{a+b}C_2} = \frac{1}{2} \Rightarrow (a+b)(a+b-1) = 4ab$$

$$\Rightarrow b^2 - (2a+1)b + a^2 - a = 0$$

but $b \in R \Rightarrow D = (2a+1)^2 - 4(a^2 - a) = 8a + 1$

$\therefore 8a + 1$ must be a perfect squre.

hence possible values of a are 3, 6, 10

Hence, the correct answer is (A, B, C).

8. $P(A).P(B) = \frac{1}{12}$ and

$$(1 - P(A))(1 - P(B)) = \frac{1}{2}$$

Solving for P(A), we get $P(A) = \frac{1}{3}$ or $\frac{1}{4}$

accordingly, $P(B) = \frac{1}{4}$ or $\frac{1}{3}$

Hence, the correct answer is (A, C).

9. $A \cap \overline{B} \le \overline{B} \Rightarrow P(A \cap \overline{B}) \le P(\overline{B})$

$A \cup B \ge B \Rightarrow P(A \cup B) \ge P(B)$

$A \cap B \le A \Rightarrow P(A \cap B) \le P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$

$\Rightarrow \quad P(A \cap B) \ge P(A) + P(B) - 1$

$\quad P(A \cap B) \ge \frac{4}{15}$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$ are used

Hence, the correct answer is (A, B, C, D).

10. Total ways 120

A) $x^2 - 25x - 150 \le 0 \Rightarrow (x - 30)(x + 5) \le 0 \Rightarrow -5 \le x \le 30$

$\quad \text{Probability} = \frac{30}{100} = 0.3$

B) $x^2 - 17x + 30 \ge 0 \Rightarrow (x - 2)(x - 15) \ge 0 \Rightarrow x \le 2$ or $x \ge 15$

$\quad \text{Probability} = \frac{88}{100} = 0.88$

C) $x(30 - x)$ is a perfect square

$\quad \Rightarrow x = 3, 6, 15, 24, 27$

$\quad$ Probability = 0.05

D) $x(30 - x) < 0 \Rightarrow x > 30$

$\quad \text{Probability} = \frac{70}{100} = 0.7$

Hence, the correct answer is (A, B, D).

11. $A \cap B = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

$B \cap C = \{(2, 1), (2, 3), (2, 5), (4, 1), (4, 3), (4, 5), (6, 1), (6, 3), (6, 5)\}$

$A \cap C = \{(1, 2), (3, 2), (5, 2), (1, 4), (3, 4), (5, 4), (1, 6), (3, 6), (5, 6)\}$

$$P(A \cap B) = \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{4}$$

$$P(B \cap C) = P(a \text{ is even and } b \text{ is odd}) = \frac{1}{4}$$

$$P(A \cap C) = P(a \text{ is odd and } b \text{ is even}) = \frac{1}{4}$$

$$P(A \cap B \cap C) = P(\phi) = 0$$

Hence, the correct answer is (A, B, C, D).

12. $1 - (1 - m)(1 - p)(1 - c) = \frac{3}{4}$ (or)

$$m + p + c - mp - mc - pc + mpc = \frac{3}{4}$$

$$mp(1 - c) + pc(1 - m) + mc(1 - p) + mpc = \frac{1}{2}$$

$$\Rightarrow mp + mc + pc - 2mpc = \frac{1}{2}$$

$$mp(1 - c) + pc(1 - m) + mc(1 - p) = \frac{4}{10}$$

$$\Rightarrow mp + mc + pc - 3mpc = \frac{2}{5}$$

$$\therefore\ mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10},\ mp + mc + pc = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$$

$$\therefore\ m + p + c = \frac{7}{10} - \frac{1}{10} + \frac{3}{4} = \frac{14 - 2 + 15}{20} = \frac{27}{20}$$

Hence, the correct answer is (B, C).

13. The probability that both will be alive for 10 years, hence i.e., the prob. That the man and his wife Both be alive 10 years hence 0.83 × 0.87 = 0.7221.

The prob, that at least one of then will be alive is 1-P (That none of them remains alive 10 years)

$$= 1 - (1 - 0.83)\ (1 – 0.87)$$

$$= 1 - 0.17 \times 0.131$$

$$= 0.9779$$

Hence, the correct answer is (A,B).

14. (A) Fixing 2, 4, 6 in the first 3 places; we can fill the remaining 6 places in 6^6 ways

Hence probability $\frac{6^6}{6^9} = \frac{1}{6^3}$

(B) Same as (A) fix number 135 in the end and fill the remaining 6 places in 6^6 ways

$$P = \frac{6^6}{6^9} = \frac{1}{6^3}$$

(C) Fix 246 and 135 at respective places. Now fill the remaining 3 places in 6^3 ways

$$P = \frac{6^3}{6^9} = \frac{1}{6^6}$$

(D) Begin with 246 or end with 135

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{6^3} + \frac{1}{6^3} - \frac{1}{6^6} = \frac{431}{6^6}$$

Hence, the correct answer is (A, B, C, D).

15. Let the probability that P_1 will pair up with P_2, P_3 and P_4 be respectively $2k$, $3k$ and $4k$ and $2k + 3k + 4k = 1$ or $k = \frac{1}{9}$.

Now, P_2 will reach the next round if it is not paired with P_1.

$\therefore$ Required probability $= 1 - \frac{2}{9} = \frac{7}{9}$

Hence, the correct answer is (B).

16. $P(\bar{A} \cap B) = (1 - P(A))\ P(B) = \frac{2}{15}$ …(1)

$P(A \cap \bar{B}) = P(A)\ (1 - P(B)) = \frac{1}{6}$ …(2)

From (1) and (2) P(B) = 4/5, P(B) = 1/6

Hence, the correct answer is (B, C).

17. P(0H or 2H or 4H)

$$Q = p^4 + 6p^2(1 - p)^2 + (1 - p)^4$$

$$= 8p^4 - 16p^3 + 12p^2 - 4p + 1$$

$$= \frac{(2p-1)^4 + 1}{2}$$

(A) $\frac{(2p-1)^4 + 1}{2} = \frac{1}{4} \Rightarrow (2p-1)^4 = -\frac{1}{2}$ (Not possible)

(B) $(2p-1)^4 = \frac{1}{2} \Rightarrow$ Two values of p

(C) $(2p-1)^4 = \frac{1}{5} \Rightarrow$ Two values of p

(D) $(2p-1)^4 = \frac{4}{5} \Rightarrow$ Two values of p

Hence, the correct answer is (A, C).

18. $P(E \cap F) = P(E)P(F) = \frac{1}{12}$

$$P(\bar{E} \cap \bar{F})1 = P(\bar{E})P(\bar{F}) = \frac{1}{2}$$

$$(1 - P(E))(1 - P(F)) = \frac{1}{2} \Rightarrow P(E) + P(F) = \frac{7}{12}$$

$$\therefore\ P(E) = \frac{1}{4}, P(F) = \frac{1}{3} \text{ or } P(E) = \frac{1}{3}, P(F) = \frac{1}{4}$$

Hence, the correct answer is (A, D).

19. $P(E_1) = {}^9C_1 \times \frac{13}{52} \times \frac{39}{51} \times \frac{38}{50} \times \frac{37}{49} \times \frac{36}{48} \times \frac{35}{47} \times \frac{34}{46} \times \frac{33}{45}$

$$\times \frac{32}{44} \times \frac{12}{43} = 3 \times \frac{{}^{39}C_8}{{}^{51}C_9}$$

$$P(E_2) = {}^7C_2 \times \frac{4}{52} \times \frac{3}{51} \times \frac{48}{50} \times \frac{47}{49} \times \frac{46}{48} \times \frac{45}{47} \times \frac{44}{46} \times \frac{2}{45}$$

$$= \frac{132}{38675}$$

$$P(E_3) = \frac{48 \times 47 \times \dots\dots \times 41}{52 \times 51 \times \dots\dots \times 45} \times \frac{4}{44} = \frac{4}{9}\left(\frac{{}^{48}C_8}{{}^{52}C_9}\right)$$

$$P(E_4) = \frac{51 \times 50 \times \dots\dots \times 41}{52 \times 51 \times \dots\dots \times 42} \times \frac{1}{41} = 1/52$$

Hence, the correct answer is (A, C, D).

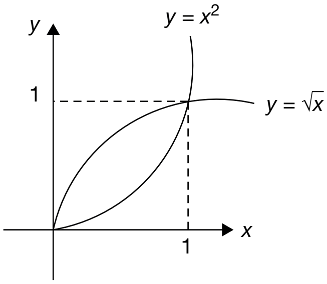
20. Probability that exactly 2 persons will get nothing

$$= {}^{n+2}C_2 \frac{n!}{(n+2)^n} = \frac{(n+2)(n+1)n!}{2(n+2)^n} = \frac{(n+1)!}{2(n+2)^{n-1}}$$

Probability that exactly 3 persons will get nothing $= {}^{n+2}C_3 . \dfrac{{}^{n-1}C_1 . {}^{n}C_2 . (n-2)!}{(n+2)^n}$

Hence, the correct answer is (A, B).

21.



$$P(A) = \frac{\int_0^1 \sqrt{x}dx}{\int_0^1 dx} = \frac{2}{3};\ P(B) = \frac{\int_0^1 \sqrt{y}dy}{\int_0^1 dy} = \frac{2}{3}$$

$$\therefore\ P(A \cap B) = \frac{\int_0^1 \sqrt{x}dx - \int_0^1 x^2dx}{\int_0^1 dx} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Hence, the correct answer is (A).

22. (A) When we select both numbers from {3, 4, … 8}

$$P = \frac{{}^8C_2 - {}^6C_2}{{}^8C_2} = \frac{13}{28}$$

(B) When we choose both numbers from {1, 2, … 5}

$$P = \frac{{}^8C_2 - {}^5C_2}{{}^8C_2} = \frac{9}{14}$$

(C) When we choose number from {1, 3, 5, 7}

$$P = \frac{{}^8C_2 - {}^4C_2}{{}^8C_2} = \frac{11}{14}$$

Hence, the correct answer is (A, B, C).

23. For $K = 2$, each element has four possibility and three out of tour are favorable for the event.

Hence, required probability $= \left(\frac{3}{4}\right)^n$.

For $K = 3$, Probability that intersection is empty is $\left(\frac{7}{8}\right)^n$

For $K = 3$, Probability that intersection is singleton $= \frac{n}{8}\left(\frac{7}{8}\right)^{n-1}$

Hence, the correct answer is (B, C).

24. If T comes in first toss then Mr. B can win in only one case that is TT.

$\Rightarrow$ Probability of Mr. B winning $= \frac{1}{4}$

$\Rightarrow$ Probability of Mr. A winning $= \frac{3}{4}$

Given first toss is head, Mr. A can win is successive tosses are T, HT, HHT, ….

$$\text{Probability} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Given first toss is head, Mr. A can win is successive tosses are HT, HHT, HHHT, ….

$$\text{Probability} = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2}$$

Hence, the correct answer is (A, B, C, D).

25.

Urn	Red Marbles	White marbles	Blue Marbles
A	5	3	8
B	3	5	0

$$P(E_1) = P(R) = \left(\frac{2}{3}\right)\left(\frac{5}{16}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{8}\right) = \frac{10}{48} + \frac{6}{48} = \frac{1}{3}$$

$$P(E_2) = P(W) = \left(\frac{2}{3}\right)\left(\frac{3}{16}\right) + \left(\frac{1}{3}\right)\left(\frac{5}{8}\right) = \frac{6}{48} + \frac{10}{48} = \frac{1}{3}$$

$$P(E_3) = P(B) = \left(\frac{2}{3}\right)\left(\frac{8}{16}\right) = \frac{1}{3}$$

(C) Let A: event that urn A is chose

$$P(A/R) = \frac{P(A \cap R)}{P(R)} = \frac{\left(\frac{2}{3}\right)\left(\frac{5}{16}\right)}{\frac{1}{3}} = \left(\frac{10}{48}\right)(3) = \frac{5}{8}$$

$\Rightarrow$ (C) is incorrect

(D) $P(A/W) = \dfrac{P(A \cap W)}{P(W)} = \dfrac{\left(\frac{2}{3}\right)\left(\frac{3}{16}\right)}{\frac{1}{3}} = \left(\dfrac{6}{48}\right)(3) = \dfrac{3}{8}$

$P(\text{facefive} / W) = \left(\frac{3}{8}\right)\left(\frac{1}{4}\right) = \frac{3}{32} \Rightarrow$ is correct

Hence, the correct answer is (A, B, D).

26. $a^2 + b^2 + c^2 = 2$ and a, b, c are integers

$\Rightarrow a = 0, b = \pm 1, c = \pm 1$ or $a = \pm 1, b = 0, c = \pm 1$

or $a = \pm 1, b = \pm 1, c = 0$. There are 12 possibilities.

(A) Now $|A| \neq 0 \Rightarrow a + b + c \neq 0 \Rightarrow 6$

possibilites $\Rightarrow$ probability $= \frac{1}{2}$

(B) $|A| = 0 \Rightarrow a + b + c = 0 \Rightarrow 6$ possibilities.

But in no case the system has infinitly many solution

(C) If $a = 0, b = 1, c = -1, a = 0, b = -1, c = 1$.

$\Rightarrow$ Two possibilities

(D) Range = {–1, 1} only

Hence, the correct answer is (A, C, D).

27. $\Delta = b^2 - 4ac = p^2 - 4q$ roots are real and equal $p^2 = 4q$

$\therefore\ (2,1)(4,4)(6,9)$

$\therefore$ probability: $\frac{3}{100}$

roots are Imaginary : $p^2 < 4q$

A	q	p	
	1	1	1
	2	1, 2	3
	3	1, 2, 3	6
	4	1, 2, 3	9
	5	1, 2, 3, 4	13
	6	1, 2, 3, 4	17
	7	1, 2, 3, 4, 5	22
	8	1, 2, 3, 4, 5	27
	9	1, 2, 3, 4, 5	32
	10	1, 2, 3, 4, 5, 6	38

Probability (roots are imaginary) $= \dfrac{38}{100} = \dfrac{19}{50}$

P(roots are real) $= \dfrac{62}{100} = \dfrac{31}{50}$

P(roots are real and distinct) $= \dfrac{59}{100}$

Hence, the correct answer is (B, C, D).

28. A, B are exhaust line events $\Rightarrow A \cup B = S$

$$\Rightarrow P(A \cup B) = 1$$

$$2P(B) = P(A) + P(A \cup B)$$

$$= K + 1 \Rightarrow P(B) = \frac{K+1}{2}$$

$$-P(A \cap B) = P(A \cup B) - P(A) - P(B)$$

$$= 1 - K - \frac{(k+1)}{2} = \frac{2 - 2k - k - 1}{2}$$

$$P(A \cap B) = \frac{3K-1}{2}$$

$$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$$

$$= 1 - \frac{3K-1}{2} = \frac{3(1-K)}{2}$$

Hence, the correct answer is (A, B, C, D).

29. Let p_1, p_2 be the chances of happening of the first and second events, $p_1 = p_2^2$ and $\dfrac{1-p_1}{p_1} = \left(\dfrac{1-p_2}{p_2}\right)^3$

By above two $\Rightarrow p_2(1+p_2) = (1-p_2)^2 \Rightarrow p_2 = \dfrac{1}{3}, p_1 = \dfrac{1}{9}$

Hence, the correct answer is (A, C).

Wrong forecast can be done by 2 ways

30. $p = {}^5C_4 . {}^2C_1 = 10$

Correct forecast for any 4 matches

Similarly for q and r

$q = {}^5C_2 . ({}^2C_1)^3 = 80; r = {}^5C_0 . ({}^2C_1)^5 = 32$

$2q = 5r, 8p = q$, and $2(p + r) > q$

Hence, the correct answer is (A, B, D).

31. Probability that atleast one ball is blue = 1 – (all red)

$$= 1 - \frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} = 1 - \frac{1}{10} = 0.9$$

Probability that exactly one blue ball

$$= P(BRR) + P(RBR) + P(RRB)$$

$$= \frac{2}{3} \times \frac{1}{4} \times \frac{2}{5} + \frac{1}{3} \times \frac{2}{4} \times \frac{2}{5} + \frac{1}{3} \times \frac{2}{4} \times \frac{2}{5} = \frac{1}{5}$$

$$P\left({}^{RRR}\!/\!_{RRR \cup BBB}\right) = \frac{P(RRR)}{P(RRR) + P(BBB)}$$

$$= \frac{\frac{1}{10}}{\frac{1}{10} + \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}} = \frac{1}{5}$$

Prob. that atleast one red ball $= 1 - P(BBB)$

$$= 1 - \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{3}{5}$$

Hence, the correct answer is (A, B, C, D).

Linked Comprehension Type Questions

1. Let E_1 = Event that A wrote a plus sign.

E_2 = Event that A wrote a minus sign.

E = Event that the reference observes a plus sign.

Given $P(E_1) = \dfrac{1}{3} \Rightarrow P(E_2) = \dfrac{2}{3}$

$P(E/E_1)$ = Probability that none of B, C, D change the sign + Probability that exactly two of B, C, D change the sign

$$= \frac{1}{27} + 3\left(\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}\right) = \frac{13}{27}$$

2. $P(E/E_2)$ = Probability that all of B, C, D change the sign + Probability that exactly one of them change the sign

$$= \frac{8}{27} + 3 \times \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}\right) = \frac{14}{27}$$

3. $$P\left({}^{E_1}\!/\!_{E}\right) = \frac{P(E_1)P\left({}^{E}\!/\!_{E_1}\right)}{P(E_1)P\left({}^{E}\!/\!_{E_1}\right) + P(E_2)P\left({}^{E}\!/\!_{E_2}\right)}$$

$$= \frac{\frac{1}{3} \times \frac{13}{27}}{\frac{1}{3} \times \frac{13}{27} + \frac{2}{3} \times \frac{14}{27}} = \frac{13}{41}$$

$$\therefore\ P(E_1 / E) = \frac{13}{41}$$

Using Bayes' theorem.

Hence, the correct answer is 1 (C), 2(B), 3(A).

4. Out of $n + 1$ matches the last match is X' s win

Out of remaining 'n' matches either one win and $n - 1$ draws or one win, one loss and $(n – 2)$ draws should be there

$$\therefore\ \text{Probability} = {}^nC_1\, a^2 b^{n-1} + (n-1)n.a^2.b^{n-2}.c$$

Hence, the correct answer is (B).

5. Probability that 'Y' wins a match in 4th game

$= 1$ W, 2 draw + 1 W, 1 L, 1 draw

$= 3.c^2 . b^2 + 6.c^2 . ba = 3c^2\, b(b + 2a)$

Hence, the correct answer is (A).

6. Let E_n = even that 'x' wins in the n^{th} game $(n \geq 2)$

probability that 'x' wins $= \sum_{n=2}^{\infty} P(E_n) = \sum_{n=1}^{\infty} P(E_{n+1})$

$$= \sum_{n=1}^{\infty} n.a^2 b^{n-1} + (n-1)n\, a^2\, b^{n-2}\, c$$

$$= a^2\left(\sum_{n-1}^{\infty} n\, b^{n-1}\right) + a^2\, c.\sum_{n=2}^{\infty} (n(n-1)b^{n-2})$$

$$= a^2 \frac{1}{(1-b)^2} + \frac{a^2 c.2}{(1-b)^3} = \frac{a^2(a+c) + 2a^2 c}{(a+c)^3} = \frac{a^3 + 3a^2 c}{(a+c)^3}$$

Hence, the correct answer is (B).

7. Exactly 3 numbers appears in two ways either (3, 1, 1) or (2, 2, 1)

$$\frac{{}^9C_3.\left({}^3C_1.{}^2C_2.\frac{5!}{3!} + {}^3C_2.{}^1C_1.\frac{5!}{2!.2!}\right)}{9^5} = \frac{{}^9C_3.(60+90)}{9^5}$$

Hence, the correct answer is (C).

8. $\begin{bmatrix} 0 & \odot^9 & \odot^9 \\ & 0 & \odot^9 \\ & & 0 \end{bmatrix}$ Required probability $= \frac{9^3}{9^9} = \frac{1}{9^6}$

Hence, the correct answer is (B).

9. The coefficient of x^0 in $\left(x^{-4} + x^{-3} + x^{-2} + \ldots\ldots\ldots\ldots + x^4\right)^5$

The coefficient of x^0 in $\left(\frac{1 + x + x^2 + \ldots\ldots\ldots + x^8}{x^4}\right)^5$

coefficient of x^{20} in $(1 - x^9)^5 . (1 - x)^{-5}$

$= {}^{24}C_{20} - 5.{}^{15}C_{11} + 10.\,{}^4C_2 = 3951$ required probability $\frac{3951}{9^5}$

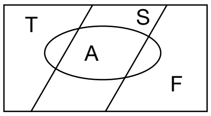
Hence, the correct answer is (D).

10. A: She gets a success

T: She studies 10 hrs: P(T) = 0.1

S: She studies 7 hrs: P(S) = 0.2

F: She studies 4 hrs: P(F) = 0.7



P(A/T) = 0.8; P(A/S) = 0.6; P(A/F) = 0.4

$P(A) = P(A \cap T) + P(A \cap S) + P(A \cap F)$

$= P(T)\, P(A/T) + P(S)\, P(A/S) + P(F)\, P(A/F)$

$= (0.1)(0.8) + (0.2)(0.6) + (0.7)(0.4)$

$= 0.08 + 0.12 + 0.28 = 0.48$

Hence, the correct answer is (C).

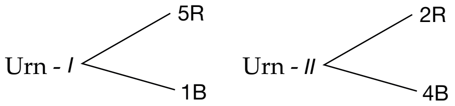
11. $P(F/A) = \frac{P(F \cap A)}{P(A)} = \frac{(0.7)(0.4)}{0.48} = \frac{0.28}{0.48} = \frac{7}{12}$

Hence, the correct answer is (B).

12. $P(F/\bar{A}) = \frac{P(F \cap \bar{A})}{P(\bar{A})} = \frac{P(F) - P(F \cap A)}{0.52} = \frac{(0.7) - 0.28}{0.52}$

$= \frac{0.42}{0.52} = \frac{21}{26}$

Hence, the correct answer is (D).

13. 

A: First two draws resulted in a blue ball

B_1: Urn - I is used $\Rightarrow P(B_1) = \frac{1}{2}$

B_2: Urn - II is used $\Rightarrow P(B_2) = \frac{1}{2}$

$$P\left(\frac{A}{B_1}\right) = \frac{1}{6}.\frac{1}{6} = \frac{1}{36} \qquad P\left(\frac{A}{B_2}\right) = \frac{4}{6}.\frac{4}{6} = \frac{4}{9}$$

$$E_1 : P\left({}^{B_1}\!/\!{}_{A}\right) = \frac{\frac{1}{2}.\frac{1}{36}}{\frac{1}{2}.\frac{1}{36} + \frac{1}{2}.\frac{4}{9}} = \frac{1}{17}$$

$$(\text{say})\, P\left(\frac{B_2}{A}\right) = \frac{\frac{1}{2}.\frac{4}{9}}{\frac{1}{2}.\frac{1}{36} + \frac{1}{2}.\frac{4}{9}} = \frac{16}{17} = P(E_2)(\text{say})$$

$\Rightarrow \quad p = \frac{1}{17},\, q = \frac{16}{17} \Rightarrow 16p$

Hence, the correct answer is (C).

14. E: third ball drawn is red

$P(E) = P(E \cap E_1) + P(E \cap E_2)$

$= \frac{1}{17}.\frac{5}{6} + \frac{16}{17}.\frac{2}{6} = \frac{37}{102}$

Hence, the correct answer is (D).

15.
1 4 7 10 13 16
2 5 8 11 14 17 Probability $= \frac{{}^6C_2 + ({}^6C_1)^2}{{}^{18}C_2} = \frac{1}{3}$
3 6 9 12 15 18

Hence, the correct answer is (A).

16. $\because \quad 2y = x + z$

So either both x and z are even or odd

1 3 5 7 9 11 13 15 17
2 4 6 8 10 12 14 16 18

Probability $= \frac{2({}^9C_2)}{{}^{18}C_3} = \frac{3}{34}$

Hence, the correct answer is (D).

17. Probability $= \frac{{}^{(18-3+1)}C_3}{{}^{18}C_3} = \frac{35}{51}$

Hence, the correct answer is (A).

CHAPTER 7

18. Clearly $P(E_1)=\frac{2}{3}$ and $P\left(\frac{E}{E_1}\right)=\frac{1}{6}$ {since when A rides H_1, all 6 horses are equally likely to win}

also $P(E_2)=1-P(E_1)=1-\frac{2}{3}=\frac{1}{3}$ and we have

$P\left(\frac{E}{E_2}\right)=3.\,P\left(\frac{E}{E_1}\right)=\frac{1}{2}.$

Hence, the correct answer is (C).

19. Now, $P(A_1)=P(E_1\cap E)=P(E_1)P\left(\frac{E}{E_1}\right)=\frac{2}{3}\cdot\frac{1}{6}=\frac{1}{9}.$

And $P(A_2)=P(E_2\cap E)=P(E_2)P\left(\frac{E}{E_2}\right)=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$

Hence, the correct answer is (B).

20. Choose any 4 numbers out of 6 by ${}^6C_4$. One of them is appearing twice. Now distribute them in 5 persons

Required probability

$$=\frac{{}^6C_4\left(4^5-{}^4C_3\,3^5+{}^4C_2\,2^5-{}^4C_1\,1^5\right)}{6^5}=\frac{3600}{7776}$$

$$\text{Alternative}=\frac{{}^6C_4\;{}^4C_1\cdot\frac{5!}{2!}}{6^5}=\frac{3600}{7776}$$

Hence, the correct answer is (B).

21. $P=\frac{{}^6C_5}{6^5}$

Hence, the correct answer is (A).

22. Favourable cases $=5,\{(6,6,6,6,5)\ldots\ldots\ldots\ldots\ldots\}$

so required probability $=\frac{5}{7776}$

Hence, the correct answer is (C).

23. As Y wins the balls must be drawn of the following 3 ways RR, RBR or BRR. This must be the case as otherwise 2 black balls will be drawn and X wins. Hence prove that Y wins.

$$Y(n)=\frac{2}{n+2}\cdot\frac{1}{n+1}+\frac{2}{n+2}\frac{n}{n+1}\cdot\frac{1}{n}+\frac{n}{n+2}\cdot\frac{2}{n+1}\frac{1}{n}$$

$$=\frac{2}{(n+1)(n+2)}+\frac{2}{(n+1)(n+2)}+\frac{2}{(n+1)(n+2)}$$

$$=\frac{6}{(n+1)(n+2)}$$

Hence, the correct answer is (B).

24. $Y(n)=6\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$

$$Y(1)+Y(2)+\ldots Y(n)=6\left[\frac{1}{2}-\frac{1}{n+2}\right]$$

$$\lim_{n\to\infty}\left[Y(1)+Y(2)+\ldots+Y(n)\right]=\lim_{n\to\infty}6\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$=\frac{6}{2}=3$$

Hence, the correct answer is (D).

25. Now $X(n)=1-Y(n)=1-\frac{6}{(n+1)(n+2)}$

$$=\frac{n^2+3n-4}{(n+1)(n+2)}=\frac{(n-1)(n+4)}{(n+1)(n+2)}$$

$$X(2)\,X(3)\ldots X(n)=\frac{1.6}{3.4}\cdot\frac{2.7}{4.5}\cdot\frac{3.8}{5.6}\cdot\frac{4.9}{6.7}\cdots\frac{(n-1)(n+4)}{(n+1)(n+2)}$$

$$=\frac{1.2.3.4\ldots(n-2)(n-1)}{3.456\ldots n(n+1)}\cdot\frac{6.7.8\ldots(n+3)(n+4)}{4.567\ldots(n+1)(n+2)}$$

$$=\frac{1.2}{n(n+1)}\cdot\frac{(n+3)(n+4)}{4.5}=\frac{1}{10}\frac{(n+3)}{n}\frac{n+4}{n+1}$$

$$\lim_{n\to\infty}X(2)X(3)\cdots X(n)=\lim_{n\to\infty}\frac{1}{10}\frac{(n+3)(n+4)}{n(n+1)}=\frac{1}{10}$$

Hence, the correct answer is (B).

26. $n(S)=6^3$

$f'(x)=3x^2+2bx+c$

$D<0\quad\Rightarrow 4b^2-12c<0\Rightarrow b^2<3c$

b	c
–	0
0,1	1
0,1,2	2
0,1,2	3
0,1,2,3	4
0,1,2,3	5

$n(E)=16\quad P(E)=\frac{16}{6\times6\times6}=\frac{2}{27}$

Hence, the correct answer is (D).

27. $n(S)=6^3$

$f'(x)=3x^2+2bx+c$ distinct roots $D>0\ \Rightarrow 4b^2-12c>0$

$\Rightarrow b^2>3c$

c	b
0	1,2,3,4,5
1	2,3,4,5
2	3,4,5
3	4,5
4	4,5
5	4,5

$n(E)=18\qquad P(E)=\frac{18}{6^3}=\frac{18}{6\times6\times6}=\frac{1}{12}$

odds in favour = 1 : 11

Hence, the correct answer is (D).

28. $P\left(\frac{B}{A}\right)=\frac{n(A\cap B)}{n(A)}=\frac{13}{18}$

c	b
1	2,3,4,5
2	3,4,5
3	4,5
4	4,5
5	4,5

$$\alpha+\beta<0, \alpha\beta>0$$

$$b>0, c>0$$

Hence, the correct answer is (D).

29. $P_1 \geq 0, P_2 \geq 0, P_3 \geq 0$ & $p_1+p_2+p_3 \leq 1$

$$\Rightarrow p \leq 1, p \geq -\frac{1}{2}, p \geq -\frac{2}{3} \text{ and } p \leq -\frac{7}{23}$$

Hence, the correct answer is (D).

30. Use $\frac{1}{3}(P_1+P_2+P_3)=(P_1 . P_2 . P_3)^{1/3}$ to obtain

$$P_1=P_2=P_3=\frac{1}{3} \Rightarrow \frac{a}{27}=\left(\frac{1}{3}\times\frac{1}{3}\right)\times 3 \Rightarrow a=9$$

Hence, the correct answer is (A).

31. Total number of choices $= 3^m$

Number of ways in which exactly one letter turns up $= 1^m + 1^m + 1^m = 3$

Number of ways in which exactly two of the three letters turn up

$$=\left(2^m-2\right)+\left(2^m-2\right)+\left(2^m-2\right)=3\left(2^m-2\right)$$

We can spell the word CAB only when we have each of the letters at least once which can therefore happen in $= 3^m - 3 - 3(2^m - 2) = 3^m - 3.2^m + 3$

Required probability $=\dfrac{3^m-3.2^m+3}{3^m}$

Hence, the correct answer is (B).

32. Total number of choices is equal to number of non-negative integral solutions to the equation $x_1+x_2+x_3=m$ i.e., ${}^{m-1}C_{3-1} = {}^{m-1}C_2$

Required probability $=\dfrac{{}^{m-1}C_2}{{}^{m+2}C_2}=\dfrac{(m-1)(m-2)}{(m+2)(m+1)}=\dfrac{m^2-3m+2}{m^2+3m+2}$

Hence, the correct answer is (C).

33. Favourable number of ways

= 3{Number of positive integral solutions to the equation $x_1+x_2=m$}

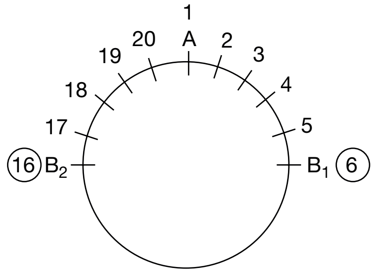
$$=3.\,{}^{m-1}C_{2-1}=3(m-1)$$

Required probability $=\dfrac{3(m-1)}{{}^{m+2}C_2}=\dfrac{6(m-1)}{m^2+3m+2}$

Hence, the correct answer is (D).

34. If A occupies any place, B has 19 places to choose. But if there are to be exactly 4 persons between A and B, B can choose only two places $B_1(6)$, $B_2(16)$ marked in the Figure.

$$\therefore p=\frac{2}{19}$$



Hence, the correct answer is (A).

35. Now B can occupy any one of the 11 places 6th to 16th

$$\therefore p=\frac{11}{19}$$

Hence, the correct answer is (C).

36. Given that $P(AB)=\frac{2}{3}, P(BC)=\frac{1}{2}, P(AC)=\frac{2}{3}$ and

$$P(ABC)=P(ACB), P(BAC)=P(BCA),$$

$$P(CAB)=P(CBA).$$

Let $P(ABC)=x, P(BAC)=y, P(CAB)=z$.

$$\therefore P(AB)=\frac{2}{3}=P(ABC)+P(ACB)+P(CAB)$$

$$\Rightarrow 2x+z=\frac{2}{3}$$

$$P(BC)=\frac{1}{2}=P(ABC)+P(BCA)+P(BAC)$$

$$\Rightarrow x+2y=\frac{1}{2}$$

$$P(AC)=\frac{2}{3}=P(ACB)+P(ABC)+P(BAC)$$

$$\Rightarrow 2x+y=\frac{2}{3}$$

$$\therefore x=\frac{5}{18}, y=z=\frac{1}{9}$$

Probability that A wins $= P(ABC)+P(ACB)=2x=\frac{5}{9}$

Hence, the correct answer is (A).

37. Probability that B wins $= P(BAC)+P(BCA)=2y=\frac{2}{9}$

Hence, the correct answer is (B).

38. $P(BA) = P(BAC) + P(BCA) + P(CBA)$

$$=2y+z=\frac{1}{3}$$

$P(CA) = P(AB) + P(BCA) + P(BCA)$

$$=2z+y=\frac{1}{3}$$

$P(CB) = P(ACB) + P(CBA) + P(CAB) = x + 2z = \frac{1}{2}$

Hence, the correct answer is (D).

39. $$P(E_i \cap A_i) = P(A_1).P\left(\frac{E_1}{A_1}\right) + P(A_2).P\left(\frac{E_2}{A_2}\right) + P(A_3).P\left(\frac{E_2}{A_3}\right) + P(A_4).P\left(\frac{E_4}{A_4}\right)$$

$$= \frac{1}{10}\times 1 + \frac{2}{10}\times\frac{1}{2} + \frac{3}{10}\times\frac{1}{3} + \frac{4}{10}\times\frac{1}{4} = \frac{4}{10} = \frac{2}{5}.$$

Hence, the correct answer is (C).

40. $$P(E_2) = P(A_2)P\left({}^{E_2}\!/_{A_2}\right) + P(A_3)P\left({}^{E_2}\!/_{A_3}\right) + P(A_4)P\left({}^{E_2}\!/_{A_4}\right)$$

$$P(E_2) = \frac{2}{10}\times\frac{1}{2} + \frac{3}{10}\times\frac{1}{3} + \frac{4}{10}\times\frac{1}{4} = \frac{3}{10}$$

$$P\left(\frac{A_3}{E_2}\right) = \frac{P(A_3 \cap E_2)}{P(E_2)} = \frac{P(A_3)(E_2/A_3)}{P(E_2)} = \frac{\frac{1}{10}}{\frac{3}{10}} = \frac{1}{3}$$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. $A \to (r);\ B \to (s);\ C \to (p);\ D \to (q)$

(A) $n(S) = \dfrac{10!}{(7!)(3!)}$

First arrange all white balls in a line by 1 way

$n(E) = {}^8C_3 = \dfrac{8!}{(3!)(5!)}$, because there are 8 places for 3 black balls.

$$\therefore P(E) = \frac{\frac{8!}{(3!)(5!)}}{\frac{10!}{(7!)(3!)}} = \frac{(8!)(7!)}{(10!)(5!)} = \frac{7.6}{10.9} = \frac{7}{15}$$

(B) $n(S) = 7!,\ n(E) = (3!)\times(4!)$

($\because$ after making 4 gentlemen sit in 3! ways, 4 ladies can sit in 4! ways in between the gentlemen)

$$P(E) = \frac{(3!)\times(4!)}{7!} = \frac{6}{7\times 6\times 5} = \frac{1}{35}$$

(C) $n(S) = {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$n(E) = n(S)$ – the number of ways in which one boy gets both the pens.

$= n(S) - {}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)$

$$\therefore P(E) = 1 - \frac{{}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)}{{}^{12}C_4 \times {}^8C_4 \times {}^4C_4} = \frac{5}{11}$$

(D) Required probability= $P(E\,E\,E\,E\,\bar{E}) + P(\bar{E}\,\bar{E}\,\bar{E}\,\bar{E}\,E)$

$$= \{P(E)\}^4.P(\bar{E}) + \{P(\bar{E})\}^4.P(E) = \frac{1}{16}$$

2. $A \to (p, q, r, s);\ B \to (p, q, r, s);\ C \to (p, r);\ D \to (r, s)$

(A) $P(A \cup B) = 1 - P(A' \cap B') = 1 - \dfrac{1}{2}\times\dfrac{4}{5} = \dfrac{3}{5}$

$P(A/B) = P(A) = \dfrac{1}{2}$ as A and B are independent

$P(A \cap B) \le \min\{P(A), P(B)\} = \dfrac{1}{5} \le \dfrac{1}{2}$

P (Exactly of A, B) $= P(A \cap B') + P(A' \cap B)$

$= P(A)P(B') + P(A')P(B)$

$= \dfrac{1}{2}\times\dfrac{4}{5} + \dfrac{1}{2}\times\dfrac{1}{5} = \dfrac{1}{2}$

(B) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.6 = P(A) + 0.2 - 0.2\,P(A)$

$\Rightarrow\ 0.4 = 0.8\,P(A) \Rightarrow P(A) = \dfrac{1}{2}$

$\therefore P(A/B) = \dfrac{1}{2}$ as A and B are independent

Also $P(A \cap B) = P(A)P(B) = 0.1 \le \dfrac{1}{2}$

Next P(Exactly one of A, B) = $\dfrac{1}{2}\times\dfrac{8}{10} + \dfrac{1}{2}\times\dfrac{2}{10} = \dfrac{1}{2}$

(c) $P(A \cup B) \ge \max\{P(A), P(B)\} = \dfrac{2}{3} \ge \dfrac{3}{5}$

$P(A \cap B) \le \min\{P(A), P(B)\} = \dfrac{1}{2}$

(d) $P(A \cup B) \ge \dfrac{1}{2}, P(A \cap B) \le \dfrac{3}{7} \le \dfrac{1}{2}$ $\quad P(A/B) = \dfrac{3}{7}$

and P(Exactly one of A, B) = $\dfrac{3}{7}\times\dfrac{1}{2} + \dfrac{4}{7}\times\dfrac{1}{2} = \dfrac{1}{2}$

3. $A \to (q);\ B \to (p);\ C \to (q);\ D \to (r)$

(A) Total no. of ways of putting the balls in boxes

$$= \frac{6!}{2!2!2!} + \frac{6!3!}{1!1!2!4!} + \frac{6!3!}{1!2!3!} = 540$$

No. of favourable ways $= \dfrac{6!}{2!2!2!} = 90$

Probability $= \dfrac{90}{540} = \dfrac{1}{6}$

(B) Probability $= \dfrac{3^6 - {}^3C_1.2^6 + 3}{3^6} = \dfrac{20}{27}$

(C) No. of ways B can throw greater than 9 is 6 i.e., {(4, 6), (6, 4), (5, 5), (6, 5), (5, 6), (6, 6),}

required prob. $= \dfrac{6}{36} = \dfrac{1}{6}$

(D) $P(A \cup \bar{B}) = 1 - P(B) + P(A \cap B) = 0.8$

$\Rightarrow\ 1 - P(B) + P(B)(0.3) = 0.8$

$\Rightarrow\ 0.2 = P(B)\times 0.7 \Rightarrow P(B) = \dfrac{2}{7}$

4. $A \to (q);\ B \to (p);\ C \to (r);\ D \to (q)$

Let E_i denotes the event that the bag contains i black and $(12 - i)$ white balls (i = 0 , 1, 2 ….., 12) and A denotes the event that the four balls drawn are all black. Then

$P(E_i) = \dfrac{1}{13}$ (i = 0, 1, 2, ….., 12); $P\left(\dfrac{A}{E_i}\right) = 0$ for i = 0 , 1 , 2, 3;

$P\left(\dfrac{A}{E_i}\right) = \dfrac{{}^iC_4}{{}^{12}C_4}$ for $i \ge 4$

(A) $P(A) = \displaystyle\sum_{i=0}^{12} P((E_i)P\left(\frac{A}{E_i}\right)$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}[{}^{4}C_4 + {}^{5}C_4 + \ldots .\ {}^{12}C_4] = \frac{{}^{13}C_5}{13 \times {}^{12}C_4} = \frac{1}{5}$$

(B) Clearly, $P\left(\frac{A}{E_{10}}\right) = \frac{{}^{10}C_4}{{}^{12}C_4} = \frac{14}{33}$

(C) By Bayes' theorem, $P\left(\frac{E_{10}}{A}\right) = \frac{P(E_{10})P\left(\frac{A}{E_{10}}\right)}{P(A)} = \frac{\frac{1}{13} \times \frac{14}{33}}{\frac{1}{5}}$

$= \frac{70}{429}$

(D) Let B denotes the probability of drawing 2 white and 2 black balls then

$P\left(\frac{B}{E_{10}}\right) = 0$ if $i = 0, 1$ or $11, 12$

$P\left(\frac{B}{E_{10}}\right) = \frac{{}^{i}C_2 \times {}^{12-i}C_2}{{}^{12}C_4}$ for $i = 2, 3, \ldots\ldots\ldots, 10$

$$\therefore P(B) = \sum_{i=0}^{12} P(E_i)P\left(\frac{B}{E_i}\right)$$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}[{}^{2}C_2 \times {}^{10}C_2 + {}^{3}C_2 \times {}^{9}C_2 + \ldots . + {}^{10}C_2 \times {}^{2}C_2]$$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}\Big[2\{{}^{2}C_2 \times {}^{10}C_2 + {}^{3}C_2 + {}^{9}C_2 \ldots . + {}^{5}C_2 \times {}^{7}C_2\} + {}^{6}C_2 \times {}^{6}C_2\Big]$$

$$= \frac{1}{13} \times \frac{1}{495}(1287) = \frac{1}{5}$$

5. A → (q); B → (r); C → (s); D → (r)

We have, $P(A \cap B) = P(A)\ .\ P(B) = \frac{1}{12}$

(A) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

(B) $P\left(\frac{A}{A \cup B}\right) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{2}{3}$

(C) $P\left(\frac{B}{A' \cap B'}\right) = \frac{P(B \cap (A' \cap B'))}{P(A' \cap B')} = \frac{P(\phi)}{P(A' \cap B')} = 0$

(D) $P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B)}{P(B)} = \frac{P(A')P(B)}{P(B)} = P(A') = \frac{2}{3}$

6. A → (q); B → (r); C → (r); D → (t)

(A) A: Number have two digits ⇒ (A) = {10, 11, 12, … 15} ⇒ $n(A) = 6$

B: Number is prime number ⇒ (B) = {2, 3, 5, 7, 11, 13}

⇒ $n(A \cap B) = 2$; {11, 13}

⇒ $P\left(B/_A\right) = \frac{2}{6} = \frac{1}{3}$

(B) The required probability

$= \lim\limits_{n \to \infty} \frac{[n/4]}{n}$, [.] denotes the greatest integers function. = 1/4.

(C) Total number of possible ways

= 100 × 100.

For $7^m + 7^n$ to be divided by 5,

(i) one of the term has to end with 9 and other with 1

m can be 2, 6, 10, 14, ..., 98 (25 values)

n can be 4, 8, 12, ..., 100 (25 values)

(ii) one of the term has to end with 3 and other with 7

Probability $= \frac{2 \times 2 \times 25 \times 25}{100 \times 100} = \frac{1}{4}$

(D) $P\left(\frac{E}{E_1}\right) = \frac{P(E).P\left(\frac{E_1}{E}\right)}{P(E).P\left(\frac{E_1}{E}\right) + P(\bar{E}).P\left(\frac{E_1}{\bar{E}}\right)}$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \left(\frac{1}{4} \times \frac{1}{5}\right)} = \frac{3}{4}$$

7. A → (r); B → (s); C → (p); D → (q)

(A) S_2 is among the losers so losers can be selected from the rest 11 players by ${}^{11}C_5$, Probability $= \frac{{}^{11}C_5}{{}^{12}C_6} = \frac{1}{2}$

(B) Let E_1 be the event that S_3 and S_4 are in same group let E_2 be the event that S_3 and S_4 are in different group

$P(E_1) = \frac{1}{11}$, $P(E_2) = 1 - P(E_1) = \frac{10}{11}$

Let E be the event that exactly one of S_3 and S_4 is among the losers, then

$P(E) = P(E_1).P(E/E_1) + P(E_2)P(E/E_2) = \frac{1}{11} \times 1 + \frac{10}{11} \times \left(\frac{1}{2}.\frac{1}{2} + \frac{1}{2}.\frac{1}{2}\right) = \frac{6}{11}$

(C) S_2 and S_4 should be different groups for both being winner

Required Probability $= \frac{10}{11}\left(\frac{1}{2}.\frac{1}{2}\right) = \frac{5}{22}$ or $\frac{{}^{10}C_4}{{}^{12}C_6} = \frac{5}{22}$

(D) S_4 and S_5 do not want to play with each other so

P = 1 – P(both play with each other)

$$= 1 - \frac{1}{11} = \frac{10}{11}$$

8. A → (r); B → (r); C → (p); D → (p)

(A) $P(A) = \frac{1}{4}, P(B) = \frac{1}{5}, P(C) = \frac{1}{6}$

$P(A') = 1 - \frac{1}{4} = \frac{3}{4}; P(B') = 1 - \frac{1}{5} = \frac{4}{5}; P(C') = 1 - \frac{1}{6} = \frac{5}{6}$

required prob. $= 1 - P(A')P(B')P(C')$

$$= 1 - \frac{3}{4}.\frac{4}{5}.\frac{5}{6} = 1 - \frac{1}{2} = \frac{1}{2}$$

(B) P(at least two heads)

= P(HHT, HTH, THH, HHH)

$= \frac{4}{8} = \frac{1}{2}$ [$\because$ total no . of equally likely cases $= 2^3 = 8$]

(C) In 365 days, there are 52 weeks and 1day. clearly there are 52 sundays in 52 weeks. the remaining 1 day, can be sunday, Monday, ……

$\therefore$ Prob. for 53 sundays $= \frac{1}{7}$

(D) Total number of equally likely cases = 8

If at least one of the coins show head, the reqd. prob.

For all heads $= \frac{1}{8-1} = \frac{1}{7}$

9. A → (p, s); B → (s, t); C → (q, s, t); D → (r)

(A) Let A_i(i = 1, 2, 3, 4) be the event that urn contains 2, 3, 4 or 5W balls and B the event that two white balls is drawn we have to find $P\left(\frac{A_4}{B}\right)$

Applying, Bayes theorem

$$P\left(\frac{A_4}{B}\right) = \frac{\frac{1}{4}\times 1}{\frac{1}{4}\times\frac{1}{10}+\frac{1}{4}\times\frac{3}{10}+\frac{1}{4}\times\frac{6}{10}+\frac{1}{4}\times 1} = \frac{1}{2}$$

(B) $n, n+3, \ldots\ldots m+12$

$n+1, \ldots\ldots$

$n+2, \ldots\ldots, n+14$

Required probability $= \frac{3\times {}^5C_3 + 5^3}{{}^{15}C_3} = \frac{31}{91}$

(C) Required probability $= \frac{3!(2)}{4\times 4\times 4} = \frac{3}{16}$

(D) For the balls in box there are three possibility

(i) All the 4 balls are red

(ii) 3 of the 4 balls are red

(iii) 2 of the 4 balls are red

Let these represented by E_1, E_2 and E_3 respectively.
Assuming that E_1, E_2 and E_3 are given to be equally likely

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let A be the event two balls drawn are red

$$P\left(\frac{A}{E_1}\right) = \frac{{}^4C_2}{{}^4C_2} = 1,\ P\left(\frac{A}{E_2}\right) = \frac{{}^3C_2}{{}^4C_2} = \frac{1}{2} \text{ and } P\left(\frac{A}{E_3}\right) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6}$$

$$P(A) = \sum_{i=1}^{3} P(E_i)P\left(\frac{A}{E_i}\right) = \frac{1}{3}\times 1 + \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} + \frac{1}{6} = \frac{5}{9}$$

Thus, $P\left(\frac{E_1}{A}\right) = \frac{\frac{1}{3}\times 1}{\frac{5}{9}} = \frac{3}{5}$, $P\left(\frac{E_2}{A}\right) = \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{5}{9}} = \frac{3}{10}$

and $P\left(\frac{E_3}{A}\right) = \frac{\frac{1}{3}\times\frac{1}{6}}{\frac{5}{9}} = \frac{1}{10}$

Now, denote the events $\frac{E_1}{A}, \frac{E_2}{A}, \frac{E_3}{A}$ by F_1, F_2, F_3 respectively, then

we have $P(F_1) = \frac{3}{5}$, $P(F_2) = \frac{3}{10}$, $P(F_3) = \frac{1}{10}$

Let B is the event that next draw is red ball.

$$P\left(\frac{B}{F_1}\right) = 1,\ P\left(\frac{B}{F_2}\right) = \frac{3}{4},\ P\left(\frac{B}{F_3}\right) = \frac{2}{4},$$

$$P(B) = \frac{3}{5}\times 1 + \frac{3}{10}\times\frac{3}{4} + \frac{1}{10}\times\frac{2}{4} = \frac{7}{8}$$

10. A → (q); B → (p); C → (s); D → (r)

If $x_i \in A$ then $x_i \in P, x_i \in Q$

$x_i \notin P,\ x_i \notin Q$

$x_i \notin P, x_i \in Q$

$x_i \in P, x_i \notin Q$

(A) $P \cap Q = \phi \Rightarrow x_i \in P, x_i \notin Q$

$x_i \notin P, x_i \notin Q$

$x_i \notin P, x_i \in Q$

$n(E) = 3^n,\ n(S) = 4^n$

$$P = \left(\frac{3}{4}\right)^n$$

(B) $P \cap Q$ is a singleton

$x_i \in P, x_i \in Q$

$n(E) = {}^nC_1(1)3^{n-1}, n(s) = 4^n$

$$P = \frac{n\cdot 3^{n-1}}{4^n}$$

(C) $P \cap Q$ contain 2 elements

$n(E) = {}^nC_2(1)^2\,3^{n-2}$

$$P = \frac{{}^nC_2\,3^{n-2}}{4^n} \quad P = \frac{n(n-1)3^{n-2}}{2\cdot 4^n}$$

(D) $n(P) = n(Q) \Rightarrow P(E)$

$$= \frac{{}^nC_0{}^nC_0 + {}^nC_1{}^nC_1 + \ldots + {}^nC_n.{}^nC_n}{2^n.2^n} = \frac{{}^{2n}C_n}{4^n}$$

Integer/Numerical Answer Type Questions

1. The probability that he gets marks $= \frac{1}{31}$

The probability that he gets marks in second trial is $\frac{30}{31}\times\frac{1}{30}$ $= \frac{1}{31}$

The probability that he gets marks in third trial is $\frac{1}{31}$

Continuing this process the probability from *r* trial is $\frac{K}{31} > \frac{1}{8}$

$$\Rightarrow K > \frac{31}{8}$$

$K = 4$

Hence, the correct answer is (4).

2. $n(X) = k + 1$

No. of ways to construct A $= 2^{k+1}$

No. of ways to construct B $= 2^{k+1}$

$\therefore$ Total ways to construct A and B $= 2^{k+1} \times 2^{k+1}$

Favourable ways to construct A $= 2^{k+1}$

Favourable ways to construct B such that $B = A^C$ is $= 1$

$\therefore$ Favourable ways $= 2^{k+1} \times 1$

$$\text{Required Probability} = \frac{2^{k+1}}{\left(2^{k+1}\right)^2} = \frac{1}{2^{k+1}}$$

$\Rightarrow m - 1 = k + 1$

$\Rightarrow m - k = 2$

Hence, the correct answer is (2).

3. The no. of ways of drawing 7 balls $= {}^{10}C_7$

For each set of 7 balls of the second draw, 3 must be common to the set of 5 balls of the draw, i.e., 2 other balls can be drawn in ${}^3C_2$ ways thus, for each set of 7 balls of the second draw, there are ${}^7C_3 \times {}^3C_2$ ways of making the first draw so that there are 3 balls common. Hence, the probability of having three balls in common $\dfrac{{}^7C_3 \times {}^3C_2}{{}^{10}C_5} = \dfrac{5}{12}$.

Hence, the correct answer is (5).

4. The total number of triangles $= {}^{20}C_3 = 1140$ there are 20 triangles with two sides of polygon there are 20×16 triangles with are side of polygon

$$\therefore \text{Required probability} = \frac{1140 - 20 - 320}{1140} = \frac{800}{1140} = \frac{40}{57}$$

Hence, the correct answer is (1).

5. All selected people have different month of Birth, so we can choose any 4 month out of 12 and arrange them by $\lfloor\!\underline{4}$ ways.

Fav. cases $= {}^{12}C_4 \cdot 4!$

Sample space have 12^4 elements.

$$\text{Required probability} = \frac{{}^{12}C_4 \lfloor\!\underline{4}}{12^4} = \frac{55}{96}$$

Hence, the correct answer is (5).

6. Total number of function $= 5^3 = 125$

Number of function satisfying $f(i) \leq, f(j)$ if $i < j$

$$= \underset{\text{Choose any 3 elements from B}}{{}^5C_3} + {}^5C_2\,\overset{\text{Any two of 3 elements of set A has same image}}{(1 + 1)} + \underset{\text{All elements has same image}}{{}^5C_1} = 35$$

$$\text{Required probability} = \frac{35}{125} = \frac{7}{25}$$

Hence, the correct answer is (7).

7. Let the sides be a, b, c

$a = b = 1, c = 1$

$a = b = 2, c = 1, 2, 3$

$a = b = 3, c = 1, 2, 3, 4, 5$

$a = b = 4, c = 1, 2, 3, 4, 5, 6$

$a = b = 5, c = 1, 2, 3, 4, 5, 6$

$a = b = 6, c = 1, 2, 3, 4, 5, 6$

The number of these triangles is $1 + 3 + 5 + 3 \times 6$

$$\text{Probability} = \frac{27}{6^3} = \frac{1}{8}$$

Hence, the correct answer is (8).

8. The total number of outcomes:

aaaa appear in $\dbinom{6}{1} = 6$ ways

aaab appear in $2\dbinom{6}{2} = 30$ ways

aabb appear in $\dbinom{6}{2} = 15$ ways

aabc appear in $3\dbinom{6}{3} = 60$ ways

abcd appear in $\dbinom{6}{4} = 15$ ways

Total $= 6 + 30 + 15 + 60 + 15 + 126$

The number of ways of primes appearing

aaaa appear in $\dbinom{3}{1} = 3$ ways

aaab appear in $2\dbinom{3}{2} = 6$ ways

aabb appear in $\dbinom{3}{2} = 3$ ways

aabc appear in $3\dbinom{3}{3} = 3$ ways

Total $= 3 + 6 + 3 + 3 = 15$

$$\text{Probability} = \frac{15}{126} = \frac{5}{42}$$

Hence, the correct answer is (5).

9. $x^3 + y^3$ is divisible by $3 \Rightarrow 3 + y$ is divisible by $3 \Rightarrow x, y$ are multiples of 3 or one leaves remainder 1 and the other 2 when divided by 3.

3, 6, 9 …. 30 are multiple of 3; 1, 4, 7, ….., 28 leave remainder 1

2, 5, 8, …, 29 leave remainder 2

$$\text{Probability} = \frac{\dbinom{10}{1}\dbinom{10}{1} + \dbinom{10}{2}}{\dbinom{30}{2}} = \frac{145}{15 \times 29} = \frac{1}{3}$$

Hence, the correct answer is (3).

10.

p	q
2	1
3	1, 2
4	1 to 4
5	1 to 6
6	1 to 9
7, 8, 9, 10	1 to 10

$p^2 \geq 4q$

The total number of pairs (p, q) is $1 + 2 + 4 + 6 + 9 + 40 = 62$

$$\text{Probability} = \frac{62}{10.10} = \frac{31}{50}$$

Hence, the correct answer is (5).

11. The sequence of consecutive heads may start with 1st toss or 2nd toss or 3rd toss ---- or at 6th toss. In any case, if it starts with r^{th} throw, the first $(r-2)$ throws may be head or tail but $(r-1)^{\text{st}}$ throw must be tail, after which again a head or tail can show up:

$$\underbrace{\text{HHH}}_{7\text{ Times}}\text{ ----- H x x x x x} \quad \text{or} \quad \underbrace{\text{x x x ----- x}}_{(r\text{-}2)}\ \underset{(r\text{-}1)\text{st}}{\text{T}}\ \text{HHH ----- H x x -----}$$

$$\therefore \quad \text{Probability} = \frac{1}{2^7} + \underbrace{\frac{1}{2}.\frac{1}{2^7} + \frac{1}{2}.\frac{1}{2^7} + \ldots\ldots\ldots + \frac{1}{2}.\frac{1}{2^7}}_{5\text{ Times}}$$

$$= \frac{1}{2^7}\left[1 + \frac{5}{2}\right] = \frac{7}{2^8}$$

Hence, the correct answer is (7).

12. Let E_1 = the toss result in a head

E_2 = the toss result in a tail

A = noted number is 7 or 8

$P(A) = P(E_1) . P(A/E_1) + P(E_2) . P(A/E_2)$

$$= \frac{1}{2} \times \frac{11}{36} + \frac{1}{2} \times \frac{2}{11} = \frac{193}{792}$$

$$\frac{P}{792} = \frac{193}{792}$$

P = 193.

Hence, the correct answer is (9).

13. Choose any two rooms from the third hotel is the favourable ways. Total number of ways will be ${}^{10}C_2$

$$P(E) = \frac{2.{}^4C_2}{10.9} = \frac{2}{15} \Rightarrow K = 2$$

Hence, the correct answer is (2).

14. If A gets 1, then B's chances are 1, 2, ... 6 = 6

If A gets 2, then B's chances are 2, 3, ... 5 = 5

$$\text{Similarly so on up to } 6 = \sum_{n=1}^{6}(n) = 21$$

$$\therefore P(E) = \frac{21}{36} = \frac{7}{12}$$

Hence, the correct answer is (7).

15. $\text{Required probability} = 1 - \frac{4}{5} \times \frac{4}{5} = \frac{9}{25}$

$\Rightarrow k = 9$

Hence, the correct answer is (9).

16. Required probability = Probability of drawing of 3 W and 3 B balls in first 6 draws and a white ball in 7th draw.

$$= {}^6C_3 \cdot \frac{1}{2^7} = \frac{5}{32}$$

Hence, the correct answer is (5).

17. Let the non-negative integers be x, y, $x = 5a + \alpha$, $y = 5b + \beta$ where

$0 \leq \alpha \leq 4, 0 \leq \beta \leq 4$

$x^2 + y^2 = 25(a^2 + b^2) + 10(a\alpha + b\beta) + \alpha^2 + \beta^2$

$\alpha^2 + \beta^2$ is divisible by 5

$\alpha, \beta \in \{(0, 0)\ (1, 2)\ (2, 1)\ (1, 3)\ (3, 1)\ (2, 4)\ (4, 2)\ (3, 4)\ (4, 3)\}$

$$\text{Probability} = \frac{9}{25} \quad \therefore k = 9$$

Hence, the correct answer is (9).

18. Total ways of choosing m and n is $n(S) = 100 \times 100$

Now, $7^1 = 7 = 5k + 2, 7^2 = 49 = 5k + 4, 7^3 = 343 = 5k + 3, 7^4 = 2301 = 5k + 1$

The same sequence will repeat for next four powers

$7^5 = 2k + 2, 7^6 = 5k + 4, 7^7 = 5k + 3, 7^8 = 5k + 1$

$\therefore 7^1, 7^5, 7^9, \ldots., 7^{97}$ are of the type $5k + 2$

$7^2, 7^6, 7^{10}, \ldots., 7^{98}$ are of the type $5k + 4$

$7^3, 7^3, 7^7, 7^{11}, \ldots., 7^{99}$ are of the type $5k + 3$

and $7^4, 7^8, 7^7, 7^{12}, \ldots., 7^{100}$ are of the type $5k + 1$

There will be only four favourable combinations in which $7^m + 7^n$ will be divisible by 5.

$7^m : 5k + 2, 5k + 4, 5k + 3, 5k + 1$

$7^n : 5k + 3, 5k + 1, 5k + 2, 5k + 4$

$\therefore$ Number of favourable cases is

$n(E) = 25 \times 25 + 25 \times 25 + 25 \times 25 + 25 \times 25 = 4 \times 625$

$$\therefore \text{Required probability} = \frac{n(E)}{n(S)} = \frac{4 \times 625}{100 \times 100} = \frac{1}{4}$$

Hence, the correct answer is (4).

19. Number of ways in which remaining '$m - 1$' cars can take their places

(excluding the car of man) $= {}^{n-1}C_{m-1}$

No. of ways in which remaining $(m - 1)$ cars can take places keeping the two places on two sides of his car vacant $= {}^{n-3}C_{m-1}$

$$\text{Prob} = \frac{{}^{n-3}C_{m-1}}{{}^{n-1}C_{m-1}} = \frac{(n-m)(n-m-1)}{(n-1)(n-2)}$$

$\Rightarrow$ A = 1, B = 2

A + B = 3

Hence, the correct answer is (3).

20. Out of 5 horses only one is the winning horse. The probability that Mr. A selected the loss horse $=\frac{4}{5}\times\frac{3}{4}$

∴ The probability that Mr. A selected the winning horse =

$1-\frac{4}{5}\cdot\frac{3}{4}=\frac{2}{5}$

Hence, the correct answer is (2).

21. It is possible when first two letters are I and third letter is T. So required probability is

$$\frac{^{10}C_1}{^{20}C_1}\times\frac{^{9}C_1}{^{19}C_1}\times\frac{^{10}C_1}{^{18}C_1}=\frac{10\times 9\times 10}{20\times 19\times 18}=\frac{5}{38}$$

Hence, the correct answer is (5).

22. Number of one-one mapping $=2n!$

and total number of mapping $=(2n)^{2n}$

Hence the probability $=\frac{2n!}{(2n)^{2n}}=1-\frac{29}{32}=\frac{3}{32}=\frac{4!}{4^4}$

$\Rightarrow 2n=4\Rightarrow n=2$

Hence, the correct answer is (2).

23. Let the second number be x (where $1<x<6$)
Then first number can be chosen in $(x-1)$ ways and third in $(6-x)$ ways

∴ Favourable cases $=\sum_{x=2}^{5}(x-1)(6-x)=20$

Probability $=\frac{20}{6^3}=\frac{5}{54}$

Hence, the correct answer is (5).

24. $x^2+2ax+10-3a>0\ \forall x\in R'$

$\Rightarrow D<0\ \therefore 4a^2-4(10-3a)<0$

or $a^2+3a-10<0\Rightarrow a\in(-5,2)$

$\Rightarrow\ \because a\in I$

$n(a)=6$ also $n(s)=21$

∴ Reqd. prob $=\frac{6}{21}=\frac{2}{7}$

Hence, the correct answer is (4).

25. If $m=3$, $n=2$
Coin is tossed 5 times, then 3 consecutive heads can come is 5 cases
{HHHTH, HHHTT, THHHT, TTHHH, HTHHH}

Probability $=\frac{5}{2^5}=\frac{2+3}{2^{3+2}}$

Hence, the correct answer is (3).

26. There are two cases
Case 1: –3 1 1 (one pocket has 3 rings)

Number of ways $={}^{3}C_1\times{}^{5}C_3\times{}^{2}C_1=60$

Case 2: –2 2 1 (one pocket has 1 ring, 2 pockets have 2 each)

Number of ways $={}^{5}C_2\times{}^{3}C_2\times{}^{3}C_1=90$

Number of total ways = 60 + 90 = 150

Number of ways without restriction $=3^5=243$

Required probability $=\frac{150}{243}=\frac{1}{P}\Rightarrow[P]=1$

Hence, the correct answer is (1).

27. $n(s)={}^{3n}C_2$

Let $A_0=\{3,6,9..........3n\}$

$A_1=\{1,4,7..........3n-2\}$

$A_2=\{5,8,11..........3n-1\}$

(x^2-y^2) divisible by 3. If both x, y should come from A_0 or A_1 or A_2 or one is from A_1 and other from A_2

$n(E)=3\ {}^{n}C_2+{}^{n}C_1\times{}^{n}C_1=\frac{n}{2}(5n-3)$

$P(E)=\frac{5n-3}{3(3n-1)},\ p=5,\ q=-3,\ r=3$

Hence, the correct answer is (5).

28. $P(A)=\frac{^{9}C_2}{^{14}C_3}=\frac{9}{91},\ P(B)=\frac{^{10}C_2}{^{14}C_3}=\frac{45}{364}$

$P(A\cap B)=\frac{^{5}C_1}{^{14}C_3}=\frac{5}{364}$

$P(A\cup B)=\frac{19}{91}$

Hence, the correct answer is (8).

29. $P(G_k)\propto k^2\Rightarrow P(G_k)=\lambda k^2$

$\sum_{k=0}^{n}P(G_k)=1$ (as these are mutually exclusive and exhaustive events)

$$\Rightarrow\lambda\sum_{k=0}^{n}k^2=1\Rightarrow\lambda=\frac{6}{n(n+1)(2n+1)}$$

$$P(A)=\sum_{k=0}^{n}P(G_k)\quad P(A/G_k)=\sum_{k=0}^{n}\lambda k^2.\frac{k}{n}$$

$$=\frac{\lambda}{n}.\frac{n^2(n+1)^2}{4}=\frac{3(n+1)}{2(2n+1)}.$$

Take $n=15$.

Hence, the correct answer is (7).

30. A number has exactly 3 factors if the number is squares of a prime number. Squares of 11, 13, 17, 19, 23, 29, 31 are 3–digit numbers

Required probability $=\frac{7}{900}$

Hence, the correct answer is (7).

31. Let R be the event that a red face appears in each of the first n throws.

E_1 : Die A is used when head has already fallen

E_2 : Die B is used when tail has already fallen.

$$P(R/E_1)=\left(\frac{2}{3}\right)^n\text{ and }P\left(\frac{R}{E_2}\right)=\left(\frac{1}{3}\right)^n$$

As per the given condition

$$\frac{\mathrm{P(E_1).P(R/E_1)}}{\mathrm{P(E_1).P(R/E_1)+P(E_2).P(R/E_2)}}=\frac{32}{33}$$

$$\Rightarrow \frac{1/2.(2/3)^n}{\frac{1}{2}.\left(\frac{2}{3}\right)^n+\frac{1}{2}\left(\frac{1}{3}\right)^n}=\frac{32}{33}$$

$$\Rightarrow \frac{2^n}{2^n+1}=\frac{32}{33}\Rightarrow n=5.$$

Hence, the correct answer is (5).

32. Let A be the event of choosing 5-digit number and E be the event the number is a palindrome (abcba).

$$\therefore n(\mathrm{A})=9\left(10^4\right)=90000$$

and $n(\mathrm{A}\cap\mathrm{B})=9\times10^2=900$

$$\therefore \mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right)=\frac{n(\mathrm{E}\cap\mathrm{A})}{n(\mathrm{A})}=\frac{1}{100}=\mathrm{P}$$

Hence, the correct answer is (3).

33. Let the number of socks be $2n$ forming n-pairs of socks

$\therefore n(s)={}^{2n}\mathrm{C}_2$ $n(\mathrm{A})={}^{n}\mathrm{C}_1$ (no. of ways of choosing exactly one pair out of n pairs)

$$\therefore \mathrm{P}(\mathrm{A})=\frac{{}^{n}\mathrm{C}_1}{{}^{2n}\mathrm{C}_2}=\frac{1}{15}\quad\Rightarrow 2n-1=15\Rightarrow n=8\ .$$

Hence, the correct answer is (8).

34. $$\mathrm{P}\left(\frac{\mathrm{B}}{\overline{\mathrm{A}}}\right)=\frac{\mathrm{P}\left(\mathrm{B}\cap\overline{\mathrm{A}}\right)}{\mathrm{P}\left(\overline{\mathrm{A}}\right)}=\frac{\mathrm{P}(\mathrm{A}\cup\mathrm{B})-\mathrm{P}(\mathrm{A})}{1-\mathrm{P}(\mathrm{A})}$$

$$=\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$$

Hence, the correct answer is (1).

35. Let the numbers are x and $2n-x$

their product $\geq \frac{3}{4}$. greatest product

$$x(2n-x)\geq\frac{3}{4}.n^2$$

$$4x^2-8nx+3n^2\leq0$$

$$x\in\left[\frac{n}{2},\frac{3n}{2}\right]$$

Required probability $=\dfrac{1}{2}$

Hence, the correct answer is (1).

36. Total number of ways in which papers of 4 students, can be checked by seven teachers $=7^4$

Now, choosing two teachers out of 7 is ${}^{7}\mathrm{C}_2=21$

The number of ways $=(21)(2^4-2)$

Required probability $=\dfrac{(21)(14)}{7^4}=\dfrac{6}{49}=A$

Hence, the correct answer is (6).

37. We have ${}^{8}\mathrm{C}_3\,p^3(1-p)^5=\dfrac{1}{25}{}^{8}\mathrm{C}_5\,p^5(1-p)^3$ or $1-p=\dfrac{(1)(p)}{5}$

So, we find $p=\dfrac{5}{6}\Rightarrow m+n-5=6$

Hence, the correct answer is (6).

38. $x+\dfrac{1}{x}+5\geq7$ $\quad y=\dfrac{147}{7}=21\qquad\Rightarrow 0<y\leq21$

Prime number in range of y {2, 3, 5, 7, 11, 13, 17, 19}

$$\therefore \text{Required probability }=\frac{{}^{8}\mathrm{C}_2\ 2!}{{}^{21}\mathrm{C}_2\ 2!}=\frac{2}{15}$$

Hence, the correct answer is (2).

39. Function is defined from set S to itself and $n(\mathrm{S})=n$

So total one-one function $=n!$

Total number of function $=n^n$

$$P=\frac{n!}{n^n}=\frac{3}{32}=\frac{8\times3}{8\times32}=\frac{24}{256}=\frac{4!}{4^4}\Rightarrow n=4$$

Hence, the correct answer is (4).

40. Let A, B, C denote the events of selecting A, B, C respectively and E be the event of obtaining 2 heads and 1 tail in 3 tosses.

$$\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right)={}^{3}\mathrm{C}_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)=\frac{3}{8}$$

$$\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{B}}\right)={}^{3}\mathrm{C}_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)=\frac{4}{9}$$

$$\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{C}}\right)={}^{3}\mathrm{C}_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)=\frac{2}{9}$$

Now, $\mathrm{P}\left(\dfrac{\mathrm{A}}{\mathrm{E}}\right)=\dfrac{\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right).\mathrm{P}(\mathrm{A})}{\sum\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right).\mathrm{P}(\mathrm{A})}=\dfrac{\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right)}{\sum\mathrm{P}\left(\frac{\mathrm{E}}{\mathrm{A}}\right)}$

Since, $\mathrm{P}(\mathrm{A})=\mathrm{P}(\mathrm{B})=\mathrm{P}(\mathrm{C})=\dfrac{1}{3}$

$$P\left(\frac{A}{E}\right)=\frac{\frac{3}{8}}{\frac{3}{8}+\frac{4}{9}+\frac{2}{9}}=\frac{9}{25}$$

Hence, the correct answer is (9).

41. Given that 2 drawn cards are found spades. So missing spades card from 11 cards.

$$\mathrm{P}=\frac{{}^{11}\mathrm{C}_n}{{}^{50}\mathrm{C}_n}=\frac{11}{245}=\frac{11}{5\times49}=\frac{11\times10}{50\times49}=\frac{{}^{11}\mathrm{C}_2}{{}^{50}\mathrm{C}_2}$$

$$\Rightarrow n=2.$$

Hence, the correct answer is (2).

42. Let 'n' be the least number of bombs required and 'X' the number of bombs that hit the bridge. Then 'X' follows a binomial distribution with parameters 'n' and $p=1/2$. Now

$$P(X \ge 2) > 0.9$$

$$\Rightarrow 1 - P(X < 2) > 0.9 \quad \Rightarrow P(X = 0) + P(X = 1) < 0.1$$

$$\Rightarrow {}^{n}C_0\left(\frac{1}{2}\right)^{n} + {}^{n}C_1\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right) < \frac{1}{10} \qquad \Rightarrow \frac{n+1}{2^n} < \frac{1}{10}$$

By trial and error, we get $n \ge 7$. Thus the least value of 'n' is 7

Hence, the correct answer is (7).

43. For $2n+1$ tickets:

Common difference of the selected number may be 1, 2, 3, … n

$$P = \frac{1+3+5+....+2n-1}{{}^{2n+1}C_3}$$

$$C = \frac{3n}{4n^2-1} = \frac{10}{133} \text{ where } n = 10$$

$$\left[\frac{1}{P} - 7\right] = 6$$

Hence, the correct answer is (6).

44. Probability of hitting the target in one fire $p = \frac{1}{5}$

Then probability of hitting the target at least once in n fires

= 1 – (probability of not hitting the target)

$$= 1 - \left(\frac{4}{5}\right)^n > \frac{3}{4}$$

$$\Rightarrow \left(\frac{4}{5}\right)^n < \frac{1}{4} \text{ as } \left(\frac{4}{5}\right)^6 > \frac{1}{4} \text{ and } \left(\frac{4}{5}\right)^7 < \frac{1}{4}$$

The least value of $n = 7$

Hence, the correct answer is (7).

45. $n(S) = {}^{100}C_3$

since AM of 3 numbers is = 25

Then their sum is 75

$$x_1 + x_2 + x_3 = 75,\ x_1 \ge 1,\ x_2 \ge 1,\ x_3 \ge 1$$

No. of +ve integral solutions ${}^{74}C_2 = 2701$

Required probability $= \frac{2701}{{}^{100}C_3}$

Hence, the correct answer is (1).

46. E_k : Box having k white balls, $1 \le i \le 9 \Rightarrow P(E_k) = \frac{1}{9}$

A : Selected ball is white $\Rightarrow P\left(\frac{A}{E_i}\right) = \frac{k}{9}, 1 \le i \le 9$

$$P\left({}^{E_k}\!/\!_{A}\right) = \frac{\frac{1}{9} \times \frac{k}{9}}{\frac{1}{9} \times \frac{1}{9} + ... + \frac{1}{9} \times \frac{9}{9}} = \frac{1}{9}$$

$$\Rightarrow \frac{k}{\left(\frac{9 \times 10}{2}\right)} = \frac{1}{9} \Rightarrow \frac{k}{45} = \frac{1}{9} \Rightarrow k = 5$$

Hence, the correct answer is (5).

47. Total number of cases = coefficient of x^n in $(1 + x + x^2 + ... + x^n)^3 = {}^{n+2}C_2$

So required probability $= \frac{3}{{}^{n+2}C_2} = \frac{6}{(n+1)(n+2)} \ge \frac{1}{6}$

$$\Rightarrow n \ge \frac{-3 + \sqrt{145}}{2}$$

$n_{\min} = 5$

So minimum number of red balls = 5.

Hence, the correct answer is (5).

48. Common letters in the given words are A, R, T, I, L, E

Probability of selecting each of there letters $= \frac{1}{7} \cdot \frac{1}{8}$

Required probability $= 6\left(\frac{1}{7} \cdot \frac{1}{8}\right) = \frac{3}{28}$

Hence, the correct answer is (3).

49. In the last throw: Out of 4, 5, 6 the favourable out come is 5,

Probability of getting 5 $= \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} + \left(\frac{1}{2}\right)^2 \frac{1}{6} + \ldots \infty$

$$= \frac{\frac{1}{6}}{1 - \frac{1}{2}} = \frac{1}{3}$$

$\therefore \quad a + b = 4$

Hence, the correct answer is (4).

50. If sum of four integers is odd then atleast one integer must be even.

This is a certain event; P = 1

Hence, the correct answer is (1).

51. i) $a > 0$ ii) $b^2 - 4ac < O$

$$\Rightarrow 4(a+4)^2 - 4(1)(64 - 5a) < 0$$

$$\Rightarrow a^2 + 16 + 8a - 64 + 5a < 0$$

$$\Rightarrow a^2 + 13a - 48 < 0$$

$$\Rightarrow a^2 + 16a - 3a - 48 < 0$$

$$\Rightarrow a(a+16) - 3(a+16) < 0$$

$$\Rightarrow (a - (-16))(a - 3) < 0$$

$$\Rightarrow -16 < a < 3$$

$a = -5, -4, -3, -2, -1, 0, 1, 2,$

$n(A) = 8 \quad n(S) = 36$

$P(A) = \frac{8}{36} = \frac{2}{9} \quad \therefore\ k = 9$

Hence, the correct answer is (9).

52.

Given letter: 1	SSC	TTT	AIIC
Given letter: 2	SSS	TT	AANI

$$P(A) = \frac{3}{10} \cdot \frac{3}{9} + \frac{3}{10} \cdot \frac{2}{9} + \frac{1}{10} \cdot \frac{2}{9} + \frac{2}{10} \cdot \frac{1}{9} = \frac{19}{90}$$

$k + 1 - 100 = 9$

Hence, the correct answer is (9).

53. P(BC) : Probability of choosing biased coin

$$\Rightarrow \quad \text{P(BC)} = \frac{1}{n+1}$$

P(F) : Probability of choosing unbiased coin

$$\Rightarrow \quad \text{P(F)} = \frac{n}{n+1}$$

$$\text{P}(\text{H}) = \text{P}(\text{BC})\text{P}\left(\frac{\text{H}}{\text{BC}}\right) + \text{P}(\text{F})\text{P}\left(\frac{\text{H}}{\text{F}}\right)$$

$$= \frac{1}{n+1}.1 + \frac{n}{n+1}, \frac{1}{2} = \frac{7}{12}$$

$$\frac{n+2}{2(n+1)} = \frac{7}{12} \Rightarrow 7n+7 = 6n+12 \Rightarrow n = 5$$

Hence, the correct answer is (5).

54. The ordered pairs (x, y) which give sum of squares is even correspond to

(1) both x and $y \in (1,3,5)$ (2) both x and $y \in (2, 4, 6)$

Number of favourable ordered pairs = 18

$$\therefore \text{P} = \frac{18}{6 \times 6} = \frac{1}{2}$$

Hence, the correct answer is (1).

55. Let A be the event that max. number on the two chosen tickets is not more than 10, and B is the event that min. number on them is 5. We have to find P(B/A).

We know that $\text{P(B / A)} = \dfrac{\text{P}(\text{B} \cap \text{A})}{\text{P}(\text{A})}$

Total ways to select two tickets out of 100 $= {}^{100}\text{C}_2$

Number of ways favourable to A

= number of ways of selecting any 2 numbers from 1 to 10 $= {}^{10}\text{C}_2 = 45$

$\text{A} \cap \text{B}$ contains one number 5 and other greater than 5 and $\leq$ 10

so ways favourable to $\text{A} \cap \text{B} = {}^5\text{C}_1 = 5$ Therefore, $\text{P(A)} = \dfrac{45}{{}^{100}\text{C}_2}$ and

$$\text{P}(\text{B} \cap \text{A}) = \frac{5}{{}^{100}\text{C}_2} \text{ Thus, } \text{P}(\text{B/A}) = \frac{5/{}^{100}\text{C}_2}{45/{}^{100}\text{C}_2} = \frac{5}{45} = \frac{1}{9}$$

Hence, the correct answer is (9).

56. WW or WB or BB

+W or + W or +W

$$\text{P}(\text{W}) = \frac{1}{3}.1 + \frac{1}{3}.\frac{2}{3} + \frac{1}{3}.\frac{1}{3}$$

$$= \frac{3}{9} + \frac{2}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$$

Hence, the correct answer is (4).

57. $$\frac{a^3}{3} - a^2b + 3ac = 0$$

$$a > 0 \Rightarrow \frac{a^2}{3} - ab + 3c = 0$$

$-ab + 3c$ is integer $\Rightarrow a = 3k \Rightarrow a = 3, 6, 9$

$$\text{R} = \{a \,/ \int_0^a (x^2 - 2bx + 3c)dx = 0\} \text{ then } n(\text{R}) = 3$$

Hence, the correct answer is (3).

58. $$\text{P}(5) = \frac{4}{36} = {}^1\!/_9, \text{P}(7) = \frac{6}{36} = {}^1\!/_6,$$

$$\text{P}(\bar{5} \cap \bar{7}) = 1 - \left(\frac{4}{36} + \frac{6}{36}\right)$$

$$\text{P}(\bar{5} \cap \bar{7}) = \frac{13}{18}$$

Required Probability =

$$\text{P}(5) + \text{P}(\bar{5} \cap \bar{7})\text{P}(5) + \text{P}(\bar{5} \cap \bar{7})\text{P}(\bar{5} \cap \bar{7})\text{P}(5) + ...$$

$$= \frac{\text{P}(5)}{1 - \text{P}(\bar{5} \cap \bar{7})} = \frac{2}{5}$$

Hence, the correct answer is (7).

59. $${}^n\text{C}_4 \frac{1}{2^n} + {}^n\text{C}_6 \frac{1}{2^n} = 2.\ {}^n\text{C}_5 \frac{1}{2^n}$$

$$\Rightarrow {}^n\text{C}_4 + {}^n\text{C}_6 = 2.\ {}^n\text{C}_5 \Rightarrow n = 7, 14; \text{ sum} = 21$$

Hence, the correct answer is (21).

60. ${}^r\text{C}_2 = 5 \cdot {}^b\text{C}_2$ and $br = 6 \cdot {}^b\text{C}_2$

$r(r-1) = 5b(b-1)$ and $r = 6\dfrac{(b-1)}{2}$

$$\Rightarrow 3(b-1)[3b-4] = 5b(b-1)$$

$$\Rightarrow 4b = 12 \Rightarrow b = 3 \Rightarrow r = 3 \times 2 = 6$$

$$\Rightarrow r - b = 3$$

Hence, the correct answer is (3).

61. $x + y + z = 20$

Number of positive solution ${}^{19}C_2$

They form a triangle if $0 < x, y, z \leq 9$

Number of solutions

= coefficient of x^{20} in$(x + x^2 + \ldots x^9)^3$

= coefficient of x^{17} in $\left(1 + x + \ldots.x^8\right)^3$

= coefficient of x^{17} in $\left(1 - x^9\right)^3 (1-x)^{-3}$

$= 1 \times {}^{19}C_2 - 3 \times {}^{10}C_2$

$$\text{Required probability} = \frac{{}^{19}C_2 - 3 \times {}^{10}C_2}{{}^{19}C_2}$$

$$= \frac{19 \times 18 - 3 \times 10 \times 9}{19 \times 18}$$

$$\text{P} = \frac{4}{19} \Rightarrow 19\text{P} = 4.$$

Hence, the correct answer is (4).

62. Factors of $x^2 - x + 1$ are $-\omega, -\omega^2$

$x^{n+1} - x^n + 1$ is divisible by $x^2 - x + 1$

$\Rightarrow (-\omega)^{n+1} - (-\omega)^n + 1 = 0$

$\Rightarrow (-1)^{n+1}\left(\omega^{n+1} + \omega^n\right) + 1 = 0$

$\Rightarrow (-1)^{n+1}\omega^n\left(-\omega^2\right) + 1 = 0$

$n + 2 = 3k$, and n is odd

$n = 6m + 1$

$P = \frac{1}{6} \Rightarrow 12P = 2$

Hence, the correct answer is (2).

63. Let drawer contains n socks, out of n, m are red

$\frac{1}{2} = \frac{{}^mC_2}{{}^nC_2}$

$\Rightarrow \frac{m(m-1)}{n(n-1)} = \frac{1}{2}$

$\frac{n(n-1)}{m(m-1)} = 2$

$\left(\frac{n}{m-1}\right)\left(\frac{n-1}{m}\right) = 2$

$\Rightarrow \frac{n}{m-1} = 2, \frac{n-1}{m} = 1$

$n = 2m - 2 = m + 1$

$m = 3, n = 4.$

Hence, the correct answer is (4).

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1. $P(H) = \frac{1}{2}$ and $P\left(\bar{H}\right) = \frac{1}{2}$

Let total 'n' bomb are required to destroy the target

$1 - {}^nC_n\left(\frac{1}{2}\right)^n - {}^nC_1\left(\frac{1}{2}\right)^n \ge \frac{99}{100} \quad \Rightarrow 1 - \frac{1}{2^n} - \frac{n}{2^n} \ge \frac{99}{100}$

$\Rightarrow \frac{1}{100} \ge \frac{n+1}{2^n} \quad \Rightarrow 2^n \ge 100(n+1)$

which is satisfied by every $n \ge 11$

minimum value of $n = 11$

Hence, the correct answer is (11.00).

2. 4 dice are independently thrown. Each die has probability to show 3 or 5 is

$p = \frac{2}{6} = \frac{1}{3}$ and $q = 1 - \frac{1}{3} = \frac{2}{3}$ (not showing 3 or 5)

$\therefore$ In one throw of 4 dices probability of showing 3 or 5 by at least twice dices is

$= p^4 + {}^4C_3\,qp^3 + {}^4C_2\,q^2p^2$

$= \left(\frac{1}{3}\right)^4 + 4.\frac{2}{3}\left(\frac{1}{3}\right)^3 + 6.\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{33}{81}$

the expected number of times, at least two dice show up a three or a five $= \frac{33}{81} \times 27 = 11$

Hence, the correct answer is (11).

3. We have, 1 – (probability of all shots result in failure) $> \frac{1}{4}$

$\Rightarrow \quad 1 - \left(\frac{9}{10}\right)^n > \frac{1}{4} \Rightarrow \frac{3}{4} > \left(\frac{9}{10}\right)^n \Rightarrow n \ge 3$

Hence, the correct answer is (3).

4. Probability that at most 2 machines are out of service

$= {}^5C_0\left(\frac{3}{4}\right)^5 + {}^5C_1\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + {}^5C_2\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

$= \left(\frac{3}{4}\right)^3\frac{17}{8}$ *we get* $\Rightarrow k = \frac{17}{8}$

Hence, the correct answer is (C).

5. (A) $p(A'/B') = P(A') = \frac{2}{3}$

(B) $P(A/B') = P(A) = \frac{1}{3}$

(C) $P(A/B) = P(A) = \frac{1}{3}$

(D) $P(A/(A \cup B)) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)}$

$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} - \frac{1}{18}} = \frac{3}{4}$

Hence, the correct answer is (B).

6. $P(A) + P(B) - 2P(A \cap B) = \frac{2}{5}$ and $P(A) + P(B) - P(A \cap B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{10}$

Hence, the correct answer is (A).

7. A: Event when card A is drawn

B: Event when card B is drawn.

$P(A) = P(B) = \frac{1}{2}$

Required probability = P(AA or (AB)A or (BA)A or (ABB)A or (BAB)A or (BBA)A)

$= \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \times 2 + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \times 3$

$= \frac{1}{4} + \frac{1}{4} + \frac{3}{16} = \frac{11}{16}$

Hence, the correct answer is (D).

8. $\sum P(X) = 1 \Rightarrow K^2 + 2K + K + 2K + 5K^2 = 1$

$\Rightarrow 6K^2 + 5K - 1 = 0 \Rightarrow (6K - 1)(K + 1) = 0$

$\Rightarrow \quad K = -1 \text{ (rejected)} \Rightarrow K = \dfrac{1}{6}$

$P(X > 2) = K + 2K + 5K^2 = \dfrac{23}{36}$

Hence, the correct answer is (A).

9. Given E_1, E_2, E_3 are pairwise independent events

$P(E_1 \cap E_2) = P(E_1).P(E_2)$, $P(E_2 \cap E_3) = P(E_2).P(E_3)$, $P(E_3 \cap E_1)$

$= P(P(E_3).P(E_1)$ & $P(E_1 \cap E_2 \cap E_3) = 0$

$$\text{Now } P\left(\frac{\bar{E}_2 \cap \bar{E}_3}{E_1}\right) = \frac{P\left[E_1 \cap \left(\bar{E}_2 \cap \bar{E}_3\right)\right]}{P(E_1)}$$

$$= \frac{P(E_1) - [P(E_1 \cap E_2) + P(E_1 \cap E_3) - P(E_1 \cap E_2 \cap E_3)]}{P(E_1)}$$

$$= \frac{P(E_1) - P(E_1).P(E_2) - P(E_1)P(E_3) - 0}{P(E_1)} = 1 - P(E_2) - P(E_3)$$

$$= [1 - P(E_3)] - P(E_2) = P\left(E_3^C\right) - P(E_2)$$

Hence, the correct answer is (A).

10. Let B_1 be the event where Box-I is selected and $B_2 \rightarrow$ where box II selected

$$P(B_1) = P(B_2) = \frac{1}{2}$$

Let E be the event where selected card is non prime.

For B_1: Prime numbers:{2, 3, 5, 7, 11, 13, 17, 19, 23, 29}

For B_2: Prime numbers:{31, 37, 41, 43, 47}

$$P(E) = P(B_1)P\left(\frac{E}{B_1}\right) + P(B_2)P\left(\frac{E}{B_2}\right) = \frac{1}{2} \times \frac{20}{30} + \frac{1}{2} \times \frac{15}{20}$$

$$\text{Required probability } P\left(\frac{B_1}{E}\right) = \frac{\frac{1}{2} \times \frac{20}{30}}{\frac{1}{2} \times \frac{20}{30} + \frac{1}{2} \times \frac{15}{20}} = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{3}{4}}$$

$$= \frac{8}{17}$$

Hence, the correct answer is (A).

11. A: Sum obtained is a multiple of 4.

A = {(1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)}

B = Score of 4 has appeared at least once.

B = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)}

$$\text{Required probability} = P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{1/36}{9/36} = \frac{1}{9}$$

Hence, the correct answer is (B).

12. Total is 6 = (1,5)(5,1)(2,4)(4,2)(3,3)

Total is 7 = (1,6)(6,1)(2,5)(5,2)(3,4)(4,3)

$$P(6) = \frac{5}{36},\ P(7) = \frac{6}{36} = \frac{1}{6}$$

P(A) = W + FFW + FFFFW + ….

$$= \frac{5}{36} + \frac{31}{36}\frac{5}{6}\frac{5}{36} + \left(\frac{31}{36}\frac{5}{6}\right)^2 \frac{5}{36} + \ldots..$$

$$= \frac{\frac{5}{36}}{1 - \frac{31}{36}.\frac{5}{6}} = \frac{30}{216 - 155} = \frac{30}{61}$$

Hence, the correct answer is (D).

13. First Case: Choose two non-zero digits ${}^9C_2$

Now, number of 5-digit numbers containing both digits = $2^5 - 2$

Second Case: Choose one non-zero & one zero as digit ${}^9C_1$.

Number of 5-digit numbers containing one non zero and one zero both = $(2^4 - 1)$

$$\text{Required probability} = \frac{\left({}^9C_2 \times \left(2^5 - 2\right) + {}^9C_1 \times \left(2^4 - 1\right)\right)}{9 \times 10^4}$$

$$= \frac{36 \times (32 - 2) + 9 \times (16 - 1)}{9 \times 10^4} = \frac{4 \times 30 + 15}{10^4} = \frac{135}{10^4}$$

Hence, the correct answer is (C).

14.

K	0	1	2	3	4	5
P(k)	$\frac{1}{32}$	$\frac{12}{32}$	$\frac{11}{32}$	$\frac{5}{32}$	$\frac{2}{32}$	$\frac{1}{32}$

$$\text{Expected value} = \sum XP(k) = -\frac{1}{32} - \frac{12}{32} - \frac{11}{32} + \frac{15}{32} + \frac{8}{32} + \frac{5}{32}$$

$$= \frac{28 - 24}{32} = \frac{4}{32} = \frac{1}{8}$$

Hence, the correct answer is (C).

15. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.8 = 0.6 + 0.4 - P(A \cap B)$; $\quad P(A \cap B) = 0.2$

$P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$

$\alpha = 1.5 - (0.2 + 0.3 + \beta) + 0.2$

$\alpha = 1.2 - \beta \in [0.85, 0.95]$

(where $\alpha \in [0.85, 0.95]$) $\beta \in [0.25, 0.35]$

Hence, the correct answer is (C).

16. Out of 11 consecutive natural numbers either 6 even and 5 odd numbers or 5 even and 6 odd numbers

when 3 numbers are selected at random then total cases = ${}^{11}C_3$

Since these 3 numbers are in A.P. let no's are a,b,c

$2b \Rightarrow$ even number

$$a + c \Rightarrow \begin{pmatrix} \text{even} + \text{even} \\ \text{odd} + \text{odd} \end{pmatrix}$$

so favourable cases = ${}^6C_2 + {}^5C_2 = 15 + 10 = 25$

$$P(\text{3 numbers are in A.P.}) = \frac{25}{{}^{11}C_3} = \frac{25}{165} = \frac{5}{33}$$

Hence, the correct answer is (C).

17. Total ways = $4^{10} = n$

Number of ways placing exactly 2 and 3 balls in two of these boxes $m = {}^4C_2 \times \dfrac{\lfloor\underline{5}}{\lfloor\underline{2}\,\lfloor\underline{3}} \times \lfloor\underline{2} \times {}^{10}C_5 \times 2^5$

Required probability $= \frac{m}{n} = \frac{945}{2^{10}}$

Hence, the correct answer is (D).

18.

X	0	50	100	–150
P(x)	$\frac{4}{27}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{8}{27}$

$$\sum P(x) = \frac{4}{27} + \frac{2}{9} + \frac{1}{3} + \frac{8}{27} = \frac{4+6+9+8}{27}$$

$$\sum P(x) = 1$$

Req. E(x) $= \sum_{i=1}^{n} x_i P(x_i) = 0 + \frac{50\times 2}{9} + \frac{100}{3} - \frac{150\times 8}{27} = 0$

Hence, the correct answer is (B).

19. Given $np = 8$; $npq = 4$

$q = \frac{1}{2} \Rightarrow p = \frac{1}{2} \Rightarrow n = 16$

$P(x \le 2) = P(x = 0) + P(x = 1) + (x = 2)$

$$= {}^{16}C_0\left(\frac{1}{2}\right)^{16}\left(\frac{1}{2}\right)^{0} + {}^{16}C_1\left(\frac{1}{2}\right)^{15}\left(\frac{1}{2}\right)^{1} + {}^{16}C_2\left(\frac{1}{2}\right)^{14}\left(\frac{1}{2}\right)^{2}$$

$= \frac{137}{2^{16}}$ $\quad\therefore\quad k = 137$

Hence, the correct answer is (C).

20.

win	+15	+12	–150
Prob.	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{26}{36}$

Probability of doublet $= \frac{6}{36}$; Probability of sum of 9 $= \frac{4}{36}$

Probability for any other outcome $= \frac{26}{36}$

Expected gain/loss $= 15 \times \frac{6}{36} + 12 \times \frac{4}{36} - 6 \times \frac{26}{36} = -\frac{1}{2}$

So, $\frac{1}{2}$ loss

Hence, the correct answer is (C).

21. Favourable cases are 7, (1, 6),(2, 5),(3, 4),(1, 2, 4)

$\because p = \frac{5}{2^{20}}$

Hence, the correct answer is (A).

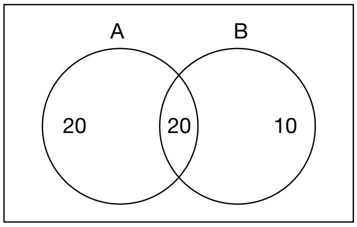
22. Sum is even = Both are even + Both are odd

$= {}^5C_2 + {}^6C_2 = 10 + 15 = 25$

Required conditional probability $= \frac{10}{25} = \frac{2}{5}$

Hence, the correct answer is (A).

23.



A opted NCC

B opted NSS

P(neither A nor B) $= \frac{10}{60} = \frac{1}{6}$

Hence, the correct answer is (B).

24. $\bar{4}\,\bar{4}\,\bar{4}\,4\,4 \rightarrow \left(\frac{5}{6}\right)^3 \frac{1}{6^2}$

$4\,\bar{4}\,\bar{4}\,4\,4 \rightarrow \left(\frac{5}{6}\right)^2 \frac{1}{6^3}$

$\bar{4}\,4\,\bar{4}\,4\,4 \rightarrow \left(\frac{5}{6}\right)^2 \frac{1}{6^3}$

Required probability $= \frac{5^3 + 5^2 + 5^2}{6^5} = \frac{175}{6^5}$

Hence, the correct answer is (D).

25. $P(X = 1) + P(X = 2)$

$= \frac{4}{52} \times \frac{48}{52} \times 2 + \frac{4}{52} \times \frac{4}{52}$

$= 2 \times \frac{12}{(13)^2} + \left(\frac{1}{13}\right)^2 = \frac{25}{169}$

Hence, the correct answer is (A).

26. 5 Red and 2 green balls

P(one red ball) $= \frac{5}{7}$

P(one green ball) $= \frac{2}{7}$

Case I:

If drawn ball is green than a red ball is added

$\begin{pmatrix} \text{6 Red} \\ \text{1 Green} \end{pmatrix}$ P(red ball) $= \frac{6}{7}$

Case II:

If drawn ball is red then a green ball is added

$\begin{pmatrix} \text{4 Red} \\ \text{3 Green} \end{pmatrix}$ P(red ball) $= \frac{4}{7}$

P(2nd red ball) $= \frac{5}{7} \times \frac{4}{7} + \frac{2}{7} \times \frac{6}{7} = \frac{32}{49}$

Hence, the correct answer is (C).

27. Sum of number of dice is 7

(1, 6), (2, 5), (3, 4)

(6, 1), (5, 2), (4, 3)

Sum of number of dice is 8

(2, 6), (3, 5), (4, 4), (6, 2), (5, 3)

$$P(\overset{\text{sum}}{7} \text{ or } 8) = \frac{11}{36}$$

(Head + dice sum 7 or 8) or (tail 1 + 7 or 8 appear)

$$P(A) = \frac{1}{2}\left(\frac{11}{36}\right) + \frac{1}{2}\left(\frac{2}{9}\right)$$

$$= \frac{11}{72} + \frac{1}{9} = \frac{19}{72}$$

Hence, the correct answer is (D).

28. $1 - \left(\frac{2}{3}\right)^n > \frac{5}{6}$

$\Rightarrow \frac{1}{6} > \left(\frac{2}{3}\right)^n$

$\Rightarrow 3^n > 2^n \times 6$

$\Rightarrow 3^{n-1} > 2^{n+1}$

$\Rightarrow n = 5$ (minimum)

Hence, the correct answer is (C).

29. $A \subset B$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

P(B) lies between (0, 1]

So that $P(A/B) \geq P(A)$

Hence, the correct answer is (B).

30. P (at least one head) = 1 – [(no one head) $\geq \frac{90}{100}$

$$\Rightarrow \quad 1 - \left(\frac{1}{2}\right)^n \geq \frac{90}{100}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n \leq \frac{10}{100} \Rightarrow 2^n \geq 10$$

Least value of n is 4

Hence, the correct answer is (A).

31. Let four persons are A, B, C, D.

Probability of Hitting target = 1 – (None of four person hit the target)

$$= 1 - P(\bar{A})\cdot P(\bar{B})\cdot P(\bar{C})\cdot P(\bar{D})$$

$$= 1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{7}{8} = \frac{25}{32}$$

Hence, the correct answer is (B).

32. $P(\text{Boy}) = P(\text{Girl}) = \frac{1}{2}$

$$\text{Required probability} = \frac{\text{All four girls}}{\text{At least two girls}}$$

$$= \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)^4 + {}^4C_3\left(\frac{1}{2}\right)^4 + {}^4C_2\left(\frac{1}{2}\right)^4} = \frac{1}{11}$$

Hence, the correct answer is (C).

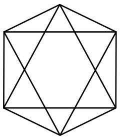
33. Required probability = 1 – P(no Head)

$$\Rightarrow \quad 1 - \left(\frac{1}{2}\right)^n > \frac{99}{100} \quad \Rightarrow \quad \left(\frac{1}{2}\right)^n < \frac{1}{100} \quad \Rightarrow \quad n = 7$$

Hence, the correct answer is (B).

34. Number of total triangles = ${}^6C_3$

Equilateral $\Delta = 2$



$$\text{Prob.} = \frac{2}{{}^6C_3} = \frac{1}{10}$$

Hence, the correct answer is (A).

35. Total problems = 50

$P(\text{Solving}) = \frac{4}{5}$; $P(\text{Not solving}) = \frac{1}{5}$

P(unable to solve less than two problems)

= P(not solving one problem) + P(not solving zero problem)

$$= {}^{50}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{50} + {}^{50}C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^{49}$$

$$= \frac{4^{50}}{5^{50}} + 50\cdot\frac{4^{49}}{5.5^{49}} = \frac{54}{5}\cdot\left(\frac{4}{5}\right)^{49}$$

Hence, the correct answer is (D).

36. Let R_t be the event of drawing red ball in 1st draw and B_t be the event of drawing black ball in 1st draw. Now, in the given bag there are 4 red and 6 black balls

$$\therefore \; P(R_1) = \frac{4}{10} \text{ and } P(B_1) = \frac{6}{10}$$

$$\text{And, } P\left(\frac{R_2}{R_1}\right) = \frac{6}{12} \text{ and } P\left(\frac{R_2}{B_1}\right) = \frac{4}{12}$$

Now, required probability

$$= P(R_1) \times P\left(\frac{R_2}{R_1}\right) + P(B_1) \times P\left(\frac{R_2}{B_1}\right)$$

$$= \left(\frac{4}{10} \times \frac{6}{12}\right) + \left(\frac{6}{10} \times \frac{4}{12}\right) = \frac{2}{5}$$

Hence, the correct answer is (A).

37. Let $A \equiv \{0, 1, 2, 3, 4, \ldots..10\}$

$n(S) = {}^{11}C_2 = 55$ where 's' denotes sample space let E be the given event

$\therefore$ E $\equiv$ {(0, 4),(0, 8),(2, 6),(2, 10),(4, 8), (6, 10)}

$\Rightarrow n(E) = 6$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{6}{55}$$

Hence, the correct answer is (B).

38. P(exactly one of A or B occurs)

$$= P(A) + P(B) - 2P(A \cap B) = \frac{1}{4} \quad \ldots(1)$$

P(Exactly one of B or C occurs)

$$= P(B) + P(C) - 2P(B \cap C) = \frac{1}{4} \quad \ldots(2)$$

P(Exactly one of C or A occurs)

$$= P(C) + P(A) - 2P(C \cap A) = \frac{1}{4} \quad \ldots(3)$$

Adding (1), (2) and (3), we get

$$2\Sigma P(A) - 2\Sigma P(A \cap B) = \frac{3}{4}$$

$$\therefore \quad \Sigma P(A) - \Sigma P(A \cap B) = \frac{3}{8}$$

Now, $P(A \cap B \cap C) = \frac{1}{16}$

$$\therefore P(A \cup B \cup C) = \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$$

$$= \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$$

Hence, the correct answer is (C).

39. $P(E_1) = \frac{1}{6}; P(E_2) = \frac{1}{6}; P(E_3) = \frac{1}{2}$

$$P(E_1 \cap E_2) = \frac{1}{36}, P(E_2 \cap E_3) = \frac{1}{12}, P(E_1 \cap E_3) = \frac{1}{12}$$

And $P(E_1 \cap E_2 \cap E_3) = 0 \neq P(E_1).P(E_2).P(E_3)$

$\Rightarrow E_1, E_2, E_3$ are not independent.

Hence, the correct answer is (A).

40. Given,

$$P(\overline{A \cup B}) = \frac{1}{6} \qquad \Rightarrow p(A \cup B) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(\bar{A}) = \frac{1}{4} \quad \Rightarrow P(A) = 1 - \frac{1}{4} = \frac{3}{4}$$

We know

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \frac{5}{6} = \frac{3}{4} + p(B) - \frac{1}{4} \quad \left(\because P(A \cap B) = \frac{1}{4}\right) \quad \Rightarrow P(B) = \frac{1}{3}$$

$$\because P(A).P(B) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4} = P(A \cap B)$$

Hence, the correct answer is (C).

41. p = P (correct answer), q = P (wrong answer)

$$\Rightarrow p = \frac{1}{3}, q = \frac{2}{3}, n = 5$$

By using Binomial distribution

$$\text{Required probability} = {}^5C_4\left(\frac{1}{3}\right)^4 \cdot \frac{2}{3} + {}^5C_5\left(\frac{1}{3}\right)^5$$

$$= 5\,\frac{2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}$$

Hence, the correct answer is (C).

42. Given sample space = {1, 2, 3, …., 8}

Let Event

A: Maximum of three numbers is 6.

B: Minimum of three numbers is 3.

This is the case of conditional probability

We have to find P (minimum) is 3 when it is given that P(maximum) is 6.

$$\therefore P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)} = \frac{{}^2C_1 / {}^8C_3}{{}^5C_2 / {}^8C_3} = \frac{{}^2C_1}{{}^5C_2} = \frac{2}{10} = \frac{1}{5}$$

Hence, the correct answer is (C).

43. In this case, $P\left(\frac{C}{D}\right) = \frac{P(C \cap D)}{P(D)} = \frac{P(C)}{P(D)}$

Where, $0 \le P(D) \le 1$, hence $P\left(\frac{C}{D}\right) \ge P(C)$

Hence, the correct answer is (D).

44. $p(\text{at least one failure}) \ge \frac{31}{32}$

$$\Rightarrow 1 - p\,(\text{no failure}) \ge \frac{31}{32}$$

$$\Rightarrow 1 - p^5 \ge \frac{31}{32} \qquad \Rightarrow p^5 \le \frac{1}{32} \qquad \Rightarrow p \le \frac{1}{2}$$

But $p \ge 0$,

Hence p lies in the interval $\left[0, \frac{1}{2}\right]$.

Hence, the correct answer is (A).

45. $n(S) = {}^9C_3$, $n(E) = {}^3C_1 \times {}^4C_1 \times {}^2C_1$

$$\text{Probability} = \frac{3 \times 4 \times 2}{{}^9C_3} = \frac{24 \times 3!}{9!} \times 6! = \frac{24 \times 6}{9 \times 8 \times 7} = \frac{2}{7}$$

Hence, the correct answer is (B).

46. $n(S) = {}^{20}C_4$

Statement-1:

Common difference is 1; total number of cases = 17

Common difference is 2; total number of cases = 14

Common difference is 3; total number of cases = 11

Common difference is 4; total number of cases = 8

Common difference is 5; total number of cases = 5

Common difference is 6; total number of cases = 2

$$\text{Probability} = \frac{17 + 14 + 11 + 8 + 5 + 2}{{}^{20}C_4} = \frac{1}{85}$$

Statement-2 is false, because common difference can be 6 also.

Hence, the correct answer is (C).

47. Let A ≡ Sum of the digits is 8

B ≡ Product of the digits is 0

Then A = {08, 17, 26, 35, 44}

B = {00, 01, 02, 03, 04, 05, 06, 07, 08, 09, 10, 20, 30, 40}

$A \cap B = \{08\}$

$$\therefore\ P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/50}{14/50} = \frac{1}{14}$$

Hence, the correct answer is (C).

48. We have

$$P(x \geq 1) \geq \frac{9}{10} \qquad \Rightarrow 1 - P(x = 0) \geq \frac{9}{10}$$

$$\Rightarrow 1 - {}^{n}C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^n \geq \frac{9}{10}$$

$$\Rightarrow 1 - \frac{9}{10} \geq \left(\frac{3}{4}\right)^n \qquad \Rightarrow \left(\frac{3}{4}\right)^n \leq \left(\frac{1}{10}\right)$$

Taking log to the base 3/4, on both sides, we get

$$n \log_{3/4}\left(\frac{3}{4}\right) \geq \log_{3/4}\left(\frac{1}{10}\right)$$

$$\Rightarrow n \geq -\log_{3/4} 10 = \frac{-\log_{10} 10}{\log_{10}\left(\frac{3}{4}\right)} = \frac{-1}{\log_{10} 3 - \log_{10} 4}$$

$$\Rightarrow n \geq \frac{1}{\log_{10} 4 - \log_{10} 3}$$

Hence, the correct answer is (C).

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Single Correct Choice Type Questions

1. Coin C_1 $P\left(H_{C_1}\right) = \frac{2}{3}; P\left(T_{C_1}\right) = \frac{1}{3}; 0 \leq \alpha \leq 2$

Coin C_2 $P\left(H_{C_2}\right) = \frac{1}{3}; P\left(T_{C_2}\right) = \frac{2}{3}; 0 \leq \beta \leq 2$

$\alpha \in W, \beta \in W$; α, β ∈ {0, 1, 2}

$D = \alpha^2 - 4\beta = 0$

Case I $\alpha = 0, \beta = 0$

$$\text{Probability} = \left(\frac{1}{3} \times \frac{1}{3}\right)\left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{4}{81}$$

Case II $\alpha = 2, \beta = 1$

$$\text{Probability} = \left(\frac{2}{3} \times \frac{2}{3}\right)\left[\frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3}\right] = \frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$$

$$\text{Required Probability} = \frac{4}{81} + \frac{16}{81} = \frac{20}{81}$$

Hence, the correct answer is (B).

2. Total number of non negative solutions of $x + y + z$ = 10 are = ${}^{12}C_2 = 66$ (using ${}^{n+r-1}C_{r-1}$)

If z is even then three can be following cases

$z = 0 \Rightarrow$ No. of ways of solving $x + y = 10 \Rightarrow {}^{11}C_1$

$z = 2 \Rightarrow$ No. of ways of solving $x + y = 8 \Rightarrow {}^{9}C_1$

$z = 4 \Rightarrow$ No. of ways of solving $x + y = 6 \Rightarrow {}^{7}C_1$

$z = 6 \Rightarrow$ No. of ways of solving $x + y = 4 \Rightarrow {}^{5}C_1$

$z = 8 \Rightarrow$ No. of ways of solving $x + y = 2 \Rightarrow {}^{3}C_1$

$z = 10 \Rightarrow$ No. of ways of solving $x + y = 0 \Rightarrow 1$

∴ Total ways when z is even = 11 + 9 + 7 + 5 + 3 + 1 = 36

$$\therefore\ \text{Required probability} = \frac{36}{66} = \frac{6}{11}$$

Hence, the correct answer is (B).

3. $P(T_1) = \frac{20}{100}, P(T_2) = \frac{80}{100}, P(D) = \frac{7}{100}$

Let $P\left(\frac{D}{T_2}\right) = x$, then $P\left(\frac{D}{T_1}\right) = 10x$

Also $P(D) = P(T_1)\,P\left(\frac{D}{T_1}\right) + P(T_2)\,P\left(\frac{D}{T_2}\right)$

$$\Rightarrow \frac{7}{100} = \frac{20}{100} \times 10x + \frac{80}{100} \times x$$

$$\Rightarrow \frac{7}{280} = x \text{ or } x = \frac{1}{40}$$

$$P\left(\frac{D}{T_1}\right) = \frac{10}{40} \text{ and } P\left(\frac{D}{T_2}\right) = \frac{1}{40}$$

$$\Rightarrow \left(\frac{\bar{D}}{T_1}\right) = \frac{30}{40} \text{ and } P\left(\frac{\bar{D}}{T_2}\right) = \frac{39}{40}$$

$$P\left(\frac{T_2}{\bar{D}}\right) = \frac{P\left(\frac{\bar{D}}{T_2}\right)P(T_2)}{P\left(\frac{\bar{D}}{T_1}\right)P(T_1) + P\left(\frac{\bar{D}}{T_2}\right)P(T_2)}$$

$$= \frac{\frac{80}{100} \times \frac{39}{40}}{\frac{20}{100} \times \frac{30}{40} + \frac{80}{100} \times \frac{39}{40}} = \frac{156}{186} = \frac{26}{31} = \frac{78}{93}$$

Hence, the correct answer is (C).

4. According to given condition, we can have the following cases

(I) G G B B B (II) B G G B B

(III) G B G B B (IV) B G B G B

(V) G B B G B

i.e., the two girls can occupy two of the first three places (case I, II, III) or second and fourth (case IV) or first and fourth (case V) places.

Thus favourable cases are = 3 × 2! × 3! + 2 × 2! × 3! = 60

Total ways in which 5 persons can be seated = 5! = 120

$$\therefore\ \text{Required probability} = \frac{60}{120} = \frac{1}{2}$$

Hence, the correct answer is (A).

5. P(at least one of them solves the problem)

= 1 – P (none of them solves it)

$$= 1 - \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}\times\frac{7}{8} = 1 - \frac{21}{256} = \frac{235}{256}$$

Hence, the correct answer is (A).

6. D_4 can show a number appearing on one of D_1, D_2 and D_3 in the following cases.

Case I: D_4 shows a number which is shown by only one of D_1, D_2 and D_3.

D_4 shows a number in ${}^6C_1$ ways.

One out of D_1, D_2 and D_3 can be selected in ${}^3C_1$ ways. The selected die shows the same number as on D_4 in one way and rest two dice show the different number in 5 ways each.

∴ Number of ways to happen case I

$= {}^6C_1 \times {}^3C_1 \times 1 \times 5 \times 5 = 450$

Case II: D_4 shows a number which is shown by only two of D_1, D_2 and D_3.

As discussed in case I, it can happen in the following number of ways

$= {}^6C_1 \times {}^3C_2 \times 1 \times 1 \times 5 = 90$

Case III: D_4 shows a number which is shown by all three dice D_1, D_2 and D_3.

Number of ways it can be done

$= {}^6C_1 \times {}^3C_3 \times 1 \times 1 \times 1 = 6$

∴ Total number of favourable ways = 450 + 90 + 6 = 546

Also total ways = 6 × 6 × 6 × 6

$$\therefore \text{ Required Probability} = \frac{546}{6\times6\times6\times6} = \frac{91}{216}$$

Hence, the correct answer is (A).

7. If ω is a complex cube root of unity then, we know that

$\omega^{3m} + \omega^{3n+1} + \omega^{3p+2} = 0$

where m, n, p are integers.

∴ r_1, r_2, r_3 should be of the form $3m$, $3n + 1$ and $3p + 2$ taken in any order. As r_1, r_2, r_3 are the numbers obtained on die, these can take any value from 1 to 6.

∴ m can take value 1 or 2, n can take values 0 or 1

p can take values 0 or 1

∴ Number of ways of selecting r_1, r_2, r_3

$= {}^2C_1 \times {}^2C_1 \times {}^2C_1 \times 3!$.

Also the total number of ways of getting r_1, r_2, r_3 on die = 6 × 6 × 6

$$\therefore \text{ Required probability} = \frac{{}^2C_1\times{}^2C_1\times{}^2C_1\times 3!}{6\times6\times6} = \frac{2}{9}$$

Hence, the correct answer is (C).

8. Let G ≡ original signal is green ⇒ P(G) = 4/5

E_1 ≡ A receives the signal correctly $P(E_1) = 3/4$

E_2 ≡ B receives the signal correctly $P(E_2) = 3/4$

E ≡ Signal received by B is green.

Then E can happen in the following ways

Original Signal	**Received at A**	**Received at B**
Red ⟶	Red ⟶	Green
Red ⟶	Green ⟶	Green
Green ⟶	Green ⟶	Green
Green ⟶	Red ⟶	Green

$$\therefore \quad P(E) = (\bar{G}\cap E_1\cap \bar{E}_2) + (\bar{G}\cap E_1\cap \bar{E}_2) + P(G\cap E_1\cap E_2) + P(G\cap \bar{E}_1\cap \bar{E}_2)$$

$$= \frac{1}{5}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{5}\times\frac{3}{4}\times\frac{1}{4} + \frac{4}{5}\times\frac{3}{4}\times\frac{3}{4} + \frac{4}{5}\times\frac{1}{4}\times\frac{1}{4}$$

$$= \frac{3+3+36+4}{80} = \frac{46}{80} = \frac{23}{40}$$

$$\therefore P(G/E) = \frac{P(G\cap E)}{P(E)} = \frac{P(G\cap E_1\cap E_2) + P(G\cap \bar{E}_1\cap \bar{E}_2)}{P(E)}$$

$$= \frac{\frac{4}{5}\times\frac{3}{4}\times\frac{3}{4} + \frac{4}{5}\times\frac{1}{4}\times\frac{1}{4}}{23/40} = \frac{40/80}{23/40} = \frac{20}{23}$$

Hence, the correct answer is (C).

9. We have n (S) = 10, n(A) = 4

Let $n(B) = x$ and $n(A \cap B) = y$

Then for A and B to be independent events

$P(A \cap B) = P(A)\, P(B)$

$$\Rightarrow \frac{y}{10} = \frac{4}{10}\times\frac{x}{10} \qquad \Rightarrow x = \frac{5}{2}y$$

⇒ y can be 2 or 4 so that x = 5 or 10

∴ $n(B)$ = 5 or 10

Hence, the correct answer is (D).

10. Let E_1 = The Indian man is seated adjacent to his wife.

E_2 = Each American man is seated ….. to his wife.

$$\text{Then } P(E_1/E_2) = \frac{P(E_1\cap E_2)}{P(E_2)}$$

Now $E_1 \cap E_2$ ≡ All men are seated adjacent to their wives.

∴ We can consider the 5 couples as single-single objects which can be arranged in a circle in 4! Ways. But for each couple, husband and wife can interchange their places in 2! ways.

∴ Number of ways when all men are seated adjacent to their wives = $4! \times (2!)^5$

Also in all 10 persons can be seated in a circle in 9! ways.

$$\therefore P(E_1\cap E_2) = \frac{4!\times(2!)^5}{9!}$$

Similarly if each American man is seated adjacent to his wife, considering each American couple as single object and Indian woman and man as separate objects there are 6 different objects which can be arranged in a circle in 5! ways. Also for each American couple, husband and wife can interchange their places in 2! ways.

So the number of ways in which each American man is seated adjacent to his wife.

$$= 5! \times (2!)^4 \qquad \therefore P(E_2) = \frac{5!\times(2!)^4}{9!}$$

So $P(E_1/E_2) = \dfrac{(4!\times(2!)^5)/9!}{(5!\times(2!)^4)/9!} = \dfrac{2}{5}$

Hence, the correct answer is (C).

11. $P(E^c \cap F^c/G) = P(E \cup F)^c/G)$

$1 - P(E \cup F/G)$

$= 1 - P(E/G) - P(F/G) + P(E \cap F/G)$

$= 1 - P(E) - P(F) + 0$

($\because$ E, F, G are pairwise independent and $P(E \cap F \cap G) = 0$)

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. | B_1 | B_2 | B_3 |
|---|---|---|
| 5R, 5G | 3R, 5G | 5R, 3G |
| $P(B_1)=\dfrac{3}{10}$ | $P(B_2)=\dfrac{3}{10}$ | $P(B_3)=\dfrac{4}{10}$ |

(1) $P(G) = P(B_1).P\left(\dfrac{G}{B_1}\right) + P(B_2).P\left(\dfrac{G}{B_2}\right) + P(B).P\left(\dfrac{G}{B_3}\right)$

$= \dfrac{3}{10}\times\dfrac{5}{10} + \dfrac{3}{10}\times\dfrac{5}{8} + \dfrac{4}{10}\times\dfrac{3}{8} = \dfrac{39}{80}$

(2) $P\left(\dfrac{G}{B_3}\right) = \dfrac{3}{8}$

(3) $P\left(\dfrac{B_3}{G}\right) = \dfrac{P(B_3 \cap G)}{P(G)} = \dfrac{\frac{4}{10}\times\frac{3}{8}}{\frac{39}{80}} = \dfrac{4}{13}$

(4) $P(B_3 \cap G) = \dfrac{4}{10}\times\dfrac{3}{8} = \dfrac{3}{20}$

Hence, the correct answer is (A, B).

2. $P(X) = \dfrac{1}{3}, P(X/Y) = \dfrac{1}{2}, P(Y/X) = \dfrac{2}{5}$

$P(X \cap Y) = P(Y/X)P(X) = \dfrac{2}{5}\times\dfrac{1}{3} = \dfrac{2}{15}$

$P(Y) = \dfrac{P(X \cap Y)}{P(X/Y)} = \dfrac{\frac{2}{15}}{\frac{1}{2}} = \dfrac{4}{15}$

$P(X'/Y) = \dfrac{P(X' \cap Y)}{P(Y)} = \dfrac{P(Y) - P(X \cap Y)}{P(Y)} = 1 - P(X/Y) = \dfrac{1}{2}$

$P(X \cup Y) = \dfrac{1}{3} + \dfrac{4}{15} - \dfrac{2}{15} = \dfrac{7}{15}$

$\therefore$ A and B are the correct options.

Hence, the correct answer is (A, B).

3. We have $P(X_1) = \dfrac{1}{2}, P(X_2) = \dfrac{1}{4}, P(X_3) = \dfrac{1}{4}$

P(X) = P(at least 2 engines are functioning)

$= P(X_1 \cap X_2 \cap X_3^C) + P(X_1 \cap X_2^C \cap X_3) + P(X_1^C \cap X_2 \cap X_3) +$
$P(X_1 \cap X_2 \cap X_3)$

$= \dfrac{1}{2}\times\dfrac{1}{4}\times\dfrac{3}{4} + \dfrac{1}{2}\times\dfrac{3}{4}\times\dfrac{1}{4} + \dfrac{1}{2}\times\dfrac{1}{4}\times\dfrac{1}{4} + \dfrac{1}{2}\times\dfrac{1}{4}\times\dfrac{1}{4} = \dfrac{1}{4}$

(A) $P(X_1^C/X) = \dfrac{P(X_1^C \cap X)}{P(X)} = \dfrac{P(X_1^C \cap X_2 \cap X_3)}{P(X)}$

$= \dfrac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}}{\frac{1}{4}} = \dfrac{1}{8}$

(B) P[Exactly two engines are functioning/X]

$= \dfrac{P[(\text{Exactly two engines are functioning}) \cap X]}{P(X)}$

$= \dfrac{P(X_1^C \cap X_2 \cap X_3) + P(X_1 \cap X_2^C \cap X_3) + P(X_1 \cap X_2 \cap X_3^C)}{P(X)}$

$= \dfrac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}}{\frac{1}{4}} = \dfrac{7}{8}$

(C) $P(X/X_2) = \dfrac{P(X \cap X_2)}{P(X_2)}$

$= \dfrac{P(X_1 \cap X_2 \cap X_3) + P(X_1^C \cap X_2 \cap X_3) + P(X_1 \cap X_2 \cap X_3^C)}{P(X_2)}$

$= \dfrac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}}{\frac{1}{4}} = \dfrac{5}{8}$

(D) $P(X/X_1) = \dfrac{P(X \cap X_1)}{P(X_1)}$

$= \dfrac{P(X_1 \cap X_2 \cap X_3) + P(X_1 \cap X_2^C \cap X_3) + P(X_1 \cap X_2 \cap X_3^C)}{P(X_1)}$

$= \dfrac{\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}}{\frac{1}{2}} = \dfrac{7}{16}$

Hence, the correct answer is (B, D).

4. We know $P(X/Y) = \dfrac{P(X \cap Y)}{P(Y)} \Rightarrow \dfrac{1}{2} = \dfrac{1/6}{P(Y)} \Rightarrow P(Y) = \dfrac{1}{3}$

Similarly, $P(Y/X) = \dfrac{P(X \cap Y)}{P(X)} \Rightarrow \dfrac{1}{3} = \dfrac{1/6}{P(X)} \Rightarrow P(X) = \dfrac{1}{2}$

$\therefore P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{6} = \dfrac{2}{3}$

Also $P(X \cap Y) = P(X)\,P(Y)$

$\Rightarrow$ X and Y are independent events.

$\therefore$ X^C and Y are also independent events.

$\therefore P(X^C \cap Y) = P(X^C) \times P(Y) = \dfrac{1}{2}\times\dfrac{1}{3} = \dfrac{1}{6}$

Hence, the correct answer is (A, B).

5. $\because$ E and F are independent events

$\therefore P(E \cap F) = P(E).P(F)$ ...(1)

Given that $P(E \cap \bar{F}) + P(\bar{E} \cap F) = \dfrac{11}{25}$

$\Rightarrow P(E)\,P(\bar{F}) + P(\bar{E})\,P(F) = \dfrac{11}{25}$

$$\Rightarrow \text{P(E) (1 – P(F)) + (1 – P(E)) P(F)} = \frac{11}{25}$$

$$\Rightarrow \text{P(E) – P(E) P(F) + P(F) – P(E) P(F)} = \frac{11}{25}$$

$$\Rightarrow \text{P(E) + P(F) – 2P(E).P(F)} = \frac{11}{25} \qquad \ldots(2)$$

$$\text{and } P(\bar{E} \cap \bar{F}) = \frac{2}{25} \Rightarrow P(\bar{E})\, P(\bar{F}) = \frac{2}{25}$$

$$\Rightarrow [1 - P(E)]\,[1 - P(F)] = \frac{2}{25}$$

$$\Rightarrow 1 - P(E) - P(F) + P(E)\,P(F) = \frac{2}{25} \qquad \ldots(3)$$

Adding equation (2) and (3) we get

$$1 - P(E)\,P(F) = \frac{13}{25} \text{ or } P(E)\,P(F) = \frac{12}{25} \qquad \ldots(4)$$

Using the result in equation (2) we get

$$P(E) + P(F) = \frac{35}{25} \qquad \ldots(5)$$

Solving (4) and (5) we get

$$P(E) = \frac{3}{5} \text{ and } P(F) = \frac{4}{5} \text{ or } P(E) = \frac{4}{5} \text{ and } P(F) = \frac{3}{5}$$

∴ (A) and (D) are the correct options.

Hence, the correct answer is (A, D).

Comprehension Type Questions

1. Number of derangements for 4 students

$$= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$= 12 - 4 + 1 = 9$

Total number of arrangements for seating of 5 students = 5! = 120

$$\therefore \text{ Required probability} = \frac{9}{120} = \frac{3}{40}$$

Hence, the correct answer is (A).

2. Total cases = 5! = 120

Favourable cases

$$\left.\begin{matrix}1,3,5,2,4\\1,4,2,5,3\end{matrix}\right\}2$$

$$\left.\begin{matrix}2,4,1,3,5\\2,5,3,1,4\\2,4,1,5,3\end{matrix}\right\}3$$

$$\left.\begin{matrix}2,4,1,3,5\\2,5,3,1,4\\2,4,1,5,3\end{matrix}\right\}3$$

$$\left.\begin{matrix}4,2,5,1,3\\4,2,5,3,1\\4,1,3,5,2\end{matrix}\right\}3$$

$$\left.\begin{matrix}5,2,4,1,3\\4,3,1,4,2\end{matrix}\right\}2$$

∴ favourable cases = 14

$$\therefore \text{ required probability} = \frac{14}{120} = \frac{7}{60}$$

Hence, the correct answer is (C).

3. (X, Y) = {(6, 0), (4, 1), (3, 3), (2, 2), (1, 4), (0, 6)}

P(X > Y) = P(T_1 wins 2 games or T_1 wins 2 games or T_1 win one game other is a draw)

$$= \frac{1}{2} \times \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Hence, the correct answer is (B).

4. P(X = Y) = P(T_1 wins 1 game loses other game or both the games draw)

$$= \left(\frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2}\right) + \frac{1}{6} \times \frac{1}{6} = \frac{1}{3} + \frac{1}{36} = \frac{13}{36}$$

Hence, the correct answer is (C).

5. Let E_1 = box I is selected

E_2 = box II is selected

E = ball draw is red

$$P(E_2/E) = \frac{\dfrac{n_3}{n_3 + n_4} \times \dfrac{1}{2}}{\dfrac{n_1}{n_1 + n_2} \times \dfrac{1}{2} + \dfrac{n_3}{n_3 + n_4} \times \dfrac{1}{2}} = \frac{1}{3}$$

$$\text{or} \quad \frac{\dfrac{n_3}{n_3 + n_4}}{\dfrac{n_1}{n_1 + n_2} + \dfrac{n_3}{n_3 + n_4}} = \frac{1}{3}$$

On checking the options, we find (A) and (B) are the correct options.

Hence, the correct answer is (A, B).

6. E_1 ≡ Red ball is selected from box I

E_2 ≡ Black ball is selected from box I

E ≡ Second ball drawn from box I is red

∴ P(E) = P(E_1) P(E/E_1) + P(E_2) P(E/E_2)

$$= \frac{n_1}{n_1 + n_2} \times \frac{n_1 - 1}{n_1 + n_2 - 1} + \frac{n_2}{n_1 + n_2} \times \frac{n_1}{n_1 + n_2 - 1}$$

On checking the options, we find (C) and (D) have the correct values.

Hence, the correct answer is (C, D).

7. $x_1 + x_2 + x_3$ will be odd

If two are even and one is odd or all three are odd.

∴ Required probability

$$= P(EEO) + P(EOE) + P(OEE) + P(OOO)$$

$$= \frac{1}{3} \times \frac{2}{5} \times \frac{4}{7} + \frac{1}{3} \times \frac{3}{5} \times \frac{3}{7} + \frac{2}{3} \times \frac{2}{5} \times \frac{3}{7} + \frac{2}{3} \times \frac{3}{5} \times \frac{4}{7}$$

$$= \frac{8+9+12+24}{105} = \frac{53}{105}$$

Hence, the correct answer is (B).

8. If x_1, x_2, x_3 are in AP then $2x_2 = x_1 + x_3$

$\because$ LHS is even, x_1 & x_3 can be both even or both odd.

x_1 and x_3 both can be even in $1 \times 3 = 3$ ways

x_1 and x_3 both can be odd in $2 \times 4 = 8$ ways

$\therefore$ Total favourable ways = 3 + 8 = 11

Also one number from each box can be drawn in $3 \times 5 \times 7$ ways

$\therefore$ Total ways = 105

Hence required probability $= \dfrac{11}{105}$

Hence, the correct answer is (C).

9.

B_1	B_2	B_3
1W	2W	3W
3R	3R	4R
2B	4B	5B

Let E_1, E_2, E_3 be the events that bag B_1, B_2 and B_3 is selected respectively.

Let E be the event that one white and one red ball is selected.

Then by Bayes' theorem,

$$P(E_2/E) = \frac{P(E/E_2)P(E_2)}{P(E/E_1)P(E_1) + P(E/E_2)P(E_2) + P(E/E_3)P(E_3)}$$

$$= \frac{\dfrac{2\times 3}{{}^9C_2}}{\dfrac{1\times 3}{{}^6C_2} + \dfrac{2\times 3}{{}^9C_2} + \dfrac{3\times 4}{{}^{12}C_2}} = \frac{55}{181}$$

Hence, the correct answer is (D).

10. Probability that all balls are of same colour

= P (all red) + P (all white) + P (all black)

$$= \frac{3}{6}\times\frac{3}{9}\times\frac{4}{12} + \frac{1}{6}\times\frac{2}{9}\times\frac{3}{12} + \frac{2}{6}\times\frac{4}{9}\times\frac{5}{12} = \frac{82}{648}$$

Hence, the correct answer is (A).

11. P (white) = P (H $\cap$ white) + P (T $\cap$ white)

= P (H) P (white/H) + P (T) P (white/T)

$$= \frac{1}{2}\left\{\frac{3}{5}\times 1 + \frac{2}{5}\times\frac{1}{2}\right\} + \frac{1}{2}\times\left\{\frac{{}^3C_2}{{}^5C_2}\times 1 + \frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3} + \frac{{}^3C_1.{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right\}$$

$$= \frac{1}{2}\times\frac{8}{10} + \frac{1}{2}\times\left(\frac{3}{10} + \frac{1}{30} + \frac{12}{30}\right) = \frac{4}{10} + \frac{11}{30} = \frac{23}{30}$$

Hence, the correct answer is (B).

12. $$P(H/\text{white}) = \frac{P(H\cap \text{white})}{P(\text{white})} = \frac{\dfrac{1}{2}\left[\dfrac{3}{5}\times 1 + \dfrac{2}{5}\times\dfrac{1}{2}\right]}{\dfrac{23}{30}}$$

$$= \frac{\dfrac{4}{10}}{\dfrac{23}{30}} = \frac{12}{23}$$

Hence, the correct answer is (D).

13. P(X = 3) = (probability of not a six in first chance) × (probability of not a six in second chance) × (probability of a six in third chance)

$$= \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \frac{25}{216}$$

Hence, the correct answer is (A).

14. $P(X \geq 3) = 1 - (X < 3) = 1 - [P(X = 1) + P(X = 2)]$

$$= 1 - \left[\frac{1}{6} + \frac{5}{6}\times\frac{1}{6}\right] = 1 - \frac{11}{36} = \frac{25}{36}$$

Hence, the correct answer is (B).

15. Let us define the events

$A \equiv X \geq 6$ and $B \equiv X > 3$ so that $A \cap B \equiv X \geq 6 \equiv A$

Now $P(A) = \left(\dfrac{5}{6}\right)^5 \times \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^6 \times \dfrac{1}{6} + \ldots.\ \infty$

$$= \left(\frac{5}{6}\right)^5 \times \frac{1}{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots.\infty\right]$$

$$= \left(\frac{5}{6}\right)^5 \times \frac{1}{6} \times \frac{1}{1-\frac{5}{6}} = \left(\frac{5}{6}\right)^5$$

and $P(B) = \left(\dfrac{5}{6}\right)^3 \times \dfrac{1}{6} + \left(\dfrac{5}{6}\right)^4 \times \dfrac{1}{6} + \ldots.\infty = \left(\dfrac{5}{6}\right)^3$

$$\therefore \quad P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\left(\frac{5}{6}\right)^5}{\left(\frac{5}{6}\right)^3} = \frac{25}{36}$$

Hence, the correct answer is (D).

Integer/Numerical Answer Type Questions

1. Let n missiles are fired probability of successful hit in one fire is $p = 0\cdot 75 = \dfrac{3}{4}$,

probability of unsuccessful hit in one fire is $q = 1 - p = \dfrac{1}{4}$

$P(\text{Destroyed}) \geq 0.95$

$\Rightarrow \quad 1 - P(\text{Not destroyed}) \geq 0.95 \Rightarrow P(\text{not destroyed}) \leq 0.05$

$$\Rightarrow \quad {}^nC_0\left(\frac{1}{4}\right)^n + {}^nC_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n-1} + {}^nC_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2} \leq 0.05$$

$$\Rightarrow \quad \frac{1}{4^n} + \frac{3n}{4^n} + \frac{\frac{n(n-1)}{2}.9}{4^n} \leq \frac{5}{100}$$

$$\Rightarrow \quad \frac{1}{4^n}\left[1 + 3n + \frac{9n(n-1)}{2}\right] \leq \frac{1}{20} \Rightarrow 4^n \geq 10[9n^2 - 3n + 2]$$

True for $n = 6$, $4^6 = 4096 \geq 10[9 \times 36 - 18 + 2] = 3080$

Hence least value is 6

Hence, the correct answer is (6).

2. Prime no. → {2, 3, 5, 7, 11}

Perfect square → {4, 9}

4 → (2, 2) (1, 3) (3, 1)

$9 \rightarrow (3, 6)\ (6, 3)\ (4, 5)\ (5, 4)$

Required probability $P = \frac{4}{7} \Rightarrow 7P = 4$

$\Rightarrow \quad 14P = 8$

Hence, the correct answer is (8).

3. For E_2

'A' will have seven 1's and two 0's. $n(E_2) = {}^9C_7 = 36$

$$\text{Ex :- } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

For E_1 when E_2 is occurring i.e, $n(E_1 \cap E_2)$

$|A| = 0$ for following cases

(1) if {1, 0, 0} are in same row or in same column.

$= (3 + 3) \times 3 = 18$

$\therefore\ n(E_1 \cap E_2) = 18$

Hence required $P\left(\frac{E_1}{E_2}\right) = \frac{18}{36} = 0.5$

Hence, the correct answer is (0.50).

4. Let $|A| = x$, $|B| = y$ and $|A \cap B| = z$

Since $|A|$ and $|B|$ are independent events

So $P(A).\ P(B) = P(A \cap B)$

$\Rightarrow \frac{x}{6}.\frac{y}{6} = \frac{z}{6}$

$\Rightarrow xy = 6z$

No. of ways

If $\quad z = 1, x = 6, y = 1 \quad \Rightarrow \quad {}^6C_6 \times {}^6C_1 = 6$

$\quad x = 3, y = 2 \quad \Rightarrow \quad {}^6C_3 \times {}^3C_1 \times {}^3C_1 = 180$

If $\quad z = 2, x = 6, y = 2 \quad \Rightarrow \quad {}^6C_6 \times {}^6C_2 = 15$

$\quad x = 4, y = 3 \quad \Rightarrow \quad {}^6C_4 \times {}^4C_2 \times {}^2C_1 = 180$

If $\quad z = 3, x = 6, y = 3 \quad \Rightarrow \quad {}^6C_6 \times {}^6C_3 = 20$

If $\quad z = 4, x = 6, y = 4 \quad \Rightarrow \quad {}^6C_6 \times {}^6C_4 = 15$

If $\quad z = 5, x = 6, y = 5 \quad \Rightarrow \quad {}^6C_6 \times {}^6C_5 = 6$

If $\quad z = 6, x = 6, y = 6 \quad \Rightarrow$ X (not possible)

Total ways = 422

Hence, the correct answer is (422.00).

5. $P(x > 2) > 0.96$

$\Rightarrow \quad 1 - P(x = 0) - P(x = 1) > 0.96$

$\Rightarrow \quad P(x = 0) + P(x = 1) < 0.04$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n + n\left(\frac{1}{2}\right)^n \leq 0.04$$

$$\Rightarrow \quad \frac{n+1}{2^n} \leq \frac{1}{25} \Rightarrow \frac{2^n}{n+1} \geq 25$$

$\Rightarrow \quad$ minimum value of n is 8.

Hence, the correct answer is (8).

6. Let $P(E_1) = x$; $P(E_2) = y$, $P(E_3) = z$

$P(\text{only } E_1) = x(1 - y)\ (1 - z) = \alpha$

$P(\text{only } E_2) = (1 - x)y\ (1 - z) = \beta$

$P(\text{only } E_3) = (1 - x)\ (1 - y)\ z = \gamma$

$P(\text{none}) = (1 - x)\ (1 - y)\ (1 - z) = p.$

Now given $(\alpha - 2\beta)p = \alpha\beta \Rightarrow x = 2y$

and $(\beta - 3\gamma)p = 2\beta r$

$\Rightarrow y = 3z \therefore x = 6z$ Hence $\frac{P(E_1)}{P(E_3)} = \frac{x}{z} = 6$

Hence, the correct answer is (6).

CHAPTER 8: PRINCIPLE OF MATHEMATICAL INDUCTION

Single Correct Choice Type Questions

1. $P(n) : 3(5^{2n+1}) + 2^{3n+1}$

   $P(1) : 3(5)^3 + 2^4 = 3(125) + 16$

   $= 375 + 16 = 391 = 17(23)$

   So, $P(1)$ is divisible by 17.

   Let $P(k) : 3(5^{2k+1}) + 2^{3k+1} = 17m$

   $\therefore \quad P(k+1) : 3(5^{2k+3}) + 2^{3k+4} = 17\lambda$

   L.H.S of $P(k+1) = 3(5^{2k+3}) + 2^{3k+4}$

   $= 3(5^{2k+1})(5^2) + (2^{3k+1})(2^3)$

   $= (17m - 2^{3k+1})\ (25) + 8(2^{3k+1})$

   [Assuming $P(k)$ to be true]

   $= 17(25m) - 25(2^{3k+1}) + 8(2^{3k+1})$

   $= 17(25m) - 17(2^{3k+1}) = 17(25m - 2^{3k+1}) = 17\lambda$

   Thus, $P(k+1)$ is divisible by 17 whenever $P(k)$ is divisible by 17. Hence $P(n)$ is divisible by 17 for all $n \in N$

   Hence, the correct answer is (A).

2. $P(n) : x^{2n-1} + y^{2n-1}$

   $P(1) : x^1 + y^1 = x + y$, which is divisible by $x + y$

   Let $P(k) : x^{2k-1} + y^{2k-1} = (x+y)m$

   $\therefore \quad P(k+1) : x^{2k+1} + y^{2k+1} = (x+y)\lambda$

   L.H.S. of $P(k+1) = x^{2k+1} + y^{2k+1} = x^2(x^{2k-1}) + y^2(y^{2k-1})$

   $= x^2[m(x+y) - y^{2k-1}] + y^2(y^{2k-1})$

   [Assuming $P(k)$ to be true]

   $= (x+y)\ (mx^2) - y^{2k+1}(x^2 - y^2)$

   $= (x+y)\ [mx^2 - y^{2k-1}(x-y)] = (x+y)\ \lambda$

   Thus, $P(k+1)$ is divisible by $(x+y)$ whenever $P(k)$ is divisible by $(x+y)$. Hence $P(n)$ is divisible by $(x+y)$ for all $n \in N$ i.e. for all positive integers.

   Hence, the correct answer is (A).

3. $P(n) = 9^n - 8^n$

   $\therefore \quad P(1) = 9 - 8 = 1$

   $\therefore \quad P(1) - 1 = 0$ which is divisible by 8

   $\therefore \quad$ 1 is the remainder when $P(n)$ is divided by 8

   Now, $P(2) = 9^2 - 8^2 = 17 = 16 + 1$.

   Remainder is 1, when divided by 8.

   Hence, the correct answer is (C).

4. $P(n) : 2^{3n} - 7^n - 1$

   $P(1) : 2^3 - 7 - 1 = 8 - 7 - 1 = 0$

   $P(2) : 2^6 - 7(2) - 1 = 64 - 14 - 1 = 49$

   which is divisible by 49

   Let $P(k) : 2^{3k} - 7k - 1 = 49m$

   $\therefore \quad P(k+1) = 2^{3k+3} - 7(k+1) - 1 = 49\lambda$

   L.H.S. of $P(k+1) = 2^{3k+3} - 7(k+1) - 1$

   $= (2^{3k})\ (2^3) - 7k - 7 - 1$

   $= (49m + 7k + 1)\ (8) - 7k - 8$

   [Assuming $P(k)$ to be true]

   $= 49(8m) + 56k + 8 - 7k - 8$

   $= 49(8m + k) = 49\lambda.$

   Thus, $P(k+1)$ is divisible by 49 whenever $P(k)$ is divisible by 49. Hence $P(n)$ is divisible by 49 for all $n \in N$.

   Hence, the correct answer is (C).

5. $P(n) : a^{2n-1} + b^{2n-1}$

   $P(1) : a^1 + b^1 = a + b$, which is divisible by itself, i.e. by $(a+b)$.

   $\therefore \quad P(n) : a^{2n-1} + b^{2n-1}$ is divisible by $(a+b)$, and is true for $n = 1$

   Let $P(K)$ be true, i.e. $P(k) : a^{2k-1} + b^{2k-1}$ is divisible by $(a+b)$

   i.e. $a^{2k-1} + b^{2k-1} = m(a+b)$

   Now, $P(k+1) = a^{2k+1} + b^{2k+1} = a^{2k-1} \cdot a^2 + b^{2k+1}$

   $= a^2[m(a+b) - b^{2k-1}] + b^{2k+1}$

   $= m(a+b)a^2 - a^2b^{2k-1} + b^{2k+1}$

   $= m(a+b)a^2 - b^{2k-1}(a^2 - b^2)$

   $= m(a+b)a^2 - (a+b)\ (a-b)\ b^{2k-1}$

   $= (a+b)\ [ma^2 - (a-b)\ b^{2k-1}]$

   $\therefore \quad P(k+1)$ is divisible by $(a+b)$ whenever $P(k)$ is divisible by $(a+b)$.

   Hence $P(n)$ is divisible by $(a+b)$ for all $n \in N$

   Hence, the correct answer is (A).

6. $f(n) = 10^n + 3{\cdot}4^{n+2} + 5$

   $f(1) = 10 + 192 + 5 = 207$, which is divisible by 3

   $f(2) = 100 + 3(256) + 5 = 105 + 768 = 873$, which is divisible by 3.

   So, $f(n) = 10^n + 3{\cdot}4^{n+2} + 5$ is divisible by 3

   Hence, the correct answer is (B).

7. Let $P(n) : x^n - 1 = \lambda(x - k)$

   Now $P(1) : x - 1 = \lambda_1\ (x - k)$

   Also, $P(2) : x^2 - 1 = \lambda_2(x - k)$

   $\Rightarrow \quad P(2) : (x-1)\ (x+1) = \lambda_2(x-k)$

   $\therefore \quad$ The least value of k for which the proposition $P(n)$ is true is $k = 1$.

   Hence, the correct answer is (A).

8. Let $\dfrac{1^3}{1} + \dfrac{1^3 + 2^3}{1+3} + \dfrac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$ (n terms)

   $$\Rightarrow \quad P(n) : \sum \frac{1^3 + 2^3 + \ldots + n^3}{1 + 3 + 5 + \ldots + (2n-1)}$$

   $$\Rightarrow \quad P(n) : \sum\left(\frac{\sum n^3}{n^2}\right) \quad \Rightarrow \quad P(n) : \sum\left[\frac{1}{4}\frac{n^2(n+1)^2}{n^2}\right]$$

   $$\Rightarrow \quad P(n) : \frac{1}{4}\sum(n^2 + 2n + 1)$$

   $$\Rightarrow \quad P(n) : \frac{1}{4}\left[\sum n^2 + 2\sum n + \sum(1)\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2} + n\right]$$

$$\Rightarrow \quad P(n) : \frac{1}{24}n[6(n+1) + (n+1)(2n+1) + 6]$$

$$\therefore \quad P(n) : \frac{1}{24}\, n(2n^2 + 9n + 13)$$

Hence, the correct answer is (B).

9. Since, $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad A^2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \text{By induction } A^{10} = \begin{pmatrix} 1 & 10x \\ 0 & 1 \end{pmatrix}$$

Hence, the correct answer is (A).

10. Let $P(n) : 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$ (n terms)

$$\Rightarrow \quad P(n) : \sum n(n+1)(n+2)$$

$$\Rightarrow \quad P(n) : \sum (n^2 + n)(n+2)$$

$$\Rightarrow \quad P(n) : \sum (n^3 + 2n^2 + n^2 + 2n)$$

$$\Rightarrow \quad P(n) : \sum (n^3 + 3n^2 + 2n)$$

$$\Rightarrow \quad P(n) : \sum n^3 + 3\sum n^2 + 2\sum n$$

$$\Rightarrow \quad P(n) : \frac{n^2(n+1)^2}{4} + 3\,\frac{1}{6}\,n(n+1)(2n+1) + 2\,\frac{1}{2}\,n(n+1)$$

$$\Rightarrow \quad P(n) : \frac{n(n+1)}{4}\{n(n+1) + 2(2n+1) + 4\}$$

$$\therefore \quad P(n) : \frac{1}{4}\,n(n+1)\,(n+2)\,(n+3)$$

Hence, the correct answer is (C).

11. Let $P(n) : 1 \cdot 3^2 + 2 \cdot 5^2 + 3 \cdot 7^2 + \ldots + (n)\,(2n+1)^2$

$$\therefore \quad P(n) : \sum n(2n+1)^2$$

$$\Rightarrow \quad P(n) : \sum n(4n^2 + 4n + 1)$$

$$\Rightarrow \quad P(n) : \sum 4n^3 + 4n^2 + n$$

$$\Rightarrow \quad P(n) : 4\sum n^3 + 4\sum n^2 + \sum n$$

$$= \frac{4 \cdot n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$\therefore \quad P(n) = \frac{1}{6}\,n(n+1)\,(6n^2 + 14n + 7)$$

Hence, the correct answer is (A).

12. Let $P(n) : \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ $(n+1)$ terms

$$t_n = \frac{1}{1+2+3+\ldots+n} = \frac{2}{n(n+1)}$$

$$\Rightarrow \quad t_n = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\Rightarrow \quad P(n) : \left\{\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right\}$$

$$\therefore \quad P(n) = 2\left(1 - \frac{1}{n+1}\right) = \frac{2n}{n+1}$$

Hence, the correct answer is (B).

13. Let $P(n) : \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$

$$\Rightarrow \quad P(1) = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105}$$

$$\Rightarrow \quad P(1) = \frac{15 + 21 + 70 - 1}{105} = 1$$

$$\Rightarrow \quad P(2) = 8\left(\frac{16}{7} + \frac{4}{5} + \frac{2}{3}\right) - \frac{2}{105} = 15$$

∴ By induction $P(n)$ is an integer $\forall\ n \in N$

Hence, the correct answer is (A).

14. Since, $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow \quad A^2 = A \times A$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} \cos^2\theta - \sin^2\theta & \begin{matrix}\sin\theta\cos\theta \\ +\cos\theta\sin\theta\end{matrix} \\ \begin{matrix}-\sin\theta\cos\theta \\ -\cos\theta\sin\theta\end{matrix} & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$\therefore \quad \text{By Induction } A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

Hence, the correct answer is (B).

15. Let $P(n) : \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7.10} + \ldots$ n terms

$$\Rightarrow \quad P(n) : \frac{1}{3}\left(1 - \frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right) + \ldots (n \text{ term})$$

$$\Rightarrow \quad P(n) : \frac{1}{3}\left\{1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \ldots + \frac{1}{(3n-2)} - \frac{1}{3n+1}\right\}$$

$$\Rightarrow \quad P(n) : \frac{1}{3}\left\{1 - \frac{1}{3n+1}\right\}$$

$$\therefore \quad P(n) : \frac{n}{3n+1}$$

Hence, the correct answer is (C).

16. Let $P(n) : 1 + 5 + 12 + 22 + 35 + \ldots$ (n terms) $= \dfrac{n^2(n+1)}{2}$

n^{th} term of LHS $= P(n) - P(n-1)$

$$\Rightarrow \quad P(n) - P(n-1) = \frac{n^2(n+1)}{2} - \frac{(n-1)^2 n}{2}$$

$$\Rightarrow \quad P(n) - P(n-1) = \frac{n}{2}\{n^2 + n - n^2 + 2n - 1\}$$

$$\therefore \quad T_n = P(n) - P(n-1) = \frac{n}{2}(3n-1)$$

Hence, the correct answer is (B).

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1. $(n+1)^7 - n^7 - 1 = {}^7C_1 n^6 + {}^7C_2 n^5 + {}^7C_3 n^4 + {}^7C_4 n^3 + {}^7C_5 n^2 + {}^7C_6 n$ is always divisible by 7

$P(n) : n^7 - n$ is divisible by 7

$P(1)$: 0 is divisible by 7 true

Let $P(k)$: $k^7 - k$ is divisible by 7 (true)

Now we need to prove $P(k+1)$: $(k+1)^7 - k - 1$ is divisible by 7

$$(k+1)^7 - k - 1 = [(k+1)^7 - k^7 - 1] + k^7 - k$$

As we know that $[(k+1)^7 - k^7 - 1]$ is divisible by 7 from statement 1 and $k^7 - k$ is divisible by 7 from $p(k)$

Hence $p(k+1)$ is also true

Hence, the correct answer is (B).

CHAPTER 9: STATISTICS

Test Your Concepts-I

1. Since the total number students = 100 and the number of boys = 70

∴ Number of girls = (100 – 70) = 30

Now, the total marks of 100 students

= 100 × 72 = 7200

And total marks of 70 boys = 70 × 75 = 5250

Total marks of girls = 7200 – 5250 = 1950

∴ Average marks of 30 girls = $\frac{1950}{30} = 65$

2. Total number of terms $n + 1$ = odd

∴ median = $\frac{n+1+1}{2}\left(\frac{n}{2}+1\right)^{th}$ terms

⇒ $^{2n}C_{n/2}$

3. $\frac{1+2+3+\ldots+99+x}{100} = 100x$

⇒ $10000x = \frac{99\times 100}{2} + x$

⇒ $9999x = \frac{99\times 100}{2}$

⇒ $101x = \frac{100}{2}$ ⇒ $x = \frac{50}{101}$.

4. Given that mean = 21 and median = 22

Using the relations

Mode = 3 Median – 2 Mean

∴ Mode = 3(22) – 2(21) = 66 – 42 = 24

5. Here the given sequence is in G.P.

∴ $a = 1, r = 3 > 1$

$$\text{A.M.} = \frac{a(r^n-1)}{n(r-1)} = \frac{1(3^n-1)}{n(3-1)} = \frac{3^n-1}{2n}$$

6. $\frac{x_1+x_2+x_3+\ldots+x_n}{n} = \bar{x}$

∴ New mean is $\bar{x}'_B = \frac{\sum_{r=1}^{n}(x_i+2r)}{n} = \bar{x} + \frac{2n(n+1)}{2n}$

$\bar{x}' = \bar{x} + (n+1)$

Test Your Concepts-II

1. If variable is decreased by 1 then there is no effect on standard deviation.

2. $\bar{x} = \frac{\text{Sum of quantities}}{n} = \frac{n/2(a+\ell)}{n}$

$= \frac{1}{2}(1 + 1 + 100d) = 1 + 50d$

$$\text{M.D.} = \frac{1}{101}(50d + 49d + \ldots + d + 0 + d + \ldots + 50d)$$

$$255 = \frac{2d}{101}\left(\frac{50\times 51}{2}\right) \Rightarrow d = \frac{255\times 101}{50\times 51} = 10.1$$

3. Since root mean square ≥ A.M.

$$\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \geq \frac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \sqrt{\frac{400}{n}} \geq 5 \Rightarrow n \leq 16$$

4. Variance of first n natural numbers is $\frac{n^2-1}{12}$

So, for $n = 20$ ⇒ $\frac{20^2-1}{12} = \frac{133}{4}$.

5. Mean = $\frac{-1+0+4}{3} = 1.$

Hence M.D. (about mean) = $\frac{|-1-1|+|0-1|+|4-1|}{3}$

$= 2$

6. We know that, S.D. of first n natural number = $\sqrt{\frac{n^2-1}{12}}$

∴ S.D. of first 10 natural numbers = $\sqrt{\frac{(10)^2-1}{12}}$

$= \sqrt{\frac{99}{12}} = \sqrt{8.25} = 2.87$

7. Given, data are 6, 5, 9, 13, 12, 8 and 10.

x_i	x_i^2
6	36
5	25
9	81
13	169
12	144
8	64
10	100
$\Sigma x_i = 63$	$\sum x_i^2 = 619$

∴ S.D. = $\sigma = \sqrt{\frac{\sum x_i^2}{N} - \left(\frac{\sum x_i}{N}\right)^2} = \sqrt{\frac{619}{7} - \left(\frac{63}{7}\right)^2}$

⇒ $= \sqrt{\frac{7\times 619 - 3969}{49}} = \sqrt{\frac{52}{7}}$.

8. $CV_1 = 60, CV_2 = 70, \sigma_1 = 21, \sigma_2 = 16$

Let $\bar{x}_1$, $\bar{x}_2$ be the means of 1st and 2nd distribution respectively.

$$\Rightarrow \quad CV_1 = \frac{\sigma_1}{\bar{x}_1} \times 100$$

$$\Rightarrow \quad \bar{x}_1 = \frac{\sigma_1}{CV_1} \times 100 = \frac{21}{60} \times 100 = 35$$

Similarly, $\bar{x}_2 = \frac{\sigma_2}{CV_2} \times 100 = \frac{16}{70} \times 100 = 22.85$

9. Data: 3, 4, 5, 7, 10, 10, 10

Median is 7 : Mean is also 7 $\left[\text{check: } \frac{49}{7} = 7\right]$ but mode is 10.

M.D. (from mode)

$$= \frac{1}{7}(7+6+5+3+0+0+0) = \frac{1}{7}(21) = 3$$

Single Correct Choice Type Questions

1. Each variable is increased by 20 so there will be no effect on standard deviation.

Hence, the correct answer is (C).

2. $$\bar{x} = \frac{0.{}^nC_0 + 1.{}^nC_1 + 2.{}^nC_2 + \ldots + n.{}^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n}$$

$$\Rightarrow \quad \bar{x} = \frac{\sum_{r=0}^{n} r.{}^nC_r}{\sum_{r=0}^{n} {}^nC_r} \Rightarrow \bar{x} = \frac{1}{2^n}\sum_{r=1}^{n} r.\frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$\Rightarrow \quad \bar{x} = \frac{n}{2^n}\sum_{2^n}^{n} {}^{n-1}C_{r-1} = \frac{n}{2^n}2^{n-1} = \frac{n}{2}.$$

Hence, the correct answer is (B).

3. According to the given condition,

$$6.80 = \frac{[(6-a)^2 + (6-b)^2 + (6-8)^2 + (6-5)^2 + (6-10)^2]}{5}$$

$$\Rightarrow \quad 34 = (6-a)^2 + (6-b)^2 + 4 + 1 + 16$$

$$\Rightarrow \quad (6-a)^2 + (6-b)^2 = 13 = 9 + 4 = 3^2 + 2^2$$

$$\Rightarrow \quad a = 3, b = 4$$

Hence, the correct answer is (A).

4. Now, $$\mu_1 = \frac{\sum_{r=0}^{n} r.{}^nC_r}{\sum_{r=0}^{n} {}^nC_r} = \frac{n \cdot 2^{n-1}}{2^n} = \frac{n}{2}$$

$$\mu_2 = \frac{\sum_{r=0}^{n} r^2.{}^nC_r}{\sum_{r=0}^{n} {}^nC_r} = \frac{n(n-1)}{2^n} \cdot 2^{n-2} + \frac{n}{2}$$

$$= \frac{n(n-1)}{4} + \frac{n}{2}$$

$\therefore$ Variance $\mu_2 = (\mu_2) - (\mu_1)^2 = \frac{n(n-1)}{4} + \frac{n}{2} - \frac{n^2}{4} = \frac{n}{4}$

Hence, the correct answer is (D).

5. Correct mean $= \frac{10 \times 16.3 + 23 - 32}{10} = \frac{154}{10} = 15.4$.

Hence, the correct answer is (B).

6. Let $x_1, x_2, \ldots, x_n$ be n items. Then $\bar{x} = \frac{1}{n}\Sigma x_i$.

Let $y_1 = x_1 + 1, y_2 = x_2 + 2, y_3 = x_3 + 3, \ldots, y_n = x_n + n$.

$\therefore$ The mean of the new series is

$$\bar{y} = \frac{1}{n}\Sigma y_i = \frac{1}{n}\Sigma(x_i + i)$$

$$\Rightarrow \quad \bar{y} = \frac{1}{n}\Sigma x_i + \frac{1}{n}(1 + 2 + 3 + \ldots + n)$$

$$\Rightarrow \quad \bar{y} = \bar{x} + \frac{1}{n}.\frac{n(n+1)}{2} = \bar{x} + \frac{n+1}{2}.$$

Hence, the correct answer is (C).

7. Let $n_1 = 7, \bar{x}_1 = 10, n_2 = 3, \bar{x}_2 = 5$.

So, combined mean $= \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{85}{10} = 8.5$.

Hence, the correct answer is (C).

8. The median is the value of 5th item when items are arranged in increasing or decreasing order $\Rightarrow$ median is unchanged.

Hence, the correct answer is (C).

9. Let $d_i = x_i - 8$ but $\sigma_x^2 = \sigma_d^2 = \frac{1}{18}\Sigma d_i^2 - \left(\frac{1}{18}\Sigma di\right)^2$

$$= \frac{1}{18} \times 45 - \left(\frac{9}{18}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}.$$

$$\therefore \quad \sigma_x = \frac{3}{2}.$$

Hence, the correct answer is (C).

10. We know that if $x_{\text{new}} \to ax_{\text{old}} + b$ then $\sigma_{\text{new}} \to |a|\sigma_{\text{old}}$.

In this case $x_{\text{new}} \to -2x_{\text{old}} - 3$

Thus $\sigma_{\text{new}} = |a|\sigma_{\text{old}} = 2(3.5) = 7$

Hence, the correct answer is (C).

11. It is true that mode can be computed from histogram and median is not independent of change of scale. But variance is independent of change of origin and not of scale.

Hence, the correct answer is (D).

12. $$\frac{\Sigma x_i^2}{n} \ge \left(\frac{\Sigma x_i}{n}\right)^2 \Rightarrow \frac{400}{n} \ge \frac{6400}{n^2} \Rightarrow n \ge 16.$$

Hence, the correct answer is (B).

13. $l = 10, f = 42, c = 28, n = 100, h = 5.$

$$\therefore \quad \text{Median} = l + \frac{N/2 - c}{f} \times h = 10 + \frac{100/2 - 28}{42} \times 5$$

$$= 10 + \frac{22 \times 5}{42} = 10 + 2.62 = 12.62.$$

Hence, the correct answer is (B).

14. There is no change in the standard deviation if each observation is increased by a constant number while the mean

is increased by that constant number while the mean is increased by that constant number.

Hence, Mean = 30 + 2 = 32, S.D. = 2

Hence, the correct answer is (A).

15. Mode $= l + \dfrac{f - f_1}{2f - f_1 - f_2} \times c = 20 + \dfrac{18-8}{36-8-10} \times 5$

$$= 20 + \frac{50}{18} = 20 + 2.78 = 22.78.$$

Hence, the correct answer is (D).

16. Mean $= \bar{x} = 0$

Standard Deviation = 2

$$= \sqrt{\left(\frac{1}{2n}\sum_{i=1}^{2n} x_i^2\right) - \bar{x}^2} = \sqrt{\left(\frac{1}{2n}\sum_{i=1}^{2n} a^2\right) - 0} = |a|$$

We get $|a| = 2$

Hence, the correct answer is (C).

17. Mean deviation is minimum when it is considered about the item, equidistant from the beginning and the end i.e., the median. In this case median is $\dfrac{101+1}{2}$th i.e., 51st item i.e., x_{51}.

Hence, the correct answer is (B).

18. Here for each $x_i = i$

Weight $w_i = i^2 + i$

Hence, the required mean $= \dfrac{\sum w_i x_i}{\sum w_i} = \dfrac{\sum_{i=1}^{n} i\left(i^2+i\right)}{\sum_{i=1}^{n}\left(i^2+i\right)}$

$$= \frac{\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i} = \frac{\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}}$$

$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2n+1}{3}\right\}}{\frac{n(n+1)}{2}\left\{\frac{2n+1}{3} + 1\right\}}$$

$$= \frac{3n^2+7n+2}{2(2n+4)} = \frac{(3n+1)(n+2)}{4(n+2)} = \frac{3n+1}{4}.$$

Hence, the correct answer is (C).

19. $\sum (x_i - \bar{x})^2 = 250$

$\therefore \quad \sigma^2 = \dfrac{250}{10} = 25$

$\therefore \quad \sigma^2 = 25 \quad \Rightarrow \quad \sigma = 5$

Or coefficient of variation is $= \dfrac{\sigma}{\bar{x}} \times 100$

$$= \frac{5}{50} \times 100 = 10\%$$

Hence, the correct answer is (A).

20. Given $\dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \bar{x}$

Then new mean is

$$\frac{x_1 + x_2 + x_3 + \ldots + x_n + 1^2 + 2^2 + \ldots + n^2}{n}$$

$$= \bar{x} + \frac{(n+1)(2n+1)}{6}$$

Hence, the correct answer is (A).

21. For group B, data are more scattered then group A, so variability of B has more than A but mean is same.

Hence, the correct answer is (C).

22. If standard deviations for data x_i is a then new S.D. for $\dfrac{a}{c}x + \dfrac{b}{c}$ is $\left|\dfrac{a}{c}\right|\sigma$ because by multiplying any constant $\dfrac{a}{c}$, S.D. has $\left|\dfrac{a}{c}\right|$ times but for addition of any constant there will be no effect.

Hence, the correct answer is (A).

23. A.M. $= \dfrac{{}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n}{n+1} = \dfrac{\frac{1}{2}\left(2^{2n+1}\right)}{n+1}$

$$= \frac{2^{2n}}{n+1}$$

Hence, the correct answer is (D).

24. Arrange in ascending order

$\Rightarrow \quad \alpha - \dfrac{7}{2}, \alpha - 3, \alpha - \dfrac{5}{2}, \alpha - 2, \alpha - \dfrac{1}{2}, \alpha + \dfrac{1}{2},$

$\alpha + 4, \alpha + 5$

$\Rightarrow \quad \dfrac{1}{2}[4^{\text{th}} + 5^{\text{th}} \text{ value}] \quad \Rightarrow \quad \dfrac{1}{2}\left[2\alpha - \dfrac{5}{2}\right] \quad \Rightarrow \quad \alpha - \dfrac{5}{4}$

Hence, the correct answer is (A).

25. Mean of series is

$$\bar{x} = \frac{a + (a+d) + (a+2d) + \ldots + (a+2nd)}{(2n+1)}$$

$$\bar{x} = a + nd$$

$$\therefore \quad \sum_{i=0}^{2n} |x_i - \bar{x}|a \quad \Rightarrow \quad \frac{2d(n)(n+1)}{2} \quad \Rightarrow \quad n(n+1)d$$

$$\therefore \quad \text{Mean deviation} = \frac{n(n+1)d}{(2n+1)}$$

Hence, the correct answer is (C).

26. $n_1 = 35,\ \bar{x}_1 = 80,\ \sigma_1 = 4$

$n_2 = 65,\ \bar{x}_2 = 60,\ \sigma_2 = 4$

$$\sigma^2 = \frac{n_2(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_1^2)}{n_1 + n_2}$$

Combined mean $= \dfrac{35 \times 80 + 65 \times 60}{100} = 67$

$\therefore \quad d_1 = 80 - 67 = 13,\ d_2 = 60 - 67 = -7$

$$\therefore \quad \sigma^2 = \frac{35(16+169) + 65(9+49)}{100}$$

$$\sigma^2 \Rightarrow 102.45$$

$$\therefore \quad \sigma = \sqrt{102.45} = 10.12$$

Hence, the correct answer is (C).

27. These is no effect due to addition of constant in variables.
Hence, the correct answer is (A).

28. Given: Number $x_1, x_2, x_3, \ldots, x_n$ and their average = M. We know that the average of n numbers

$$M = \frac{x_1+x_2+x_3+\ldots+x_n}{n}$$

$$\text{or} \quad \frac{nM+x}{n} = \frac{x_1+x_2+x_3+\ldots+x_n+x}{n}$$

$$\text{or} \quad \frac{nM+x}{n} = \frac{x_1}{n} + \frac{x_2+x_3+\ldots+x_n+x}{n}$$

$$\text{or} \quad \frac{nM+x-x_1}{n} = \frac{x_2+x_3+\ldots+x_n+x}{n}$$

or new average

$$= \left(\frac{x_2+x_3+\ldots+x_n+x}{n}\right) = \frac{nM-x_1+x}{n}$$

Hence, the correct answer is (D).

29. Mean of n items is $\bar{x}$
New mean

$$= \frac{x_1+x_2+\ldots+x_n}{n} + \frac{3+3^2+3^3+\ldots+3^n}{n}$$

$$\Rightarrow \quad \bar{x} + \frac{3(3^n-1)}{2n}$$

Hence, the correct answer is (D).

30. Average speed $= \dfrac{120+120+120}{\dfrac{120}{30}+\dfrac{120}{25}+\dfrac{120}{50}}$

$$= \frac{3}{\dfrac{1}{30}+\dfrac{1}{25}+\dfrac{1}{50}} \text{ km/hr.}$$

Hence, the correct answer is (C).

31. Required mean

$$= \frac{(ax_1+b)+(ax_2+b)+\ldots+(ax_n+b)}{n}$$

$$= \frac{a(x_1+x_2+\ldots+x_n)+nb}{n} = a\bar{x} + b$$

$$\left[\because \frac{x_1+x_2+\ldots+x_n}{n} = \bar{x}\right]$$

Hence, the correct answer is (D).

32. Weighted mean $= \dfrac{1.1^2+2.2^2+\ldots+n.n^2}{1^2+2^2+\ldots+n^2}$

$$= \frac{\sum n^3}{\sum n^2} = \frac{\dfrac{n(n+1)}{2}\dfrac{n(n+1)}{2}}{\dfrac{n(n+1)(2n+1)}{6}} = \frac{3n(n+1)}{2(2n+1)}$$

Hence, the correct answer is (B).

33. Mean $= \dfrac{1.1+\dfrac{1}{2}.2+\dfrac{1}{3}.3+\dfrac{1}{4}.4+\dfrac{1}{5}.5+\ldots+\dfrac{1}{n}.n}{1+2+3+\ldots+n}$

$$= \frac{1+1+1+1+\ldots+1}{\dfrac{n(n+1)}{2}} = \frac{n}{\dfrac{n(n+1)}{2}} = \frac{2}{n+1}$$

Hence, the correct answer is (D).

34. $\dfrac{x_1+x_2+\ldots+x_{10}}{10} = 4.5$

$$\Rightarrow \quad x_1+x_2+\ldots+x_{10} = 45$$

$$\text{and} \quad \frac{x_{11}+x_{12}+\ldots+x_{40}}{30} = 3.5$$

$$\Rightarrow \quad x_{11}+x_{12}+\ldots+x_{40} = 105$$

$$\therefore \quad x_1+x_2+\ldots+x_{40} = 150$$

$$\therefore \quad \frac{x_1+x_2+\ldots+x_{40}}{40} = \frac{150}{40} = \frac{15}{4}$$

Hence, the correct answer is (B).

35. $\dfrac{x_1+x_2+\ldots+x_{16}}{16} = 16$

If $x_1 = 16$

$$\frac{x_1+x_2+\ldots+x_{16}-16+3+4+5}{18}$$

$$= \frac{16\times16-16+12}{18} = \frac{240+12}{18} = \frac{252}{18} = 14$$

Hence, the correct answer is (D).

36. Since $x > 0$,

$$\frac{x}{5}, \frac{x}{4}, \frac{x}{3}, \frac{x}{2}, x \text{ is in ascending order.}$$

$$\therefore \quad \text{Median} = \frac{x}{3}$$

Also, median = 8. So, $x = 24$.

Hence, the correct answer is (A).

37. No impact on Standard Deviation.
Hence, the correct answer is (A).

38. Standard deviation of numbers 2, 3, a and 11 is 3.5

$$\therefore \quad (3.5)^2 = \frac{\sum x_1^2}{4} - (\bar{x})^2$$

$$\Rightarrow \quad (3.5)^2 = \frac{4+9+a^2+121}{4} - \left(\frac{2+3+a+11}{4}\right)^2$$

On solving, we get $3a^2 - 32a + 84 = 0$

Hence, the correct answer is (A).

39. Given, observations are 3, 10, 10, 4, 7, 10 and 5.

$$\therefore \quad \bar{x} = \frac{3+10+10+4+7+10+5}{7} = \frac{49}{7} = 7$$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
3	4
10	3
10	3
4	3
7	0
10	3
5	2
Total	$\Sigma d_i = 18$

Now, M.D. $= \dfrac{\sum d_i}{N} = \dfrac{18}{7} = 2.57$

Hence, the correct answer is (B).

40. Since, marks obtained by 9 students in Mathematics are 50, 69, 20, 33, 53, 39, 40, 65 and 59 Rewrite the given data in ascending order 20, 33, 39, 40, 50, 53, 59, 65, 69.
Here, $n = 9$ [odd]

$\therefore$ Median $= \left(\dfrac{9+1}{2}\right)$ term $= 5^{\text{th}}$ term

Median $= 50$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
20	30
33	17
39	11
40	10
50	0
53	3
59	9
65	15
69	19
Total	$\Sigma d_i = 114$

$\therefore$ M.D. $= \dfrac{114}{9} = 12.67$

Hence, the correct answer is (C).

41. Given observation are a, b, c, d and e.

Mean $= m = \dfrac{a+b+c+d+e}{5}$

$\Sigma x_i = a+b+c+d+e = 5m$

Now, mean $= \dfrac{a+k+b+k+c+k+d+k+e+k}{5}$

$$\Rightarrow \frac{(a+b+c+d+e)+5k}{5} = m+k$$

$$\therefore \text{ S.D.} = \sqrt{\frac{\sum (x_i+k)^2}{n} - (m+k)^2}$$

$$= \sqrt{\frac{\sum (x_i^2+k^2+2kx_i)}{n} - (m^2+k^2+2mk)}$$

$$= \sqrt{\frac{\sum x_i^2}{n} - m^2 + \frac{2k\sum x_i}{n} - 2mk}$$

$$= \sqrt{\frac{\sum x_i^2}{n} - m^2 + 2mk - 2mk} \qquad \left[\because \frac{\sum x_i}{n} = m\right]$$

$$= \sqrt{\frac{\sum x_i^2}{n} - m^2} = s$$

Hence, the correct answer is (A).

42. Here, $m = \dfrac{\sum x_i}{5}$, $s = \sqrt{\dfrac{\sum x_i^2}{5} - \left(\dfrac{\sum x_i}{5}\right)^2}$

$$\therefore \text{ S.D.} = \sqrt{\frac{k^2\sum x_i^2}{5} - k^2\left(\frac{\sum x_i}{5}\right)^2}$$

$$= k\sqrt{\left(\frac{\sum x_i^2}{5}\right) - \left(\frac{\sum x_i}{5}\right)^2} = ks$$

Hence, the correct answer is (C).

43. $w_i = x_i + k$, $\bar{x}_i = 48$, $sx_i = 12$, $w_i = 55$ and $sw_i = 15$

Then, $\overline{w}_i = \bar{x}_i + k$

Now, mean of w_i = Mean of x_i

$\Rightarrow$ $55 = \ell \cdot 48 + k$ …(1)

Now, S.D. of w_i = S.D. of x_i

$\Rightarrow$ $15 = 12\ell$

$\Rightarrow$ $\ell = \dfrac{15}{12} = 1.25$ …(2)

From equation (1) and (2),

$\Rightarrow$ $k = 55 - 1.25 \times 48 = -5$

Hence, the correct answer is (A).

44. Given numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. If 1 is added to each number, then observations will be 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.

$\therefore$ $\Sigma x_i = 2 + 3 + 4 + \ldots + 11$

$$= \frac{10}{2}[2\times 2 + 9 \times 1] = 5[4+9] = 65$$

and $\sum x_i^2 = 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2$

$$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2) - (1^2)$$

$$= \frac{11 \times 12 \times 23}{6} - 1$$

$$\Rightarrow \quad \sum x_i^2 = \frac{11\times 12\times 23-6}{6} = 505$$

$$\therefore \quad \sigma^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2 = \frac{505}{10} - \left(\frac{65}{10}\right)^2$$

$$= 50.5 - (6.5)^2 = 50.5 - 42.25 = 8.25$$

Hence, the correct answer is (D).

45. Since, the first 10 positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.
On multiplying each number by –1 we get
$-1, -2, -3, -4, -5, -6, -7, -8, -9, -10$
On adding 1 in each number, we get
$0, -1, -2, -3, -4, -5, -6, -7, -8, -9$

$$\therefore \quad \Sigma x_i = 0 - 1, 2, 3, 4, 5, 6, 7, 8, 9 = \frac{-9\times 10}{2} = -45$$

and $\sum x_i^2 = 0^2 + (-1)^2 + (-2)^2 + \ldots + (-9)^2$

$$= \frac{9\times 10\times 19}{6} = 285$$

$$\therefore \quad \text{S.D.} = \sqrt{\frac{285}{10} - \left(\frac{-45}{10}\right)^2} = \sqrt{\frac{285}{10} - \frac{2025}{100}}$$

$$= \sqrt{\frac{2850-2025}{100}} = \sqrt{8.25}$$

Now variance $= (\text{S.D.})^2 = (\sqrt{8.25})^2 = 8.25$

Hence, the correct answer is (A).

46. Given, $\frac{\sum x_i}{50} = 38$,

$\therefore \quad \Sigma x_i = 1900$

New value of $\Sigma x_i = 1900 - 55 - 45 = 1800$, $n = 48$

$$\therefore \quad \text{New mean} = \frac{1800}{48} = 37.5$$

Hence, the correct answer is (B).

47. $\frac{2+4+6+a+b}{5} = 4 \quad \Rightarrow \quad a+b = 8$ …(1)

$$\frac{4+16+36+a^2+b^2}{5} - 16 = 5.2$$

$$56 + a^2 + b^2 = 21.2\times 5$$

$a^2 + b^2 = 50$ …(2)

Solve (1) and (2) we get

$a = 1, b = 7.$

Hence, the correct answer is (D).

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

1. Let the total number of students be x.
Then according to question,
Average of remaining $(x - 12)$ students = 72

$$\therefore \quad \frac{10(100)+2(0)+72(x-12)}{x} = 76$$

$$\Rightarrow \quad 1000 + 72x - 864 = 76x \quad \Rightarrow \quad x = 34.$$

Hence, the correct answer is 34.

2. $\sigma_c = 5 \quad \Rightarrow \quad \frac{5}{9}(F-32) = C$

$$F = \frac{9C}{5} + 32$$

$$\sigma_F = \frac{9}{5}\sigma_c = \frac{9}{5}\times 5 = 9$$

Here, $\sigma_F^2 = (9)^2 = 81.$

Hence, the correct answer is 81.

3. Let $x_1, x_2, x_3, \ldots x_9$ are 9 items they

$$\Rightarrow \quad \frac{x_1+x_2+x_3+\ldots+x_9}{9} = 15$$

$$\therefore \quad \sum_{i=1}^{9} x_i = 135$$

If one more item x_{10} is added then

$$\sum_{i=1}^{10} x_i = 135 + x_{10}$$

$$\therefore \quad \text{Mean } \bar{x} = \frac{135+x_{10}}{10} = 16$$

$$\therefore \quad x_{10} = 160 - 135 = 25$$

Hence, the correct answer is 25.

4. $$\text{Mean} = \frac{0\cdot{}^nC_0 + 2\cdot{}^nC_1 + 4\cdot{}^nC_2 + \ldots + 2n\cdot{}^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n} = \frac{91\times 8}{2^n}$$

$$\Rightarrow \quad \frac{3^n-1}{2^n} = \frac{91\times 8}{2^n} \quad \Rightarrow \quad 3^n - 1 = 728 \quad \Rightarrow \quad n = 6$$

Hence, the correct answer is 6.

5. Data is multiplied by 5 then variance should be 25 times
So new variance = 25 × 9 = 225
Hence, the correct answer is 225.

6.

Marks (x)	Frequency	Cumulative Frequency
0–10	10	10
10–20	20	30
20–30	30	60
30–40	50	110
40–50	40	150
50–60	30	180

$N = 180$

then $\dfrac{N}{2} = 90^{\text{th}}$ item

Median in Class $= 30 - 40$, $\ell = 30$, $C = 60$, $f = 50$ and $i = 10$

$$\text{Median} = \ell + \frac{\left(\frac{N}{2} - c\right)}{f} \times i = 30 + \left(\frac{90 - 60}{50}\right)10 = 36$$

Hence, the correct answer is 36.

7. Since, the lives of 5 bulbs are 1357, 1090, 1666, 1494 and 1623.

$\therefore \quad \text{Mean} = \dfrac{1357 + 1090 + 1666 + 1494 + 1623}{5}$

$\Rightarrow \quad \dfrac{7230}{5} = 1446$

x_i	$d_i = \lvert x_i - \bar{x}\rvert$
1357	89
1090	356
1666	220
1694	248
1623	177
Total	$\Sigma d_i = 890$

$$\text{M.D.} = \frac{\sum d_i}{N} = \frac{890}{5} = 218.$$

Hence, the correct answer is 218.

8. $\bar{x} = 50$, $n = 100$ and $\sigma = 5$, now $\sum x_i^2 = ?$

$\therefore \quad \bar{x} = \dfrac{\sum x_i}{n} \quad \Rightarrow \quad 50 = \dfrac{\sum x_i}{100}$

$\therefore \quad \sum x_i = 50 \times 100 = 5000$

$$\text{Now, } \sigma = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$\Rightarrow \quad \sigma^2 = \dfrac{\sum x_i^2}{n} - (\bar{x})^2$

$\Rightarrow \quad 25 = \dfrac{\sum x_i^2}{100} - (50)^2 \quad \Rightarrow \quad 25 = \dfrac{\sum x_i^2}{100} - 2500$

$\Rightarrow \quad 2525 = \dfrac{\sum x_i^2}{100}$

$\therefore \quad \sum x_i^2 = 252500.$

Hence, the correct answer is 252500.

9. $\text{Variance} = \dfrac{\sum x_i^2}{n} - \left(\dfrac{\sum x_i}{n}\right)^2$

$$= \frac{18000}{60} - \left(\frac{960}{60}\right)^2 = 300 - 256 = 44.$$

Hence, the correct answer is 44.

10. According to question,

Mean $= 6 \quad \Rightarrow \quad a + b + 8 + 5 + 10 = 30$

$\Rightarrow \quad a + b = 7$ …(1)

Also, variance $= 6.8$

$\Rightarrow \quad (a - 6)^2 + (b - 6)^2 + 2^2 + 1^2 + 4^2 = (6.8) \times 5$

$\Rightarrow \quad a^2 + b^2 = 25$ …(2)

On solving (1) and (2), we get $ab = 12$

Hence, the correct answer is 12.

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1.

x	0	2	4	8		2^n
f	${}^nC_0$	${}^nC_1$	${}^nC_2$	${}^nC_3$		${}^nC_n$

$$\text{Mean} = \frac{\sum x_1 f_1}{\sum f_1} = \frac{\sum_{r=1}^{n} 2^r\, {}^nC_r}{\sum_{r=0}^{n} {}^nC_r} = \frac{(1+2)^n - {}^nC_0}{2^n} = \frac{728}{2^n}$$

$\Rightarrow \quad \dfrac{3^n - 1}{2^n} = \dfrac{728}{2^n} \quad \Rightarrow \quad 3^n = 729 \quad \Rightarrow \quad n = 6$

Hence, the correct answer is (6.00).

2. $\because$ Variance is independent of shifting of origin

$\Rightarrow \quad x_1 : 15 \quad 25 \quad 35 \quad \text{or} \quad -10 \quad 0 \quad 10$

$f_1 : 2 \quad x \quad 2 \qquad\qquad 2 \quad x \quad 2$

$\Rightarrow \quad \text{Variance } (\sigma^2) = \dfrac{\Sigma x_i^2 f_i}{\Sigma f_1} - (\bar{x})^2$

$\Rightarrow \quad 50 = \dfrac{200 + 0 + 200}{x + 4} - 0 \qquad \{\bar{x} = 0\}$

$\Rightarrow \quad x = 4$

Hence, the correct answer is (4).

3. Let a be the first term and d be the common difference of the given A.P. Where $d > 0$

$$\bar{X} = a + \frac{0 + d + 2d + \ldots + 10d}{11} = a + 5d$$

$\Rightarrow \quad \text{variance} = \dfrac{\Sigma\left(\bar{X} - x_i\right)^2}{11}$

$\Rightarrow \quad 90 \times 11 = (25d^2 + 16d^2 + 9d^2 + 4d^2 + d^2)2$

$\Rightarrow \quad d = \pm 3 \quad \Rightarrow \quad d = 3$

Hence, the correct answer is (3.00).

4. Variance of first 'n' natural numbers

$$= \frac{\sum_{r=1}^{n} r^2}{n} - \left(\frac{\sum_{r=1}^{n} r}{n}\right)^2 = \frac{(n+1)(2n+1)}{6} - \left(\frac{\frac{n(n+1)}{2}}{n}\right)^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{(n+1)^2}{4} = \frac{4n^2+6n+2-3n^2-6n-3}{12}$$

$$= \frac{n^2-1}{12} = 10$$

$\Rightarrow \quad n = 11$

variance of first 'm' even natural numbers

$$= 4\left(\frac{m^2-1}{12}\right) \Rightarrow \frac{m^2-1}{3} = 16 \Rightarrow m = 7$$

$m + n = 18$

Hence, the correct answer is (18).

5. $\dfrac{3+7+9+12+13+20+x+y}{8} = 10 \Rightarrow x + y = 16$

$3^2 + 7^2 + 9^2 + 13^2 + 20^2 + x^2 + y^2 = 1000$

$\Rightarrow \quad x^2 + y^2 = 148$

$xy = 54$

Hence, the correct answer is (54.00).

6. $20p - q = 10$ …(1)

and $2|p| = 1 \Rightarrow p = \pm\frac{1}{2}$ …(2)

so, $p = -\frac{1}{2}$ and $q = -20$

Hence, the correct answer is (D).

7. Mean $= \dfrac{\sum x_i}{20} = 10 \Rightarrow \Sigma x_i = 200$ …(1)

Variance $= \dfrac{\sum x_i^2}{20} - 100 = 4 \Rightarrow \Sigma x_i^2 = 2080$ …(2)

Actual mean $= \dfrac{200-9+11}{20} = \dfrac{202}{20}$

Variance $= \dfrac{2080-81+121}{20} - \left(\dfrac{202}{20}\right)^2 = 3.99$

Hence, the correct answer is (C).

8. $\sum_{i=1}^{10}(x_i - 5) = 10$

$\Rightarrow$ Mean of observation $x_i - 5 = \dfrac{1}{10}\sum_{i=1}^{3}(x_i - 5) = 1$

μ = mean of observation $(x_i - 3)$

= (mean of observation $(x_i - 5)$) + 2 = 1 + 2 = 3

Variance of observation $x_i - 5$

$$= \frac{1}{10}\sum_{i=1}^{10}(x_i-5)^2 - (\text{Mean of } (x_i-5))^2 = 3$$

$\Rightarrow \quad \lambda$ = variance of observation $(x_i - 3)$

= variance of observation $(x_i - 5) = 3$

$\therefore \quad (\mu, \lambda) = (3, 3)$

Hence, the correct answer is (C).

9. Mean $= \mu = \dfrac{\sum_{x=1}^{17}(ax+b)}{17} = 17 \Rightarrow 9a + b = 17$ …(1)

Variance $= \sigma^2 = 216 = \dfrac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$

$$= \frac{\sum_{x=1}^{17}(ax+b-17)^2}{17} = 216$$

$$\Rightarrow \frac{\sum_{x=1}^{17} a^2(x-9)^2}{17} = \frac{a^2\sum_{x=1}^{17}(x^2-18x+81)}{17}$$

$$= \frac{a^2\left(\frac{17.18.35}{6} - 18.\frac{17.18}{2} + 81.17\right)}{17}$$

$= 216$

$\Rightarrow \quad a^2(3.35 - 18 \times 9 + 81) = 216$

$\Rightarrow \quad a^2 = \dfrac{216}{24} = 9 \Rightarrow a = 3\ (a > 0)$

From (1), $b = -10$ So we get $a + b = -7$

Hence, the correct answer is (A).

10. $\because \quad \sigma^2 \le \dfrac{1}{4}(M - m)^2$

Where M and m are upper and lower bounds of values of any random variable.

$\therefore \quad \sigma^2 < \dfrac{1}{4}(10 - 0)^2 \Rightarrow 0 < \sigma < 5$

$\therefore \quad \sigma \ne 6$

Hence, the correct answer is (D).

11. Variance $= \dfrac{\Sigma(x_i-p)^2}{n} - \left(\dfrac{\Sigma(x_i-p)}{n}\right)^2 = \dfrac{9}{10} - \left(\dfrac{3}{10}\right)^2$

$= \dfrac{81}{100}$

S.D. $= \dfrac{9}{10}$

Hence, the correct answer is (C).

12. $\bar{x} = 10$

$\Rightarrow \quad \bar{x} = \dfrac{63+a+b}{8} = 10 \Rightarrow a + b = 17$ …(1)

Since, variance is independent of origin.
So, we subtract 10 from each observation.

So, $\sigma^2 = 13.5 = \dfrac{79+(a-10)^2+(b-10)^2}{8} - (10-10)^2$

$\Rightarrow \quad a^2 + b^2 - 20(a+b) = -171 \Rightarrow a^2 + b^2 = 169$ …(2)

From (1) and (2); $a = 12$ and $b = 5$

Hence, the correct answer is (A).

13. Mean $= \dfrac{3+5+7+a+b}{5} = 5 \Rightarrow a+b=10$ …(1)

$$\text{S.d.} = \sqrt{\frac{\sum_{i=1}^{5}(x_i-\bar{x})^2}{5}} = 2$$

$\Rightarrow (3-5)^2+(5-5)^2+(7-5)^2+(a-5)^2+(b-5)^2 = 20$

$\Rightarrow 4+0+4+(a-5)^2+(b-5)^2 = 20$

$\Rightarrow a^2+b^2-10(a+b)+50 = 12$

$\Rightarrow (a+b)^2-2ab-100+50 = 12 \Rightarrow ab = 19$ …(2)

Equation is $x^2-10x+19=0$

Hence, the correct answer is (B).

14. $\bar{x} = \dfrac{2+4+10+12+14+x+y}{7} = 8 \Rightarrow x+y=14$ …(1)

$$(\sigma)^2 = \frac{\sum(x_i)^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

$$\Rightarrow 16 = \frac{4+16+100+144+196+x^2+y^2}{7} - 8^2$$

$$\Rightarrow 16+64 = \frac{460+x^2+y^2}{7}$$

$\Rightarrow 560 = 460+x^2+y^2 \Rightarrow x^2+y^2 = 100$ …(2)

Clearly by (1) and (2),

$(x+y)^2+(x-y)^2 = 2(x^2+y^2) = 200$

$\Rightarrow 196+(x-y)^2 = 200 \Rightarrow |x-y| = 2$

Hence, the correct answer is (A).

15. $$\text{S.D.} = \sqrt{\frac{\sum_{i=1}^{n}(x_1-a)^2}{n} - \left(\frac{\sum_{i=1}^{n}(x_1-a)}{n}\right)^2} = \sqrt{\frac{na}{n}-\left(\frac{n}{n}\right)^2}$$

$= \sqrt{a-1}$

$\left\{\text{Given } \sum_{i=1}^{n}(x_1-a) = n;\ \sum_{i=1}^{n}(x_1-a)^2 = na\right\}$

Hence, the correct answer is (B).

16. $\dfrac{x_1+x_2+x_3+x_4+x_5}{5} = 150$ cm

$\Rightarrow x_1+x_2+x_3+x_4+x_5 = 150\times 5 = 750$

$$\sigma = \frac{1}{5}\sum_{i=1}^{5}x_i^2 - \bar{x}^2$$

$$\Rightarrow 18 = \frac{1}{5}(x_1^2+x_2^2+x_3^2+x_5^2)-(150)^2$$

$$\Rightarrow 18+(150)^2 = \frac{1}{5}\left(\sum_{i=1}^{5}x_1^2\right)$$

$$\Rightarrow 112590 = \sum_{i=1}^{5}x_1^2 \qquad \text{from (1)}$$

Now new student of height 156 cm

$$\sigma = \frac{1}{6}\sum_{i=1}^{6}x_1^2-(\bar{x})^2$$

$$= \frac{1}{6}[x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2] - \left(\frac{750+156}{6}\right)^2 \quad \text{…(2)}$$

Now (2) $\dfrac{1}{6}[112590+(156)^2] - \left(\dfrac{750+156}{6}\right)^2$

$= 22821-(151)^2 = 22821-22801 = 20$

Hence, the correct answer is (A).

17. $\sum_{i=1}^{n}(x_1+1)^2 = 9n \Rightarrow \sum_{i=1}^{n}x_i^2+2\sum_{i=1}^{n}x_i+n = 9n$ …(1)

$\sum_{i=1}^{n}(x_i-1)^2 = 5n \Rightarrow \sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i+n = 5n$ …(2)

(1) – (2)

we get $4\sum_{i=1}^{n}x_i = 4n \Rightarrow \sum_{i=1}^{n}x_i = n;$

(1) + (2) we get $\sum_{i=1}^{n}x_i^2 = 6n$

$$\text{S.D. } \sigma = \sqrt{\frac{\sum x_1^2}{n}-(\bar{x})^2} = \sqrt{\frac{6n}{n}-(1)} = \sqrt{5}$$

Hence, the correct answer is (B).

18. $\mu = 5$; variance $= 9.20$

$\mu = \dfrac{1+3+8+x_1+x_2}{5} = 5 \Rightarrow x_1+x_2 = 13$

$9.20 = \dfrac{1}{n}\sum x_1^2-\mu^2 \Rightarrow \sum x_1^2 = (9.20+25)5$

$1+9+64+x_1^2+x_2^2 = 171$

$x_1^2+x_2^2 = 97 \Rightarrow x_1^2+(13-x_1)^2 = 97$

$(x_1-9)(x_1-4) = 0$

$x_1 = 9; x_2 = 4$ or $x_1 = 4; x_2 = 9$

$x_1 : x_2 = 4:9$ or $9:4$

Hence, the correct answer is (C).

19. S.D. = 3

$$\text{variance} = 9 = \frac{1}{5}\sum_{i=1}^{5}x_1^2-(\bar{x})^2$$

$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 5\times 109$

variance of x_1, x_2, x_3, x_4, x_5 and –50

$$\text{variance} = \frac{1}{6}\sum_{i=1}^{6}x_1^2-(\bar{x})^2$$

$$= \frac{1}{6}\left(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+2500\right)-(\bar{x})^2$$

$(\because\ \bar{x} = 0$ of $x_1, x_2, x_3, x_4, x_5, -50)$

$$= \frac{1}{6}(5 \times 109 + 2500) = 507.5$$

Hence, the correct answer is (B).

20. Variance remains some if same number is subtracted from each observation. (subtract $\frac{1}{2}$ from each observation)

$$\therefore \quad \frac{10(-d)^2 + 10(0)^2 + 10(d)^2}{30} - \left(\frac{10(-d) + 10(0) + 10(d)}{30}\right)^2 = \frac{4}{3}$$

$$\Rightarrow \quad \frac{20d^2}{30} = \frac{4}{3} \quad \Rightarrow \quad d^2 = 2 \quad \Rightarrow \quad d = \sqrt{2}$$

Hence, the correct answer is (D).

21. P(probability of getting white ball) $= \frac{30}{40}$

$$q = \frac{1}{4}, n = 16$$

$$\text{mean} = np = 16 \times \frac{3}{4} = 12$$

$$(\text{S, D}) = \sqrt{npq} = \sqrt{3}$$

$$\frac{\text{Mean of } x}{(\text{standard deviation of } x)} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$$

Hence, the correct answer is (B).

22. $$\sum_{i=1}^{50}(x_1 - 30) = 50 \quad \Rightarrow \quad \sum_{i=1}^{50} x_i = 1500 + 50$$

$$(\bar{x}) = \frac{1550}{50} = 31$$

Hence, the correct answer is (A).

23. $$\frac{3+4+4+x+y}{5} = 4$$

$$\Rightarrow \quad 11 + x + y = 20 \quad \Rightarrow \quad x + y = 9$$

$$\Rightarrow \quad \frac{1}{5}(9 + 16 + 16 + x^2 + y^2) - 16 = 5.2$$

$$\Rightarrow \quad 41 + x^2 + y^2 = 106$$

$$\Rightarrow \quad x^2 + y^2 = 65 \quad \Rightarrow \quad (x+y)^2 - 2xy = 65$$

$$\Rightarrow \quad 81 - 65 = 2xy \quad \Rightarrow \quad xy = 8$$

$$(x-y)^2 - x^2 + y^2 - 2xy = 65 - 2 \times 8 = 49$$

$$\therefore \quad |x - y| = 7$$

Hence, the correct answer is (B).

24. Let unknown observation are x_1 and x_2

$$\frac{2 + 4 + 10 + 12 + 14 + x_1 + x_2}{7} = 8$$

$$x_1 + x_2 = 14 \qquad \ldots(1)$$

$$\frac{1}{7}\Sigma x_1^2 - \bar{x}^2 = 16 \quad \text{(given)}$$

$$\Rightarrow \quad \frac{1}{7}(2^2 + 4^2 + 10^2 + 12^2 + 14^2 + x_1^2 + x_2^2) - (8) = 16$$

$$\Rightarrow \quad x_1^2 + x_2^2 = 100$$

$$\Rightarrow \quad (x_1 + x_2)^2 - 2x_1x_2 = 100 \quad \Rightarrow \quad 2x_1x_2 = 96$$

$$\Rightarrow \quad x_1x_2 = 48$$

Hence, the correct answer is (B).

25. Let unknown observation is x

$$\frac{45 + 54 + 41 + 57 + 43 + x}{6} = 48 \quad \Rightarrow \quad x = 48$$

$$\sigma^2 = \left(\frac{1}{6}(45^2 + 54^2 + 41^2 + 51^2 + 43^2 + 48^2)\right) - (48)^2$$

$$= \frac{1}{6} \times (14024) - (48)^2 = \frac{100}{3} \quad \Rightarrow \quad \sigma^2 = \frac{10}{\sqrt{3}}$$

Hence, the correct answer is (A).

26. $$\text{S.D.} = \sqrt{\frac{1}{n}\sum x_1^2 - (\bar{x})^2}$$

Now mean $\bar{x} = \frac{-1+0+1+k}{4} = \frac{k}{4}$;

Given that S.D. $= \sqrt{5}$

$$\Rightarrow \quad \sqrt{5} = \sqrt{\frac{1}{4}(1 + 0 + 1 + k^2) - \frac{k^2}{16}}$$

$$\Rightarrow \quad 5 = \frac{2 + k^2}{4} - \frac{k^2}{16}$$

$$\Rightarrow \quad 80 = 8 + 4k^2 - k^2 \quad \Rightarrow \quad 3k^2 = 72 \quad \Rightarrow \quad k^2 = 24$$

$$\Rightarrow \quad k = \pm 2\sqrt{6} \text{ acceptable value } k = 2\sqrt{6} \qquad (\because k > 0)$$

Hence, the correct answer is (D).

27. Mean $= 42$

$$\Rightarrow \quad \frac{10 + 22 + 26 + 29 + 34 + x + 42 + 67 + 70 + y}{10} = 45$$

$$\Rightarrow \quad 420 = 300 + x + y \quad \Rightarrow \quad x + y = 120 \qquad \ldots(1)$$

and medium $= 35$

$$\Rightarrow \quad \frac{34 + x}{2} = 35 \quad \Rightarrow \quad x = 36$$

From (1) $y = 84$

$$\frac{y}{x} = \frac{84}{36} = \frac{7}{3}$$

Hence, the correct answer is (D).

28. $$\because \quad \sum f_i = (x+1)^2 + (2x - 5) + (x^2 - 3x) + x = 20$$

$$\Rightarrow \quad x = 3, -4 \text{ (rejected)}$$

$$\therefore \quad \text{Mean } \bar{x} = \frac{\sum x_i f_i}{\sum f_i} = 2.8$$

Hence, the correct answer is (B).

29. $$\text{Mean } (\mu) = \frac{\sum x_i}{50} = 16$$

$$\therefore \quad \sum x_i = 16 \times 50$$

$$\text{S.D. } (\sigma) = \sqrt{\frac{\sum x_i^2}{50} - (\mu)^2} = 16 \quad \Rightarrow \quad \frac{\sum x_i^2}{50} = 256 \times 2$$

$$\text{Required mean} = \frac{\sum (x_i - 4)^2}{50}$$

$$= \frac{\sum x_i^2 + 16 \times 50 - 8\sum x_i}{50}$$

$$= 256 \times 2 + 16 - 8 \times 16 = 400$$

Hence, the correct answer is (A).

30. $x_1 + x_2 + x_3 + x_4 = 44$ and $x_5 + x_6 \ldots x_{10} = 96$

$$\sum x_i = 140$$

$$\bar{x} = \frac{140}{40} = 14; \sum \bar{x}^2 = 2000$$

$$\text{SD} = \sqrt{\frac{\sum \bar{x}^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{2000}{10} - \left(\frac{140}{10}\right)^2}$$

$$= \sqrt{200 - 196} = 2$$

Hence, the correct answer is (B).

31. Given $\sum_{i=1}^{9} (x_i - 5) = 9 \Rightarrow \sum_{i=1}^{9} x_1 = 54$ …(1)

Also $\sum_{i=1}^{9} (x_i - 5)^2 = 45$

$$\Rightarrow \sum_{i=1}^{9} x_i^2 - 10\sum_{i=1}^{9} x_1 + 9(25) = 45 \quad \ldots(2)$$

From (1) and (2) we get, $\sum_{i=1}^{9} x_i^2 = 360$

$$\text{Since, variance} = \frac{\sum x_1^2}{9} - \left(\frac{\sum x_1}{9}\right)^2$$

$$= \frac{360}{9} - \left(\frac{54}{9}\right)^2 = 40 - 36 = 4$$

∴ Standard deviation = $\sqrt{\text{Variance}} = 2$

Hence, the correct answer is (B).

32. $\bar{x} = \dfrac{2 + 3 + a + 11}{4} = \dfrac{a}{4} + 4$

$$\sigma = \sqrt{\sum \frac{x_i^2}{n} - (\bar{x})^2}$$

$$\Rightarrow 3.5 = \sqrt{\frac{4 + 9 + a^2 + 121}{4} - \left(\frac{a}{4} + 4\right)^2}$$

$$\Rightarrow \frac{49}{4} = \frac{4(134 + a^2) - (a^2 + 256 + 32a)}{16}$$

$$\Rightarrow 3a^2 - 32a + 84 = 0$$

Hence, the correct answer is (D).

33. Sum of 16 observations = 16 × 16 = 256

Sum of resultant 18 observations = 256 – 16 + (3 + 4 + 5)

= 252

Mean of observations = $\dfrac{252}{18} = 14$

Hence, the correct answer is (B).

34. First 50 even natural numbers are 2, 4, 6, …, 100

$$\text{Variance} = \frac{\sum x_i^2}{N} - (\bar{x})^2$$

$$\Rightarrow \sigma^2 = \frac{2^2 + 4^2 + \ldots + 100^2}{50} - \left(\frac{2 + 4 + \ldots + 100}{50}\right)^2$$

$$= \frac{4(1^2 + 2^2 + 3^2 + \ldots + 50^2)}{50} - (51)^2$$

$$= 4\left(\frac{50 \times 51 \times 101}{50 \times 6}\right) - (51)^2 = 3434 - 2601$$

$$\Rightarrow \sigma^2 = 833$$

Hence, the correct answer is (D).

35. If initially all marks were x_i then

$$\sigma_i^2 = \frac{\sum_i (x_i - \bar{x})^2}{N} = \sigma_i^2$$

Hence, variance will not change even after the grace marks were given.

Hence, the correct answer is (D).

36. Key Concept: If each observation is multiplied by k, mean gets multiplied by k and variance gets multiplied by k^2. Hence the new mean should be $2\bar{x}$ and new variance should be $k^2\sigma^2$.

So statement-1 is true and statement-2 is false.

Hence, the correct answer is (D).

37. Median is the mean of 25th and 26th observation

$$\therefore \quad M = \frac{25a + 26a}{2} = 25.5a$$

$$\text{M.D. } (M) = \frac{\sum |x_i - M|}{N}$$

$$\Rightarrow 50 = \frac{1}{50}[2 \times |a| \times (0.5 + 1.5 + 2.5 + \ldots 24.5)]$$

$$\Rightarrow 2500 = 2|a| \times \frac{25}{2}(25) \Rightarrow |a| = 4$$

Hence, the correct answer is (B).

38. $\sigma_x^2 = 4,\ \sigma_y^2 = 5, x = 2, y = 4$

$$\frac{1}{5}\sum x_i^2 - (2)^2 = 4; \frac{1}{5}\sum y_i^2 - (4)^2 = 5$$

$$\sum x_i^2 = 40; \sum y_i^2 = 105$$

$$\Rightarrow \sum \left(x_i^2 + y_i^2\right) = 145$$

$$\Rightarrow \sum (x_i + y_i) = 5(2) + 5(4) = 30$$

Variance of combined data

$$= \frac{1}{10}\sum\left(x_i^2 + y_i^2 - \left(\frac{1}{10}\sum (x_i + y_i)\right)\right)^2$$

$$= \frac{145}{10} - 9 = \frac{11}{2}$$

Hence, the correct answer is (A).

39. For the numbers 2, 4, 6, 8, …, $2n$

$$\bar{x} = \frac{2[n(n+1)]}{2n} = (n+1)$$

And $\text{Var} = \dfrac{\sum (x-\bar{x})^2}{2n} = \dfrac{\sum x^2}{n} - (\bar{x})^2$

$$= \frac{4\sum n^2}{n} - (n+1)^2 = \frac{4n(n+1)(2n+1)}{6n} - (n+1)^2$$

$$= \frac{2(n+1)(n+1)}{3} - (n+1)^2 = (n+1)\left[\frac{4n+2-3n-3}{3}\right]$$

$$= \frac{(n+1)(n-1)}{3} = \frac{n^2-1}{3}$$

∴ Statement-1 is false. Clearly, statement-2 is true.

Hence, the correct answer is (C).

40. $\text{Mean} = \dfrac{101 + d(1+2+3+\ldots+100)}{101}$

$$= 1 + \frac{d \times 100 \times 101}{101 \times 2} = 1 + 50d$$

∵ Mean deviation from the mean = 255

$\Rightarrow$ $\dfrac{1}{101}[\,|1-(1+50d)| + |(1+d)-(1+50d)|$

$+ (1+2d) - (1+50d)| + \ldots + |(1+100d)-(1+50d)|]$

$= 255$

$\Rightarrow$ $2d[1+2+3+\ldots+50] = 101 \times 255$

$\Rightarrow$ $2d \times \dfrac{50 \times 51}{2} = 101 \times 255$

$\Rightarrow$ $d = \dfrac{101 \times 255}{50 \times 51} = 10.1$

Hence, the correct answer is (B).

CHAPTER 10: MATHEMATICAL REASONING

Test Your Concepts-I

1. Sentences which are either True or False always are statement. Sentences which can be True or False both are not statement.

2. Let p be true then statement
$(p \wedge \sim q) \vee (q \wedge r) = (T \wedge F) \vee (T \wedge T)$
$= F \vee T$
$= T.$
also let p be false then statement
$(p \wedge \sim q) \vee (q \wedge r) = (F \wedge F) \vee (T \wedge T)$
$= F \vee T$
$= T.$
$\therefore$ p may be true or false.

3. Let p : 101 is not a multiple of 3.
$\sim p$: 101 is a multiple of 3.

4. (i), (ii), (iii) are mathematically acceptable statements with truth values F, F, T respectively.

5. Negation of P is "For no rational number x, $x^2 = 2$". Hence (iii) is not a negation of P.

6.

p	q	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee p$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

Hence statement $(p \wedge \sim q) \vee p$ is logically equal to statement p.

Test Your Concepts-II

1. Contrapositive of $a \Rightarrow b$ is $\sim b \Rightarrow \sim a$
$\sim(\sim p \wedge q) \Rightarrow \sim p$
$(p \vee \sim q) \Rightarrow \sim p$

2. We know that
$\sim(p \Leftrightarrow q) = (p \wedge \sim q) \vee (\sim p \wedge q)$
$(p \Leftrightarrow q) = (\sim p \vee q) \wedge (p \vee \sim q)$

3.

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \Leftrightarrow \sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	F
F	F	F	T	F

Hence the statement is neither tautology nor contradiction.

4.

p	q	$p \vee q$	$p \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Hence the statement is a tautology.

5.

p	q	$\sim q$	$p \to \sim q$	$p \wedge q$	$(p \to \sim q) \Leftrightarrow p \wedge q$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F

Hence the statement is a contradiction.

6. We know that $p \wedge q$ is true only when both p and q are true so option (A) is not correct
We know that $p \to q$ is false only when p is true and q is false so option (B) is not correct
We know that p q is true only when either p and q both are true or both are false so option (C) is not correct
We know that $\sim(p \vee q)$ is true only when $(p \vee q)$ is false i.e. p and q both are false
So option (D) is correct.

7. $\sim(p \to (q \wedge r)) \equiv p \wedge \sim(q \wedge r) \equiv p \wedge (\sim q \vee \sim r)$
$(\because \sim(p \to q) \equiv p \wedge \sim q)$

8. Let p and q be the statements as given below
p : a quadrilateral is a square
q : a quadrilateral is a rhombus
the given statement is $p \to q$
$\because \quad \sim(p \to q) \equiv p \wedge \sim q$
Therefore the negation of the given statement is a quadrilateral is a square and it is not a rhombus.

9. $\sim\{p \to (q \wedge r)\} \equiv \sim\{\sim p \vee (q \wedge r)\}$
$\equiv p \wedge (\sim q \vee \sim r) \equiv (p \wedge \sim q) \vee (p \wedge \sim r)$

10. The inverse of given conditional is $\sim p \to \sim(\sim q \wedge \sim r) \equiv \sim p \to (q \vee r)$. This is false implies that $\sim p$ is true and $(q \vee r)$ is false $\therefore$ p is false and each of q, r is false.

11. $S - \text{I} : \sim(\sim p \leftrightarrow \sim q) \equiv \sim\{(\sim p \to \sim q) \wedge (\sim q \to \sim p)\}$
$\equiv \sim\{(p \vee \sim q) \wedge (q \vee \sim p)\}$
$\equiv (\sim p \wedge q) \vee (\sim q \wedge p)$
But $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
$\therefore$ S – I is false
S – II : $\sim p \leftrightarrow \sim q \equiv (\sim p \to \sim q) \wedge (\sim q \to \sim p) \equiv (p \vee \sim q) \wedge (q \vee \sim p)$
$\therefore$ S – II is true

12. $((p \to q) \wedge (\sim p \to q)) \to q$

$\equiv (\sim p \vee q) \wedge (p \vee q) \to q$

$\equiv (\sim p \wedge p) \vee q \to q \equiv c \vee q \to q \equiv q \to q \equiv \sim q \vee q \equiv t$

13. (A) $p \to (q \to p) \equiv p \to (\sim q \vee p) \equiv \sim p \vee (\sim q \vee p) \equiv (\sim p \vee p) \vee \sim q \equiv t \vee \sim q \equiv t$

(B) $p \to (p \vee q) \equiv \sim p \vee (p \vee q) \equiv t \vee q \equiv t$

(C) $p \vee q \to (\sim p \wedge \sim q) \equiv \sim(p \vee q) \vee (\sim(p \vee q)) \equiv \sim(p \vee q)$

(D) $(p \vee \sim p) \to (q \wedge \sim q) \equiv t \to c \equiv (\sim t \vee c) \equiv c \vee c \equiv c$

Single Correct Choice Type Questions

1. Where are you going? is an interrogative sentence which is not considered as statement
Hence, the correct answer is (C).

2. 'The sun is a star' is scientifically proven hence it is a statement
Hence, the correct answer is (C).

3. The statement is true only when p, q and r are all true.
Hence, the correct answer is (B).

4. We have,

$\sim(p \vee q) \vee (\sim p \wedge q)$

$= (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$= \sim p \wedge (\sim q \vee q)$

$\cong \sim p \wedge t$, where t is a tautology

$\cong \sim p$

Hence, the correct answer is (A).

5. $\sim p$: It is not that the earth is round
$\sim q$: It is not that $3 + 4 = 7$.
$(\sim p) \vee (\sim q)$: It is not that the earth is round or it is not that $3 + 4 = 7$
Hence, the correct answer is (C).

6. Use the property

$\sim(a \wedge b) = \sim a \vee \sim b$

Hence, the correct answer is (B).

7. Come here is not a statement. Since, no sentence can be called a statement, if it an order.
Hence, the correct answer is (D).

8. Let p : A circle is an ellipse;
$\sim p$: A circle is not an ellipse.
Hence, the correct answer is (C).

9. Let p : 72 is divisible by 2 and 3.
Let q : 72 is divisible by 3.

r : 72 is divisible by 3.

$\sim q$: 72 is not divisible by 2.

$\sim r$: 72 is not divisible by 3.

$\sim(q \wedge r) : \sim q \vee \sim r$

$\Rightarrow$ 72 is not divisible by 2 or 72 is not divisible by 3.

Hence, the correct answer is (A).

10. Let p : x^2 is not even.
and q : x is not even.
Converse of the statement $p \to q$ is $q \to p$.
i.e., If x is not even, then x^2 is not even.
Hence, the correct answer is (B).

11. $p \to q$ is same as p only if q.
Hence, the correct answer is (C).

12. p : roses are red

$\sim p$: roses are not red

$(\sim p) \vee q$; roses are not red or sun is a star.

Hence, the correct answer is (D).

13. Use the property

$\sim(p$ and $q) = \sim p$ or $\sim q$

Hence, the correct answer is (D).

14. Use the property

$\sim(a \vee b) = \sim a \wedge \sim b$

Hence, the correct answer is (C).

15. $p \to q$ is false only when p is true and q is false
Hence, the correct answer is (B).

16.

p	q	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	T	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	T	T

$\therefore$ neither tautology nor contradiction.
Hence, the correct answer is (C).

17. It is obvious.
Hence, the correct answer is (D).

18. $p \Rightarrow q$ and its contrapositive logically equivalent
Hence, the correct answer is (B).

19. It is obvious.
Hence, the correct answer is (A).

20. $\sim(p \vee q) \quad \Rightarrow \quad \sim p \wedge \sim q$
Hence, the correct answer is (C).

21. $s = q \wedge \sim q$ is contradiction

p	s	$p \to s$
T	F	F
F	F	T

$\therefore$ Neither tautology nor contradiction
Hence, the correct answer is (C).

22. $s = \sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$

p	q	$\sim(p \wedge q)$	$\sim(q \Leftrightarrow p)$	s
T	T	F	F	F
T	F	T	T	T
F	T	T	T	T
F	F	T	F	T

Hence, the correct answer is (C).

23.

p	q	$\sim p$	$p \Leftrightarrow q$	$\sim p \wedge (p \Leftrightarrow q) = s$	$\sim s = p \vee q$
T	T	F	T	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	T	T	F

Hence, the correct answer is (A).

24. $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$\equiv \sim p \wedge (\sim q \vee q) \equiv \sim p \wedge t \equiv \sim p$
Hence, the correct answer is (A).

25.

A	T	F	T	F
B	F	T	T	F
$A \vee B$	T	T	T	F
$A \wedge B$	F	F	T	F
$A \wedge (A \vee B)$	T	F	T	F
$A \vee (A \wedge B)$	T	F	T	F
$A \rightarrow B$	F	T	T	T
$A \wedge (A \rightarrow B)$	F	F	T	F
$[A \wedge (A \rightarrow B)] \rightarrow B$	T	T	T	T
$B \rightarrow [A \wedge (A \rightarrow B)]$	T	F	T	T

Hence, the correct answer is (C).

26. Use the property
$\sim(p \rightarrow q) = p \wedge \sim q$
Hence (B) is correct option.
Hence, the correct answer is (B).

27. $F \rightarrow (T \wedge F) \vee (F \wedge T)$
$F \rightarrow (F \vee F)$
$F \rightarrow F = T$
Hence, the correct answer is (A).

28. Contra positive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$
$\therefore$ Contra positive of $(p \vee q) \Rightarrow r$ is
$\sim r \Rightarrow \sim(p \vee q)$ i.e., $\sim r \Rightarrow (\sim p \wedge \sim q)$.
Hence, the correct answer is (C).

29. $p \Rightarrow q$ is false when p is true and q is false.
$\therefore$ $p \Rightarrow q \vee r$ is false when p is true and $q \vee r$ is false, and $q \vee r$ is false when both q and r are false.
Hence, truth values of p, q and r are respectively T, F, F.
Hence, the correct answer is (A).

30. $[(p \wedge \sim q) \vee q] \vee (\sim p \wedge q)$
$= (p \vee q) \wedge (\sim q \vee q) \vee (\sim p \wedge q)$
$= (p \vee q) \wedge [t \vee (\sim p \wedge q)]$
$= (p \vee q) \wedge t = p \vee q$.
Hence, the correct answer is (B).

31. Let p : Plants take in CO_2 and given out O_2.
Let q : Plants take in CO_2
r : Plants give out O_2.
$\sim q$: Plants do not take in CO_2.
$\sim r$: Plants do not give out O_2.
$\sim(q \wedge r)$: Plants do not take in CO_2 or do not give out O_2.
Hence, the correct answer is (B).

32. Let p : Jaipur is capital of Rajasthan
and q : Jaipur is in India.
$\sim p$: Jaipur is not capital of Rajasthan.
$\sim q$: Jaipur is not in India.
Contrapositive of the statement $p \rightarrow q$ is
If $(\sim q)$, then $(\sim p)$.
If Jaipur is not in India, then Jaipur is not the capital of Rajasthan.
Hence, the correct answer is (A).

33. The negation of the above statement is 'it is false that the product of 3 and 4 is 9'.
Hence, the correct answer is (A).

34. The false negation of the given statement is "It is false that a natural number is not greater than zero".
Hence, the correct answer is (C).

35. p : A number is a prime.
q : It is odd. We have, $p \Rightarrow q$
The inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$
i.e., if a number if not a prime then it is not odd.
Hence, the correct answer is (B).

36. Use the property
$\sim(a \rightarrow b) = a \wedge \sim b$
Hence (C) is correct option
Hence, the correct answer is (C).

37. $A \rightarrow B$ means $(\sim A) \vee B$
So, $p \rightarrow (\sim q) = (\sim p) \vee (\sim q)$
$= (\sim q) \vee (\sim p)$. $[\because A \vee B = B \vee A]$
Hence, the correct answer is (B).

38. $(\sim q \wedge p) \vee (p \vee \sim q)$

q	$\sim q$	p	$\sim p$	$\sim q \wedge p$	$p \vee \sim p$	$(\sim q \vee p) \vee (p \vee \sim p)$
T	F	T	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	T	T	T
F	T	F	T	F	T	T

Hence, the correct answer is (C).

39. $\sim\{(p \vee (\sim q)) \wedge q\} = (\sim(p \vee (\sim q))) \vee (\sim q)$
[By De Morgan's Law, $\sim(A \wedge B) = (\sim A) \vee (\sim B)$]
$= (\sim p \wedge q) \vee (\sim q)$ $[\because \sim(\sim q) = q]$
Hence, the correct answer is (C).

40. The given statement can be written as, $A \rightarrow B$,
where $A \equiv$ "It is raining."
$B \equiv$ "I will not come."

The contrapositive for $A \to B$ is, $(\sim B) \to (\sim A)$.

Now, $\sim B \equiv$ "I will come." $\bar{A} \equiv$ "It is not raining."

So, $(\sim B) \to (\sim A)$, can be written as,

"If I will come, then it is not raining."

Hence, the correct answer is (A).

41. Let 'p' : Ram secures 100 marks in mathematics

q : Ram will get a mobile

Converse of statement p and q

If 'p', then 'q' $\Rightarrow$ if 'q', then 'p'

$\therefore$ Converse of above statement is

If Ram will get a mobile, then he secures 100 marks in mathematics.

Hence, the correct answer is (C).

42. It is obvious.

Hence, the correct answer is (A).

43. $p \to q$ is false only when p is true and q is false.

$p \to (\sim p \vee q)$ is false only when p is true and $(\sim p \vee q)$ is false.

$\sim p \vee q$ is false if q is false, because $\sim p$ is false.

Hence, the correct answer is (B).

44. Let p : Rajesh or Rajni lived in Bengaluru.

and q : Rajesh lived in Bengaluru.

r : Rajni liven in Bengaluru

$\sim q$: Rajesh did not live in Bengaluru.

$\sim r$: Rajni did not live in Bengaluru.

$\sim(q \wedge r)$: Rajesh did not live in Bengaluru and Rajni did not live in Bengaluru.

Hence, the correct answer is (C).

45. Let $p : x > y$

$q : x + a > y + a$

Converse of the above statement is

$q \to p$

i.e., If $x + a > y + a$, then $x + y$

Hence, the correct answer is (B).

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1. $(p \to q) \wedge (q \to \sim p) = (\sim p \vee q) \wedge (\sim q \vee \sim p) = \sim p \vee (q \wedge \sim q)$

$= \sim p \vee c = \sim p$

Hence, the correct answer is (C).

2. Contrapositive of $p \to q$ is $\sim q \to \sim p$

$(A \subseteq B) \wedge (B \subseteq D) \to (A \subseteq C)$

Contrapositive is $\sim(A \subseteq C) \to \sim(A \subseteq B) \vee \sim(B \subseteq D)$

$A \not\subseteq C \to (A \not\subseteq B) \vee (B \not\subseteq D)$

Hence, the correct answer is (B).

3. (A) $P \wedge (P \vee Q) \equiv P$

(B) $P \vee (P \wedge Q) \equiv P$

(C) $Q \to (P \wedge (P \to Q)) \equiv Q \to (P \wedge (\sim P \vee Q))$

$\equiv Q \to (P \wedge Q) \equiv (\sim Q) \vee (P \wedge Q) \equiv (P \vee (\sim Q))$

(D) $(P \wedge (P \to Q)) \to Q \equiv (P \wedge (\sim P \vee Q)) \to Q$

$\equiv (P \wedge Q) \to Q$

$\equiv ((\sim P \vee (\sim Q)) \vee Q \equiv \sim P \vee t \equiv t$

Hence, the correct answer is (D).

4. $\sim(p \vee \sim q) \to p \vee q = (\sim p \wedge q) \to p \vee q = \sim\{(\sim p \wedge q) \wedge (\sim p \wedge \sim q)\}$

$= \sim(\sim p \wedge f) = \sim f = t$

Hence, the correct answer is (A).

5. Let $p = \sqrt{5}$ is an integer and q : 5 is irrational

$\sim(p \vee q) \equiv \sim p \wedge \sim q = \sqrt{5}$ is not an integer and 5 is not irrational.

Hence, the correct answer is (B).

6. $p \to (p \wedge \sim q)$ is F $\Rightarrow$ p is T and $p \wedge \sim q$ is F $\Rightarrow$ q is T

$\therefore$ p is T, q is T

Hence, the correct answer is (D).

7. $\sim p \wedge (p \vee q) \to q = (\sim p \wedge p) \vee (\sim p \wedge q) \to q$

$= c \vee (\sim p \wedge q) \to q$

$= (\sim p \wedge q) \to q = \sim((\sim p \wedge q) \wedge \sim q)$

[using $a \to b = \sim(a \wedge \sim b) = \sim a \vee b$]

$= \sim(\sim p \wedge q) \vee q = (p \vee \sim q) \vee q = p \vee (\sim q \vee q) = p \vee t = t$

so $\sim p \wedge (p \vee q) \to q$ is a tautology

Option (B) [using $a \to b = \sim(a \wedge \sim b) = \sim a \vee b$]

$(q \to p) \vee \sim(p \to q) = \sim(p \to q) \vee (q \to p)$

$= (p \to q) \to (q \to p)$

If p is false and q is true, the statement is false (not a tautology)

Option (C)

$(p \to q) \wedge (q \to p)$ If p is false and q is true, the statement is false (not a tautology)

Option (D)

$(\sim q) \vee (p \wedge q) \to q$

Clearly if q is false, irrespective of P the statement is False, (not a tautology)

Hence, the correct answer is (A).

8. p : I reach the station in time.

q : I will catch the train.

Contrapositive of $p \to q$ is $\sim q \to \sim p$

$\sim q \to \sim p$: if I will not catch the train then I do not reach the station in time.

Hence, the correct answer is (C).

9. $p \to \sim(p \wedge \sim q)$ using $a \to b = \sim(a \wedge \sim b) = \sim a \vee b$

$= \sim p \vee \sim(p \wedge \sim q)$

$= \sim p \vee (\sim p \vee q) = \sim p \vee q$

Hence, the correct answer is (A).

10. p = function is differentiable at a

q = function is continuous at a

contrapositive of statement $p \to q$ is $\sim q \to \sim p$

Hence, the correct answer is (C).

11. $(p \wedge q) \to (\sim q \vee r)$ = false

when $(p \wedge q) = T$ and $(\sim q \vee r) = F$

So $(p \wedge q) = T$ is possible when $p = q$ = true

∴ $\sim q$ = False (q = true)

So $(\sim q \vee r)$ = False is possible if r is false hence $p = T$, $q = T$, $r = F$

Hence, the correct answer is (C).

12. Let TV(r) denotes truth value of a statement r.

Now, if TV(p) = TV(q) = T $\Rightarrow$ TV(S_1) = F

Also, if TV(p) = T and TV(q) = F $\Rightarrow$ TV(S_2) = T

Hence, the correct answer is (C).

13.

p	**q**	**q → p**	**p → (q → p)**	**p ∨ q**	**p → p ∨ q**	**(p → (q → p)) → (p → (q ∨ p))**
T	T	T	T	T	T	T
T	F	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T

Hence, the correct answer is (C).

14. $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

$x \leftrightarrow \sim y \equiv (x \to \sim y) \wedge (\sim y \to x)$

∵ $(p \to q \equiv \sim p \vee q)$

$x \leftrightarrow \sim y \equiv (\sim x \vee \sim y) \wedge (y \vee x)$

$\sim (x \leftrightarrow \sim y) \equiv (x \wedge y) \vee (\sim x \wedge \sim y)$

Hence, the correct answer is (C).

15. Contrapositive of $(p \to q)$ is $\sim q \to \sim p$

For an integer n, if n is even then $(n^3 - 1)$ is odd

Hence, the correct answer is (B).

16. Negation of $p \vee (\sim p \wedge q)$

$p \vee (\sim p \wedge q) = (p \vee \sim p) \wedge (p \vee q) = (t) \wedge (p \vee q) = (p \vee q)$

now negation of $(p \vee q)$ is $\sim (p \vee q) = \sim p \wedge \sim q$

Hence, the correct answer is (C).

17. $(p \oplus q) \wedge (\sim p \odot q) = p \wedge q$

$\oplus, \odot \in \{\wedge, \vee\}$

option (3) satisfy $(p \wedge q) \wedge (\sim p \vee q)$

p	**q**	**p ∧ q**	**~p**	**~p ∨ q**	**(p ∨ q) ∧ (~p ∨ q)**	**p ∧ q**
T	T	T	F	T	T	T
T	F	F	F	F	F	F
F	T	F	T	T	F	F
F	F	F	T	T	F	F

Hence, the correct answer is (C).

18. $[\sim(\sim p \vee q)] \wedge (p \wedge r)] \wedge (\sim q \wedge r)]$

$= [(p \wedge \sim q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$

$= [p \wedge (\sim q \vee r)] \wedge (\sim q \wedge r)$

$= p \wedge (\sim q \wedge r) = (p \wedge r) \wedge \sim q$

Hence, the correct answer is (C).

19. $P(n) = n^2 - n + 41$

$P(5) = 25 - 5 + 41 = 61$

$P(3) = 9 - 3 + 41 = 47$

Both true

Hence, the correct answer is (B).

20. P is true, Q is false, R is true

by option

$P \vee (\sim Q \wedge R) = T \vee (T \vee T) = T$

Hence, the correct answer is (B).

21. If q is false $p \wedge q \leftrightarrow r$ is true

p	**q**	**r**	**p ∧ q**	**p ∧ q ↔ r**
T	F	T	F	F
T	F	F	F	T
F	F	T	F	F
F	F	F	F	T

check option $(p \wedge r) \to (p \vee r)$

If $p \wedge q \leftrightarrow r$ is true then

p	**q**	**r**	**p ∧ r**	**p ∧ r**	**(p ∧ r) → (p ∧ r)**
T	F	F	F	T	T
F	F	F	F	F	T

Hence, the correct answer is (D).

22. Check by truth table

p	**~p**	**q**	**~q**	**(p ∧ q)**	**(p ∨ ~q)**	**(~p ∨ ~q)**	**[(p ∧ q) ∨ (p ∨ ~q)]**	**[(p ∧ q) ∨ (p ∨ ~q)] ∧ [(~p ∧ ~q)]**
T	F	T	F	T	T	T	T	F
T	F	F	T	F	T	T	T	F
F	T	T	F	F	F	F	F	F
F	T	F	T	F	T	T	T	T

Hence, the correct answer is (D).

23.

p	q	$\sim p$	$\sim p \to q$	$\sim(p \to p)$	$(\sim p \wedge \sim q)$
T	T	T	T	F	F
F	T	T	T	F	F
T	F	T	T	F	F
F	F	F	F	T	T

Hence, the correct answer is (B).

24. Contrapositive of given statement is. If you are not a citizen of India, then you are not born in India.
Hence, the correct answer is (C).

25.

p	q	$p \vee q$	$\sim q$	$p \vee \sim q$	$(p \vee q) \to (p \vee \sim q)$
T	T	T	F	T	T
T	F	T	T	T	T
F	T	T	F	F	F
F	F	F	T	T	T

Hence, the correct answer is (D).

26. $\sim(p \vee (\sim p \wedge q)) = \sim p \wedge (p \vee \sim q)$
$= (\sim p \wedge p) \vee (\sim p \vee \sim q) = c \vee (\sim p \wedge \sim q) = \sim p \wedge \sim q$
Hence, the correct answer is (D).

27. $p \Rightarrow (q \vee r)$ is false

$(\because\ T \vee F = F)$

So, $P = T$, $q = F$ and $r = F$
Hence, the correct answer is (B).

28. From options
$(p \vee q) \wedge (\sim p \vee \sim q) \equiv (p \vee q) \wedge \sim (p \wedge q)$
$\to$ Not a tautology
$(p \vee q) \vee (p \vee \sim q) \equiv p \vee (q \vee \sim q) \to$ tautology
$(p \wedge q) \vee (p \vee \sim q) \equiv p \wedge (q \vee \sim q) \to$ Not a tautology
$(p \vee q) \wedge (p \vee \sim q) \equiv p \vee (q \wedge \sim q) \to$ Not a tautology
Hence, the correct answer is (B).

29. $\sim (\sim s \vee (\sim r \wedge s)) = s \wedge (r \vee \sim s) = (s \wedge r) \vee (s \wedge \sim s)$
$= (s \wedge r) \vee (c) = (s \wedge r)$
Hence, the correct answer is (C).

30. $p \to (\sim q \vee r)$ is false
It is true when $p \to T$ and $(\sim q \vee r) =$ false
It will true : $\sim q$ false and r false
$\sim q \to F \mid r \to F \quad \Rightarrow \quad q \to T$
Truth value of $p, q, r \quad \Rightarrow \quad T, T, F$
Hence, the correct answer is (C).

31. $\sim(p \to \sim q) = p \wedge q$
Hence, the correct answer is (A).

32. Let p : two numbers are not equal
q : there squares are not equal
Contrapositive of $p \to q$ is
$\sim q \to \sim p$
If the squares of two numbers are equal, then the number are equal.
Hence, the correct answer is (B).

33. $\sim(p \vee q) \vee (\sim p \wedge q)$
$(\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$\Rightarrow \quad \sim p \wedge (\sim q \vee q) \quad \Rightarrow \quad \sim p \wedge t \equiv \sim p$
Hence, the correct answer is (D).

34.

p	q	$\sim p$	$p \to q$	$\sim p \to q$	$(\sim p \to q) \to q$	$(p \to q) \to ((\sim p \to q) \to q)$
T	F	F	F	T	F	T
T	F	F	T	T	F	T
F	F	T	T	F	T	T
F	T	T	T	T	T	T

$\therefore$ It is tautology
Hence, the correct answer is (B).

35. $(p \wedge \sim q) \vee q\ (\sim p \wedge q)$
$\Rightarrow \quad \{(p \vee q) \wedge (\sim q \vee q)\} \vee (\sim p \wedge q)$
$\Rightarrow \quad \{(p \vee q) \wedge T\} \vee (\sim p \wedge q)$
$\Rightarrow \quad (p \vee q) \vee (\sim p \wedge q)$
$\Rightarrow \quad \{(p \vee q) \vee \sim p\} \wedge (p \vee q \vee q)$
$\Rightarrow \quad T \wedge (p \vee q)$
$\Rightarrow \quad p \vee q$
Hence, the correct answer is (A).

36. $\sim[\sim s \vee (\sim r \wedge s)]$
$= \sim s \wedge \sim(\sim r \wedge s)$
$= s \wedge (r \vee \sim s)$
$= (s \wedge r) \vee (s \wedge \sim s)$
$= (s \wedge r) \vee 0$
$= s \wedge r$
Hence, the correct answer is (B).

37.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
F	F	T	F	T
F	T	F	T	F
T	F	T	T	F
T	T	F	F	T

Clearly equivalent to $p \leftrightarrow q$
Hence, the correct answer is (C).

38. Statement-2: $(p \to q) \leftrightarrow (\sim q \to \sim p)$
$\equiv (p \to q) \leftrightarrow (p \to q)$
which is always true.
So statement 2 is true
Statement-1: $(p \wedge \sim q) \wedge (\sim p \wedge q) = p \wedge \sim q \wedge \sim p \wedge q$
$= p \wedge \sim p \wedge \sim p \wedge q = f \wedge f \wedge f$
So statement-1 is true
Hence, the correct answer is (B).

39. Let p : I become a teacher.

q : I will open a school

Negation of $p \to q$ is $\sim(p \to q) = p \wedge \sim q$

i.e. I will become a teacher and I will not open a school.

Hence, the correct answer is (A).

40. Suman is brilliant and dishonest if and only if Suman is rich is expressed as

$$Q \leftrightarrow (P \wedge \sim R)$$

Negation of it will be $\sim (Q \leftrightarrow (P \wedge \sim R))$

Hence, the correct answer is (A).

41. P : there is a rational number $x \in S$ such that $x > 0$

$\sim P$: Every rational number $x \in S$ satisfies $x > 0$

Hence, the correct answer is (B).

42. The truth table for logical statements, involved is statement 1, is as follows :

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(q \leftrightarrow \sim p)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

We observe the columns for $\sim(p \leftrightarrow \sim q)$ and $p \leftrightarrow q$ are identical, therefore

$\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

But $\sim(p \leftrightarrow \sim q)$ is not a tautology as all entries in its column are not T.

$\therefore$ Statement-1 is true but statement-2 is false.

Hence, the correct answer is (B).

CHAPTER 11: DETERMINANT

Test Your Concepts-I

1. $$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 & 3/3 \\ 3 & 4/3 & 6/3 \\ 5 & 4 & 9/3 \end{vmatrix} = \begin{vmatrix} 1/3 & 2/3 & 1 \\ 3 & 4/3 & 2 \\ 5 & 4 & 3 \end{vmatrix}$$

2. The minors are $M_{11} = \begin{vmatrix} 4 & 5 \\ 1 & -3 \end{vmatrix} = -17$,

$$M_{12} = \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = -13, \qquad M_{13} = \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} = -7$$

$$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} = 3, \qquad M_{22} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -12,$$

$$M_{23} = \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = 6 \qquad M_{31} = \begin{vmatrix} -2 & 3 \\ 4 & 5 \end{vmatrix} = -22,$$

$$M_{32} = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 7, \qquad M_{33} = \begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} = 10,$$

The cofactors are

$C_{11} = (-1)^{1+1} M_{11} = M_{11} = -17$,
$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = 13$,
$C_{13} = (-1)^{1+3} M_{13} = M_{13} = -7$,
$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -3$,
$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -12$,
$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -6$,
$C_{31} = (-1)^{3+1} M_{31} = M_{31} = -22$,
$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -7$,
$C_{33} = (-1)^{3+3} M_{33} = M_{33} = 10$

3. $$\text{L.H.S.} = \begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \Psi \\ \lambda & \mu & \nu \end{vmatrix} = \begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \Psi & \nu \end{vmatrix}$$

(Interchanging rows and columns)

$$= (-1)\begin{vmatrix} \alpha & \lambda & \theta \\ \beta & \mu & \phi \\ \gamma & \nu & \Psi \end{vmatrix} = (-1)^2\begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \Psi \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & \nu & \Psi \end{vmatrix} = \text{R.H.S.}$$

4. $$\begin{vmatrix} 1 & -2 & 1 \\ 4 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = (4 - 4 - 6) - (2 - 3 - 16)$$

$$= -6 + 17 = 11$$

5. Consider $\begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ x & y & z \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ xyz & xyz & xyz \\ x^2 & y^2 & z^2 \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} = \text{LHS}$$

Hence Proved

6. $$\sum_{r=1}^{3} \Delta_r = \begin{vmatrix} 1+2+3 & 1 \\ 1^2+2^2+3^2 & 2 \end{vmatrix} = \begin{vmatrix} 6 & 1 \\ 14 & 2 \end{vmatrix} = 12 - 14 = -2$$

7. $$\Delta_a = \begin{vmatrix} (a-1) & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^3-3n \end{vmatrix}$$

Then $$\sum_{a=1}^{n} \Delta_a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^3-3n \end{vmatrix}$$

$$+ \begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^3-3n \end{vmatrix} + \ldots + \begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+3+\ldots\ldots+(n-1) & n & 6 \\ 1^2+2^2+3^2+\ldots\ldots+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+\ldots\ldots+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{n(n-1)}{2} & n & 6 \\ \dfrac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\dfrac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{12}\begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix} = 0$$

[$\because$ C_1 and C_3 are identical]

Test Your Concepts-II

1. $$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Using $R_1 \to R_1 - R_3$; $R_2 \to R_2 - R_3$]

$$\Delta = \begin{vmatrix} 0 & x-z & x^2-z^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-z)\,(y-z)\begin{vmatrix} 0 & 1 & x+z \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$ expanding along first column

$$= (x - z)(y - z)[0 - 0 + 1(y + z - x - z)]$$

$$= (x - z)(y - z)(y - x) = (x - y)(y - z)(z - x)$$

2. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$

Using $C_1 \to C_1 - C_3; C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ x - z & y - z & z \\ x^3 - z^3 & y^3 - z^3 & z^3 \end{vmatrix}$$

using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$= (x - z)(y - z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x^2 + xz + z^2 & y^2 + yz + z^2 & z^3 \end{vmatrix}$$

Expanding along first row

$$= (x - z)(y - z)[0 - 0 + 1(y^2 + yz + z^2 - x^2 - xz - z^2)]$$

$$= (x - z)(y - z)[(y^2 - x^2) + z(y - x)]$$

$$= (x - z)(y - z)(y - x)(y + x + z)$$

$$= (x - y)(y - z)(z - x)(x + y + z)$$

3. $\Delta = \begin{vmatrix} 1 & a^2 & b^2 + c^2 \\ 1 & b^2 & c^2 + a^2 \\ 1 & c^2 & a^2 + b^2 \end{vmatrix}$ using $C_3 \to C_3 + C_2$

$$\Delta = \begin{vmatrix} 1 & a^2 & a^2 + b^2 + c^2 \\ 1 & b^2 & a^2 + b^2 + c^2 \\ 1 & c^2 & a^2 + b^2 + c^2 \end{vmatrix} = (a^2 + b^2 + c^2)\begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} = 0$$

4. $\Delta = \begin{vmatrix} a - b & b - c & c - a \\ b - c & c - a & a - b \\ c - a & a - b & b - c \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 0 & b - c & c - a \\ 0 & c - a & a - b \\ 0 & a - b & b - c \end{vmatrix} = 0$$

5. $\Delta = \begin{vmatrix} b + c & a + b & a \\ c + a & b + c & b \\ a + b & c + a & c \end{vmatrix}$

Using $C_1 \to C_1 + C_3; C_2 \to C_2 - C_3]$

$$\Delta = \begin{vmatrix} a + b + c & b & a \\ a + b + c & c & b \\ a + b + c & a & c \end{vmatrix} = (a + b + c)\begin{vmatrix} 1 & b & a \\ 1 & c & b \\ 1 & a & c \end{vmatrix}$$

$$= (a + b + c)[c^2 - ab - b(c - b) + a(a - c)]$$

$$= (a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca)$$

6. $\Delta = \begin{vmatrix} p & q - y & r - z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$

using $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$pyz - (q - y)(-xz) + (r - z)xy = 0$$

$$pyz + qzx + rxy = 2xyz$$

$$\therefore \quad \frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$$

$$\therefore \quad \frac{p}{x} + \frac{q}{y} + \frac{r}{z} + 2 = 4$$

7. We have, $\begin{vmatrix} b + c & c & b \\ c & c + a & a \\ b & a & a + b \end{vmatrix}$ [Applying $R_1 \to R_1 - (R_2 + R_3)$]

$$= \begin{vmatrix} 0 & -2a & -2a \\ c & c + a & a \\ b & a & a + b \end{vmatrix} \qquad \text{[Taking } -2a \text{ common from } R_1]$$

$$= -2a\begin{vmatrix} 0 & 1 & 1 \\ c & c + a & a \\ b & a & a + b \end{vmatrix} = -2a\begin{vmatrix} 0 & 1 & 0 \\ c & c + a & -c \\ b & a & b \end{vmatrix}$$

$= (-2a)(-bc - \text{bc})$ [Expanding along R_1]

$$\Delta = 4abc$$

So, $\dfrac{\Delta}{abc} = 4$

8. LHS $= \begin{vmatrix} \cos A \cos P + \sin A \sin P & \cos A \cos Q + \sin A \sin Q & \cos A \cos R + \sin A \sin R \\ \cos B \cos P + \sin B \sin P & \cos B \cos Q + \sin B \sin Q & \cos B \cos R + \sin B \sin R \\ \cos C \cos P + \sin C \sin P & \cos C \cos Q + \sin C \sin Q & \cos C \cos R + \sin C \sin R \end{vmatrix}$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0 \times 0 = 0$$

9. Given determinant $= \begin{vmatrix} a + b + 2c & a & b \\ c & b + c + 2a & b \\ c & a & c + a + 2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we obtain

$$= \begin{vmatrix} 2(a + b + c) & a & b \\ 2(a + b + c) & b + c + 2a & b \\ 2(a + b + c) & a & c + a + 2b \end{vmatrix}$$

$$= 2(a + b + c)\begin{vmatrix} 1 & a & b \\ 1 & b + c + 2a & b \\ 1 & a & c + a + 2b \end{vmatrix}$$

using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(a + b + c)\begin{vmatrix} 1 & a & b \\ 0 & b + c + a & 0 \\ 0 & a & c + a + b \end{vmatrix}$$

$$= 2(a + b + c)\{(b + c + a)(c + a + b) - (0 \times 0)\}$$

$$= 2(a + b + c)^3$$

10. $\Delta = \begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix}$

Taking α, β, γ common from C_1, C_2, C_3 respectively

$\Delta = \alpha\beta\gamma \begin{vmatrix} -\alpha & \alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & \gamma & -\gamma \end{vmatrix}$ Now taking α, β, γ common from R_1, R_2, R_3 respectively

$\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ Now applying,

$R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$.

we have $\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$

Now expanding from C_1, $\Delta = \alpha^2\beta^2\gamma^2(-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$

$$= -\alpha^2\beta^2\gamma^2(0-4) = 4\alpha^2\beta^2\gamma^2$$

11. L.H.S. $= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

Multiplying C_1, C_2, C_3 by a, b, c respectively

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Now taking common a, b, c from R_1, R_2, R_3 respectively

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

Using $[C_1 \to C_1 + C_2 + C_3]$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

using $[R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

$= (1+a^2+b^2+c^2)\,(1)$

$= 1+a^2+b^2+c^2 =$ R.H.S.

12. $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

Using $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 12\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}$$

using $R_3 \to R_3 - R_2$.

$= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix}$ expanding it along third row

$\Rightarrow \quad 0 - 1\,(6x-12-3x+4) + 4(3x-6-2x+3) = 0$

$\Rightarrow \quad -3x+8+4(x-3) = 0$

$\Rightarrow \quad x = 4$

13. $$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{1}{x}\begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} = \begin{vmatrix} \cos x & 1 & 1 \\ 2\sin x & x & 2x \\ \tan x & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

14. $\Delta = \sqrt{5}.\sqrt{5}\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$

$= \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}-\sqrt{6} & \sqrt{5}-2\sqrt{2} & 0 \\ 3-\sqrt{15} & \sqrt{3}-\sqrt{5} & 0 \end{vmatrix}$, $R_2 \to R_2 - \sqrt{2}R_1$
$R_3 \to R_3 - \sqrt{5}R_1$

$$= \begin{vmatrix} \sqrt{15}-\sqrt{6} & \sqrt{5}-2\sqrt{2} \\ 3-\sqrt{15} & \sqrt{3}-2\sqrt{5} \end{vmatrix}$$

$$= 5\sqrt{3}\begin{vmatrix} \sqrt{5}-\sqrt{2} & \sqrt{5}-2\sqrt{2} \\ \sqrt{3}-\sqrt{5} & \sqrt{3}-2\sqrt{5} \end{vmatrix}$$

$$= 5\sqrt{3}\left\{(\sqrt{5}-\sqrt{2})(\sqrt{3}-2\sqrt{5})-(\sqrt{3}-\sqrt{5})(\sqrt{5}-2\sqrt{2})\right\}$$

$$= 5\sqrt{3}\left\{\sqrt{15}-10-\sqrt{6}+2\sqrt{10}-\sqrt{15}+2\sqrt{6}+5-2\sqrt{10}\right\}$$

$$= 5\sqrt{3}(\sqrt{6}-5)$$

15. Here $D = n!(n+1)!(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$

$$= n!.(n+1).(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 0 & 1 & (n+2)2 \\ 0 & 1 & (n+3)2 \end{vmatrix},$$

$R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$

$= n!(n+1)!(n+2)![(n+3)2-(n+2).2]$

$= n!(n+1)!(n+2)!.2$

$$\therefore \quad \frac{D}{(n!)^3} = 2(n+1)^2(n+2) = 2n^3 + 8n^2 + 10n + 4$$

$$\therefore \quad \left\{\frac{D}{(n!)^3} - 4\right\} \text{ is divisible by } n.$$

Test Your Concepts-III

1. For non trivial solution $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ \lambda & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow \quad 2(-3) - (-1) + \lambda = 0 \Rightarrow \lambda = 5.$

2. We have $\Delta = \begin{vmatrix} 3 & -7 & 5 \\ 3 & 1 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 5\begin{vmatrix} 3 & -7 & 1 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

Apply, $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = 5\begin{vmatrix} 0 & -8 & 0 \\ 1 & -2 & 0 \\ 2 & 3 & 1 \end{vmatrix}$$

Now, expanding along C_3, we get

$$\Delta = 5\,(0\,(-2) - 1\,(-8)) = 40 \neq 0$$

$$\Delta_x = \begin{vmatrix} 3 & -7 & 5 \\ 7 & 1 & 5 \\ 5 & 3 & 5 \end{vmatrix} = 120 \neq 0$$

$\Rightarrow$ System is consistent with unique non-trivial solution

3. For non-zero solutions, $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$

Apply, $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - C_3$, we get

$$\begin{vmatrix} 0 & 2a-3b & a-b \\ 0 & 3b-4c & b-c \\ 1 & 4c & c \end{vmatrix} = 0$$

Now expanding along R_1 we get

$\Rightarrow \quad (2a - 3b)\,(b - c) - (a - b)\,(3b - 4c) = 0$

$\Rightarrow \quad (2ab - 2ac - 3b^2 + 3bc) - (3ab - 4ac - 3b^2 + 4bc) = 0$

$\Rightarrow \quad ab + 2ac - bc = 0 \Rightarrow 2ac = b\,(a + c)$

$\Rightarrow \quad \dfrac{2}{b} = \dfrac{a+c}{ac} \Rightarrow \dfrac{2}{b} = \dfrac{1}{c} + \dfrac{1}{a}$ or $b = \dfrac{2}{\dfrac{1}{a} + \dfrac{1}{c}}$

$\Rightarrow a, b, c$ in H.P.

4. $\begin{vmatrix} 1-p & 2 & 1 \\ -3 & p & -2 \\ 2 & -2 & 1+p \end{vmatrix} = 0 \Rightarrow p^3 - 9p = 0 \Rightarrow p = 0, 3, -3$

5. For a homogenous system, we have $D_x = D_y = D_z = 0$
Now for system of equation to have non zero solution

then $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\,(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow \quad (a + b + c)[\,(a - b)^2 + (b - c)^2 + (c - a)^2\,] = 0$

As $\;a + b + c \neq 0$ then $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

Then we get $a = b = c$

$$at^2 + bt + c = 0 \text{ or } a\,(t^2 + t + 1) = 0$$

Discriminant $D = 1 - 4 = -3 < 0$

Roots of given equations are imaginary

6. Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix}$

applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\therefore \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 2 & -1 & -3 \end{vmatrix} = 1.(-9 + 5) = -4$$

$$\Delta_x = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} \qquad \text{Applying } C_2 \to C_2 + C_3$$

$$\therefore \quad \Delta_x = \begin{vmatrix} 9 & 2 & 1 \\ 52 & 12 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1\,(108 - 104) = -4$$

$$\Delta_y = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} \qquad \text{Applying } C_1 \to C_1 + 2C_3$$

$$\therefore \quad \Delta_y = \begin{vmatrix} 3 & 9 & 1 \\ 16 & 52 & 7 \\ 0 & 0 & -1 \end{vmatrix} = -1(156 - 144) = -12 \text{ and}$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} \qquad \text{Applying } C_1 \to C_1 - 2C_2$$

$$= \begin{vmatrix} -1 & 1 & 9 \\ -8 & 5 & 52 \\ 0 & 1 & 0 \end{vmatrix} = -1(-52 + 72) = -20$$

$\therefore$ By Cramer's rule

$$x = \frac{\Delta_x}{\Delta} = \frac{-4}{-4} = 1;\; y = \frac{\Delta_y}{\Delta} = \frac{-12}{-4} = 3;\; z = \frac{\Delta_z}{\Delta} = \frac{-20}{-4} = 5$$

$\therefore \quad x = 1, y = 3, z = 5$

Single Correct Choice Type Questions

1. We have, $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log_z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{\log x \cdot \log y \cdot \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \mathrm{1og}y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

[$\because$ all rows are identical]

Hence, the correct answer is (A).

2. We have, $\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-\beta & \alpha - x & 0 \\ 0 & x-\gamma & 0 \\ \beta & \gamma & 1 \end{vmatrix} = 0$

[Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$]

$\Rightarrow \quad (x-\beta)(x-\gamma) = 0.$ [Expanding along C_3]

Hence, the correct answer is (B).

3. We have,

$$|A| = \begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & x-2y+z \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$$

[Applying $R_1 \to R_1 - 2R_2 + R_3$]

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} \; [\because x, y, z \text{ are in A.P.},$$

$\therefore \quad x - 2y + z = 0] = 0$

Hence, the correct answer is (C).

4. We have,

$$f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

[Applying $C_1 \to C_1 - \sin\theta\, C_3$, $C_2 \to C_2 + \cos\theta\, C_3$]

$$= \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ 0 & 0 & 1 \end{vmatrix} \; [\text{Applying } R_3 \to R_3 - \sin\theta\, R_1 + \cos\theta\, R_2]$$

$= 1$ for all θ.

Hence, the correct answer is (A).

5. Here we can observe that the elements of R_3 are the derivatives of the elements of R_2 and they in turn are proportional to the derivatives of the elements of R_1. Therefore,

$$f'(x) = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix} = 0,$$

$\forall\, x \in R$ and $f(x) =$ constant

As $f(0) = \begin{vmatrix} 6 & 10 & 16 \\ -2 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 2 : f(x) = 2$ always constant

Hence, the correct answer is (D).

6. The given determinants Δ_2 is obtained by corresponding cofactors of determinants Δ_1.

$\Delta_2 = \Delta_1{}^2$

$\Delta_1\Delta_2 = \Delta_1{}^3$

Hence, the correct answer is (A).

7. We have,

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & x & 2^n - 1 \\ \sum_{r=1}^{n} 2.3^{r-1} & y & 3^n - 1 \\ \sum_{r=1}^{n} 4.5^{r-1} & z & 5^n - 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2^n - 1 & x & 2^n - 1 \\ 3^n - 1 & y & 3^n - 1 \\ 5^n - 1 & z & 5^n - 1 \end{vmatrix} = 0$$

[$\because \sum_{r=1}^{n} 2^{r-1} = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$, similarly

$\sum_{r=1}^{n} 2.3^{r-1} = 2.\dfrac{3^n - 1}{3 - 1} = 3^n - 1$ and $\sum_{r=1}^{n} 4.5^{r-1}$

$= 4.\dfrac{5^n - 1}{5 - 1} = 5^n - 1$]

Hence, the correct answer is (D).

8. $$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = \begin{vmatrix} x^3 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} + \begin{vmatrix} 1 & x^2y & x^2z \\ 0 & y^3+1 & y^2z \\ 0 & yz^2 & z^3+1 \end{vmatrix}$$

$$= x\begin{vmatrix} x^2 & x^2y & x^2z \\ y^2 & y^3+1 & y^2z \\ z^2 & yz^2 & z^3+1 \end{vmatrix} + (y^3+1)(z^3+1) - y^3z^3$$

$C_2 \to C_2 - yC_1$ and $C_3 \to C_3 - zC_1$

$$= x\begin{vmatrix} x^2 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix} + y^3z^3 + z^3 + y^3 + 1 - y^3z^3$$

$= x^3 + y^3 + z^3 + 1$

Given, $x^3 + y^3 + z^3 + 1 = 30 \quad \Rightarrow x^3 + y^3 + z^3 = 29$

As, $29 = 3^3 + 1^3 + 1^3$, then solutions are $(3, 1, 1), (1, 3, 1), (1, 1, 3)$

Sum of all the possible values of x is $= 3 + 1 + 1 = 5$

Hence, the correct answer is (A).

9. We are given $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is constant.

We get, $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$;

$$f''(x)=\begin{vmatrix}6x & -\sin x & -\cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix};$$

$$f'''(x)=\begin{vmatrix}6 & -\cos x & \sin x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$$

$$f'''(0)=\begin{vmatrix}6 & -1 & 0\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=0 \qquad (\because R_1\equiv R_2)$$

$\frac{d^3}{dx^3}$ [$f(x)$] is independent of p.

Hence, the correct answer is (D).

10. $$f(x)=\begin{vmatrix}1 & x & x+1\\ 2x & x(x-1) & (x+1)x\\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1)\end{vmatrix}$$

Apply $R_2\to R_2-xR_1$ and $R_3\to R_3-(x-1)\,R_2$

$$=\begin{vmatrix}1 & x & x+1\\ x & x(x-1)-x^2 & 0\\ x(x-1) & x(x-1)(x-2)-x(x-1)^2 & 0\end{vmatrix}$$

$$=\begin{vmatrix}1 & x & x+1\\ x & -x & 0\\ x(x-1) & -x(x-1) & 0\end{vmatrix}$$

$$\Rightarrow\quad f(x)=x^2(x-1)\begin{vmatrix}1 & x & x+1\\ 1 & -1 & 0\\ 1 & -1 & 0\end{vmatrix}=0$$

$\therefore\quad f(100)=0$

Hence, the correct answer is (A).

11. Breaking the given determinant into two determinants, we get

$$\begin{vmatrix}3^2+k & 4^2 & 3^2+3+k\\ 4^2+k & 5^2 & 4^2+4+k\\ 5^2+k & 6^2 & 5^2+5+k\end{vmatrix}=\begin{vmatrix}3^2+k & 4^2 & 3^2+k\\ 4^2+k & 5^2 & 4^2+k\\ 5^2+k & 6^2 & 5^2+k\end{vmatrix}$$

$$+\begin{vmatrix}3^2+k & 4^2 & 3\\ 4^2+k & 5^2 & 4\\ 5^2+k & 6^2 & 5\end{vmatrix}=0$$

$$\Rightarrow\quad 0+\begin{vmatrix}9+k & 16 & 3\\ 16+k & 25 & 4\\ 25+k & 36 & 5\end{vmatrix}=0$$

[Applying $R_3\to R_3-R_2$ and $R_2\to R_2-R_1$ in second det.]

$$\Rightarrow\quad 0+\begin{vmatrix}9+k & 16 & 3\\ 7 & 9 & 1\\ 9 & 11 & 1\end{vmatrix}=0 \qquad [\text{Applying } C_2\to C_2-C_1]$$

$$\Rightarrow\quad \begin{vmatrix}9+k & 7-k & 3\\ 7 & 2 & 1\\ 9 & 2 & 1\end{vmatrix}=0 \qquad [\text{Applying } R_3\to R_3-R_2]$$

$$\Rightarrow\quad \begin{vmatrix}9+k & 7-k & 3\\ 7 & 2 & 1\\ 2 & 0 & 0\end{vmatrix}=0$$

$\Rightarrow\quad 2(7-k-6)=0\Rightarrow k=1.$

Hence, the correct answer is (B).

12. Expanding the det., we get

$\Delta=-(b-a)^2\,[0-(a-c)^2\,(c-b)^2]+(c-a)^2[(a-b)^2\,(b-c)^2-0]$

$=2(a-b)^2\,(b-c)^2\,(c-a)^2.$

Hence, the correct answer is (C).

13. $$\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta\end{vmatrix}=0$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$,

we get $$\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}=0$$

$\Rightarrow\quad 2+4\sin 4\theta=0 \qquad \Rightarrow \sin 4\theta=-1/2$

$\Rightarrow\quad 4\theta=n\pi+(-1)^n\,(-\pi/6) \qquad \Rightarrow \theta=n\pi/4+(-1)^n\,(-\pi/24)$

$\therefore\quad \theta=7\pi/24,\ 11\pi/24$

Hence, the correct answer is (C).

14. $$\therefore\quad \frac{d}{dx}(\Delta_1)=\begin{vmatrix}\frac{d}{dx}(x) & \frac{d}{dx}(b) & \frac{d}{dx}(b)\\ a & x & b\\ a & a & x\end{vmatrix}$$

$$+\begin{vmatrix}x & b & b\\ \frac{d}{dx}(a) & \frac{d}{dx}(x) & \frac{d}{dx}(b)\\ a & a & x\end{vmatrix}+\begin{vmatrix}x & b & b\\ a & x & b\\ \frac{d}{dx}(a) & \frac{d}{dx}(a) & \frac{d}{dx}(x)\end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ a & x & b\\ a & a & x\end{vmatrix}+\begin{vmatrix}x & b & b\\ 0 & 1 & 0\\ a & a & x\end{vmatrix}+\begin{vmatrix}x & b & b\\ a & x & b\\ 0 & 0 & 1\end{vmatrix}$$

$$=\begin{vmatrix}x & b\\ a & x\end{vmatrix}+\begin{vmatrix}x & b\\ a & x\end{vmatrix}+\begin{vmatrix}x & b\\ a & x\end{vmatrix}=3\begin{vmatrix}x & b\\ a & x\end{vmatrix}=3\Delta_2.$$

Hence, the correct answer is (C).

15. Here $\Delta=(n!)^3$

$$\begin{vmatrix}1 & n+1 & (n+2)(n+1)\\ n+1 & (n+2)(n+1) & (n+3)(n+2)(n+1)\\ (n+2)(n+1) & (n+3)(n+2)(n+1) & (n+4)(n+3)(n+2)(n+1)\end{vmatrix}$$

$$=(n!)^3\,(n+1)^2\,(n+2)$$

$$\begin{vmatrix}1 & 1 & 1\\ n+1 & n+2 & (n+3)\\ (n+2)(n+1) & (n+3)(n+2) & (n+4)(n+3)\end{vmatrix}$$

operating $C_2\to C_2-C_1$, $C_3\to C_3-C_2$ and expanding

$$=(n!)^3\,(n+1)^2\,(n+2).\begin{vmatrix}1 & 0 & 0\\ n+1 & 1 & 1\\ (n+2)(n+1) & 2(n+2) & 2(n+3)\end{vmatrix}$$

$$= (n!)^3\,(n+1)^2\,(n+2).2$$

$$= (n!)^3\,((2n^3 + 8n^2 + 10n + 4) \text{ as on simplification.}$$

Hence, the correct answer is (A).

16. Applying $R_2 \to R_2 - (R_1 + R_3)$

$$\text{Determinant} \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Expanding it along R_2

Determinant $= 4\,((x+1)-(x-2))\,(2x-1) = 12(2x-1)$

$= 24x - 12$

We get $P = 24$.

Hence, the correct answer is (A).

17. $$\begin{vmatrix} x^2 & (x-1)^2 & (x-2)^2 \\ (x-1)^2 & (x-2)^2 & (x-3)^2 \\ (x-2)^2 & (x-3)^2 & (x-4)^2 \end{vmatrix}$$

Apply $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} (2x-1) & (2x-3) & (x-2)^2 \\ (2x-3) & (2x-5) & (x-3)^2 \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 2 & 2 & (2x-5) \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

Apply $R_1 \to R_1 - R_2$,

$$= \begin{vmatrix} 0 & 0 & 2 \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix} = -8$$

$\therefore$ Value of determinant is independent of x.
No value of 'x' satisfy the equation.
Hence, the correct answer is (D).

18. Let

$$f(x) = \begin{vmatrix} -1 + x\sin\alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ 13 + x\sin\beta & \cos(x+\beta) & \sin(x+\beta) \\ -12 + x\sin\gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$$

$$\therefore \quad f'(x) = \begin{vmatrix} \sin\alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ \sin\beta & \cos(x+\beta) & \sin(x+\beta) \\ \sin\gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$$

$$+ \begin{vmatrix} -1 + x\sin\alpha & -\sin(x+\alpha) & \sin(x+\alpha) \\ 13 + x\sin\beta & -\sin(x+\beta) & \sin(x+\beta) \\ -12 + x\sin\gamma & -\sin(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$$

$$+ \begin{vmatrix} -1 + x\sin\alpha & \cos(x+\alpha) & \cos(x+\alpha) \\ 13 + x\sin\beta & \cos(x+\beta) & \cos(x+\beta) \\ -12 + x\sin\gamma & \cos(x+\gamma) & \cos(x+\gamma) \end{vmatrix}$$

$\Rightarrow \quad f'(x)$

$$= \begin{vmatrix} \sin\alpha & \cos x\cos\alpha - \sin x\sin\alpha & \sin x\cos\alpha + \cos x\sin\alpha \\ \sin\beta & \cos x\cos\beta - \sin x\sin\beta & \sin x\cos\beta + \sin x\sin\beta \\ \sin\gamma & \cos x\cos\gamma - \sin x\sin\gamma & \sin x\cos\gamma + \cos x\sin\beta \end{vmatrix}$$

$+ 0 + 0$

This determinant can be expressed as the sum of four determinants in each of which two columns coincide after taking cosx, sinx common.
$\therefore$ The value of the determinant = 0
i.e., $f'(x) = 0 \Rightarrow$ f(x) = a constant function i.e., independent of x.
Hence, the correct answer is (A).

19. $$\text{Given that } f(-x) = \begin{vmatrix} -x^5 & |\sin x| & 2x^4 \\ -\tan^3 x & 1 & \sec 2x \\ -\sin^3 x & x^4 & 5 \end{vmatrix} = -f(x)$$

So $f(x)$ is an odd function

$$\therefore \quad \int_{-\pi/2}^{\pi/2} f(x)\,dx = 0$$

Hence, the correct answer is (C).

20. We have,

$$\sum_{n=1}^{k} U_n = \begin{vmatrix} \sum_{n=1}^{k} 1 & k & k \\ 2\sum_{n=1}^{k} n & k^2+k+1 & k^2+k \\ 2\sum_{n=1}^{k} n - \sum_{n=1}^{k} 1 & k^2 & k^2+k+1 \end{vmatrix}$$

$$= \begin{vmatrix} k & k & k \\ k(k+1) & k^2+k+1 & k^2+k \\ k^2 & k^2 & k^2+k+1 \end{vmatrix}$$

$$= \begin{vmatrix} k & 0 & k \\ k^2+k & 1 & k^2+k \\ k^2 & 0 & k^2+k+1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1]$$

$$= k(k^2 + k + 1) - k^3 = k(k + 1) = 110$$

$\Rightarrow \quad k = 10$

Hence, the correct answer is (A).

21. The given determinant breaks in the product of two determinants

$$= \begin{vmatrix} \tan A & \cot A & 0 \\ \tan B & \cot B & 0 \\ \tan C & \cot C & 0 \end{vmatrix} \times \begin{vmatrix} \cot A & \tan A & 0 \\ \cot B & \tan B & 0 \\ \cot C & \tan C & 0 \end{vmatrix} = 0.$$

Hence, the correct answer is (C).

22. If $f(x)$ is polynomial of degree at most 6 in x
If $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_6x^6$. Then
$f'(x) = a_1 + 2a_2x + \ldots + 6a_6x^5$
$\Rightarrow \quad a_1 = f'(0)$

$$\text{Now, } f'(x) = \begin{vmatrix} 4 & (x-2)^2 & x^3 \\ 8 & \left(x-2\sqrt{2}\right)^2 & (x+1)^3 \\ 12 & \left(x-2\sqrt{3}\right)^2 & (x-1)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 4x-4 & 2(x-2) & x^3 \\ 8x-4\sqrt{2} & 2\left(x-2\sqrt{2}\right) & (x+1)^3 \\ 12x-4\sqrt{3} & 2\left(x-2\sqrt{3}\right) & (x-1)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 4x-4 & (x-2)^2 & 3x^2 \\ 8x-4\sqrt{2} & \left(x-2\sqrt{2}\right)^2 & 3(x+1)^2 \\ 12x-4\sqrt{3} & \left(x-2\sqrt{3}\right)^2 & 3(x-1)^2 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = 0 + 0 + \begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 3 \\ -4\sqrt{3} & 12 & 3 \end{vmatrix}$$

$$\Rightarrow \quad f'(0) = 48\left(1+\sqrt{2}-\sqrt{3}\right)$$

Hence, the correct answer is (D).

23. $a_{pq} = (i^p)^q$

$$\Rightarrow \quad \Delta = \begin{vmatrix} i & i^2 & i^3 \\ i^2 & i^4 & i^6 \\ i^3 & i^6 & i^9 \end{vmatrix} = i \cdot i^2 \cdot i^3 \begin{vmatrix} 1 & 1 & 1 \\ i & i^2 & i^3 \\ i^2 & i^4 & i^6 \end{vmatrix}$$

$$= i^6\, i^3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & i & i^2 \\ 1 & i^2 & i^4 \end{vmatrix} = i^9 \begin{vmatrix} 1 & 1 & 1 \\ 1 & i & -1 \\ 1 & -1 & 1 \end{vmatrix} = -4i$$

Hence, the correct answer is (D).

24. For non-trivial solutions,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0 \Rightarrow \alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma$$

$$\therefore \left(\alpha+\beta+\gamma\right)\left(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha\right) = 0$$

$$(\alpha+\beta+\gamma)\left((\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\right) = 0$$

$$(-a)(a^2-3b) = 0$$

$\Rightarrow a^3 - 3ab = 0$. Given $a^3 - 6c = 0$

We get $2c = ab$

Hence, the correct answer is (A).

25. $$\begin{vmatrix} x+1 & x+1 & x+1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0 \Rightarrow (x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$ and expanding $(x+1)(x-2)^2 = 0$, $x = 2$, $x = -1$

Hence, the correct answer is (B).

26. $$\begin{vmatrix} a+p & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{a}{a-(a+p)} + \frac{b}{b-(b+q)} + \frac{c}{c-(r+c)} = 1$$

$$\Rightarrow \quad \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1$$

Hence, the correct answer is (D).

27. $$\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $2(a+b+c)$ common from first column

$$\text{Det.} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

[by $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$= 2(a+b+c)^3$$

Hence, the correct answer is (C).

28. $$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

By the operation $C_3 \to C_3 - (\alpha C_1 + C_2)$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+2b\alpha+c) \end{vmatrix} = 0$$

$$\Rightarrow \quad -(a\alpha^2+2b\alpha+c)\,(ac-b^2) = 0$$

$$\Rightarrow \quad b^2 = ac$$

Hence a, b, c are in G.P.

Hence, the correct answer is (B).

29. Possible values are $-2, -1, 0, 1, 2$.

i.e. $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$, $\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0$, $\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$,

$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$, $\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$.

Positive distinct values of det = 2

Hence, the correct answer is (A).

30. $$\begin{vmatrix} p & b & c \\ p-a & b-q & 0 \\ 0 & q-b & c-r \end{vmatrix} = 0$$

$$p((b-q)\,(c-r)-0) - (p-a)\,(b(c-r) - c(q-b)) = 0$$

$$\frac{p}{p-a} = \frac{b}{b-q} + \frac{c}{c-r} - 2 + 2$$

$$\frac{p}{p-a} = \frac{q}{b-q} + \frac{r}{c-r} + 2$$

$$\Rightarrow \quad \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Hence, the correct answer is (C).

31. Here the cofactors of $\lambda, c, -b, \dots$ in are $a^2 + \lambda^2, ab + c\lambda, ca - b\lambda$, … respectively.
Therefore the value of Δ_o is Δ^2.
Hence, the correct answer is (B).

32. $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2+4\sin 2x & \cos^2 x & 2\sin 2x \\ 2+4\sin 2x & 1+\cos^2 x & 4\sin 2x \\ 2+4\sin 2x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$= (2 + 4\sin 2x) \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$$= (2 + 4 \sin 2x)$$

$$f(x)_{\max} = 6$$

Hence, the correct answer is (C).

33. We have

$$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = a^3 + b^3$$

$$\therefore \quad \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0 \Rightarrow a^3 + b^3 = 0 \Rightarrow (a/b)^3 = -1$$

$\Rightarrow$ a/b is one of the cube roots of –1.

Hence, the correct answer is (D).

34. $$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = \begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0$$

or $bc + 2ca - ab - 2bc = 0$
or $2ca - ab - bc = 0$

or $b = \dfrac{2ac}{a+c}$

Hence, the correct answer is (B).

35. $$\Delta_1 = 2a^2 \begin{vmatrix} p^2 & p & 1 \\ q^2 & q & 1 \\ r^2 & r & 1 \end{vmatrix} = 2a^2 \begin{vmatrix} p^2 & p & 1 \\ q^2-p^2 & q-p & 0 \\ r^2-p^2 & r-p & 0 \end{vmatrix}$$

$$= 2a^2(q-p)(r-p) \begin{vmatrix} p^2 & p & 1 \\ q+p & 1 & 0 \\ r+p & 1 & 0 \end{vmatrix}$$

$$= 2a^2(q-p)\,(r-p)\,(q+p-r-p) = 2a^2(p-q)\,(q-r)\,(r-p)$$

$$\Delta_2 = a^2 \begin{vmatrix} pq & p+q & 1 \\ qr & q+r & 1 \\ rp & r+p & 1 \end{vmatrix} = a^2 \begin{vmatrix} pq & p+q & 1 \\ q(r-p) & r-p & 0 \\ p(r-q) & r-q & 0 \end{vmatrix}$$

$$= a^2(r-p)\,(r-q)\,(q-p)$$

$$= a^2(p-q)\,(q-r)\,(r-p)$$

$\Delta_1 = 2\Delta_2$

Hence, the correct answer is (C).

36. $C_1 \to C_1 + C_2 + C_3$ gives

$$\Delta = (x+y)^2 \begin{vmatrix} 1 & x^2 & y^2 \\ 1 & y^2 & 2xy \\ 1 & 2xy & x^2 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\Delta = (x+y)^2 \begin{vmatrix} y^2-x^2 & 2xy-y^2 \\ 2xy-x^2 & x^2-y^2 \end{vmatrix}$$

$$= (x+y)^2\,[-(x^2-y^2)^2 - (2xy-x^2)\,(2xy-y^2)]$$

$$= -(x+y)^2\,[x^2 - xy + y^2]^2 = -(x^3+y^3)^2.$$

Hence, the correct answer is (D).

37. $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$

$$\Delta = \begin{vmatrix} x & 2y-z & -z \\ y-x & 2(x-y) & 0 \\ z-y & 2(y-x) & 2(x-y) \end{vmatrix}$$

$$= (x-y) \begin{vmatrix} x & 2y-z & -z \\ -1 & 2 & 0 \\ z-y & 2(y-x) & 2(x-y) \end{vmatrix}$$

So $(x-y)$ is a factor of Δ
Expand determinant
Hence, the correct answer is (A).

38. Multiply by y, z and x in rows 1,2 and 3 respectively and then take common y, z and x from column 1, 2 and 3 respectively, then

$$\begin{vmatrix} y^3+1 & y^3 & y^3 \\ z^3 & z^3+1 & z^3 \\ x^3 & x^3 & x^3+1 \end{vmatrix} = 11$$

$(C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3)$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & y^3 \\ -1 & 1 & z^3 \\ 0 & -1 & x^3+1 \end{vmatrix} = 11$$

$$\Rightarrow \quad 1\left(x^3 + 1 + z^3\right) + y^3(1) = 11 \Rightarrow x^3 + y^3 + z^3 = 10$$

So solution are (1, 1, 2), (1, 2, 1) or (2, 1, 1)
Hence, the correct answer is (C).

39. Subtracting $R_1 + 2R_2$ from R_3(i.e.,$R_3 \to R_3 - R_1 - 2R_2$) we have

$$f(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{simplifying we get}$$

$$f(x) = 2ax(b+1) - b(2ax-1) = 2ax + b$$

Therefore $f^{-1}(x) = \dfrac{x-b}{2a} (\because a \neq 0)$

Hence, the correct answer is (C).

40. Since α, β are the roots of the equation

$x^2 - 5x + 10 = 0$ so $\alpha + \beta = 5$, $\alpha\beta = 10$

$$\Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

(Considering Row – Row multiplication)

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & \beta-1 \\ 0 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$$= (\alpha - 1)^2 (\beta - 1)^2 (\beta - \alpha)^2$$

$$= (\alpha + \beta - \alpha\beta - 1)^2 \{(\alpha + \beta)^2 - 4\alpha\beta\}$$

$$= (5 - 10 - 1)^2 (-15) = -540$$

Hence, the correct answer is (B).

41. Consider the terms independently we get

$$\sum_{i=1}^{n} 2^{i-1} = 1 + 2 + 2^2 + \ldots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}\left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\sum_{i=1}^{n} \sin i\theta = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$$

$$= \frac{\sin\left(n + \frac{1}{2}\right)\theta \sin\frac{n\theta}{2}}{2\sin\frac{\theta}{2}}$$

Now, $\sum_{i=1}^{n} \Delta_i =$

$$\begin{vmatrix} 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \\ \sum_{r=1}^{n} \frac{r(r+1)}{2} & \sum_{r=1}^{n} \frac{r(r+1)(2r+1)}{6} & \sum_{r=1}^{n}\left(\frac{r(r+1)}{2}\right)^2 \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \end{vmatrix}$$

$= 0$ ($\because$ First and third row are identical)

Hence, the correct answer is (C).

42. If A is a square matrix and X is a column matrix, then the matrix equation $AX = 0$ has non-zero solution if det A is equal to zero. Therefore $\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$$1(1+1) + k(-k+1) - 1(k+1) = 0$$

$$-k^2 + 1 = 0 \Rightarrow k = \pm 1$$

Hence, the correct answer is (D).

43. Req. $= \begin{vmatrix} \sum_{r=1}^{n} r & x & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n} (2r-1) & y & n^2 \\ \sum_{r=1}^{n} (3r-2) & z & \frac{n(3n-1)}{2} \end{vmatrix}$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix} = 0$$

Hence, the correct answer is (A).

44. G.E. is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 + x_3 & y_2 + y_3 & 2 \end{vmatrix} = 0$ (or)

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2 + x_3}{2} & \frac{y_2 + y_3}{2} & 1 \end{vmatrix} = 0$$

Hence, the correct answer is (B).

45. $\because \begin{vmatrix} c-a & c-b & ab-c^2 \\ a-b & a-c & bc-a^2 \\ b-c & b-a & ac-b^2 \end{vmatrix}$ is determinant of cofactors of

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}$$

$$P = \begin{vmatrix} c-a & c-b & ab-c^2 \\ a-b & a-c & bc-a^2 \\ b-c & b-a & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}^2 = (2010)^2$$

$$= 2 \times 3 \times 5 \times (67)^2 = 2^2 3^2 5^2 (67)^2$$

No of divisors of $p = (2 + 1)\ (2 + 1)\ (2 + 1)\ (2 + 1) = 81$

Hence, the correct answer is (D).

46. $R_2 \to R_2 - R_3$

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ (\alpha-1)^2 & (\beta-1)^2 & (\gamma-1)^2 \end{vmatrix} = 0$$

Apply $R_3 \to R_3 - R_1$

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma^2 \\ 1-2\alpha & 1-2\beta & 1-2\gamma \end{vmatrix} = 0$$

$R_3 \to R_3 + 2R_2$

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 \\ \alpha & \beta & \gamma \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow (\beta-\alpha)(\gamma-\alpha)(\beta-\gamma)=0$$

Hence, the correct answer is (B).

47. $$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3(\lambda-1) \end{vmatrix} = 0 \Rightarrow \lambda = 0 \text{ or } 3\,.$$

If $\lambda = 0$, the equations become $-x + y = 0$, $-x - 2y + 3z = 0$ and $2x + y - 3z = 0$,

$$\therefore \frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4}$$

Hence, the correct answer is (D).

48. $\Delta.\Delta^2 = 64$ (∵ Δ and its adjoint)

$\Rightarrow \Delta^3 = 64$

$\Rightarrow \Delta = 4$

G.E. $= [(2\times3\times5)+(3\times5\times4)]\Delta$

$= (30+60)\Delta$

$= 90(4)$

$= 360$

(∵ splitting into 8 determinants only 2 will survive)

Hence, the correct answer is (C).

49. If we interchange any 2 rows of a determinant in the set *B*, its value becomes –1 and hence it is in *C*. Likewise, for every determinant in *C*, there is a corresponding determinant in *B*. *B* and *C* have the same number of elements.

$A \neq B \cup C$

Hence, the correct answer is (B).

50. We use the pascal rule $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ or

$$\binom{n+1}{r+1} - \binom{n}{r} = \binom{n}{r+1}$$

repeatedly in transforming the determinant

$$\begin{vmatrix} \binom{x}{r} & \binom{x+1}{r+1} & \binom{x+2}{r+2} \\ \binom{y}{r} & \binom{y+1}{r+1} & \binom{y+2}{r+2} \\ \binom{z}{r} & \binom{z+1}{r+1} & \binom{z+2}{r+2} \end{vmatrix} \begin{matrix} C_3 - C_2 \to C_3 \\ C_2 - C_1 \to C_2 \end{matrix}$$

$$= \begin{vmatrix} \binom{x}{r} & \binom{x}{r+1} & \binom{x+1}{r+2} \\ \binom{y}{r} & \binom{y}{r+1} & \binom{y+1}{r+2} \\ \binom{z}{r} & \binom{z}{r+1} & \binom{z+1}{r+2} \end{vmatrix} C_3 - C_2 \to C_3$$

$$= \begin{vmatrix} \binom{x}{r} & \binom{x}{r+1} & \binom{x}{r+2} \\ \binom{y}{r} & \binom{y}{r+1} & \binom{y}{r+2} \\ \binom{z}{r} & \binom{z}{r+1} & \binom{z}{r+2} \end{vmatrix} \therefore \lambda = 1$$

Hence, the correct answer is (B).

51. $$\begin{vmatrix} \alpha & \beta \\ \gamma & -\alpha \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2 \Rightarrow -\alpha^2 - \beta\gamma = 1$$

or $\alpha^2 + \beta\gamma + 1 = 0$

Hence, the correct answer is (A).

52. $$\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ x & y & z \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

$\therefore \quad \alpha + \beta + \gamma = xy + yz + zx$

Hence, the correct answer is (C).

53. $$\begin{vmatrix} a^2 + 2^{n+1} + 2p & b^2 + 2^{n+2} + 3q & c^2 + p \\ 2^n + p & 2^{n+1} + q & 2q \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{vmatrix}$$

$R_1 \to R_1 - R_3$ and $2q = p + r$

$$= \begin{vmatrix} 2^{n+1} - 2^n + p & 2^{n+2} - 2^{n+1} + q & p + r \\ 2^n + p & 2^{n+1} + q & 2q \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{vmatrix}$$

$$= \begin{vmatrix} 2^n(2-1) + p & 2^{n+1}(2-1)q & p + r \\ 2^n + p & 2^{n+1} + q & 2q \\ a^2 + 2^n + p & b^2 + 2^{n+1} + 2q & c^2 - r \end{vmatrix} = 0$$

$[R_1 = R_2]$.

Hence, the correct answer is (B).

54. $$\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} x-m & n-y & 0 \\ 0 & y-n & r-z \\ m & n & z \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{z}{z-r} + \frac{m}{x-m} + \frac{n}{y-n} = 0$$

$$\frac{x}{x-m} + \frac{y}{y-n} + \frac{z}{z-r} = 2$$

Now $A.M. \geq G.M.$

$$\frac{\frac{z}{z-r} + \frac{x}{x-m} + \frac{y}{y-n}}{3} \geq \left(\frac{z}{(z-r)} \frac{x}{(x-m)} \frac{y}{(y-n)} \right)^{1/3}$$

$$\Rightarrow \frac{z}{z-r}\frac{x}{x-m}\frac{y}{y-n} \le \frac{8}{27}$$

Hence, the correct answer is (B).

55. $a_r = \frac{20}{r}, r = 1,2,3,....9$

The value of determinant is 50/21

Hence, the correct answer is (C).

56. $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1+10b_1+100a_1 & c_2+10b_2+100a_2 & c_3+10b_3+100a_3 \end{vmatrix}$$

$$(\text{By } R_3 \Rightarrow R_3 + 10R_2 + 100R_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 19m_1 & 19m_2 & 19m_3 \end{vmatrix} = 19\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 19n$$

Hence, the correct answer is (C).

57. Note that $\Delta(x)$ is a polynomial of degree at most 6 in x.

If $\Delta(x) = a_0 + a_1x + a_2x^2 + ... + a_6x^6$, then

$\Delta'(x) = a_1 + 2a_2x + ... + 6a_6x^5$

$\Rightarrow \quad a_1 = \Delta'(0)$.

Now,

$$\Delta'(\mathrm{x}) = \begin{vmatrix} 1 & (\mathrm{x}-1)^2 & \mathrm{x}^3 \\ 1 & \mathrm{x}^2 & (\mathrm{x}+1)^3 \\ 1 & (\mathrm{x}+1)^2 & (\mathrm{x}+2)^3 \end{vmatrix} + \begin{vmatrix} \mathrm{x}-2 & 2(\mathrm{x}-1) & \mathrm{x}^3 \\ \mathrm{x}-1 & 2\mathrm{x} & (\mathrm{x}+1)^3 \\ \mathrm{x} & 2(\mathrm{x}+1) & (\mathrm{x}+2)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} \mathrm{x}-2 & (\mathrm{x}-1)^2 & 3\mathrm{x}^2 \\ \mathrm{x}-1 & \mathrm{x}^2 & 3(\mathrm{x}+1)^2 \\ \mathrm{x} & (\mathrm{x}+1)^2 & 3(\mathrm{x}+2)^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta'(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 8 \end{vmatrix} + \begin{vmatrix} -2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & 2 & 8 \end{vmatrix} + \begin{vmatrix} -2 & 1 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 12 \end{vmatrix}$$

$$= -8 - 12 + 18 = -2$$

Hence, the correct answer is (B).

58. $$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow \quad (3\sin\theta - 4\sin^3\theta) + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow \quad 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \quad \sin\theta = 0 \text{ or } \theta = n\pi + (-1)^n\frac{\pi}{6}, \forall n \in Z$

Hence, the correct answer is (A).

59. $$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}, a - 2b + c = 1; (a-b) + (c-b) = 1$$

Apply the operation,

$R_1 \to R_1 - 2R_2 + R_3$

$R_3 \to R_3 - R_2$, the determinant reduces to

$$\begin{vmatrix} 0 & 0 & 1 \\ x+2 & x+3 & x+b \\ 1 & 1 & c-b \end{vmatrix} = -1$$

Hence, the correct answer is (C).

60. $p + q + r = 0 \quad \Rightarrow p^3 + q^3 + r^3 = 3\,pqr$

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr(a^3 + b^3 + c^3 - 3abc)$$

$pqr \Rightarrow k = pqr$

Hence, the correct answer is (C).

61. $$\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix} - \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

By applying $C_2 \to C_2 + C_1$, $C_3 \to C_3 + C_2$

$$\begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix} - \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

Now apply $C_3 \to C_3 + C_2$

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix} - \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

$\therefore \quad \text{Ans} = 0$

Hence, the correct answer is (A).

62. We have

$(a^2 + b^2 + c^2)\,x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$

$\Rightarrow \quad (ax - b)^2 + (bx - c)^2 + (cx - d)^2 \le 0$

$\Rightarrow \quad (ax - b)^2 = (bx - c)^2 = (cx - d)^2 = 0$

$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = x$

$\Rightarrow \quad b^2 = ac$ or $2\log b = \log a + \log c$,

$$\text{Now, } \begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = \begin{vmatrix} 130 & 54 & \log a + \log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$$

[Apply $R_1 \to R_1 + R_3$]

$$\begin{vmatrix} 0 & 0 & 0 \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = 0$$

[Apply $R_1 \to R_1 - 2R_2$]

Hence, the correct answer is (C).

63. $$\Delta = \begin{vmatrix} X & SX & tX \\ X_1 & SX_1 + S_1X & tX_1 + t_1X \\ X_2 & SX_2 + 2S_1X_1 + S_2X & tX_2 + 2t_1X_1 + t_2X \end{vmatrix}$$

$$\begin{pmatrix} C_2 \to C_2 - SC_1 \\ C_3 \to C_3 - tC_1 \end{pmatrix}$$

$$\Delta = \begin{vmatrix} X & 0 & 0 \\ X_1 & S_1X & t_1X \\ X_2 & 2S_1X_1 + S_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X^2 \begin{vmatrix} S_1 & t_1 \\ 2S_1X_1 + S_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X^3 \begin{vmatrix} S_1 & t_1 \\ S_2 & t_2 \end{vmatrix} (R_2 \to R_2 - 2X_1R_1)$$

$\therefore \quad n = 3.$

Hence, the correct answer is (C).

64. $f(x) - x > 0$ or $f(x) - x < 0 \ \forall \ x \in R$

$f(f(x)) - f(x) > 0$ or $f(f(x)) - f(x) < 0$

Adding, $f(f(x)) - x > 0$ or $f(f(x)) - x < 0$

$\Rightarrow$ roots of $f(f(x)) - x = 0$ are imaginary.

$$\text{Let } z = \begin{vmatrix} 2 & \alpha & \delta \\ \beta & 0 & \alpha \\ \gamma & \beta & 1 \end{vmatrix}$$

$$\bar{z} = \begin{vmatrix} 2 & \bar{\alpha} & \bar{\delta} \\ \bar{\beta} & 0 & \bar{\alpha} \\ \bar{\gamma} & \bar{\beta} & 1 \end{vmatrix} = \begin{vmatrix} 2 & \beta & \gamma \\ \alpha & 0 & \beta \\ \delta & \alpha & 1 \end{vmatrix} = z$$

Hence, the correct answer is (B).

65. $100A + 80 + 8 = 72\lambda_1$

$600 + 10B + 8 = 72\lambda_1 \qquad \lambda_1, \lambda_2, \lambda_3 \in I.$

$800 + 60 + C = 72\lambda_3$

$$\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix} (R_3\ R_3 + 10R_2 + 100R_1)$$

$$= \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 72\lambda_1 & 72\lambda_2 & 72\lambda_3 \end{vmatrix} = 72 \begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}$$

$A88$ is div. by 72

$\Rightarrow$ $A88$ is div. by 9

$\Rightarrow$ $A + 8 + 8$ is div. by 9

$\therefore A = 2$

$6B8$ is div. by 9 $\Rightarrow B = 4$

So determinant is divisible by $72 \times 4 = 288$ or 144

Hence, the correct answer is (B).

66. $\text{LHS} = \dfrac{d^3\Delta}{dx^3} - \dfrac{d^2\Delta}{dx^2} = (24mx + 6n) - (12mx^2 + 6nx + 2r)$

$x = 0 \Rightarrow 6n - 2r$

$\Rightarrow$ $2(3n - r)$

Hence, the correct answer is (B).

67. $$\Delta = x^ky^kz^k \begin{vmatrix} 1 & ar & a^2r^2 \\ 1 & ar^2 & a^2r^4 \\ 1 & ar^3 & a^2r^6 \end{vmatrix}$$

$$= x^ky^kz^kar(a^2r^2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & r & r^2 \\ 1 & r^2 & r^4 \end{vmatrix}$$

$$= a^{3(k+1)} \cdot r^{3(2k+1)} \begin{vmatrix} 1 & 1 & 1 \\ 0 & r-1 & r^2-1 \\ 0 & r^2-1 & r^4-1 \end{vmatrix}$$

$a^{3(k+1)}.r^{3(2k+1)}[(r - 1)(r^4 - 1) - (r^2 - 1)^2] \Rightarrow k = -1$

Hence, the correct answer is (A).

68. $$\Delta = \frac{1}{ab} \begin{vmatrix} \left(1 + a^2 - b^2\right)b & 2ab^2 & -2b^2 \\ 2a^2b & a\left(1 - a^2 + b^2\right) & 2a^2 \\ 2b & -2a & 1 - a^2 - b^2 \end{vmatrix}$$

by $(R_1 \times b, R_2 \times a)$

$$\Delta = \frac{1}{ab} \cdot ab \begin{vmatrix} 1 + a^2 - b^2 & 2b^2 & -2b^2 \\ 2a^2 & 1 - a^2 + b^2 & 2a^2 \\ 2 & -2 & 1 - a^2 - b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + a^2 + b^2 & 0 & -2b^2 \\ 1 + a^2 + b^2 & 1 + a^2 + b^2 & 2a^2 \\ 0 & -(1 + a^2 + b^2) & 1 - a^2 - b^2 \end{vmatrix}$$

$(C_1 \to C_1 + C_2, C_2 \to C_2 + C_3)$

$= (1 + a^2 + b^2)^3$

Hence, the correct answer is (C).

69. If $\Delta = \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$ other determinant (say Δ_1) is the cofactor determinant

$\Delta\Delta_1 = \Delta^3$ (for 3rd order det)

$\Delta = \lambda(\lambda^2 + a^2 + b^2 + c^2)$

by comparing $\lambda = 1$

Hence, the correct answer is (B).

70. For constant term put $x = 0$

$$f(0) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = 1$$

Hence, the correct answer is (C).

71. For the given system to have a non-trivial solution, we must have

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3 - 2k & -10 \end{vmatrix} = 0$$

[Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$]

$\Rightarrow \quad 20k + 11(3 - 2k) = 0 \Rightarrow k = \dfrac{33}{2}$

Hence, the correct answer is (C).

72. $$\begin{vmatrix} pq & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\left(a^3+b^3+c^3\right) - abc\left(p^3+q^3+r^3\right)$$

$$= pqr\left(3\,abc\right) - abc\left(3\,pqr\right) = 0$$

(Using Sarrus Rule)

Hence, the correct answer is (A).

73. Let $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$c_2 \to c_2 - \frac{a_{12}}{a_{11}}C_1 \qquad C_3 \to \quad C_3 - \frac{a_{13}}{a_{11}}C_1$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \dfrac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \dfrac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \dfrac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \dfrac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

so minimum value = –4 for $\begin{vmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}$

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. Applying $R_1 \to R_1 + \sin\phi(R_2) + \cos\phi\,(R_3)$,

$$\Delta = \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\theta & \sin\theta & \cos\phi \end{vmatrix}$$

$$= (\cos 2\phi + 1)\,(\sin^2\theta + \cos^2\theta) = (1 + \cos 2\phi) = 0$$

θ can be all real $\phi = \pi/2$

Hence, (A) and (B) are correct.

2. $$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}$$

Apply $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

Apply $C_1 \to C_1 + C_2$, we get

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0 \Rightarrow 2 + 4\sin 4\theta = 0$$

$$\Rightarrow \quad \sin 4\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right) \Rightarrow 4\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \theta = [6n - (-1)^n]\,\frac{\pi}{24} \qquad \Rightarrow \text{for } n = 1, 2$$

$$\theta = \frac{7\pi}{24} \text{ and } \frac{11\pi}{24} \in \left(0, \frac{\pi}{2}\right)$$

Hence, (A) and (C) are correct.

3. $$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 3 \end{vmatrix}$$

$2P_1 + P_2 = P_3$

$2 + \alpha = \alpha^2 \Rightarrow \alpha^2 - \alpha - 2 = 0 \Rightarrow (\alpha - 2)\,(\alpha + 1) = 0$

$\alpha = 2, -1$ $\quad\infty$ soln.

$\alpha \neq 2, -1$ $\quad$ No soln.

Hence, (A) and (C) are correct.

4. For the system to have nontrivial solution

$$\begin{vmatrix} \sin\theta & -2\cos\theta & -a \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

ie $\quad \sin\theta + 4\cos\theta = 3a$

$$\Rightarrow \quad -\frac{\sqrt{17}}{3} \le a \le \frac{\sqrt{17}}{3} \Rightarrow a \text{ has three integer values.}$$

Hence, (B), (C) and (D) are correct.

5. By the row operation $R_1 \to R_1 - R_2$ and taking $x - 2$ common from R_1, we get

$$f(x) = (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}$$

$$f(x) = (x-2)\begin{vmatrix} 1 & 3 & -1 \\ 0 & -3x-6 & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

(by $R_2 \to R_2 - 2R_1, R_3 \to R_3 + 3R_1$)

$$f(x) = (x-2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & -3x-6 & x-1 \\ 0 & 2x+9 & x-1 \end{vmatrix}$$

(by $C_2 \to C_2 - 3C_1, C_3 \to C_3 + C_1$)

Therefore

$$f(x) = (x-2)(x-1)(-5x-15)$$

$$f(x) = -5(x+3)(x-1)(x-2)$$

Hence we get

$$\alpha = -3, \beta = 1, \gamma = 2$$

Hence, (A), (B) and (C) are correct.

6. $$\begin{vmatrix} 1+\cos^2 x & \sin^2 x & \cos 2x \\ \cos^2 x & 1+\sin^2 x & \cos 2x \\ \cos^2 x & \sin^2 x & 1+\cos 2x \end{vmatrix}$$

$$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$= \begin{vmatrix} 1+\cos^2 x & \sin^2 x & \cos 2x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \left(1+\cos^2 x\right) - \sin^2 x(-1) + \cos 2x(1)$$

$$= 1+\cos^2 x + \sin^2 x + \cos 2x$$

$\therefore \quad \alpha = 3, \beta = 1$

Hence, (A), (B) and (C) are correct.

7. $$\Delta = \begin{vmatrix} a\alpha+b & a & b \\ b\alpha+c & b & c \\ 0 & a\alpha+b & b\alpha+c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a\alpha+b & a & b \\ b\alpha+c & b & c \\ -\left(a\alpha^2+2b\alpha+c\right) & 0 & 0 \end{vmatrix} = 0$$

$R_3 \to R_3 - \alpha R_1 - R_2$

$\Rightarrow \quad -(a\alpha^2+2b\alpha+c) = 0 \qquad$ (or) $ac = b^2$

α is a root of $ax^2+2bx+c=0$ or a, b, c are in G.P.

Hence, (B) and (D) are correct.

8. Applying $C_3 \to C_3 - xC_2, \quad C_2 \to C_2 - xC_1$

$$f(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2+2a^2 & 4a^2x & 6a^2x^2+2a^4 \end{vmatrix}$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & 2x \\ 3x^2+2a^2 & 2x & 3x^2+a^2 \end{vmatrix} \qquad [C_3 \to C_3 - xC_2]$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2+2a^2 & 2x & x^2+a^2 \end{vmatrix} \qquad [C_1 \to C_1 - 3C_3]$$

$$= 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -a^2 & 2x & x^2+a^2 \end{vmatrix} = 4a^6$$

Hence, (A), (B), (C) and (D) are correct.

9. $$\Rightarrow f'(x) = \begin{vmatrix} 2x-5 & 2x-5 & 3 \\ 6x+1 & 6x+1 & 9 \\ 14x-6 & 14x-6 & 21 \end{vmatrix}$$

$$+ \begin{vmatrix} x^2-5x+3 & 2 & 3 \\ 3x^2+x+4 & 6 & 9 \\ 7x^2-6x+9 & 14 & 21 \end{vmatrix} = 0+0=0$$

$\Rightarrow f(x)$ is a constant polynomial and $f(0) \neq 0 \;\Rightarrow \mathrm{d} \neq 0$

Hence, (A), (B) and (C) are correct.

10. Replace every element of Δ by its cofactors we get $(A) = \Delta^2$.

$$\text{Also } \Delta^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ -c & -a & -b \\ b & c & a \end{vmatrix} = \text{(B)}$$

Here $\Delta = 3abc - a^3 - b^3 - c^3 = 0$

$$\frac{-(a+b+c)}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) = 0$$

$\Rightarrow \quad$ either $a = b = c$ or $a + b + c = 0$.

Hence, (A), (B), (C) and (D) are correct.

11. Expand third column and express it as sum of two determinants then two columns become identical determinant value becomes equal to zero

$$\Delta = \begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha)g(\theta)+g(\alpha)f(\theta) \\ f(\beta) & g(\beta) & f(\beta)g(\theta)+g(\beta)f(\theta) \\ f(\gamma) & g(\gamma) & f(\gamma)g(\theta)+f(\theta)g(\gamma) \end{vmatrix}$$

$\Delta = 0$

Hence, (A), (B), (C) and (D) are correct.

12. $$\begin{vmatrix} 1 & a & -1 \\ 2 & -1 & a \\ a & 1 & 2 \end{vmatrix} = 0 \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - aR_1 \end{matrix}$$

$$\begin{vmatrix} 1 & a & 1 \\ 0 & -1-2a & a+2 \\ 0 & 1-a^2 & 2+a \end{vmatrix} = 0 \Rightarrow (a+2)\left(a^2-2a-2\right) = 0$$

$\Rightarrow \quad a = -2, 1 \pm \sqrt{3}$

Hence, (B), (C) and (D) are correct.

13. (A) Put $x = 0 \Rightarrow f = 1$

(B) Diff. both sides and put $x = 0 \Rightarrow e = 3$.

(C), (D) put $x = 1, x = -1$ solve

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = a+b+c+d+e+f = 4$$

$$\begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = -a+b-c+d-e+f = 0$$

We get $a + c = -1, b + d = 1$

Hence, (A), (B), (C) and (D) are correct.

14. $c_3 \to c_3 - \left(c_2 - c_1\right)$

$c_4 \to c_4 - \left(c_1 + c_2\right)$

$$\begin{vmatrix} a & c & 0 & 0 \\ c & b & 0 & 0 \\ a-b & b-c & a+c-2b & 0 \\ x & y & z+x-y & 1 \end{vmatrix} = \left(a+c-2b\right)\left(ab-c^2\right)$$

Hence, (A) and (C) are correct.

15. $$\Delta'(x) = \begin{vmatrix} 2x+4 & 2x+4 & 13 \\ 4x+5 & 4x+5 & 26 \\ 16x-6 & 16x-6 & 104 \end{vmatrix} + \begin{vmatrix} x^2+4x-3 & 2 & 13 \\ 2x^2+5x-9 & 4 & 26 \\ 8x^2-6x+1 & 16 & 104 \end{vmatrix}$$

$$= 0 + 2 \times 13 \times (0) = 0$$

$\Rightarrow \quad \Delta(x) = \text{constant}$

$\Rightarrow \quad a = 0, b = 0, c = 0$

Hence, (B) and (C) are correct.

16. Let $\begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} = A + Bx + Cx^2 + \ldots$

putting $x = 0$, we get $A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Now differentiating both sides w.r.t. x and putting $x = 0$, we get

$$B = \begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = 0$$

Hence coefficient of x is 0.

Hence, (A) and (B) are correct.

17. Open it along R_1

$$\sin((n+1)x - nx) - a\sin((n+1)x - (n-1)x)$$
$$+ a^2 \sin(nx - (n-1)x)$$
$$= \sin x - a\sin 2x + a^2 \sin x$$
$$\sin x(1 - 2a\cos x + a^2) = 0$$
$$\Rightarrow \quad \sin x = 0 \ \cos x = \frac{1+a^2}{2a}$$

Hence, (C) and (D) are correct.

18. $\begin{vmatrix} 1 & -\cos\theta & \cos 2\theta \\ \cos 2\theta & -1 & \cos\theta \\ \cos 2\theta & -\cos\theta & 1 \end{vmatrix} = 0$

$$\Rightarrow \quad \Delta = -1 + \cos^2\theta + \cos\theta(\cos 2\theta - \cos\theta\cos 2\theta)$$
$$+ \cos 2\theta(-\cos\theta\cos 2\theta + \cos 2\theta)$$
$$\Rightarrow \quad -1 + \cos^2\theta + \cos\theta\cos 2\theta(1 - \cos\theta) + \cos^2 2\theta(1 - \cos\theta) = 0$$
$$\Rightarrow \quad -1 + \cos^2\theta + (1 - \cos\theta)(\cos\theta + \cos 2\theta)\cos 2\theta = 0$$
$$\Rightarrow \quad (\cos\theta - 1)[\cos\theta + 1 - \cos\theta\cos 2\theta - \cos^2 2\theta] = 0$$
$$\Rightarrow \quad (1 - \cos\theta)(\cos 2\theta - 1)\,(2\cos^2\theta + \cos\theta) = 0$$
$$\Rightarrow \quad \theta = \frac{2\pi}{3}$$

Hence, (A) and (C) are correct.

19. $A = \int_1^{\sin\theta} \frac{t}{1+t^2}\,dt$ put $t = 1/x$ then $dt = -dx/x^2$

$$A = \int_1^{\operatorname{cosec}\theta} \frac{1/x}{1+1/x^2}\,\frac{-dx}{x^2} = -\int_1^{\operatorname{cosec}\theta} \frac{1}{x(x^2+1)}\,dx = -B$$

$$A + B = 0 \Rightarrow B = -A$$

Determinant $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix} = \begin{vmatrix} A & A^2 & -A \\ 1 & A^2 & -1 \\ 1 & 2A^2 & -1 \end{vmatrix}$

$= 0$ (As C_1 and C_3 are proportional)

Hence, the correct answer is (D).

20. As, p, q, r, s are in A.P.

So, let common difference of A.P. $= d$

Now, $p = p =$ first term

$q = p + d$

$r = p + 2d$ and $s = p + 3d$

Now,

$$F(x) = \begin{vmatrix} p+\sin x & p+\sin x+d & p-(p+2d)+\sin x \\ p+\sin x+d & p+2d+\sin x & -1+\sin x \\ p+\sin x+2d & p+\sin x+3d & (p+3d)-(p+d)+\sin x \end{vmatrix}$$

Apply, $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$F(x) = \begin{vmatrix} -d & -d & -2d+1 \\ -d & -d & -(2d+1) \\ p+2d+\sin x & p+3d+\sin x & 2d+\sin x \end{vmatrix}$$

Apply, $R_1 \to R_1 - R_2$, we get

$$F(x) = 2\begin{vmatrix} -d & -d \\ p+2d+\sin x & p+3d+\sin x \end{vmatrix}$$

$$= -2d\begin{vmatrix} 1 & 1 \\ p+2d+\sin x & p+3d+\sin x \end{vmatrix}$$

$$\Rightarrow \quad F(x) = -2d^2$$

Hence, $F(x) = -2d^2$

As, $\int_0^2 F(x)dx = -4$ (given)

$$\Rightarrow \quad \int_0^2 (-2d^2)\,dx = -4 \quad \Rightarrow \quad \int_0^2 (d^2)dx = 2$$

$$\Rightarrow \quad 2d^2 = 2 \qquad \Rightarrow d^2 = 1$$

$\therefore$ Common difference of A.P. $\Rightarrow \quad d = \pm 1$

Hence, (A) and (C) are correct.

Linked Comprehension Type Questions

1. $a + b + c = p$, $ab + bc + ca = 0$

$$\therefore \quad a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$$
$$= p^2 - 0 = p^2$$

If $\quad \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$\therefore \quad \Delta^c = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \Delta^{3-1}$$

$$= \Delta^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & b \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} p^2 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p^2 \end{vmatrix} = p^6$$

Hence, the correct answer is (C).

2. $\because\ x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow\ (x-1)^3 + 8 = 0$

$\Rightarrow\ (x-1)^3 = (-2)^3 \Rightarrow \left(\dfrac{x-1}{-2}\right)^3 = 1$

$\Rightarrow\ \dfrac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$

$\Rightarrow\ x - 1 = -2, -2\omega, -2\omega^2$

or $\ x = -1, 1-2\omega, 1-2\omega^2$

$\therefore\ a = -1, b = 1-2\omega, c = 1-2\omega^2$

$$\because \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & c^2 \end{vmatrix}$$

(row by row)

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$

$$= \{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\}^2$$

$$= \frac{1}{4}(a+b+c)^2\,\{(a-b)^2+(b-c)^2+(c-a)^2\}^2$$

$$= \frac{9}{4}\{-12(1+\omega+\omega^2)\} = 0$$

Hence, the correct answer is (D).

3. $$\begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix} = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = 49$$

$$\therefore \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -7;\ a+b+c > 0$$

Hence, the correct answer is (A).

4. $C_1 \to C_1 - C_2$

$$f'(x) = \begin{vmatrix} 1 & 2ax-1 & 2ax+b+1 \\ -1 & b+1 & -1 \\ -1 & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2ax-1 & 2ax+b+1 \\ 0 & 2ax+b & 2ax+b \\ 0 & 2(2ax+b) & 2(2ax+b)+1 \end{vmatrix} = 2ax+b$$

$\Rightarrow f(x) = ax^2 + bx + c$

$f(0) = 2, f(1) = 1 \Rightarrow c = 2, a + b = -1$

$f'\left(\dfrac{5}{2}\right) = 0 \Rightarrow 5a + b = 0 \Rightarrow a = \dfrac{1}{4}, b = \dfrac{-5}{4}$

$\therefore y = \dfrac{x^2}{4} - \dfrac{5x}{4} + 2$

i.e. $x^2 - 5x + 8 = 4y$

i.e. $\left(x - \dfrac{5}{2}\right)^2 + \dfrac{7}{4} = 4y$

i.e. $\left(x - \dfrac{5}{2}\right)^2 = 4\left(y - \dfrac{7}{16}\right)$

Equation of direction is $y - \dfrac{7}{16} + 1 = 0$

i.e. $16y + 9 = 0$

Hence, the correct answer is (D).

5. $f(x) = \dfrac{7}{2} \Rightarrow x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6$ and $x = -1$

Hence, the correct answer is (D).

6. $$f(x) = \begin{vmatrix} 5 & 7 & 3 \\ 2 & 9g^2(x) & 2x^2 \\ x & 3g(x)-x^2 & 1 \end{vmatrix} - \begin{vmatrix} 4 & 5 & 2 \\ 2 & 9g^2(x) & 2x^2 \\ x & 3g(x)-x^2 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & -2 & -1 \\ 2 & 9g^2(x) & 2x^2 \\ x & 3g(x)-x^2 & 1 \end{vmatrix}$$

$$f(x) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 9g^2(x) & 2x^2 \\ x & 3g(x)-x^2 & 1 \end{vmatrix}$$

$f(x) = [9g^2(x) - 6x^2 g(x) + 2x^4]$

$$m = \int_0^1 \left[\left\{3g(x) - x^2\right\}^2 + x^4\right] dx$$

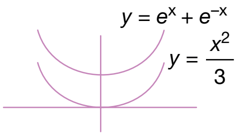
$$= \int_0^1 \left\{3g(x) - x^2\right\}^2 dx + \frac{1}{5} \text{ is min}$$

when $g(x) = \dfrac{x^2}{3}$; and $m = \dfrac{1}{5}$

Hence, the correct answer is (C).

7. $$Q(x) = \int_0^x \left(\int_0^t \frac{y^2}{3}\,dy - (e^t - e^{-t})\right) dt$$

$$Q(x) = \int_0^x \left\{\frac{t^3}{9} - e^t + e^{-t}\right\} dt$$



$$Q'(x) = \frac{x^3}{9} - e^x + e^{-x}$$

$$Q''(x) = \frac{x^2}{3} - e^x - e^{-x}$$

$$\Rightarrow \frac{x^2}{3} = e^x + e^{-x} \text{ No solution}$$

Hence, the correct answer is (A).

8. $b_1 . C_{31} + b_2 . C_{32} + b_3 . C_{33} =$

$$b_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - b_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + b_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

Hence, the correct answer is (A).

9. Value of new determinant $= 2^3\Delta = 8\Delta$
Hence, the correct answer is (B).

10. $a_3 M_{13} - b_3 . M_{23} + d_3 . M_{33} = a_3 C_{13} + b_3 . C_{23} + d_3 . C_{33} = \Delta$ by definition
Hence, the correct answer is (D).

Matrix Match/Column Match Type Questions

1. A → (q, r, s); B → (p); C → (q, r); D → (p)

Here $\Delta = \begin{vmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{vmatrix} = 0$ for all a, b, c

$\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0 \Rightarrow 3a + b - c = 0$

(A) a, b, c are as (0, 0, 0) then system has infinite solution, non-trivial solution, trivial solution
(B) a, b, c are as (–1, 1, 1) as $3a - b + c \neq 0$ then system has no solution,
(C) a, b, c are as (1, 1, 4) as $3a - b + c = 0$ then system has infinite solution, non-trivial solution
(D) a, b, c are as (2, 1, –1) $3a - b + c \neq 0$ then system has no solution

2. A → (s); B → (t); C → (p); D → (q)

$$(x-1)^3 = (-2)^3$$

$$\alpha - 1 = -2 \quad \Rightarrow \alpha = -1$$

$$\beta - 1 = -2\omega \quad \Rightarrow \beta = 1 - 2\omega$$

$$\gamma - 1 = -2\omega^2 \quad \Rightarrow \gamma = 1 - 2\omega^2$$

(A) $\alpha\beta\gamma = -1(1 - 2\omega)(1 - 2\omega^2)$

$$= -(1 - 2\omega^2 - 2\omega + 4\omega^3) = -7$$

(B) $\dfrac{\alpha-1}{\beta-1} + \dfrac{\beta-1}{\gamma-1} + \dfrac{\gamma-1}{\alpha-1} = 3\omega^2$

(C) $(\alpha - 1)(\beta - 1)(\gamma - 1) = (-2)(-2\omega)(-2\omega^2) = -8$

(D) $\begin{vmatrix} -2 & -2\omega & -2\omega^2 \\ -2\omega & -2\omega^2 & -2 \\ -2\omega^2 & -2 & -2\omega \end{vmatrix} = -8\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$

3. A → (q); B → (s); C → (r); D → (p)

(A) Expand along C_1 to obtain

$p(\theta) = \left(-\sqrt{2}\right)(-1) + (-1)(-2\sin\theta\cos\theta) + (-1)$
$(\sin^2\theta - \cos^2\theta)$

$$= \sqrt{2} + \sin 2\theta + \cos 2\theta = \sqrt{2} + \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$$

$\therefore$ range of $p(\theta)$ is $\left[0, 2\sqrt{2}\right]$.

(B) Applying $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 + 7R_1$, we get

$$q(\theta) = \begin{vmatrix} \sin 2\theta & -1 & 1 \\ \cos 2\theta + 4\sin 2\theta & 0 & 1 \\ 2 + 7\sin 2\theta & 0 & 2 \end{vmatrix}$$

$$= 2\cos 2\theta + 8\sin 2\theta - 2 - 7\sin 2\theta$$

$$= 2\cos 2\theta + \sin 2\theta - 2$$

As $2\cos 2\theta + \sin 2\theta$ lies between $-\sqrt{5}$ to $\sqrt{5}$, we get range of $q(\theta)$ is $\left[-\sqrt{5} - 2, \sqrt{5} - 2\right]$.

(C) Using $C_1 \to C_1 + C_3$, we get

$$r(\theta) = 2\cos\theta\begin{vmatrix} 1 & \sin\theta & \cos\theta \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{vmatrix} = 2\cos\theta$$

$\therefore$ range of $r(\theta)$ is [–2, 2]

(D) Taking $\sec^2\theta$ common from R_1, we get

$$s(\theta) = \sec^2\theta\begin{vmatrix} 1 & \cos^2\theta & \cos^2\theta \\ \cos^2\theta & \cos^2\theta & \operatorname{cosec}^2\theta \\ 1 & \cos^2\theta & \cot^2\theta \end{vmatrix}$$

$R_3 \to R_3 - R_1$, we get

$$s(\theta) = \sec^2\theta\begin{vmatrix} 1 & \cos^2\theta & \cos^2\theta \\ \cos^2\theta & \cos^2\theta & \operatorname{cosec}^2\theta \\ 0 & 0 & \cot^2\theta - \cos^2\theta \end{vmatrix}$$

$$= \sec^2\theta(\cot^2\theta - \cos^2\theta)(\cos^2\theta - \cos^4\theta)$$

$$= (\cot^2\theta - \cos^2\theta)\sin^2\theta = \cos^2\theta - \cos^2\theta\sin^2\theta - \cos^4\theta$$

$\therefore$ Range of $s(\theta)$ is [0, 1]

4. A → (p); B → (p, q, r, s); C → (r, s); D → (p, q, r)

(A) $C_1 \to C_1 + C_3 - 2\cos(by)\, C_2$

$$\Rightarrow \begin{vmatrix} 1 + x^2 - 2x\cos by & x & x^2 \\ 0 & \cos ay & \cos(a+b)y \\ 0 & \sin ay & \sin(a+b)y \end{vmatrix}$$

$$= (1 + x^2 - 2x\cos by)(\sin by)$$

(B) $\Delta = \begin{vmatrix} x^2 + y^2 & ax + by & x + y \\ ax + by & a^2 + b^2 & a + b \\ x + y & a + b & 2 \end{vmatrix}$

$$= \begin{vmatrix} x & y & 0 \\ a & b & 0 \\ 1 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} x & a & 1 \\ y & b & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(C) $\begin{vmatrix}1 & a & a^2+b\\ 1 & b & b^2+a\\ 1 & 1 & 1+ab\end{vmatrix}=\begin{vmatrix}1 & a & a^2+b\\ 0 & b-a & b^2-a^2+a-b\\ 0 & 1-a & 1-a^2+ab-b\end{vmatrix}$

$=(b-a)(1-a)\begin{vmatrix}1 & a & a^2+b\\ 0 & 1 & b+a-1\\ 0 & 1 & 1+a-b\end{vmatrix}$ dependent on a and b

(D) (i) multiply a, b, x in C_1, C_2, C_3 respectively and after that take a, b, x common from R_1, R_2, R_3 respectively

(ii) $C_1 \to C_1 + C_2 + C_3$

(iii) $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\Delta = \frac{(a^2+b^2+x^2)^3}{(a^2+b^2+x^2)^3}.\cos^2 y = \cos^2 y$$

5. A → (p); B → (q); C → (r); D → (p)

(A) $|A| = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$

$|B| = \begin{vmatrix}a_{11} & k^{-1}a_{12} & k^{-2}a_{13}\\ ka_{21} & a_{22} & k^{-1}a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}$

$=\frac{1}{k^3}\begin{vmatrix}k^2a_{11} & ka_{12} & a_{13}\\ k^2a_{21} & ka_{22} & a_{23}\\ k^2a_{31} & ka_{32} & a_{33}\end{vmatrix}=|A|$

$k_1|A|+k_2|B|=0$

$k_1+k_2=0$

(B) $\begin{vmatrix}a & h & g\\ x & b & f\\ y & z & c\end{vmatrix}=abc+xzg+yhf-ybg-azf-xch$

all numbers can be 1, –1
For maximum
let first 3 terms as +1, $a=b=c=x=y=z=g=1$
$h=f=-1$

$\begin{vmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=4$

(C) $\begin{vmatrix}1 & \cos\alpha & \cos\beta\\ \cos\alpha & 1 & \cos\gamma\\ \cos\beta & \cos\gamma & 1\end{vmatrix}=\begin{vmatrix}0 & \cos\alpha & \cos\beta\\ \cos\alpha & 0 & \cos\gamma\\ \cos\beta & \cos\gamma & 0\end{vmatrix}$

$\Rightarrow \sin^2\gamma-\cos\alpha(\cos\alpha-\cos\beta\cos\gamma)$
$\qquad+\cos\beta(\cos\alpha\cos\gamma-\cos\beta)$

$=-\cos\alpha(-\cos\beta\cos\gamma)+\cos\beta(\cos\alpha\cos\gamma)$

$\Rightarrow \sin^2\gamma-\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma-\cos^2\beta$
$=2\cos\alpha\cos\beta\cos\gamma$

$\Rightarrow \sin^2\gamma=\cos^2\alpha+\cos^2\beta$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

(D) $R_2 \to R_2-(R_1+R_3)$

$\Rightarrow \begin{vmatrix}x^2+x & x+1 & x-2\\ -4 & 0 & 0\\ x^2+2x+3 & 2x-1 & 2x-1\end{vmatrix}=4\begin{vmatrix}x+1 & x-2\\ 2x-1 & 2x-1\end{vmatrix}$

$=4\begin{vmatrix}x+1 & -3\\ 2x-1 & 0\end{vmatrix}=(24x-12)$

$\therefore A=24, B=-12$

$\therefore A+2B=0$

6. A → (q, r); B → (p); C → (s); D → (p, q, r)

$\Delta=\begin{vmatrix}2 & -3 & 5\\ 3 & 1 & \lambda\\ 1 & -7 & 8\end{vmatrix}=2(8+7\lambda)+3(24-\lambda)+5(-21-1)$

$=16+72-105+14\lambda-3\lambda-5=88-110+11\lambda=11\lambda-22$

$\Delta_z=\begin{vmatrix}2 & -3 & 12\\ 3 & 1 & \mu\\ 1 & -7 & 17\end{vmatrix}$

$=2(17+7\mu)+3(51-\mu)+12(-21-1)$

$=34+14\mu+153-3\mu-264$

$=187-264+11\mu=-77+11\mu$

(A) If ($\mu=7$ and $\lambda\neq 2$) or ($\lambda\neq 2$ and $\mu\neq 7$) then unique solution

(B) If $\lambda=2$ and $\mu=7$ then infinite solution

(C) If $\lambda=2$ and $\mu\neq 7$ then no solution

(D) For p, q and r Consistent system of equation.

7. A → (s); B → (p); C → (q); D → (s)

(A) Coefficient of x in $f(x)$ is coefficient of x in $\begin{vmatrix}x & 1 & 1\\ 1 & x & 2\\ x^2 & 1 & 0\end{vmatrix}=$

$x(0-2)-1(-1)+x^2(2-x)$

Therefore, coefficient of x is -2

(B) Let $D=\begin{vmatrix}1 & 3\cos\theta & 1\\ \sin\theta & 1 & 3\cos\theta\\ 1 & \sin\theta & 1\end{vmatrix}=(3\cos\theta-\sin\theta)^2$

$\Delta_{\max}=10$

(C) $f'(x)=\begin{vmatrix}a & x^2+1 & 1\\ b & 2x^2-1 & 1\\ c & 3x^2-2 & 1\end{vmatrix}+\begin{vmatrix}x+a & 2x & 1\\ x+b & 4x & 1\\ x+c & 6x & 1\end{vmatrix}+0$

$f'(x)=0 \quad\Rightarrow\quad f'(0)=0$

(D) $a_0=\begin{vmatrix}0 & 2 & 0\\ 1 & 0 & 6\\ 0 & 0 & 1\end{vmatrix}=-2(1)=-2$

8. A → (s); B → (r); C → (q); D → (p)

(A) The given determinant is

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} x & x & x+1 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix}$$

[Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$]

$$= 2\begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$]

$= -2$ [Expanding along R_3]

(B) $$\begin{vmatrix} 7 & 6 & x^2-13 \\ 2 & x^2-13 & 2 \\ x^2-13 & 3 & 7 \end{vmatrix}$$

Let $x^2 - 13 = t$. Then $(9 + t)(9t - 14 - t^2) = 0$

$\Rightarrow t = -9, 2, 7 \quad \Rightarrow x = \pm 2, \pm\sqrt{2}, \pm\sqrt{15}$

Hence, sum of other five roots is 2.

(C) $$\Delta = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6} \\ \sqrt{12} & \sqrt{3}+\sqrt{8}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{18} & \sqrt{2}+\sqrt{12}i & \sqrt{27}+2i \end{vmatrix}$$

Taking $\sqrt{6}$ common from C_1, we get

$$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ \sqrt{2} & \sqrt{3}+2\sqrt{2}i & 3\sqrt{2}+\sqrt{6}i \\ \sqrt{3} & \sqrt{2}+2\sqrt{3}i & 3\sqrt{3}+2i \end{vmatrix}$$

Applying $R_2 \to R_2 - \sqrt{2}\ R_1$ and $R_3 \to R_3 - \sqrt{3}\ R_1$

we get $$\Delta = \sqrt{6}\begin{vmatrix} 1 & 2i & 3+\sqrt{6} \\ 0 & \sqrt{3} & \sqrt{6}i-2\sqrt{3} \\ 0 & \sqrt{2} & 2i-3\sqrt{2} \end{vmatrix}$$

$$= \sqrt{6}\begin{vmatrix} \sqrt{3} & \sqrt{6}i-2\sqrt{3} \\ \sqrt{2} & 2i-3\sqrt{2} \end{vmatrix} = \sqrt{6}\begin{vmatrix} \sqrt{3} & -2\sqrt{3} \\ \sqrt{2} & -3\sqrt{2} \end{vmatrix} = -6$$

[Applying $C_1 \to C_2 - \sqrt{2}iC_1$]

(D) $$f(\theta) = \begin{vmatrix} \cos^2\theta & \cos\theta\sin\theta & -\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + (\sin\theta)\, R_3$

and $R_2 \to R_2 - (\cos\theta)\, R_3$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

$$= \sin^2\theta + \cos^2\theta = 1 \Rightarrow f\left(\frac{\pi}{3}\right) = 1$$

9. A → (p, q, r); B → (q, r); C → (r, s); D → (p, q, r)

(A) Multiplying C_1 by a, C_2 by b and C_3 by c, we obtain

$$\Delta = \frac{1}{abc}\begin{vmatrix} \frac{a}{c} & \frac{b}{c} & \frac{-(a+b)}{c} \\ \frac{-(b+c)}{a} & \frac{b}{a} & \frac{c}{a} \\ \frac{-b(b+c)}{ac} & \frac{b(a+2b+c)}{ac} & \frac{-b(a+b)}{ac} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} 0 & \frac{b}{c} & \frac{-(a+b)}{c} \\ 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & \frac{b(a+2b+c)}{ac} & \frac{-b(a+b)}{ac} \end{vmatrix}$$

This shows that Δ is independent of a, b and c.

(B) Applying $C_1 \to C_1 - (\cot b)\, C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin a\sin b & \cos a \\ 0 & \cos a\sin b & -\sin a \\ \frac{-\sin a}{\sin b} & \sin a\cos b & 0 \end{vmatrix}$$

[Expanding along C_1]

$= \sin a$

(C) Taking $\frac{1}{\sin a\cos b}, \frac{1}{\sin a\sin b}, \frac{1}{\cos a}$ common from C_1, C_2, C_3, respectively, we get

$$\Delta = \frac{1}{\sin^2 a\cos a\sin b\cos b}\Delta_1$$

where $$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ -\cot a & -\cot a & \tan a \\ \tan b & -\cot b & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 0 & -\cot a & \tan a \\ \frac{1}{\sin b\cos b} & -\cot b & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \frac{1}{\sin b\cos b}[\tan a + \cot a] = \frac{1}{\sin a\ \cos a\sin b\cos b}$$

(D) $$\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a\sin B & a\sin C \\ a\sin B & 1 & \cos A \\ a\sin C & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1-\sin^2 B & \cos A - \sin B\sin C \\ \sin C & \cos A - \sin B \sin C & 1-\sin^2 C \end{vmatrix}$$

[Applying $C_2 \to C_2 - (\sin B)C_1$ and $C_3 \to C_3 - (\sin C)C_1$]

$= a^2[\cos^2 B \cos^2 C - (\cos A - \sin B \sin C)^2]$

$= a^2[\cos^2 B \cos^2 C - (\cos(B + C) + \sin B \sin C)^2]$

$= a^2[\cos^2 B \cos^2 C - \cos^2 B \cos^2 C] = 0$

Integer/Numerical Answer Type Questions

1. Given that the system of equations has non-trivial

$$\Rightarrow \begin{vmatrix} \lambda & b-a & c-a \\ a-b & \lambda & c-b \\ a-c & b-c & \lambda \end{vmatrix} = 0$$

From observation if we Put $\lambda = 0$ we get

$$\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = 0$$

This is skew-symmetric determinants of odd order, which is equal to zero. Thus, $\lambda = 0$

Hence, the correct answer is (0).

2. If Δ denotes the coefficient matrix, then

$$\Delta = \begin{vmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{vmatrix} = \lambda^2(\lambda + 3)$$

For $\lambda = 0$ and $= \lambda = -3$, the system is inconsistent.

Hence, the correct answer is (2).

3. If A denotes the co-efficient matrix, then

$$A = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix}$$

$$|A| = \lambda^3 + 2 - 3\lambda = (\lambda - 1)^2 (\lambda + 2)$$

For $\lambda = 1$, the system has infinite number of solutions.

For $\lambda = 2$, the system has no solution.

Hence, the correct answer is (1).

4. $\Delta = 1(y) - 1(x - 4z)$

$= y - x + 4z$

$= (10a + 1) - x + 4(10c)$

$\Delta + 1 = 10a + 40c - x + 2$ which is divisible by 10 $\Rightarrow$ The units place of x is 2

Hence, the correct answer is (2).

5. After simplification

$C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$D = \begin{vmatrix} \begin{array}{c}(\tan B + \tan C - \tan A)\\(\tan B + \tan C + \tan A)\end{array} & 0 & \tan^2 A \\ \begin{array}{c}(\tan B - \tan A - \tan C)\\(\tan B + \tan A + \tan C)\end{array} & \begin{array}{c}(\tan A + \tan C - \tan B)\\(\tan A + \tan C + \tan B)\end{array} & \tan^2 B \\ 0 & \begin{array}{c}(\tan C - \tan A - \tan B)\\(\tan C + \tan A + \tan B)\end{array} & (\tan A + \tan B)^2 \end{vmatrix}$$

$D = 2(\tan A + \tan B + \tan C)(\tan A + \tan B + \tan C)^3$

$= 2(\tan A + \tan B + \tan C)^4$

as $\tan A + \tan B + \tan C \geq 3\sqrt{3}$

so, $\dfrac{D}{1000} \geq 1.458$

so least integral value is 2.

Hence, the correct answer is (2).

6. On solving we get

$(2\lambda + 15) f(x+1) - (\lambda + 10) f(x+8) - f(x+1) = 0$

$(2\lambda + 14) f(x+1) = (\lambda + 10) f(x+8)$

Since f is periodic with period 7

$\therefore \quad f(x + 1) = f(x + 8)$

$\Rightarrow \quad 2\lambda + 14 = \lambda + 10 \Rightarrow |\lambda| = 4$

Hence, the correct answer is (4).

7. Consider

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_2 l_1 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$|\Delta| = 1$

Hence, the correct answer is (1).

8. $\begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & \lambda \end{vmatrix} = 0 \Rightarrow -3 + 0 + 3\lambda = 0 \Rightarrow \lambda = 1$

Hence, the correct answer is (1).

9. We have $\displaystyle\sum_{k=1}^{n} \Delta_k = \begin{vmatrix} \displaystyle\sum_{k=1}^{n} 1 & n & n \\ \displaystyle\sum_{k=1}^{n} 2k & n^2 + n + 1 & n^2 + n \\ \displaystyle\sum_{k=1}^{n} (2k-1) & n^2 & n^2 + n + 1 \end{vmatrix}$

$$= \begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix}$$

$$= \begin{vmatrix} n & 0 & 0 \\ n(n+1) & 1 & 0 \\ n^2 & 0 & n+1 \end{vmatrix} = n(n+1) = 56 = 7 \times 8,$$

we get $n = 7$

Hence, the correct answer is (7).

10. $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \quad (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow \quad (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & \cos x - \sin x & \sin x - \cos x \end{vmatrix} = 0$$

$$\Rightarrow \quad (\sin x + 2\cos x)(\sin^2 x + \cos^2 x - 2\sin x \cos x) = 0$$

since $-{}^{\pi}\!/_{4} \le x \le {}^{\pi}\!/_{4}$, the only solution is $\sin x - \cos x = 0$ or $\tan x = 1$

Hence, the correct answer is (1).

11. $c_1 \to c_1 + c_2$

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2 + 4\sin 4\theta = 0 \Rightarrow \sin 4\theta = \frac{-1}{2}$$

Hence, the correct answer is (2).

12. Clearly $f'(x) = 0 \Rightarrow f(x) = K \ \forall\ x$

$$\therefore f(2) = \frac{1}{4} \Rightarrow f(x) = \frac{1}{4}\ \forall\ x$$

$$\sum_{r=1}^{15} f(x_r) = \frac{15}{4} \Rightarrow \left[\frac{15}{4}\right] = 3$$

Hence, the correct answer is (3).

13. $$\begin{vmatrix} 0 & -8 & 10 \\ \lambda & -14 & 15 \\ 1 & 2 & -3 \end{vmatrix} = 0 \Rightarrow -\lambda[24-20] + 1[-120+140] = 0$$

$$\Rightarrow \quad -4\lambda + 20 = 0$$

$$\begin{vmatrix} 3 & -2 & 1 \\ \lambda & -14 & 15 \\ 1 & 2 & -3 \end{vmatrix} = 0 \Rightarrow \lambda = 5$$

Hence, the correct answer is (5).

14. $$\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix}\begin{vmatrix} 0 & 3 & 0 \\ 0 & -3 & 1-\lambda \\ 0 & 1 & \lambda \end{vmatrix} = 0 \ \Rightarrow \lambda = 1$$

Hence, the correct answer is (1).

15. $$\Delta = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix} = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \Rightarrow k = 4$$

Hence, the correct answer is (4).

16. Using $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\begin{vmatrix} [x]+1 & [y] & [z] \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = [x]+[y]+[z]+1$$

$$k_{\min} = -2 - 1 - 3 + 1 = -5$$

Hence, the correct answer is (5).

17. Let Δ be the given determinant. Taking a, b, c common from C_1, C_2 and C_3, respectively, we get

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Now the column operation gives $C_3 \to C_3 - (C_1 + C_2)$

$$\Delta = \begin{vmatrix} a & c & 0 \\ a+b & b & -2b \\ b & b+c & -2b \end{vmatrix}(abc) = (abc)(-2b)\begin{vmatrix} a & c & 0 \\ a+b & b & 1 \\ b & b+c & 1 \end{vmatrix}$$

$$= -2b(abc)[a(b-b-c) - c(a+b-b)] = -2b(abc)(-2ac)$$

$$= 4a^2b^2c^2$$

Hence, the correct answer is (6).

18. The system has a nonzero solution

$\therefore |A| = 0$

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 2 & k \\ 4 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & k+1 \\ 1 & -1 & 1-k \end{vmatrix} = 0 \ \Rightarrow K = 0$$

The solution is $(2a, -3a, -5a)$

Least positive integer value of z is 5.

Hence, the correct answer is (5).

19. We have $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$

Since $a_n = \dfrac{20}{n}$; $d = \dfrac{1}{20}$

$$\text{Hence } D = \begin{vmatrix} 20 & \frac{20}{2} & \frac{20}{3} \\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6} \\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9} \end{vmatrix} = \frac{(20)^3}{4\times 7}\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{4}{5} & \frac{4}{6} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$= \frac{(20)^3}{4\times 7}\begin{vmatrix} 0 & \frac{-3}{10} & \frac{-1}{3} \\ 0 & \frac{-3}{40} & \frac{-1}{9} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix} = \frac{50}{21} = \frac{p}{q} \Rightarrow p = 50, q = 21$$

$$P - 2q = 50 - 42 = 8$$

Hence, the correct answer is (8).

20. $f'(x) = \left\{\frac{d}{dx}(\Delta)\right\}_{x=0} = 0 \Rightarrow coff.of\ x = 0$

$(\because f(x)$ is polynomial)

Hence, the correct answer is (0).

21. Let Δ be the given determinant. Then apply column transformation $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} x & 0 & 1 \\ y & 2y & 1 \\ -2z & -2z & 1+3z \end{vmatrix} = 0$$

$$\Delta = \begin{vmatrix} x & 0 & 0 \\ y & 2y & 1-\frac{y}{x} \\ -2z & -2z & 1+3z+\frac{2z}{x} \end{vmatrix} = 0 \text{ (by } C_3 \to C_3 - \frac{1}{x}C_1)$$

Solving we get

$$x\left[2y\left(1+3z+\frac{2z}{x}\right)+2z\left(1-\frac{y}{x}\right)\right] = 0$$

$$\Rightarrow \quad [2xy + 6zxy + 4yz + 2zx - 2yz] = 0$$

$$\Rightarrow \quad 2(xyz)\left[\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right] = 0$$

$$-\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 3$$

Hence, the correct answer is (3).

22. $$\sum_{x=1}^{n} f(x) = \begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{(n-1)n(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{(n-1)^2n^2}{4} & 3n^2 & 3n^2-3n \end{vmatrix}$$

$$= \frac{(n-1)n}{12}\begin{vmatrix} 6 & n & 6 \\ 4n-2 & 2n^2 & 4n-2 \\ 3n^2-3n & 3n^2 & 3n^2-3n \end{vmatrix}$$

$$= 0$$

Hence, the correct answer is (0).

23. $$\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2 = \frac{(a+b+c)^2(b^2-4ac)}{a^4}$$

$$\Rightarrow p + q + r = 2 + 1 + 4 = 7$$

Hence, the correct answer is (7).

24. $$f(x) = \begin{vmatrix} x & 3 & x \\ x^2 & x & 5 \\ x & x & 5 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$$

$$f(1) = A + B + C + D + E = 0 \to (1)$$

$$f'(1) = 4A + 3B + 2C + D$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 5 \\ 1 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 5 \\ 1 & 1 & 0 \end{vmatrix} = 0 - 24 + 10$$

$$4A + 3B + 2C + D = 14 \to (2)$$

from (1) and (2)

$$5A + 4B + 3C + 2D + E + 19 = -14 + 19 = 5$$

Hence, the correct answer is (5).

25. We have,

$$\begin{vmatrix} 2 & 5 & 8 \\ a23 & b53 & c83 \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 & 6 \\ 100a+20+3 & 100b+50+3 & 100c+80+3 \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 & 8 \\ 100a & 100b & 100c \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2 & 5 & 8 \\ 20 & 50 & 80 \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2 & 5 & 8 \\ 3 & 3 & 3 \\ a & b & c \end{vmatrix}$$

$$= 100\begin{vmatrix} 2 & 5 & 8 \\ a & b & c \\ a & b & c \end{vmatrix} + 10\begin{vmatrix} 2 & 5 & 8 \\ 2 & 5 & 8 \\ a & b & c \end{vmatrix} + 3\begin{vmatrix} 2 & 5 & 8 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= 0 + 0 + 3\begin{vmatrix} 2 & 3 & 6 \\ 1 & 0 & 0 \\ a & b-a & c-a \end{vmatrix}$$

(Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$)

$$= -3[3(c-a) - 6(b-a)] = -9[c - a - 2b + 2a]$$

$$= -9(a - 2b + c) = 0 \qquad [\because a, b, c \text{ are in A.P.}, \therefore 2b = a + c]$$

Hence, the correct answer is (0).

26. Applying $C_1 \to C_1 - \sin\theta\, C_3$ and $C_2 \to C_2 - \cos\theta\, C_3$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

Applying $R_3 \to R_3 - \sin\theta R_1 + \cos\theta R_2$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{vmatrix} = 1$$

Thus, $f\left(\frac{\pi}{6}\right) = 1$

Hence, the correct answer is (1).

27. On applying $R_1 \to R_1 - R_2 - R_3$

$$D = \begin{vmatrix} 0 & -2y^2 & -2x^2 \\ x^2 & y^2+z^2 & x^2 \\ y^2 & y^2 & z^2+x^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & y^2 & x^2 \\ x^2 & z^2 & 0 \\ y^2 & 0 & z^2 \end{vmatrix}$$

$$= 4x^2y^2z^2 = 4\,(1.\text{w}.\text{w}.^2)^2 = 4$$

Hence, the correct answer is (4).

28. $C_1 \to C_1 - bc_3;\ C_2 \to C_2 + aC_3$

$$\begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)\left[1(1-a^2-b^2+2a^2) - 0 + b(0+2b)\right]$$

$$= (1+a^2+b^2)(1+a^2-b^2+2b^2)$$

$$= (1+a^2+b^2)^3 \Rightarrow \ell = 1, m = 3$$

Hence, the correct answer is (4).

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1. $\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$ using $R_1 \to R_1 - R_2$ and

$R_2 \to R_2 - R_3$

$$\Rightarrow \begin{vmatrix} 0 & 3-\lambda & \lambda-3 \\ \lambda-3 & \lambda-3 & -2(\lambda-3) \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)^2\begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0$$

$\Rightarrow (\lambda-3)^2[(3\lambda+1)+(3\lambda-1)] = 0 \Rightarrow 6\lambda(\lambda-3)^2 = 0$

$\Rightarrow \lambda = 0, 3$

Required sum $= 3$

Hence, the correct answer is (3).

2. $D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 1 & -7 & a \end{vmatrix} = 0 \Rightarrow a = 8$

and $\begin{vmatrix} 1 & -2 & 9 \\ 2 & 1 & b \\ 1 & -7 & 24 \end{vmatrix} = 0 \Rightarrow b = 3$

Hence, $a - b = 8 - 3 = 5$

Alternate Solution

They represent family of planes $3P_1 = P_2 + P_3$ we get $a = 8$, $b = 3$

Hence, the correct answer is (5).

3. Given system of equation has infinite solution

$$x - 2y + 5z = 0;\ -2x + 4y + z = 0,\ -7x + 14y + 9z = 0$$

we have $z = 0$ and $x = 2y$

$$15 \le x^2 + y^2 + z^2 \le 150 \Rightarrow 15 \le 4y^2 + y^2 + 0 \le 150$$

$\Rightarrow\ 3 \le k^2 \le 30 \Rightarrow k = \pm 2, \pm 3, \pm 4, \pm 5$

Number of elements in $S = 8$

Hence, the correct answer is (8).

4. System has infinitely many solution

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & \lambda-2 \end{vmatrix} \Rightarrow \lambda = 1$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 1 \end{vmatrix} = 0\ \Rightarrow \mu = 14 \text{ we get } \mu - \lambda^2 = 13$$

Hence, the correct answer is (13).

5. For non-zero solution

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0$$

$\Rightarrow\ (3b - 2a)(c - a) - (b - a)(4c - 2a) = 0$

$\Rightarrow\ 3bc - 2ac - 3ab + 2a^2 - (4bc - 2ab - 4ac + 2a^2) = 0$

$\Rightarrow\ 2ac = bc + ab \Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Hence, the correct answer is (C).

6. Computing given equations as 2(ii) –2(i) – (iii)

We get $0 = 2\mu - 2 - \delta \Rightarrow \delta = 2(\mu - 1)$ true for consistent system

For inconsistent system $\delta \ne 2(\mu - 1)$

Hence, the correct answer is (D).

7. $$\Delta = \begin{vmatrix} \lambda & 2 & 2 \\ 2\lambda & 3 & 5 \\ 4 & \lambda & 6 \end{vmatrix} = (\lambda + 8)\,(2 - \lambda)$$

for $\lambda = 2$; $\Delta_x \neq 0$

Hence, no solution for $\lambda = 2$

Hence, the correct answer is (A).

8. For planes to intersect of a line

$\Rightarrow$ there should be infinite solution of the given system of equations

for infinite solutions

$$\Delta = \begin{vmatrix} 1 & 4 & -2 \\ 1 & 7 & -5 \\ 1 & 5 & \alpha \end{vmatrix} = 0 \Rightarrow 3\alpha + 9 = 0 \Rightarrow \alpha = -3$$

$$\Delta_z = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 7 & \beta \\ 1 & 5 & 5 \end{vmatrix} = 0 \Rightarrow 13 - \beta = 0 \Rightarrow \beta = 13$$

Also for $\alpha = -3$ and $\beta = 13$ $\Delta_x = \Delta_y = 0$

$\therefore \quad \alpha + \beta = -3 + 13 = 10$

Hence, the correct answer is (C).

9. $R_1 \to R_1 + R_3 - 2R_2$

$$f(x) = \begin{vmatrix} a + c - 2b & 0 & 0 \\ x + b & x + 3 & x + 2 \\ x + c & x + 4 & x + 3 \end{vmatrix} = (a + c - 2b)\,((x + 3)^2$$

$$-\,(x + 2)\,(x + 4)) = x^2 + 6x + 9 - x^2 - 6x - 8 = 1$$

$\Rightarrow \quad f(x) = 1 \Rightarrow f(50) = 1$

Hence, the correct answer is (B).

10. $7x + 6y - 2z = 0$ …(1)

$3x + 4y + 2z = 0$ …(2)

$x - 2y - 6z = 0$ …(3)

$$\Delta = \begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} = 0 \Rightarrow \text{infinite solutions}$$

Now (1) + (2) $\Rightarrow y = -x$ put in (1), (2) and (3) all will lead to $x = 2z$

Hence, the correct answer is (B).

11. For no solution:

$$\Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & \lambda \\ 1 & \lambda & 1 \end{vmatrix} = 0 \Rightarrow 2(-2 - \lambda^2) + 1(1 - \lambda) + 2(\lambda + 2) = 0$$

$$\Rightarrow \quad -2\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{2}$$

$$\Delta_x = \begin{vmatrix} 2 & -1 & 2 \\ -4 & -2 & \lambda \\ 4 & \lambda & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & 2 \\ -2 & -2 & \lambda \\ 2 & \lambda & 1 \end{vmatrix} = 2(-\lambda^2 - 6\lambda + 4)$$

which is not equal to zero for $\lambda = 1, -\frac{1}{2}$

Hence we have no solution for two values of λ

Hence, the correct answer is (C).

12. Using $R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_2$

$$\Delta = \begin{vmatrix} x - 2 & 2x - 3 & 3x - 4 \\ x - 1 & x - 1 & x - 1 \\ x - 2 & 2(x - 2) & 6(x - 2) \end{vmatrix}$$

$$= (x - 1)\,(x - 2)\begin{vmatrix} x - 2 & 2x - 3 & 3x - 4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix}$$

$$= -3(x - 1)^2\,(x - 2) = -3x^3 + 12x^2 - 15x + 6$$

$\therefore \quad B + C = 12 - 15 = -3$

Hence, the correct answer is (C).

13. For infinite solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\text{Now } \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 3 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \frac{9}{2}$$

$$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ 6 & 4 & -1 \\ \mu & 2 & -\frac{9}{2} \end{vmatrix} = 0 \Rightarrow \mu = 5$$

For $\lambda = \frac{9}{2}$ and $\mu = 5$, $\Delta_y = \Delta_z = 0$, Now check option $2\lambda + \mu = 14$

Hence, the correct answer is (C).

14. Given system of equations has infinite solution

$x + y + 3z = 0$ …(1)

$x + 3y + k^2 z = 0$ …(2)

$3x + y + 3z = 0$ …(3)

(1) – (3) $\Rightarrow -2x = 0 \Rightarrow x = 0$

Now from (1) $\Rightarrow y + 3z = 0 \Rightarrow \frac{y}{z} = -3$

$x + \frac{y}{z} = -3$

Hence, the correct answer is (B).

15. Using $C_3 \to C_3 - C_1$

$$\begin{vmatrix} x & a + y & a \\ y & b + y & b \\ z & c + y & c \end{vmatrix} \text{ using } C_2 \to C_2 - C_3$$

$$= \begin{vmatrix} x & y & a \\ y & y & b \\ z & y & c \end{vmatrix} \text{ using } R_3 \to R_3 - R_1,\ R_2 \to R_2 - R_1$$

$$= \begin{vmatrix} x & y & a \\ y - x & 0 & b - a \\ z - x & 0 & c - a \end{vmatrix} = (-y)[(y - x)\,(c - a) - (b - a)\,(z - x)]$$

using $a + x = b + y = c + z + 1$

$$= (-y)[(a - b)\,(c - a) + (a - b)\,(a - c - 1)]$$

$$= (-y)[(a - b)\,(c - a) + (a - b)\,(a - c) + b - a)$$

$$= -y(b - a) = y(a - b)$$

Hence, the correct answer is (B).

16. $C_3 \to C_3 - (C_1 - C_2)$

$$f(\theta) = \begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 0 \\ -\cos^2\theta & -1-\cos^2\theta & 0 \\ 12 & 10 & -4 \end{vmatrix}$$

$$= -4[(1 + \cos^2\theta)\sin^2\theta - \cos^2\theta\,(1 + \sin^2\theta)]$$

$$= -4[\sin^2\theta + \sin^2\theta\cos^2\theta - \cos^2\theta - \cos^2\theta\sin^2 q]$$

$$f(\theta) = 4\cos 2\theta \text{ where } \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \text{ or } 2\theta \in \left[\frac{\pi}{2}, \pi\right]$$

$f(\theta) \in [-4, 0]$

$(m, M) = (-4, 0)$

Hence, the correct answer is (D).

17. $$\Delta = \begin{vmatrix} 2 & -4 & \lambda \\ 1 & -6 & 1 \\ \lambda & -10 & 4 \end{vmatrix} = 2(3\lambda + 2)(\lambda - 3)$$

$\Delta_x = -2(\lambda - 3)$

$\Delta_y = -2(\lambda + 1)(\lambda - 3)$

$\Delta_z = -2(\lambda - 3)$

When $\lambda = 3$, then $\Delta = \Delta_x = \Delta_y = \Delta_z = 0 \Rightarrow$ Infinite many solution

when $\lambda = -\frac{2}{3}$ then $\Delta_x, \Delta_y, \Delta_z$ none of them is zero so equations are inconsistent $\therefore\ \lambda = -\frac{2}{3}$

Hence, the correct answer is (A).

18. For infinite many solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\text{Now } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = 0 \Rightarrow 1.(2\lambda - 9) - 1.(\lambda - 3) + 1.(3 - 2) = 0$$

$\therefore\quad \lambda = 5$

$$\text{Now, } \Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 2 & 3 \\ \mu & 3 & 5 \end{vmatrix} = 0$$

$\Rightarrow\quad 2(10 - 9) - 1(25 - 3\mu) + 1(15 - 2\mu) = 0 \Rightarrow \mu = 8$

Hence, the correct answer is (D).

19. $$\begin{vmatrix} \cos^2 x & 1+\sin^2 x & \sin 2x \\ 1+\cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$$= \begin{vmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

$= -1(\sin^2 x) - 1(1 + \sin 2x + \cos^2 x) = -\sin 2x - 2$

$m = -3, M = -1$

Hence, the correct answer is (A).

20. $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & (a^2-1) \end{vmatrix} = 3(a^2 - 1) - 6 - 2(a^2 - 1) + 4 + 0 = a^2 - 3$$

For unique solution $\Delta \neq 0 \Rightarrow a \neq \pm\sqrt{3}$

$\therefore$ For inconsistent $|a| = \sqrt{3}$

Hence, the correct answer is (C).

21. $$D = \begin{vmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{vmatrix} = [(-27 + 25) - 2(20 - 21)] = 0$$

If system is consistent then $\Delta_x = \Delta_y = \Delta_z = 0$

$$\begin{vmatrix} 1 & -4 & g \\ 0 & 3 & h \\ -2 & 5 & k \end{vmatrix} = 0 \text{ or } 2g + h + k = 0$$

Hence, the correct answer is (D).

22. $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \alpha \end{vmatrix} = 2\alpha + 3 + 3 - 2 - 9 - \alpha = 0 \Rightarrow \alpha = 5$$

Here $P_1 + P_3 = 2P_2$

For ∞ solution or $\Delta_x = \Delta_y = \Delta_z = 0$

$$\frac{5+\beta}{2} = 9 \Rightarrow \beta = 13 \therefore \beta - \alpha = 8$$

Hence, the correct answer is (A).

23. For any value of r determinant is zero

$R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

Now R_2 and R_3 are same.

Hence, the correct answer is (B).

24. $$\text{For non trivial solution } \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 3 & 7 \\ -1 & 4 & 7 \\ \sin 3\theta & \cos 2\theta & 2 \end{vmatrix} = 0$$

$\Rightarrow\quad 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow\quad \sin\theta = 0 \text{ or } \sin\theta = \frac{1}{2} \text{ or } \sin\theta = \frac{3}{2}$

For $\theta \in (0, \pi)$

$\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ are satisfy the equation

Number of values of $\theta = 2$

Hence, the correct answer is (A).

25. $p_1 : 2x + 2y + 3z = a,\ p_2 : 3x - y + 5z = b,\ p_3 : x - 3y + 2z = c$

using the family of plane concept

we find $p_1 + p_3 = p_2$

$(2x + 2y + 3z - a) + (x - 3y + 2z - c) = (3x - y + 5z - b)$

$\Rightarrow\quad a + c = b$

Hence, the correct answer is (A).

26. $R_1 \to R_1 + R_2 + R_3$

$$D = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-b-a \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-b-a \end{vmatrix}$$

$C_2 \to C_2 - C_1; C_3 \to C_3 - C_1$

$D = (a+b+c)\,(a+b+c)^2 = (a+b+c)^3$

Now, $(a+b+c)\,(x+a+b+c)^2 = (a+b+c)^3$

$(x+a+b+c)^2 = (a+b+c)^2$

When $(x+a+b+c) = -(a+b+c) \Rightarrow x = -2(a+b+c)$

Hence, the correct answer is (A).

27. For unique solution if $\Delta \neq 0 \Rightarrow \begin{vmatrix} 1+\alpha & \beta & 1 \\ \alpha & 1+\beta & 1 \\ \alpha & \beta & 2 \end{vmatrix} \neq 0$

$\Rightarrow \quad \alpha + \beta \neq -2$

Hence, the correct answer is (D).

28. Equations are $(\lambda-1)x + 2y + 2z = 0$, $x + (2-\lambda)y + z = 0$ and $x + y + \lambda z = 0$

$$\begin{vmatrix} \lambda-1 & 2 & 2 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (\lambda-1)\,(2\lambda-\lambda^2-1) - 2(\lambda-1) + 2(1-2+\lambda) = 0$

$\Rightarrow \quad (\lambda-1)^3 = 0$

$\lambda = 1$ is a singleton

Hence, the correct answer is (C).

29. $\begin{vmatrix} 1 & -c & -c \\ c & -1 & c \\ c & c & -1 \end{vmatrix} = 0 \Rightarrow 1(1-c^2) + c(-c-c^2) - c(c^2+c) = 0$

$\Rightarrow \quad 1 - c^2 - c^2 - c^3 - c^3 - c^2 = 0$

$\Rightarrow \quad 2c^3 + 3c^2 - 1 = 0 \Rightarrow c = -1 \quad \Rightarrow \quad c = \dfrac{1}{2}$

Maximum value of $c = \dfrac{1}{2}$

Hence, the correct answer is (A).

30. Roots of equation $x^2 + x + 1 = 0$ are α and β

$$\alpha, \beta = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$\Rightarrow \quad \alpha = \omega, \beta = \omega^2$ (complex cube root of unity)

$$\Delta = \begin{vmatrix} y+1 & \omega & \omega^2 \\ \omega & y+\omega^2 & 1 \\ \omega^2 & 1 & y+\omega \end{vmatrix} \qquad R_1 \to R_1 + R_2 + R_3$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} y & y & y \\ \omega & y+\omega^2 & 1 \\ \omega^2 & 1 & y+\omega \end{vmatrix} \qquad (\because 1 + \omega + \omega^2 = 0)$$

$$\Rightarrow \quad \Delta = y\begin{vmatrix} 1 & 1 & 1 \\ \omega & y+\omega^2 & 1 \\ \omega^2 & 1 & y+\omega \end{vmatrix} = y(y^2) = y^3$$

Hence, the correct answer is (B).

31. Given system of equations has non-trival solution

$$\therefore \quad \Delta = 0 \Rightarrow \begin{vmatrix} 2 & 3 & -1 \\ 1 & K & -2 \\ 2 & -1 & 1 \end{vmatrix} = 0 \Rightarrow K = \frac{9}{2}$$

so equation are

$2x + 3y - z = 0 \qquad \text{...(1)}$

$x + \dfrac{9}{2}y - 2z = 0 \qquad \text{...(2)}$

$2x - y + z = 0 \qquad \text{...(3)}$

$(1) - (3) \Rightarrow 4y - 2z = 0 \Rightarrow 2y = z \Rightarrow \dfrac{y}{z} = \dfrac{1}{2} \qquad \text{...(4)}$

From equation (1) and (4)

$\Rightarrow \quad 2x + y = 0 \Rightarrow \dfrac{x}{y} = \dfrac{-1}{2}$ or $\dfrac{z}{x} = -4;\ \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x} + k = \dfrac{1}{2}$

Hence, the correct answer is (A).

32. $\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = x\,(-x^2-1) - \sin\theta\,(-x\sin\theta - \cos\theta)$

$+ \cos\theta\,(-\sin\theta + x\cos\theta) = -x^3$

$$\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix} = -x^3$$

$\Delta_1 + \Delta_2 = -2x^3$

Hence, the correct answer is (D).

33. $\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 4 & \lambda & -\lambda \\ 3 & 2 & -4 \end{vmatrix} = 0$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 4 & \lambda-4 & -\lambda-4 \\ 3 & -1 & -7 \end{vmatrix} = 0$$

$\Rightarrow \quad -7\lambda + 28 - \lambda - 4 = 0$

$\Rightarrow \quad \lambda = 3$

On solving we get $\lambda = 3$

Hence, the correct answer is (B).

34. Expand

$x(-3x\,(x+2) - 2x(x-3)) - (-6)\,(2(x+2) + 3(x+3)) + (-1)\,(4x+3)\,(-3x))$

$\Rightarrow \quad -5x^3 + 25x + 78 = 0 \Rightarrow 5x^3 - 25x - 78 = 0$

Sum of roots $= 0$

Hence, the correct answer is (B).

35. $\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix}$

using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$=\begin{vmatrix}1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix} \text{ using } C_1\to C_1+C_2$$

$$=\begin{vmatrix}2 & \sin^2\theta & 4\cos 6\theta\\ 0 & 1 & 0\\ -1 & 0 & 1\end{vmatrix}=0\Rightarrow \cos 6\theta=-\frac{1}{2}$$

$$\Rightarrow \quad 6\theta=\frac{2\pi}{3}\Rightarrow\theta=\frac{\pi}{9}$$

Hence, the correct answer is (B).

36. Here, $\begin{vmatrix}x-4 & 2x & 2x\\ 2x & x-4 & 2x\\ 2x & 2x & x-4\end{vmatrix}=(A+Bx)\,(x-A)^2$

Put $x=0\Rightarrow\begin{vmatrix}-4 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{vmatrix}=A^3\Rightarrow A^3=(-4)^3\Rightarrow A=-4$

Put $x=1\Rightarrow\begin{vmatrix}-3 & 2 & 2\\ 2 & -3 & 2\\ 2 & 2 & -3\end{vmatrix}=25(B-4)\Rightarrow B=5$

∴ ordered pair (A, B) is $(-4, 5)$

Hence, the correct answer is (B).

37. For non zero solution of the system of linear equations.

$$\begin{vmatrix}1 & k & 3\\ 3 & k & -3\\ 2 & 4 & -3\end{vmatrix}=0\Rightarrow k=11$$

Now equations become

$$x+11y+3z=0 \qquad \ldots(1)$$
$$3x+11y+2z=0 \qquad \ldots(2)$$
$$2x+4y+3z=0 \qquad \ldots(3)$$

Adding equations (1) and (3) we get

$$3x+15y=0\Rightarrow x=-5y$$

Now put $x=-5y$ in equation (1), we get

$$-5y+11y+3z=0\Rightarrow z=-2y$$

$$\therefore \quad \frac{xz}{y^2}=\frac{(-5\,y)(-2\,y)}{y^2}=10$$

Hence, the correct answer is (A).

38. $D=\begin{vmatrix}1 & 1 & 1\\ 1 & a & 1\\ a & b & 1\end{vmatrix}=0\Rightarrow 1[a-b]-1[1-a]+1[b-a^2]=0$

$\Rightarrow \quad (a-1)^2=0\Rightarrow a=1$

For $a=1$, First two equations are identical ie. $x+y+z=1$
To have no solution with $x+by+z=0$
$b=1$ So $b=\{1\}\Rightarrow$ It is singleton set.

Hence, the correct answer is (A).

39. For trivial solution, $\begin{vmatrix}1 & \lambda & -1\\ \lambda & -1 & -1\\ 1 & 1 & -\lambda\end{vmatrix}=0$

$\Rightarrow \quad -\lambda(\lambda+1)(\lambda-1)=0\Rightarrow\lambda=0,+1,-1$

Hence, the correct answer is (A).

40. Rewriting given equations

$$(2-\lambda)x_1-2x_2+x_3=0, 2x_1-(3+\lambda)x_3+2x_3=0,$$
$$-x_1+2x_2-\lambda x_3=0$$

For non-trivial solution, $\Delta=0$

$$\text{i.e.}\begin{vmatrix}2-\lambda & -2 & 1\\ 2 & -(3+\lambda) & 2\\ -1 & 2 & -\lambda\end{vmatrix}=0$$

$\Rightarrow \quad (2-\lambda)[\lambda(3+\lambda)-4]+2[-2\lambda+2]+1[4-(3+\lambda)]=0$

$\Rightarrow \quad \lambda^3+\lambda^2-5\lambda+3=0\Rightarrow\lambda=1, 1, 3$

Hence, λ has 2 values.

Hence, the correct answer is (A).

41. $\begin{vmatrix}3 & 1+f(1) & 1+f(2)\\ 1+f(1) & 1+f(2) & 1+f(3)\\ 1+f(2) & 1+f(3) & 1+f(4)\end{vmatrix}$

$$=\begin{vmatrix}1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2\\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3\\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}\times\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}$$

$$=\begin{vmatrix}1 & 1 & 1\\ 1 & \alpha & \beta\\ 1 & \alpha^2 & \beta^2\end{vmatrix}^2=(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

So, $k=1$

Hence, the correct answer is (C).

42. $\Delta=\begin{vmatrix}k+1 & 8\\ k & k+3\end{vmatrix}=k^2+4k+3-8k=k^2-4k+3$

$=(k-3)\,(k-1)$

$\Delta_x=\begin{vmatrix}4k & 8\\ 3k-1 & k+3\end{vmatrix}=4k^2+12k-24k+8=4k^2-12k+8$

$=4(k^2-3k+2)=4(k-2)\,(k-1)$

$\Delta_y=\begin{vmatrix}k+1 & 4k\\ k & 3k-1\end{vmatrix}=3k^2+2k-1-4k^2=-k^2+2k-1$

$=-(k-1)^2$

As given no solution,
$k=3$, for which $\Delta=0$ and $\Delta_1, \Delta_2\neq 0$

Hence, the correct answer is (B).

43. $\Delta=0\Rightarrow\begin{vmatrix}4 & k & 2\\ k & 4 & 1\\ 2 & 2 & 1\end{vmatrix}=0$

$\Rightarrow \quad 4(4-2)-\text{k}(k-2)+2(2k-8)=0$

$\Rightarrow \quad 8-k^2+2k+4k-16=0 \quad\Rightarrow\quad k^2-6k+8=0$

$\Rightarrow \quad (k-4)(k-2)=0,\ k=4,2$

Hence, the correct answer is (D).

44. $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 0 \qquad \Delta_x = \begin{vmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & 2 \end{vmatrix} \neq 0$

$\Rightarrow$ Given system, does not have any solution.

$\Rightarrow$ No solution

Hence, the correct answer is (D).

45. $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$

(Taking transpose of second determinant)

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & a-1 & (-1)^{n+2}a \\ b+1 & b-1 & (-1)^{n+1}b \\ c-1 & c+1 & (-1)^n c \end{vmatrix} = 0$$

$C_1 \Leftrightarrow C_3$

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} - \begin{vmatrix} (-1)^{n+2}a & a-1 & a+1 \\ (-1)^{n+2}(-b) & b-1 & b+1 \\ (-1)^{n+2}c & c+1 & c-1 \end{vmatrix} = 0$$

$C_2 \Leftrightarrow C_3$

$$\Rightarrow \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+2} \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0$$

$$\Rightarrow \left[1+(-1)^{n+2}\right] \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1+(-1)^{n+2}=0$, n should be an odd integer.

Hence, the correct answer is (A).

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Multiple Correct Choice Type Problems

1. $\alpha x + 2y = \lambda$

$3x - 2y = \mu$

For infinite solution, both lines are coincident

$$\frac{\alpha}{3} = \frac{2}{-2} = \frac{\lambda}{\mu}$$

We get $\alpha = -3$ and $\lambda + \mu = 0$ for infinite solutions

If $\alpha \neq -3$ then they are intersecting line at exactly one point (unique solution)

if $\lambda + \mu \neq 0$ and $\alpha = -3$ they are parallel straight line so no point of intersection hence no solution

Hence, (B), (C) and (D) are correct.

2. We get $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$

(using $R_3 \to R_3 - R_2$; $R_2 \to R_2 - R_1$)

$$\Rightarrow \begin{vmatrix} \alpha^2-2 & 4\alpha^2-2 & 9\alpha^2-2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

(using $R_1 \to R_1 - R_2$; $R_3 \to R_3 - R_2$)

$$\Rightarrow \begin{vmatrix} -3\alpha^2 & -5\alpha^2 & 9\alpha^2-2 \\ -2\alpha & -2\alpha & 3+6\alpha \\ 0 & 0 & 2 \end{vmatrix} = -648\alpha$$

(using $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$\Rightarrow \quad -8\alpha^3 = -648\alpha \Rightarrow \alpha = \pm 9$

Alternate Solution

Given determinant can be written as

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \begin{vmatrix} 1 & 2\alpha & \alpha^2 \\ 4 & 4\alpha & \alpha^2 \\ 9 & 6\alpha & \alpha^2 \end{vmatrix} = 2\alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix}$$

$$= -2\alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = -2\alpha^3 \times 4$$

$\Rightarrow \quad -8\alpha^3 = -648\alpha \Rightarrow \alpha = \pm 9$

Hence, (B) and (C) are correct.

Reasoning Based Questions

1. The given equation are

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

Here $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$

and $\Delta_y = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = k-3 \neq 0$ if $k \neq 3$

$\therefore$ If $k \neq 3$, the system has no solutions.

Hence statements-I is true and statements-2 is a correct explanation for statements-I.

Hence, the correct answer is (A).

Integer/Numerical Answer Type Questions

1. Simplifying the given determinant

$$= x.x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix}$$

using $R_2 \to R_2 - 2\,R_1$, $R_3 \to R_3 - 3\,R_1$,

$$= x.x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & 6x^3-1 \\ 0 & 6 & 24x^3-2 \end{vmatrix} = 10 \text{ or } x^3(12x^3+2) = 10$$

$$6x^6 + x^3 - 5 = 0 \Rightarrow (6x^3-5)(x^3+1) = 0$$

$\Rightarrow \quad x = -1, x = \left(\frac{5}{6}\right)^{1/3}$ so number of solutions = 2

Hence, the correct answer is (2).

2. $(xyz)\sin 3\theta - y\cos 3\theta - z\cos 3\theta = 0$

$(xyz)\sin 3\theta - 2y\sin 3\theta - 2z\cos 3\theta = 0$

$(xyz)\sin 3\theta - y(\cos 3\theta + \sin 3\theta) - 2z\cos 3\theta = 0$

For non-zero solutions

$$\Delta = \begin{vmatrix} \sin 3\theta & -\cos 3\theta & -\cos 3\theta \\ \sin 3\theta & -2\sin 3\theta & -2\cos 3\theta \\ \sin 3\theta & -(\sin 3\theta + \cos 3\theta) & -2\cos 3\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad \sin 3\theta \cos 3\theta \begin{vmatrix} 1 & \cos 3\theta & 1 \\ 1 & 2\sin 3\theta & 2 \\ 1 & \sin 3\theta + \cos 3\theta & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \sin 3\theta \cos 3\theta (\sin 3\theta - \cos 3\theta) = 0$$

CASE-1: $\sin 3\theta = 0$: from 2nd given equation we get $\frac{2}{y} = 0$ not possible

CASE-2: $\cos 3\theta = 0$: from Ist equation we get $x = 0$ as $y, z \neq 0$ then from 2nd eq. we get $\frac{1}{z} \neq 0$ not possible

CASE-3: $\cos 3\theta = \sin 3\theta \Rightarrow \tan 3\theta = 1$

$\theta \in (0, \pi)$ or $3\theta \in (0, 3\pi)$ $\quad 3\theta = n\pi + \frac{\pi}{4}$

$n = 0;\ \theta = \frac{\pi}{12},\ n = 1;\ \theta = \frac{5\pi}{12},\ n = 2;\ \theta = \frac{3\pi}{4}$

$\therefore$ 3 acceptable solution

Hence, the correct answer is (3).

CHAPTER 12: MATRICES

Test Your Concepts-I

1. Since $20 = 1 \times 20 = 2 \times 10 = 4 \times 5$

   $= 5 \times 4 = 10 \times 2 = 20 \times 1$

   Out of this number of rows is even so possible cases are 2×10, 4×5, 10×2 and 20×1

   So number of different order matrices are 4.

2. Since there are 4 elements then possible orders are 1×4, 2×2, 4×1matrix and we have given with 4 elements so number of different matrices is $= 3 \times 4! = 3 \times 24 = 72$

3. $$A + A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 5 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 10 & 4 \\ 5 & 4 & 6 \end{bmatrix}$$

   Sum of second row $= 2 + 10 + 4 = 16$

4. $$4A - 3B = 4\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

   $$= \begin{bmatrix} 4 & 8 & 12 \\ -4 & 0 & 8 \\ 4 & -12 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 15 & 18 \\ -3 & 0 & 3 \\ 6 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -8 & -7 & -6 \\ -1 & 0 & 5 \\ -2 & -15 & -2 \end{bmatrix}$$

5. $$X = 5B - 3A = 5\begin{bmatrix} 1 & -2 \\ -2 & 2 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 3 & -1 \\ -5 & 2 \end{bmatrix}$$

   $$= \begin{bmatrix} 5 & -10 \\ -10 & 10 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ -9 & 3 \\ 15 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -10 \\ -19 & 13 \\ 20 & -1 \end{bmatrix}$$

6. $\because A + 2X = B$

   $$\Rightarrow \quad X = \frac{1}{2}(B - A)$$

   $$\Rightarrow \quad X = \frac{1}{2}\left(\begin{bmatrix} -2 & 5 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 1 & -6 \end{bmatrix}\right)$$

   $$\Rightarrow \quad X = \begin{bmatrix} -\frac{5}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{7}{2} \end{bmatrix}$$

Test Your Concepts-II

1. A is a 2×3 matrix and B is a 3×2 matrix

   $\therefore$ AB is defined and it will be a 2×2 matrix

   $$\text{Now } AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

   $$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

   Since B is a 3×2 matrix and A is a 2×3 matrix

   $\therefore$ BA is defined and it will be a 3×3 matrix.

   $$\text{Again } BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

   $$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

   Clearly $AB \neq BA$

2. $A = 3 \times 3$ matrix $\begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix}$, $B = 3 \times 2$ matrix $\begin{bmatrix} 4 & -1 \\ 2 & -2 \\ 5 & 3 \end{bmatrix}$ then

   $$AB = \begin{bmatrix} 1.4 + 0.2 + 5.5 & 1(-1) + 0(-2) + 5.3 \\ (-1)4 + 2.2 + 4.5 & (-1)(-1) + 2(-2) + 4.3 \\ 3.4 + (-2)2 + 6.5 & 3(-1) - 2(-2) + 6.3 \end{bmatrix}$$

   $$= \begin{bmatrix} 29 & 14 \\ 20 & 9 \\ 38 & 19 \end{bmatrix}$$

   Here BA is not defined because number of columns in B is not equal to the number of rows in A.

3. Since $P^2 = I - P$ (given) ...(1)

   $\Rightarrow$ $P^3 = P(I - P)$

   $\Rightarrow$ $P^3 = P - P^2 = P - I + P$ (using (1))

   $\Rightarrow$ $P^3 = 2P - \text{I}$

   Similarly $P^4 = 2P^2 - P = 2I - 3P$

   $P^5 = 2P - 3P^2 = 5P - 3\text{I}$ $\quad$ $P^6 = 5P^2 - 3P = 5I - 8P$

   $\therefore$ $n = 6$

4. Here, $A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

   We get $\alpha^2 = 1$ or $\alpha = 1, -1$

   and $\alpha + 1 = 5$ or $\alpha = 4$

   Now we get no common value of α

5. 1^{st} consider $AB = \begin{bmatrix} 1 & 2 & 8 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = [4 + 10 + 48] = [62]$

   Now consider $CD = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} = [0 + 4 + 0] = [4]$

   So, $AB + CD = [62] + [4] = [66]$

6. $X = BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

   $\Rightarrow$ $X = BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{I}$

   So, $X^n = I^n = I$

7. $\begin{bmatrix} x-2 & -10 & 2x-8 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = [0]$

$$x^2-2x-40+2x-8=0$$
$$x^2-48=0$$
$$x=\pm 4\sqrt{3}$$

8. $AB = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x+y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$x+y=0$$

9. $|3AB| = 3^3|A|\,|B| = 27\times(-1)\times(3) = -81$

10. $2ABC$ is not defined

∴ there is no solution

11. We have $A^T = -A$

$$(A^4)^T = (A.A.A.A.)^T = A^T A^T A^T A^T$$
$$\Rightarrow \quad (-A)(-A)(-A)(-A)$$
$$= (-1)^4 A^4 = A^4$$

12. $AB = A$ and $BA = B$

$A(BA) = A^2$ and $B(AB) = B^2$

$AB = A^2$ and $BA = B^2$

$A = A^2$ and $B = B^2$

$A^2B^2 = AB = A$

Test Your Concepts-III

1. Let B be the matrix whose elements are cofactors of the corresponding elements of the matrix A. Then

$$B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = B' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

2. $\therefore\ |A| = 0(2-3)-1(1-9)+2(1-6) = 0+8-10 = -2 \neq 0$

Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

if C be the matrix of cofactors of the elements in $|A|$

$$\therefore \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1;\ C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8;\ C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1;\ C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6;\ C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1;\ C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2;\ C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$\therefore \quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} \quad \therefore \text{ Adj } A = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{Adj\ A}{|A|} = -\dfrac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

3. Let B be the matrix whose elements are co-factors of the corresponding elements of A, then

$$B = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}$$

$$\therefore \quad \text{adj } A = B' = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1.2 + 2(-3) + 5.5 = 21$$

$$\therefore \quad A^{-1} = \frac{adj\ A}{|A|} = \frac{adj\ A}{21} = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

Also $A^{-1}A = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad A^{-1}A = \begin{bmatrix} \dfrac{2+6+13}{21} & \dfrac{4+9-13}{21} & \dfrac{10+3-13}{21} \\ \dfrac{-3+12-9}{21} & \dfrac{-6+18+9}{21} & \dfrac{-15+6+9}{21} \\ \dfrac{5-6+1}{21} & \dfrac{10-9-1}{21} & \dfrac{25-3-1}{21} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

4. $\left|adjA^{-1}\right| = \left|A^{-1}\right|^2 = \dfrac{1}{|A|^2}$

$$\left|\left(adjA^{-1}\right)^{-1}\right| = \frac{1}{\left|adj\ A^{-1}\right|} = |A|^2 = 2^2 = 4$$

5. Adj $(4A) =$ 4^2Adj(A)=16Adj(A)

$\Rightarrow \quad |\text{Adj } 4\text{A}| = 16^3.|\text{AdjA}| = 16^3.5^2$

6. $AB = BA$

Previous and post multiplying both sides by A^{-1}.

$$A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$$

$$(A^{-1}A)\,(BA^{-1}) = A^{-1}\,B(AA^{-1}) \quad\Rightarrow\quad BA^{-1} = A^{-1}B$$

$\Rightarrow$ $(BA^{-1})' = (A^{-1}B)' = (A^{-1})'B'$ (reversal laws)

$= A^{-1}B$ (as $B = B'$, $(A^{-1})' = A^{-1}$)

$\Rightarrow$ $A^{-1}B$ is symmetric

Similarly for $A^{-1}B^{-1}$.

7. $$\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & -100\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -(1+2)\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & -100\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -(1+2+3)\\0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & -100\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & -(1+2+3+....+100)\\0 & 1\end{bmatrix}$$

$$a = -\frac{100(101)}{2} = -5050,\ b = 0$$

$$a + b = -5050$$

8. Clearly, $A = \begin{pmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{pmatrix} \quad\Rightarrow\quad |A| = 3^3 = 27$

$\therefore$ det. (adj. (adj. A)) $= |A|^4 = (27)^4$

$$= (25+2)^4 = 25\lambda + 16$$

Where λ is some positive integer.

$$\therefore \quad \left\{\frac{\det(adj.(adj.A))}{5}\right\} = \left\{\frac{25\lambda+16}{5}\right\}$$

$$= \left\{\frac{16}{5}\right\} = \left\{3+\frac{1}{5}\right\} = \left\{\frac{1}{5}\right\}$$

9. $|A| = 3\alpha - 4;\ |B| = 9 - 7 = 2$

$$4|A|^9\frac{1}{|B|} = 4\times(3\alpha-4)^9\cdot\frac{1}{2} = -2 \quad\Rightarrow\quad 3\alpha - 4 = -1$$

$\Rightarrow$ $\alpha = 1$

Test Your Concepts-IV

1. For $PP' = I$

$$\begin{bmatrix}2/3 & 3k & a\\-1/3 & -4k & b\\2/3 & -5k & c\end{bmatrix}\begin{bmatrix}2/3 & -1/3 & 2/3\\3k & -4k & -5k\\a & b & c\end{bmatrix}$$

$$= \begin{bmatrix}\frac{4}{9}+9k^2+a^2 & -\frac{2}{9}-12k^2+ab & \frac{4}{9}-15k^2+ac\\ -\frac{2}{9}-12k^2+ab & \frac{1}{9}+16k^2+b^2 & -\frac{2}{9}+20k^2+bc\\ \frac{4}{9}-15k^2+ac & -\frac{2}{9}+20k^2+bc & \frac{4}{9}+25k^2+c^2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$\frac{4}{9}+9k^2+a^2 = 1;\ -\frac{2}{9}-12k^2+ab = 0;\ \frac{4}{9}-15k^2+ac = 0;$$

$$\frac{1}{9}+16k^2+b^2 = 1;\ -\frac{2}{9}+20k^2+bc = 0;\ \frac{4}{9}+25k^2+c^2 = 1$$

Solving above equations

$$a^2 + b^2 - c^2 = \frac{8}{9}$$

2. Let $A = \begin{bmatrix}5 & 3 & 1\\2 & 1 & 3\\1 & 2 & 4\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}16\\19\\25\end{bmatrix}$

Then the matrix equation of the given system of equations become $AX = B$

Now $|A| = \begin{vmatrix}5 & 3 & 1\\2 & 1 & 3\\1 & 2 & 4\end{vmatrix} = 5\,(4-6) - 3\,(8-3) + 1\,(4-1)$

$$= -22 \neq 0$$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1}B$

Let C be the matrix whose elements are the cofactors of the corresponding elements of A, then

$$C = \begin{bmatrix}-2 & -5 & 3\\-10 & 19 & -7\\8 & -13 & -1\end{bmatrix}$$

$$\therefore \quad \text{adj } A = C' = \begin{bmatrix}-2 & -10 & 8\\-5 & 19 & -13\\3 & -7 & -1\end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{-22}\begin{bmatrix}-2 & -10 & 8\\-5 & 19 & -13\\3 & -7 & -1\end{bmatrix}$$

$$= \begin{bmatrix}2/22 & 10/22 & -8/22\\5/22 & -19/22 & 13/22\\-3/22 & 7/22 & 1/22\end{bmatrix}$$

$$\therefore \quad X = A^{-1}B = \begin{bmatrix}2/22 & 10/22 & -8/22\\5/22 & -19/22 & 13/22\\-3/22 & 7/22 & 1/22\end{bmatrix}\begin{bmatrix}16\\19\\25\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\5\end{bmatrix}$$

$$\therefore \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\5\end{bmatrix}$$

$\therefore$ $x = 1, y = 2, z = 5$

3. $A^3 = AA^2 = A(5A - 7I)$

$$= 5A^2 - 7A = 5(5A - 7I) - 7A = 18A - 35I$$

$$A^4 = A.A^3 = A(18A - 35I) = 18(5A - 7I) - 35A = 55A - 126I$$

$$\begin{aligned} A^5 &= A.A^4 = A(55A - 126I) \\ &= 55A^2 - 126A \\ &= 55(5A - 7I) - 126A \\ &= 275A - 385I - 126A \end{aligned}$$

$$A^5 = 149A - 385I$$

$$a = 149,\ b = -385$$

$$2a + b = -87$$

4. $A.A^T = I \ \Rightarrow\ \dfrac{1}{9}\begin{bmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & x+2-2y \\ x+4+2y & 2x+2-2y & x^2+y^2+4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x + 4 + 2y = 0,\ 2x + 2 - 2y = 0 \quad\Rightarrow\quad x = -2,\ y = -1$$

5. Since $AB = B$ and $BA = A$

$\therefore$ A and B both are idempotent

$$(A - B)^2 = A^2 - AB - BA + B^2 = A - B - A + B = 0$$

$\therefore$ $A - B$ is nilpotent

6. $AA' = \text{I}$

$\Rightarrow$ $|A| = \pm 1$

$\therefore$ $|\text{adj. (adj. } A)| = |A|^{(n-1)^2} = \pm 1.$

7. To have a non-trivial solution

$$\begin{vmatrix} t & t+1 & t-1 \\ t+1 & t & t+2 \\ t-1 & t+2 & t \end{vmatrix} = 0$$

$\Rightarrow$ $2t + 1 = 0 \Rightarrow t = -\dfrac{1}{2}$

8. Given matrix A is invertible $\Rightarrow$ det $A \neq 0$

$\Rightarrow$ the given system of equation has only one solution i.e., (0, 0, 0)

Single Correct Choice Type Questions

1. $P^3 = P(I - P) = P - P^2 = P - (I - P) = 2P - 1$

$$P^4 = P(2P - I) = 2(I - P) - P = 2I - 3P$$

$$P^5 = P(2I - 3P) = 2P - 3(I - P) = 5P - 3I$$

$$P^6 = P(5P - 3I) = 5(I - P) - 3P = 5I - 8P$$

Hence, the correct answer is (C).

2. $a_{ij} = \lim\limits_{x\to 0}\dfrac{\sin(ix)}{\tan(jx)} = \dfrac{i}{j}$

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix} \quad\Rightarrow\quad A^2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 2 & 1 & \frac{2}{3} \\ 3 & \frac{3}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \frac{3}{2} & 1 \\ 6 & 3 & 2 \\ 9 & \frac{9}{2} & 3 \end{bmatrix}$$

$\Rightarrow$ $A^2 = 3A$

Hence, the correct answer is (B).

3. $|\text{Adj}(\text{adj}(A))| = |A|^{(n-1)^2} = 5^4;\ n = 3$

Hence, the correct answer is (C).

4. $|A| \neq 0\ |B| \neq 0$

$$9A^2B - 6AB = 9\left\{B - \begin{bmatrix} 2 & -3 \\ 2 & 1 \end{bmatrix}\right\} - B = \begin{bmatrix} 9 & 27 \\ -9 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix};$$

$$(9A^2 - 6A)\,B = \begin{bmatrix} 6 & 27 \\ -10 & -1 \end{bmatrix}$$

$$(9A^2 - 6A) = \begin{bmatrix} 6 & 27 \\ -10 & -1 \end{bmatrix} B^{-1}$$

$$9A^2 - 6A = \begin{bmatrix} 6 & 27 \\ -10 & -1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 27 \\ -3 & -1 \end{bmatrix}$$

$$9A^2 - 6A + I = \begin{bmatrix} -6 & 27 \\ -3 & 0 \end{bmatrix}$$

$$(3A - I)^2 = \begin{bmatrix} -6 & 27 \\ -3 & 0 \end{bmatrix}$$

$$|3A - I|^2 = 81$$

$$|3A - I| = \pm 9$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $\begin{vmatrix} 3a-1 & 3b \\ 3c & 3d-1 \end{vmatrix} = \pm 9$

$$(3a - 1)\,(3d - 1) - 9bc = \pm 9$$

$$9ad - 3a - 3d + 1 - 9bc = 9 \text{ or } -9$$

$$9(ad - bc) - 3(a + d) = 8 \text{ or } -10$$

$$9(ad - bc) - 10 = 8 \text{ or } -10$$

$$9(ad - bc) = 18 \text{ or } 0 \text{ (rejected)} \qquad \because\ |A| \neq 0$$

$$ad - bc = 2$$

Hence, the correct answer is (B).

5. Given question becomes

$$= \det A^{-1}.\ \det\left(adj.\left(\frac{adj\,B}{|B|}\right)\right).\ \det\left(adj.\left(2\frac{adj.A}{|A|}\right)\right)$$

$$= \frac{1}{|A|}\ |\text{adj.(adj. } B)| \cdot |\text{adj. adj}(-A)|$$

$$= \frac{1}{|A|} \cdot |B|^4\ |-A|^4 = -8$$

Hence, the correct answer is (B).

6. Its characteristic equation is $|M - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} \Rightarrow (1-\lambda)(2-\lambda)^2 + 1 - 2(2-\lambda) = 0$$

$\Rightarrow$ $\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$

$\therefore \quad M^3 = (5M - I)(M - I) \quad \Rightarrow \quad \alpha = 5, \beta = 1$

Hence, the correct answer is (C).

7. $AA^T = 2I$ $\qquad \ldots(1)$

$\Rightarrow \quad |A|^2 = 8|I| = 8$

Given $A^{-1} = A^T - A \text{ adj. } (2B^{-1})$ $\qquad \ldots(2)$

Multiply by A

$$I = AA^T - A^2 \text{ adj.}(2B^{-1})$$

$$I = 2I - A^2 \text{ adj.}(2B^{-1})$$

$\therefore \quad A^2 \text{ adj. } (2B^{-1}) = \text{I}$

$$|A^2| \; |\text{adj. } (2B^{-1})| = 1$$

$\Rightarrow \quad 8|2B^{-1}|^2 = 1 \quad \Rightarrow \quad 8 \cdot \dfrac{(2^3)^2}{|B|^2} = 1$

$\Rightarrow \quad |B|^2 = 64 \times 8$

$\therefore \quad |B| = \pm 8 .2\sqrt{2} = \pm 16\sqrt{2}$

Hence, the correct answer is (D).

8. $\Delta = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{vmatrix} = 0$ (condition for trivial solution)

$\Rightarrow \quad 1(3k - 16) - 2(k - 12) + 2(4 - 9) = 0$

$\Rightarrow \quad 3k - 16 - 2k + 24 - 10 = 0 \quad \Rightarrow \quad k = 2$

Hence, the correct answer is (C).

9. Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad (\text{tr.}(M))^2 - \text{tr.}(M^2)$

$$= (a + d)^2 - (a^2 + bc + bc + d^2)$$

$$= a^2 + d^2 + 2ad - a^2 - bc - bc - d^2$$

$$= 2(ad - bc)$$

$$= 2 \det M$$

Hence, the correct answer is (B).

10. $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad B^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}; B^5 = \begin{bmatrix} \cos 5\theta & \sin 5\theta \\ -\sin 5\theta & \cos 5\theta \end{bmatrix}$

$\therefore \quad T_r(B^5) = 2\cos 5\theta = 2\cos 5\left(\dfrac{3\pi}{5}\right) = -2$

Hence, the correct answer is (B).

11. $M = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, M^2 = MM = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$

$M^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, M^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, M^{4k} = I \;\forall\; k \in N.$

So, $I + M + M^2 + M^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad I + M + M^2 + M^3 + M^4 + M^5 + \ldots + M^{2010}$

$$= (I + M + M^2 + M^3) + M^4(I + M + M^2 + M^3) + \ldots + M^{2008}(I + M + M^2)$$

$$= O + O + O + \ldots + I(I + M + M^2)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = M \text{ (itself)}$$

Hence, the correct answer is (B).

12. $AB = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix}$

$$B^T AB = \begin{bmatrix} -x & 0 & x \\ -y & y & -y \\ z & 2z & z \end{bmatrix} \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix}$$

$$= \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 42z^2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{bmatrix}$$

equating coefficients $x^2 = 4$; $y^2 = 9$ and $z^2 = 1$

$\Rightarrow \quad x = \pm 2, y = \pm 3$ and $z = \pm 1$

Number of solutions $= 2 \times 2 \times 2 = 8$

Hence, the correct answer is (C).

13. Characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} \dfrac{1}{2} - \lambda & \dfrac{1}{2} \\ a & b - \lambda \end{vmatrix} = 0 = \lambda^2 - p\lambda + q$$

Where $p = b + \dfrac{1}{2}, \; q = \dfrac{b - a}{2}$

According to Caley Hamilton theorem

$$A^2 - pA + q\,I = 0$$

$\Rightarrow \quad A^3 = pA^2 - qA = p(pA - qI) - qA$

$$A^3 = A \;\Rightarrow (p^2 - q - 1)A = pq\,I$$

$\Rightarrow \quad p^2 - q - 1 = pq = 0 \Rightarrow q = p^2 - 1 \;\&\; p(p^2 - 1) = 0$

$\Rightarrow \quad (p, q) = (-1, 0)\ (0, -1)\ (1, 0)$

Correspondingly we get

$$(a, b) = \left(\frac{-3}{2}, \frac{-3}{2}\right), \left(\frac{3}{2}, \frac{-1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hence, the correct answer is (C).

14. $|A| = 2;\ A^{2011} - 5A^{2010} = A^{2010}(A - 5I)$

$\therefore \left|A^{2011} - 5A^{2010}\right| = 2^{2010}\begin{vmatrix} -2 & 11 \\ 2 & 3 \end{vmatrix} = 2^{2010}(-28) = -2^{2012}7$

Hence, the correct answer is (A).

15. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} a - b \\ c - d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\therefore \quad a - b = -1$ and $c - d = 2$ $\qquad \ldots(1)$

Similarly, $A^2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\Rightarrow \quad A\,A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \quad\Rightarrow\quad A\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}-a+2b\\-c+2d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$

$\therefore \quad -a+2b=1$ and $-c+2d=0$ …(2)

$\therefore$ We get $a=-1, b=0, c=4, d=2$

$\Rightarrow \quad (a+b+c+d)=5$

Hence, the correct answer is (D).

16. Let $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}$

Given, $a_{11}=a_{22}=a_{33}=(a+b)$;

$a_{12}=a_{23}=ab$; $a_{21}=a_{32}=1$ and all other elements are zero.

$\therefore \quad |A|=\det. A=\begin{vmatrix}a+b & ab & 0\\1 & a+b & ab\\0 & 1 & a+b\end{vmatrix}$

Now, expanding along R_1, we get

$=[(a+b)((a+b)^2-ab)-ab(a+b)]=(a+b)((a+b)^2-2ab)$

$\therefore \quad \det. A=(a+b)(a^2+b^2)$

Hence, the correct answer is (D).

17. We have

Det. $(D_2)=\begin{vmatrix}0 & 1\\2 & 0\end{vmatrix}=0-2=-2$

(As, $f(2)=2 \Rightarrow$ unity as the element of 1st row and 2nd column)

similarly, Det. $(D_3)=\begin{vmatrix}0 & 3 & 1\\3 & 0 & 3\\3 & 3 & 0\end{vmatrix}$ (Expanding along R_1)

(As, $f(3)=3 \Rightarrow$ unity as element 1st row and 3rd column)

$=0-3(0-9)+1(9-0)$

$=0+27+9=36$

$\Rightarrow \quad$ Det. $(D_2)+$ Det. $(D_3)=-2+36=34$

Hence, the correct answer is (B).

18. We have $A=\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$

Now, $\prod_{r=1}^{50}\begin{bmatrix}1 & 2r-1\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 5\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 7\\0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & 99\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & \underbrace{1+3+5+....+97+99}_{A.P.}\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & \frac{50}{2}(1+99)\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & 2500\\0 & 1\end{bmatrix}$

Hence, the correct answer is (D).

19. As, A is an involutory matrix,

So, $A^2=I \quad\Rightarrow\quad A=A^{-1}$

Now, $A^2=\left(\frac{A}{2}\right)(2A)=I \quad\Rightarrow\quad \left(\frac{A}{2}\right)^{-1}=2A$

Hence, the correct answer is (A).

20. We have $A=\begin{bmatrix}3 & 1\\-6 & -2\end{bmatrix}$

Now, $A^2=AA=\begin{bmatrix}3 & 1\\-6 & -2\end{bmatrix}\begin{bmatrix}3 & 1\\-6 & -2\end{bmatrix}=\begin{bmatrix}3 & 1\\-6 & -2\end{bmatrix}$

$\Rightarrow \quad A^2=A$

$\therefore \quad A$ is an idempotent matrix.

So, $A^n=A, \forall\, n\ge 2$ and $n\in N$ As, $A=A^2=A^3=A^4=\ldots=A^{99}$

$\therefore (I+A)^{99}={}^{99}C_0I+{}^{99}C_1A+{}^{99}C_2A^2+\ldots+{}^{99}C_{98}A^{98}+{}^{99}C_{99}A^{99}$

$\Rightarrow \quad (I+A)^{99}=I+({}^{99}C_1+{}^{99}C_2+\ldots+{}^{99}C_{99})A$

$=I+(2^{99}-1)\,A.$

Hence, the correct answer is (D).

21. Given, $AB=I$

$AB=\begin{bmatrix}2 & 0 & 7\\0 & 1 & 0\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}-\alpha & 14\alpha & 7\alpha\\0 & 1 & 0\\\alpha & -4\alpha & -2\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}5\alpha & 0 & 0\\0 & 1 & 0\\0 & 10\alpha-2 & 5\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\therefore \quad$ We get, $5\alpha=1 \quad\Rightarrow\quad \alpha=\frac{1}{5}$

Now, $\mathrm{tr}(B)=(-\alpha)+1+(-2\,\alpha)$

$=1-3\alpha=1-\frac{3}{5}=\frac{2}{5}$

Hence, the correct answer is (B).

22. We have $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$

Now, $\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 6\\0 & 1\end{bmatrix}$

$\therefore \quad \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\cdots\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & 1+2+3+\ldots+n\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & \frac{n(n+1)}{2}\\0 & 1\end{bmatrix}$

But, we are given $\begin{bmatrix}1 & \frac{n(n+1)}{2}\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 378\\0 & 1\end{bmatrix}$

$\therefore \quad \frac{n(n+1)}{2}=378 \quad\Rightarrow\quad n(n+1)=27\times 28 \quad\Rightarrow\quad n=27$

Hence, the correct answer is (B).

23. *M-I* : "Cayley Hamilton theorem"
i.e., everysquare matrix satisfies its characteristic equation.
i.e., $|A - \lambda I| = 0$

$$\text{Now, } A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$\therefore \quad |A - \lambda I| = 0 \quad \Rightarrow \quad \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad (a - \lambda)(d - \lambda) - bc = 0$

$\Rightarrow \quad \lambda^2 + ad - \lambda(a + d) - bc = 0$

$\therefore$ On comparing, we get $k = (ad - bc)$

Hence, the correct answer is (C).

24. $A.B = O \quad \Rightarrow \quad A.B.B^{-1} = O.B^{-1}$

$\Rightarrow \quad A.I = O$

$\Rightarrow \quad A = O$

Hence, the correct answer is (B).

25. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\text{Now } X^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore$ We must have

$a^2 + bc = 1$ ...(1)

$b(a + d) = 0$ ...(2)

$c(a + d) = 0$ ...(3)

$bc + d^2 = 1$ ...(4)

CASE-1: When $a = d$

$\therefore$ From (2), $2ab = 0$ and from (3), $2ac = 0$

Now, if $a = 0$, then $d = 0$ and from (1), $c = 1/b$

$$\therefore \quad X = \begin{pmatrix} 0 & 2 \\ \frac{1}{b} & 0 \end{pmatrix}$$

$\Rightarrow$ infinite number of metrics are possible with $b \in R - \{0\}$

e.g. If $b = 2$ then $X = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$

$$\text{Now, } X^2 = XX = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Verified and so on.

CASE-2: When $a = -d$

Then from (1) and (iv) we get $d^2 = 1 - bc$

or $\quad bc = 1 - d^2 \quad \therefore X = \begin{pmatrix} -\alpha & \beta \\ \gamma & \alpha \end{pmatrix}$

where $\beta\gamma = 1 - a^2$

Hence, the correct answer is (D).

26. If matrix A is singular, so $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$

Apply $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$,

$$\text{we get } \begin{vmatrix} 1-x & -2 & 1-x \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

Apply $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} 1-x & -2 & 3-x \\ 0 & -x & 0 \\ -2 & -4 & 3-x \end{vmatrix} = 0 \Rightarrow -(3-x)\begin{vmatrix} 1-x & +2 & 1 \\ 0 & +x & 0 \\ -2 & +4 & 1 \end{vmatrix} = 0$$

Now, expanding along R_1, we get

$\Rightarrow \quad (x - 3)[(1 - x)\cdot(x - 0) - 2(0 - 0) + 1(0 + 2x)] = 0$

$\Rightarrow \quad (x - 3)(x - x^2 + 2x) = 0 \quad \Rightarrow \quad (3x - x^2)(x - 3) = 0$

$\Rightarrow \quad -x(x - 3)^2 = 0 \;\therefore\; x = 0 \text{ or } 3$

Hence, the correct answer is (B).

27. A diagonal matrix is commutative with every square matrix if it is scalar matrix so every diagonal element is 4.

$\therefore \quad |A| = 64$

Hence, the correct answer is (A).

28. Here, det. A is twice the area of the triangle with vertices $A(l, p)$, $B(m, q)$, $C(n, r)$ with sides 3, 4, 5
It is right angle triangle .

$\Rightarrow \quad \Delta = \frac{1}{2}3.4 = 6 \quad \Rightarrow \quad \Delta = 6$

Now, det. $A = 2\Delta = 2(6) = 12$

det.B = det. $(A^2) = (\text{det.}(A))^2 = (12)^2 = 144$

Hence, the correct answer is (C).

29. $I + A + A^2 + \ldots \infty = (I - A)^{-1}$

$$|I - A| = \begin{vmatrix} 2 & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = 1 + \frac{3}{4} = \frac{7}{4}$$

$$(I - A)^{-1} = \frac{\text{adj}(I - A)}{|I - A|}$$

$$(I - A)^{-1} = \begin{bmatrix} 2 & \frac{-3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \frac{4}{7}\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{-1}{2} & 2 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

Hence, the correct answer is (C).

30. $AA^T = I \quad \Rightarrow \quad (3A)(3A^T) = 9I$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$\Rightarrow \quad x + 4 + 2y = 0$ and $2x + 2 - 2y = 0$

solving $x = -2$, $y = -1$.

Hence, the correct answer is (C).

31. A is skew symmetric matrix

$|a_{ij}| \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$a_{ij} \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9\}$

$\therefore$ number of such matrix $= 19 \times 19 \times 19 = 6859$

Hence, the correct answer is (D).

32. $$A = \begin{bmatrix} \alpha+\beta & \alpha^2+\beta^2 & \alpha^3+\beta^3 \\ \alpha^2+\beta^2 & \alpha^3+\beta^3 & \alpha^4+\beta^4 \\ \alpha^3+\beta^3 & \alpha^4+\beta^4 & \alpha^5+\beta^5 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & \beta & 0 \\ \alpha^2 & \beta^2 & 0 \\ \alpha^3 & \beta^3 & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ x & y & z \end{bmatrix}$$

$|A| = 0 \quad \Rightarrow \quad$ Singular matrix

Hence, the correct answer is (B).

33. $AA^T = 3I \quad \Rightarrow \quad A^{-1} = \dfrac{A^T}{3}$ and $|\mathrm{A}| = -\,3\sqrt{3}$

$$|\text{adj.}(\text{adj.}A)| = |A|^4$$

$$\frac{|A^{-1} + A^T|}{|adj.(adj.A)|} = \frac{\left|\dfrac{A^T}{3} + A^T\right|}{|A|^4} = \frac{\left|\dfrac{4A^T}{3}\right|}{|A|^4}$$

$$= \left(\frac{4}{3}\right)^3 \frac{|A^T|}{|A|^4} = \frac{64}{27 \times |A|^3}$$

$$= \frac{64}{27 \times (-3\sqrt{3})^3} = -\frac{64}{3^7\sqrt{3}} = -\frac{2^6\sqrt{3}}{3^8}$$

Hence, the correct answer is (A).

34. $$A = \begin{vmatrix} i & \omega^2 & \omega^2 \\ \omega & i & \omega^2 \\ \omega & \omega & i \end{vmatrix}$$

$= i(-2) - \omega^2\,(i\,\omega - 1) + \omega^2\,(\omega^2 - \omega\, i) = -1 - 4i$

Hence, the correct answer is (D).

35. Also, equation $x^3 - 6x^2 + px - 8 = 0$ has roots a_{12}, a_{23}, a_{31}

$\therefore \quad a_{12} + a_{23} + a_{31} = 6;\ a_{12}\,a_{23}\,a_{31} = 8$

A.M. = G.M. = 2 $\quad \Rightarrow \quad a_{12} = a_{23} = a_{31} = 2$

Clearly $A + A^T$ is a symmetric matrix

$$\therefore \quad \left.\begin{array}{l} a_{21} + a_{12} = 4 \\ a_{31} + a_{13} = 4 \\ a_{23} + a_{32} = 4 \end{array}\right\} \quad \Rightarrow \quad a_{21} = a_{13} = a_{23} = 2$$

and for diagonal elements.

$2a_{11} = 6,\ 2a_{22} = 10,\ 2a_{33} = 8$

$\Rightarrow \quad a_{11} = 3,\ a_{22} = 5,\ a_{33} = 4$

$$\therefore \quad A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$\Rightarrow \quad |A| = 3(20 - 4) - 2(8 - 4) + 2(4 - 10) = 48 - 8 - 12 = 28$

Hence, the correct answer is (A).

36. (A) $AB = -\,BA$

$A^{-1}\,AB = -\,A^{-1}\,BA$

$B = -\,A^{-1}\,BA$

$\Rightarrow \quad BA^{-1} = -\,A^{-1}\,B \qquad \ldots(1)$

(B) $(AB)^T = B^TA^T = -\,BA = AB$

(C) $A^{-1}B^{-1} = (BA)^{-1} = (-AB)^{-1} = -B^{-1}A^{-1}$

(D) $(A^{-1}B)^T = B^T(A^T)^{-1} = -\,BA^{-1} = A^{-1}B$ using (1)

Hence $A^{-1}B$ is symmetric

Hence, the correct answer is (D).

37. $|A| = (1 + 2) - 2\,(-1 - 4) - (1 - 2)$

$= 3 + 10 + 1 = 14$

$\therefore \quad \det(\text{Adj}\,(\text{Adj}\,A)) = |\text{Adj}\,A|^2 = |A|^4 = (14)^4$

Hence, the correct answer is (A).

38. Here $a_{ij} + a_{ji} = 0 \Rightarrow A^T = -A$ and $b_{ij} - b_{ji} = 0 \Rightarrow B^T = B$ and A, B are 3×3 matrices, Hence

$|A| = 0, \Rightarrow \left|A^4B^3\right| = 0 \Rightarrow A^4B^3$ is singular matrix.

Hence, the correct answer is (C).

39. $(AB).\,(AB) = A(BA)B$

$= A^3B^2$

$(AB)(AB)(AB) = A^3B^2AB$

$= A^3B(BA)B = A^3B(A^2B)B$

$= A^3(BA)AB^2 = A^3(A^2B)AB^2$

$= A^5(BA)B^2 = A^5(A^2B)B^2$

$= A^7B^3$

so $(AB)^n = A^{2^n - 1}B^n$

so $k = 2^{10} - 1 = 1023$

Hence, the correct answer is (B).

40. $AB = BA$

$(A(A + B)^{-1}B)^{-1}(AB) = B^{-1}(A + B)A^{-1}(AB)$

$= B^{-1}(A + B)\,B$

$= (B^{-1}A + I)\,B$

$= B^{-1}AB + B \qquad \because AB = BA$

$= B^{-1}BA + B$

$= A + B$

Hence, the correct answer is (A).

41. $A^2 = A$

$(I - 0.4\,A)^{-1} = I - \lambda A$

$$I + (-1)\,(-0.4\mathrm{A}) + \frac{(-1)(-1-1)(-0.4A)^2}{2!} + \ldots$$

$= I + (0.4)A + (0.4)^2\,A^2 + (0.4)^3\,A^3 + \ldots$

$= I + (0.4)A(1 + (0.4) + (0.4)^2 + \ldots)$

$$= I + (0.4)A\left(\frac{1}{1 - 0.4}\right)$$

$$= I + \frac{4A}{6} = I + \frac{2A}{3}$$

$$\Rightarrow \quad \lambda = -\frac{2}{3}$$

Hence, the correct answer is (C).

42. By the property of inverse matrices a is true.

$$A^{-1} = A^{-1}I = A^{-1}(AB) = (A^{-1}A)B = IB = B \quad \Rightarrow \quad b \text{ is true.}$$

Similarly $B^{-1} = A \quad \Rightarrow$ option c is true.

Hence, the correct answer is (D).

43. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A + A^T = I \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad 2a = 1, b + c = 0, 2d = 1$$

$$a = \frac{1}{2}, c = -b, d = \tfrac{1}{2}$$

$$A = \begin{pmatrix} \tfrac{1}{2} & b \\ -b & \tfrac{1}{2} \end{pmatrix}$$

Now $AA^T = I \Rightarrow \begin{pmatrix} \tfrac{1}{2} & b \\ -b & \tfrac{1}{2} \end{pmatrix}\begin{pmatrix} \tfrac{1}{2} & -b \\ b & \tfrac{1}{2} \end{pmatrix} = I$

$$\Rightarrow \begin{pmatrix} \frac{1}{4} + b^2 & 0 \\ 0 & b^2 + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow b^2 + \frac{1}{4} = 1 \Rightarrow b = \pm\frac{\sqrt{3}}{2}$$

$$\therefore A = \begin{pmatrix} \tfrac{1}{2} & \tfrac{\sqrt{3}}{2} \\ -\tfrac{\sqrt{3}}{2} & \tfrac{1}{2} \end{pmatrix} \& \begin{pmatrix} \tfrac{1}{2} & -\tfrac{\sqrt{3}}{2} \\ \tfrac{\sqrt{3}}{2} & \tfrac{1}{2} \end{pmatrix}$$

No. of matrices = 2

Hence, the correct answer is (C).

44. $S^3 = (ABCD)\,(ABCD)\,(ABCD)$

$$= (ABC)\,(DAB)\,(CDA)\,(BCD)$$

$$= D^TC^TB^TA^T = D^T(ABC)^T = D^TD\,(ABC)D = S$$

$$\therefore S^3 = S \;\Rightarrow S^4 = S^2$$

Hence, the correct answer is (C).

45. $a_{ij} = i^2 - j^2$

$$a_{ji} = j^2 - i^2$$

$$\Rightarrow a_{ij} = -a_{ji}$$

Hence, the correct answer is (A).

46. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $AA^T = I \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$

$$\Rightarrow \quad a^2 + b^2 = 1, ac + bd = 0, c^2 + d^2 = 1$$

$$\Rightarrow \quad a = 1, b = 0, c = 0, d = 1 \text{ and}$$

$$a = 0, b = 1, c = 1, d = 0$$

$\Rightarrow$ two matrices

Hence, the correct answer is (C).

47. Put $\tan(\theta/2) = a$ so that $B = \begin{bmatrix} \frac{1-a^2}{1+a^2} & \frac{-2a}{1+a^2} \\ \frac{2a}{1+a^2} & \frac{1-a^2}{1+a^2} \end{bmatrix}$

Therefore $(I - A)B = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}\begin{bmatrix} \frac{1-a^2}{1+a^2} & \frac{-2a}{1+a^2} \\ \frac{2a}{1+a^2} & \frac{1-a^2}{1+a^2} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1-a^2}{1+a^2} + \frac{2a^2}{1+a^2} & \frac{-2a}{1+a^2} + \frac{a(1-a^2)}{1+a^2} \\ \frac{-a(1-a^2)}{1+a^2} + \frac{2a}{1+a^2} & \frac{2a^2}{1+a^2} + \frac{1-a^2}{1+a^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} = I + A$$

Hence, the correct answer is (A).

48. $AB = B, BA = A \Rightarrow A^2 = A, B^2 = B$

$$(A+B)^2 = A^2 + AB + BA + B^2 = A + B + A + B = 2(A+B)$$

$$(A+B)^3 = 2(A+B)(A+B) = 2.2(A+B) = 2^2(A+B)$$

$$(A+B)^4 = 2^2(A+B)(A+B) = 2^3(A+B)$$

$$\therefore (A+B)^n = 2^{n-1}(A+B)$$

Hence, the correct answer is (B).

49. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$A^2 - 5A - 2I = 0$ By Caley Hamilton theorem

$$\Rightarrow A^4 - 5A^3 - A^2 - 4A - 2I$$

$$= A^2(A^2 - 5A - 2I) + A^2 - 4A - 2I$$

$$= A^2(0) + A^2 - 5A - 2I + A = A$$

Hence, the correct answer is (B).

50. Since $A \neq P$, we have

$$\left(A^2 + P^2\right)(A - P) = A^3 - A^2P + P^2A - P^3 = 0_n$$

Hence $A^2 + P^2$ has zero divisor.

Hence, the correct answer is (B).

51. $A = \begin{vmatrix} \omega^4 - \omega^2 & -(\omega^2 - \omega) & \omega - \omega^2 \\ -(\omega^2 - \omega) & \omega^2 - \omega & \omega - 1 \\ \omega - \omega^2 & -(\omega - 1) & \omega^2 - 1 \end{vmatrix}^T$

Cofactor matrix of $A = (\omega - \omega^2)\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$

$$\text{adj. } A = (\omega-\omega^2)\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = \left(\omega-\omega^2\right)\overline{A}\ .$$

Hence, the correct answer is (B).

52. Clearly 1st element of $A_{10} = a_{1^2+2^2+\ldots+9^2+1} = a_{286}$

And last element of $A_{10} = a_{1^2+2^2+\ldots+10^2} = 385$

$$\therefore \quad T_r\left(A_{10}\right) = \left[\log_2 286\right] + \ldots + \left[\log_2 385\right]$$

$$= 8 + 8 + \ldots + 8 = 80.$$

Hence, the correct answer is (B).

53. $AB = A,\ BA = B$

$\Rightarrow \quad (AB)B = (A)B = A$

$\Rightarrow \quad A(AB^2) = A.\ A = A^2 = A^2B^2$ …(1)

$A(BA) = A(B)$

$\Rightarrow \quad A^2 = A$ …(2)

from (1) and (2) $A^2B^2 = A$.

Hence, the correct answer is (A).

54. For the given system of equations to have non-trival solutions $\begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0$ which is a cubic in t with coefficient of t^3 as unity and the constant term as $-\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Hence product of all the values of $t = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Hence, the correct answer is (B).

55. $(XY)^T = Y^TX^T$.

$Y^T = (ABC - CBA)^T = C^TB^TA^T - A^TB^TC^T$

$= -CBA + ABC = y$

$X^T = (ABC + CBA)^T = C^TB^TA^T + A^TB^TC^T = -CBA - ABC = -X$

Hence, the correct answer is (D).

56. $A^{-1}.3A^3 + A^{-1}.2A^2 + A^{-1}.5A + IA^{-1} = 0$

$\therefore A^{-1} = -3A^2 - 2A - 5I$

Hence, the correct answer is (B).

57. $AB = A \Rightarrow A^2 = A \Rightarrow A^n = A$

and $BA = B \Rightarrow B^2 = B \Rightarrow B^n = B$

Now $(A^{2010} + B^{2010})^{2011} = (A + B)^{2011}$

$(A + B)^2 = A^2 + B^2 + AB + BA$

$= 2\,(A + B)$

$(A + B)^k = 2^k\,(A + B)$

Hence, the correct answer is (D).

58. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $X'AX = 0$

$$[x_1\ x_2\ x_3]\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{23} + a_{32})x_2x_3 = 0$$

It is true every x_1, x_2, x_3 then $a_{11} = a_{22} = a_{33} = 0$

$a_{12} + a_{21} = 0 \quad a_{13} + a_{31} = 0 \quad a_{23} + a_{32} = 0$

$\therefore \quad A$ is a skew symmetric matrix

$a_{21} = -2008,\ a_{31} = -2010,\ a_{32} = 2012$

Hence, the correct answer is (C).

59. $A^5(AB^2) = A^5\,BA.$

$\Rightarrow \quad B^2 = A^5BA$

$\Rightarrow \quad B^4 = (A^5BA)\,(A^5BA) = A^5B^2A = A^5(A^5BA)A$

$\Rightarrow \quad B^4 = A^4BA^2$

$\Rightarrow \quad B^8 = (A^4BA^2)\,(A^4BA^2) = A^4B^2A^2 = A^4(A^5BA)A^2$

$\Rightarrow \quad B^8 = A^3BA^3$

$\Rightarrow \quad B^{16} = (A^3BA^3)\,(A^3BA^3) = A^3B^2A^3 = A^3 = (A^5BA)\,A^3BA^4$

$\Rightarrow \quad B^{32} = (A^2BA^4)\,(A^2BA^4) = A^2B^2A^4 = A^2 = (A^5BA)\,A^4$

$= ABA^5$

$B^{64} = (ABA^5)\,(ABA^5) = AB^2A^5 = A(A^5BA)\,A^5 = B$

$\Rightarrow \quad B^{63} = I$

Hence, the correct answer is (C).

60. $A(z) = A\left(\dfrac{x+y}{1+xy}\right)$

$$= \left[\frac{1+xy}{(1-x)(1-y)}\right]\begin{bmatrix} 1 & -\left(\dfrac{x+y}{1+xy}\right) \\ -\left(\dfrac{x+y}{1+xy}\right) & 1 \end{bmatrix}$$

$\therefore \quad A(x).A(y) = A(z)$

Hence, the correct answer is (C).

61. $(A^{-1}BA)^2 = (A^{-1}BA)\,(A^{-1}BA) = A^{-1}B(AA^{-1})BA = A^{-1}\,BIBA$

$= A^{-1}B^2A$

$\Rightarrow \quad (A^{-1}BA)^3 = (A^{-1}B^2A)\,(A^{-1}BA) = A^{-1}B^2(AA^{-1})BA$

$= A^{-1}\,B^2IBA = A^{-1}B^3A$ and so on

$\Rightarrow \quad (A^{-1}BA)^n = (A^{-1}B^n\,A$

Hence, the correct answer is (C).

62. $A^3 = B^3$ …(1)

$A^2B = B^2A$ …(2)

$(A^2 + B^2)\,(A - B) = 0$

$\because \quad |A - B| \neq 0$

$A^2 + B^2 = O$

Hence, the correct answer is (A).

63. Let $B = I + A + A^2 + \ldots + A^{m-1}$

$\Rightarrow \quad B\,(I - A) = (I + A + A^2 + \ldots + A^{m-1})\,(I - A)$

$= I - A^m = I$

$\Rightarrow \quad B = (I - A)^{-1} \quad \Rightarrow \quad n = -1.$

Hence, the correct answer is (D).

64. We have $A = iB$

$$\Rightarrow \quad A^2 = (iB)^2 = i^2B^2 = -B^2 = -\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = -2B$$

$\Rightarrow \quad A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow \quad (A^4)^2 = (8B)^2 \quad \Rightarrow \quad A^8 = 64B^2 = 128B$

Hence, the correct answer is (B).

65. $A^2 = I$

$(I + A)(I - A) = I - A^2 = I - I = O$

Hence, the correct answer is (C).

66. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$AA^T = \text{I} \qquad (1)$

Now, $C = ABA^T$

$\Rightarrow \quad A^TC = BA^T \qquad (2)$

Now $A^TC^nA = A^TC.C^{n-1}A = BA^TC^{n-1}A$ (from (2))

$= BA^TC.C^{n-2}A = B^2A^TC^{n-2}A = \ldots$

$= B^{n-1}A^TCA = B^{n-1}BA^TA = B^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

Hence, the correct answer is (D).

67. $A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, A^4 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}, A^8 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix},$

$A^{16} = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$

Hence, the correct answer is (B).

68. $AB = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}, \text{BA} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

$AB = BA \quad \Rightarrow \quad a = b$

Hence, the correct answer is (D).

69. $\left(B^TAB\right)^T = B^TA^T\left(B^T\right)^T = B^TA^TB$

$= B^TAB$ iff A is symmetric

$\therefore \quad B^TAB$ is symmetric iff A is symmetric

Also $\left(B^TAB\right)^T = B^TA^TB = (-B)A^TB$

$\therefore \quad B^TAB$ is not skew symmetric if B is skew symmetric

Hence, the correct answer is (A).

70. $A^2 = I \Rightarrow A^2 - I = O$

$\Rightarrow \quad (A + I)\,(A - I) = O$

$\therefore$ either $|A + I| = 0$ or

$|A - I| = 0$

If $|A - I| \neq 0$, then $(A + I)\,(A - I) = O \Rightarrow A + I = O$ which is not so

$\therefore \quad |A - I| = 0$ and $A - I \neq O$.

Hence, the correct answer is (C).

71. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}(-1)$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence, the correct answer is (A).

72. We have

$$A = \begin{pmatrix} 0 & \sin\alpha & \sin\alpha\cdot\sin\beta \\ -\sin\alpha & 0 & \cos\alpha\cdot\cos\beta \\ -\sin\alpha\cdot\sin\beta & -\cos\alpha\cdot\cos\beta & 0 \end{pmatrix}$$

As, matrix A is skew-symmetric matrix of odd order

$\Rightarrow \quad |A| = 0 \Rightarrow$ A is a singular matrix

$\therefore \quad A^{-1}$ does not exists.

$\Rightarrow \quad |A|$ is independents of α and β.

Hence, the correct answer is (A).

Multiple Correct Choice Type Questions

1. $M = \{A/A$ is 3×3 matrix whose entries are -1 and $1\}$

Det $A = p_1 + p_2 + p_3 - p_4 - p_5 - p_6$. Where $|P_i| = 1$

For example

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{vmatrix} = 7. \text{ Evaluation process } \begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & 4 & 1 & 3 \\ 2 & 4 & 1 & 2 & 4 \end{array}$$

$p_1 + p_2 + p_3 - p_4 - p_5 - p_6 = 3 + 16 + 12 - 18 - 16 - 2 = -5$

Since each element of a matrix are either -1 or 1. Product of any three numbers is -1

but modulus of all p_i are equal to 1.

$$|\det A| \leq \sum_{i=1}^{3} |p_i| = 6 \Rightarrow -6 \leq \det A \leq 6.$$

Hence first option is correct.

$C_2 \to C_2 + C_3$ and $C_3 \to C_3 + C_1$ we get second and third columns whose entries are even numbers so det $A = 4$ det B where B is a matrix with integer entries.

Hence 4 divides detA $\Rightarrow$ detA $\in \{-4, 0, 4\}$.

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 4, \begin{vmatrix} -1 & 1 & -1 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -4$$

Every element from A can be chosen from either '–1' or '1'. Hence number of possible matrices are 2^9.

Hence, (A), (B) and (C) are correct.

2. $A \in M_n(C)$ (Where M matrix of order $n \times n$).is non-singular matrix such that

$3ABA^{-1} + A = 2A^{-1}BA$

$\Rightarrow \quad 3A\,BA^{-1} + A + 2A = 2A^{-1}BA + 2A$

$\Rightarrow \quad 3A(BA^{-1} + I) = 2(A^{-1}B + I)A$

$\Rightarrow \quad 3A(B + IA)A^{-1} = 2A^{-1}(B + AI)A$. Let $B + AI = X$

$3AXA^{-1} = 2A^{-1}XA \quad \Rightarrow \quad 3^n|A|\,|X|\,|A^{-1}|$

$= 2^n|A^{-1}|\,|X|\,|A|, 3^n|X| = 2^n|X|$

(Cancellation is allowed because A is non singular)

$3^n|X| = 2^n|X| \Rightarrow |X| = 0$ *i.e* $|A + B| = 0 \qquad \ldots(1)$

$M = ABA^{-1} - A^{-1}BA$

$AM = A^2BA^{-1} - BA \Rightarrow BA = A^2BA^{-1} - AM$

$3ABA^{-1} + A = 2A^{-1}BA = 2A^{-1}(A^2BA^{-1} - AM)$

$= 2ABA^{-1} - 2M$

$\Rightarrow \quad ABA^{-1} + A = -2M \Rightarrow A(BA^{-1} + I) = -2M$

Taking determinants both sides we get $|-2M|$

$= |A|\,|A+B|\,|A^{-1}| = 0$ from (1)

$\Rightarrow \quad |ABA^{-1} - AB^{-1}A| = 0$

Hence, (B) and (C) are correct.

3. As skew-symmetric matrix A,

$a = c = 0$, b can be selected in 10 ways.

$|A| = ac + b^2 = b^2$ so $b \neq 0$ for invertible matrix

For divisibility by 3, $b = 0, 3, 6, 9$ (four)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad |A+I| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

Hence, (A), (B), (C) and (D) are correct.

4. $$P = \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \omega^2 - \omega \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix} = (\omega^2 - \omega)(\omega - \omega^2) = 3$$

$|P| = 3$ and $PP' \neq \text{I}$

$\therefore$ P is not orthogonal matrix

$\because$ P is symmetric

$$\Rightarrow P \text{ is symmetric and } P^{-1} = \frac{1}{3}\begin{bmatrix} 1-\omega^2 & 0 & \omega - \omega^2 \\ 0 & 1-\omega & \omega^2 - \omega \\ \omega - \omega^2 & \omega^2 - \omega & 0 \end{bmatrix}$$

Hence, (A), (C) and (D) are correct.

5. $$A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}; \text{adj. } A = \begin{pmatrix} d_2d_3 & 0 & 0 \\ 0 & d_1d_3 & 0 \\ 0 & 0 & d_1d_2 \end{pmatrix}$$

Let $C = \text{adj. } A$ satisfy $x^3 - 9x + px - 27 = 0$

$\therefore$ Tr. $(C) = 9$ and

det. $C = 27 \quad \Rightarrow \quad |\text{adj. } A| = 27$

$\Rightarrow \quad |A|^2 = 27 \quad \Rightarrow \quad |A| = 3\sqrt{3}$

Also, $|\text{adj.(adj. } A)| = |A|^4 = 3^6$

Now, Tr. (adj. A) = 9 given

$\therefore \quad d_1d_2 + d_2d_3 + d_3d_1 = 9$ …(1)

and $|A| = d_1d_2d_3 = 3\sqrt{3}$ …(2)

$$\Rightarrow \quad \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} = \frac{9}{3\sqrt{3}} = \sqrt{3} = \text{Tr}(A^{-1})$$

Also, GM of d_1 ; d_2 ; d_3 is $\sqrt{3}$

and H.M. of d_1, d_2, d_3 is $\sqrt{3}$

$\therefore \quad d_1 = d_2 = d_3 = \sqrt{3}$

$\Rightarrow \quad$ Tr. $(A) = 3\sqrt{3}$

Hence, (A), (B) and (D) are correct.

6. $$A_n = \begin{pmatrix} 1 & n & n\ldots & n \\ n & 2 & n\ldots & n \\ n & n & 3\ldots & n \\ n & n & n\ldots & n \end{pmatrix}$$

$\therefore \quad A_n$ is non-singular for all 'n'

$|A_n| = (-1)^{n+1}.n!$

Hence, (B) and (C) are correct.

7. Given $A^2 = A$ and $B^2 = B$

$(A - B)(A + B) = A - B$

det$(A - B)$.det $(A + B)$ = det$(A - B)$ …(1)

also $(A+B)^2 = A^2 + B^2 + \underbrace{AB + BA}_{null\ matrix} = A^2 + B^2 = A + B$

and $(A-B)^2 = A^2 + B^2 - (AB + BA) = A^2 + B^2 = A + B$

hence $(A+B)^2 = (A-B)^2$

(det. $(A+B)^2$ = (det. $(A-B))^2$

det. $(A + B)$ = + det $(A - B)$ or – det. $(A - B)$ …(2)

(rejecting (–) sign)

substituting in (1)

det.$(A - B)$ det. $(A - B)$ = det. $(A - B)$

det. $(A - B)$ [det. $(A - B)$ –1] = 0

either det. $(A - B) = 0$ or det. $(A - B) = 1$

Hence, (B) and (C) are correct.

8. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a \neq b \neq c \neq d > 0$

$$A + A^T = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$$

$\left|A + A^T\right| = 4ad - (b+c)^2 = 0 \Rightarrow b + c = 2\sqrt{ad}$

$\Rightarrow b + c = 2\sqrt{ad} > 2\sqrt{bc}$ (A.M > G.M)

$\Rightarrow ad > bc, \quad b \neq c$

$\Rightarrow ad - bc > 0$

$\Rightarrow \det A > 0$

$$\left|A - A^T\right| = \left|\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}\right| = 0 + (b-c)^2 > 0$$

$\left|AA^T\right| = |A|\left|A^T\right| = |A|^2 = (\det A)^2 > 0$

Hence, (A), (C) and (D) are correct.

9. $A(adjA) = |A|I \quad \therefore |A| = 4$

$|adjA| = |A|^{n-1} = 16; |adj(adjA)| = |A|^{(n-1)^2} = |A|^4 = 256$

$adjKA = K^{n-1}adjA$

$\therefore |adjKA| = \left(K^{n-1}\right)^n |adjA|$

$\therefore |adj2A| = 2^6.16$.

Hence, (A), (B) and (C) are correct.

10. $$|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix} = -\begin{vmatrix} q & b & y \\ -p & -a & -x \\ r & c & z \end{vmatrix} \text{ (operate } C_2)$$

$$= \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} = -\begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix} = -\begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} = -\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$(R_1 \leftrightarrow R_2)$

$\Rightarrow \quad |B| = -|A|$

Hence, (B) and (D) are correct.

11. $\because |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) - 2(-3+4) + 0 = 1$

$\because \quad adj(adj\ A) = |A|^{3-2} A = A$ and $|\text{adj}(\text{adj } A)| = |A| = 1$

Also, $|adj\ A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

Hence, (A), (B) and (C) are correct.

12. $A^T = A \quad \Rightarrow \quad A = \text{adj.}(2A) \quad \Rightarrow \quad A = 4 \text{ adj. } A$

$\Rightarrow \quad |A| = 64|A|^2 \quad \Rightarrow \quad |A| = \dfrac{1}{64}$

Now, $A^{-1} = \dfrac{adj.A}{|A|} = \dfrac{A.64}{4.1} = 16A$

and adj. $A^{-1} = \text{adj.}(16A) = 256 \text{ adj.}A$

$= 256.\dfrac{A}{4} = 64A$

Hence, (B) and (C) are correct.

13. $|\text{adj. } A| = |A|^2$

$|\text{adj. } A| = -2 + 42 + 24$ (expanding along R_1)

$|\text{adj. } A| = 64$

$|A| = \pm 8$

$A^{-1} = \dfrac{adj.A}{|A|}$

$\text{tr}(A^{-1}) = \pm\dfrac{6}{8} = \pm 0.75$

Hence, (A), (B), (C) and (D) are correct.

14. $A^3 + B^3 + C^3 = A^3 + B^3 + C^3 - 3ABC$

$= (A+B+C).(A^2 + B^2 + C^2 - AB - BC - CA)$

$= (A+B+C).(A + \omega B + \omega^2 C).(A + \omega^2 B + \omega C)$

(ω is complex cube root of unity)

$= (A+B+C)\ (A + \omega B + \omega^2 C)\ \overline{(A + \omega B + \omega^2 C)}$

$\therefore \quad \det.(A^3 + B^3 + C^3).\det.(A+B+C)$

$\lambda = \{\det.(A+B+C)\}^2.\{\det(A + \omega B + \omega^2 C)\}^2$

$\lambda = \{\det.(A+B+C)\}^2[\det(A + \omega B + \omega^2 C)]^2 \geq 0$

Hence, (A), (B) and (D) are correct.

15. $A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = 0$

So $A^4 = O = A^5 = \ldots = A^{19}$

$x \in \{3, 4, 5, \ldots 19\}$

Greatest value of $x = 19$, sum $= 3 + 4 + \ldots + 19 = 187$,

Difference of values $= 19 - 3 = 16$

Hence, (A), (C) and (D) are correct.

16. $A + \text{adj.}(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ …(1)

$A^T - \text{adj.}(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$\Rightarrow \quad A - (\text{adj. } B)^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ($\because$ take transpose)

$\Rightarrow \quad A - (\text{adj. } B^T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ …(2)

add (1) and (2)

$\Rightarrow \quad 2A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \Rightarrow \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \quad |A| = 0$

$\Rightarrow \quad A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2A$

Similarly $A^n = nA$

$\Rightarrow \quad A + A^2 + A^3 + A^4 + A^5 = 15A$

$\Rightarrow \quad |A + A^2 + A^3 + A^4 + A^5| = |15A| = 0$

Now, subtract (2) from (1)

$\Rightarrow \quad 2\text{adj.}B^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$

$\Rightarrow \quad (\text{adj.}B)^T = I \quad \Rightarrow \quad \text{adj.}B = I \quad \Rightarrow \quad B = I$

$|B + B^2 + B^3 + B^4 + B^5| = |5I| = 5^3|I| = 5^3$

Hence, (A) and (C) are correct.

17. $M = \begin{bmatrix} 1 & \lambda^3 & 0 \\ 0 & 1 & \mu^3 \\ \gamma^3 & 0 & 1 \end{bmatrix}$

$[M - xI] = \begin{bmatrix} 1-x & \lambda^3 & 0 \\ 0 & 1-x & \mu^3 \\ \gamma^3 & 0 & 1-x \end{bmatrix}$

$|M - xI| = (1-x)^3 + \lambda^3\mu^3\gamma^3$

so equation is $(1-x)^3 + (\lambda\mu\gamma)^3 = 0$

$(x-1)^3 = (\lambda\mu\gamma)^3$

$x = 1 + \lambda\mu\gamma, 1 + (\lambda\mu\gamma)\omega, 1 + (\lambda\mu\gamma)\omega^2$

$\Rightarrow$ all are correct.

Hence, (A), (B), (C) and (D) are correct.

18. A is skew symmetric matrix $\Rightarrow \quad A + A^T = 0$

$B = A + A^3 + A^5 + A^7 + A^9$

$B^T = -(A + (A^3) + (A^5) + A^7 + A^9)$

$B + B^T = 0 \quad \Rightarrow \quad B$ is skew symmetric matrix

Let $C = A + B$ …(1)

$C^T = A^T + B^T$

$C^T = -A - B$ …(2)

(1) and (2) $\Rightarrow \quad C + C^T = 0$

$A + B$ is skew symmetric matix

Hence, (A), (B) and (C) are correct.

19. (A) $A=\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$

where $\frac{1}{2}\left(A+A^T\right)$ is symmetric and $\frac{1}{2}\left(A-A^T\right)$ is skew - symmetric

(B) $A(\text{Adj}A) = |A|\ I$

$\Rightarrow\quad |A|\,|\text{Adj}A| = |A|^{n} \quad\Rightarrow\quad |\text{Adj}A| = |A|^{n-1}$

(C) $A^{-1}=\frac{AdjA}{|A|}\rightarrow\left|A^{-1}\right|=\frac{1}{|A|^n}|A|^{n-1}=\frac{1}{|A|}$

(D) product AB is possible when no. of columns of 'A' is equal to no. of rows of 'B'

Hence, (A), (B) and (C) are correct.

20. $$A=\begin{bmatrix} 0 & 1^2-2^2 & 1^2-3^2.... \\ 2^2-1^2 & 0 & 2^2-3^2 \\ 3^2-1^2 & 3^2-2^2 & 0 \end{bmatrix}$$

$\therefore$ A is skew-symmetric of even order

$\therefore$ $|A|$ = perfect square

Hence, (A) and (B) are correct.

21. $$A^2=\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Now, $A^3 = A^2 - A = I$

$\Rightarrow\quad A^{3n} = I$

and $A^{-1} = A^2$ Also $A^4 = A$

$\because\quad A^3 = I \quad\Rightarrow\quad |A|^3 = 1$

$\Rightarrow\quad |A| = 1$

$\Rightarrow\quad |\text{Adj}\ A| = |A|^{n-1} = 1$

Hence, (A), (B), (C) and (D) are correct.

22. $B = -A^{-1}BA$

$AB = -AA^{-1}BA = -BA$

$\Rightarrow\quad AB + BA = 0$

$\Rightarrow\quad (A + B)^2 = A^2 + B^2$

Hence, (A) and (B) are correct.

23. $\because\quad (AB)^2 = ABAB = AABB$

$(AB)^2 = A^2B^2$

$(AB)^3 = (A^2B)^2AB = A^2\mathrm{B}^2AB = A^2BABB$

$(AB)^3 = A^2B^3$

$\because\quad (AB)^n = A^nB^n$ A is correct

$A^2B = AAB = ABA = BAA = BA^2$

$\Rightarrow\quad A^nB = BA^n$

$\because\quad (A^n - B^n)\ (A^n + B^n) = A^{2n} + A^nB^n - B^nA^n - B^{2n} = A^{2n} - B^{2n}$

$\because\quad A^nB^n = B^nA^n$

$(A + B)^n = A^n + {}^nC_1\ A^{n-1}B + {}^nC_2A^{-2}B^2 + \ldots + B^n$

So, A,B,C,D are correct choices .

Hence, (A), (B), (C) and (D) are correct.

24. $$A^2=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^3-A^2=\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, A-I=\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$\Rightarrow\quad A^3 - A^2 = A - I$

$\Rightarrow\quad Det\left|A^n-I\right|=Det\left(\left(A-I\right)\left(1+A+A^2+...+A^{n-1}\right)\right)$

$= \text{Det}\ (A - I)\ \text{Det}\ (1 + A + A^2 + \ldots + A^{n-1}) = 0$

$A^3 - A^2 = A - I$ ...(1)

$\Rightarrow\quad A^4 - A^3 = A^2 - A$ ...(2)

$A^5 - A^4 = A^3 - A^2 = A - I$ (Using (1))

If n is even $A^n - A^{n-1} = A^2 - A$ ...(3)

If n is odd $A^n - A^{n-1} = A - I$ ...(4)

Consider n is even

$A^n - A^{n-1} = A^2 - A$ (Using (3))

$A^{n-1} - A^{n-2} = A - I$ (Using (4))

$\Rightarrow\quad A^n - A^{n-2} = A^2 - I$

$\Rightarrow\quad A^n = A^{n-2} + A^2 - I$

$=\left(A^{n-4}+A^2-I\right)+A^2-I$

$=\left(A^{n-6}+A^2-I\right)+2\left(A^2-I\right)$

$=\left(A^2\right)+\frac{n-2}{2}\left(A^2-I\right)$

$A^n=\left(\frac{n}{2}\right)A^2-\left(\frac{n-2}{2}\right)I$

$A^{50} = 25A^2 - 24I$

$$=25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}-24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Hence, (A), (B) and (C) are correct.

25. $AB = \lambda B \Rightarrow \left(A - \lambda I\right).B = 0$

$$\Rightarrow\quad \begin{bmatrix} -3-\lambda & -7 & -5 \\ 2 & 4-\lambda & 3 \\ 1 & 2 & 2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}=0$$

$\Rightarrow\quad \det\left(A-\lambda I\right)=0\Rightarrow\left(\lambda-1\right)^3=0\Rightarrow\lambda=1$

$\therefore\quad 4a+7b+5=0, 2a+3b+3=0$

$\Rightarrow\quad a=-3, b=1, a+b=-2$

Hence, (A), (B), (C) and (D) are correct.

26. $A^2 + A + 2I = 0$

$A^2 + A = -2I$

$\left|A^2+A\right|=\left|-2I\right|$

$$|A||A+I|=(-2)^n$$

$$|A|\neq 0$$

'A' is non singular. Hence its inverse exists. Also multiplying the given equation both sides with A^{-1} we get $A^{-1}=-\frac{1}{2}(A+I)$.

Hence, (A), (C) and (D) are correct.

27. $\operatorname{adj}(AB)=\operatorname{adj}B\,\operatorname{adj}A$

$$=(|B|B^{-1})(|A|A^{-1})$$

$$=|B|\,|A|B^{-1}A^{-1}$$

$$adj(AB)=|AB|(AB)^{-1}=|A||B|\left(B^{-1}A^{-1}\right)$$

Hence, (A), (B) and (D) are correct.

28. $(A+B)^2=A^2+AB+BA+B^2=A+B$

$\Rightarrow AB+BA=0$ $\qquad(\because A^2=A,\ B^2=B)$

$$A^2B+ABA=0\Rightarrow A^2B+(-BA)A=0$$

$\Rightarrow AB-BA=0$

Hence, (A) and (D) are correct.

29. $$A^2-4A-5I_3=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$$

$$-4\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}-5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}+\begin{bmatrix}-4&-8&-8\\-8&-4&-8\\-8&-8&-4\end{bmatrix}+\begin{bmatrix}-5&0&0\\0&-5&0\\0&0&-5\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

$\therefore\quad A^2-4A-5I_3=0$

or $\quad A^{-1}A^2-4A^{-1}A-5A^{-1}I_3=0$

or $\quad \left(A^{-1}A\right)A-4I_3-5A^{-1}=0$

or $\quad IA-4I_3-5A^{-1}=0$

$\therefore\quad A^{-1}=\frac{1}{5}(A-4I_3)$

Also, $$\left|A^2\right|=\begin{vmatrix}9&8&8\\8&9&8\\8&8&9\end{vmatrix}$$

$$=9(81-64)-8(72-64)+8(64-72)$$

$$=9\times 17-8\times 8+8\times(-8)$$

$$=133-128=5\neq 0$$

$\therefore\quad A^2$ is invertible

and $A^3=A.\,A^2=A.\,(4A-5I_3)=4A^2-5A$

$$=\begin{bmatrix}36&32&32\\32&36&32\\32&32&36\end{bmatrix}+\begin{bmatrix}-5&-10&-10\\-10&-5&-10\\-10&-10&-5\end{bmatrix}=\begin{bmatrix}31&22&22\\22&31&22\\22&22&31\end{bmatrix}$$

$\therefore\quad \left|A^3\right|\neq 0$

$\therefore\quad A^3$ is invertible.

Hence, (A), (B) and (D) are correct.

30. $$A^2=\begin{bmatrix}3&3&3\\3&3&3\\3&3&3\end{bmatrix}=3A$$

$A^3=A^2A=3A.A=3A^2=3.(3A)=9A$ and $|A|=0$

$\therefore A^{-1}$ does not exist

Hence, (A) and (D) are correct.

31. $B=P^{-1}$ A.P.

$$B^{-1}=(P^{-1}AP)^{-1}=P^{-1}A^{-1}P$$

$$PB^{-1}P^{-1}=A^{-1}$$

So, A is invertible when B is also invertible.

$B^2=P^{-1}APP^{-1}$ A.P.

$$B^2=P^{-1}A^2P$$

$$\Rightarrow\quad B^n=P^{-1}A^nP$$

$$B-\lambda I=P^{-1}AP-\lambda I$$

$$=P^{-1}AP-\lambda P^{-1}P$$

$$=P^{-1}(AP-\lambda P)$$

$$=P^{-1}(A-\lambda I)P$$

Hence, (A), (B) and (C) are correct.

32. Observe that 1, 2, 3, 4 are roots of $n^4-wn^3-zn^2-yn-x=0$

$$w=10,\ z=-35,\ y=50,\ x=-24$$

Clearly $(|y|,|x|)$ and $(|w|,|x|)$ have HCF 2

Hence, the correct answer is (C).

33. (A) $$\begin{vmatrix}a_1&a_1+d&a_1+2d\\a_1+3d&a_1+4d&a_1+5d\\a_1+4d&a_1+5d&a_1+6d\end{vmatrix}=\begin{vmatrix}a_1&d&d\\a_1+3d&d&d\\a_1+4d&d&d\end{vmatrix}=0$$

(B) $$\begin{vmatrix}a_1&a_1+d&a_1+2d\\a_1+d&a_1+2d&a_1+3d\\a_1+d&a_1+3d&a_1+5d\end{vmatrix}=\begin{vmatrix}a_1&d&d\\a_1+d&d&d\\a_1+d&2d&2d\end{vmatrix}=0$$

(C) $$\begin{vmatrix}a_1&a_2&a_3\\a_3&a_1&a_2\\a_4&a_5&a_6\end{vmatrix}(C_3\rightarrow C_3+C_1)$$

$$\Rightarrow\begin{vmatrix}a_1&a_2&a_1+a_3\\a_3&a_1&a_2+a_3\\a_4&a_5&a_4+a_6\end{vmatrix}=\begin{vmatrix}a_1&a_2&2a_2\\a_3&a_1&a_4+a_3\\a_4&a_5&2a_5\end{vmatrix}\neq 0$$

(D) $$\begin{vmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\\a_7&a_8&a_9\end{vmatrix}(C_3\rightarrow C_3+C_1)$$

$$\Rightarrow\begin{vmatrix}a_1&a_2&a_1+a_3\\a_4&a_5&a_4+a_6\\a_7&a_8&a_7+a_9\end{vmatrix}=\begin{vmatrix}a_1&a_2&2a_2\\a_4&a_5&2a_5\\a_7&a_8&2a_8\end{vmatrix}=0$$

Determinants in (A), (B) and (D) are zero, and in (C) the determinant is non zero.

Hence, (A), (C) and (D) are correct.

34. $(2n+1)^2-(2m+1)^2=4(m+n+1)(n-m)$

$$= \text{multiple of } 8$$

We can apply operations $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

We have 8 common from column second and third each

Hence Det(A) is divisible by 64

Hence, (C) and (D) are correct.

35. $A^2=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}$,

$$A^3=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 6 & 1\end{bmatrix}$$

Hence, $A^n=\begin{bmatrix}1 & 0\\ 2n & 1\end{bmatrix}$

Hence, (A), (B) and (C) are correct.

36. (A) $A=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$; $B=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

is a counter example. Therefore, (A) is wrong.

(B) We have $(A^2+B^2)(A-B)=A^3-B^3-A^2B+B^2A=0$

and $A-B\neq 0 \Rightarrow A^2+B^2$ is not invertible. Therefore (B) is also wrong.

(C) $A^T=A$

$$\left(\left(A^{-1}\right)^{-1}\right)^{-1}=A^{-1}$$

$$\left(A^{-1}\right)^T=\left(A^T\right)^{-1}=A^{-1}$$

So, $\left(\left(A^{-1}\right)^{-1}\right)^{-1}$ is also symmetric. Therefore (C) is also wrong.

(D) $|A|\neq 0, |B|\neq 0 \Rightarrow |AB|\neq 0$

So, AB is invertible. Therefore (D) is correct.

Hence, (A), (B) and (C) are correct.

37. We have, $\begin{vmatrix}\cos(\theta+\alpha) & -\sin(\theta+\alpha) & \cos 2\alpha\\ \sin\theta & \cos\theta & \sin\alpha\\ -\cos\theta & \sin\theta & \lambda\cos\alpha\end{vmatrix}$

$$=\frac{1}{\sin\alpha\cos\alpha}\begin{vmatrix}\cos(\theta+\alpha) & -\sin(\theta+\alpha) & \cos 2\alpha\\ \sin\theta\sin\alpha & \cos\theta\sin\alpha & \sin^2\alpha\\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha & \lambda\cos^2\alpha\end{vmatrix}$$

[Multiplying R_2 and R_3 by sin α and cos α, respectively]

$$=\frac{1}{\sin\alpha\cos\alpha}\begin{vmatrix}0 & 0 & \cos 2\alpha+\sin^2\alpha+\lambda\cos^2\alpha\\ \sin\theta\sin\alpha & \cos\theta\sin\alpha & \sin^2\alpha\\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha & \lambda\cos^2\alpha\end{vmatrix}$$

[Applying $R_1 \to R_1+R_2+R_3$]

$$=\frac{\cos 2\alpha+\sin^2\alpha+\lambda\cos^2\alpha}{\sin\alpha.\cos\alpha}\begin{vmatrix}\sin\theta\sin\alpha & \cos\theta\sin\alpha\\ -\cos\theta\cos\alpha & \sin\theta\cos\alpha\end{vmatrix}$$

$$=(\cos^2\alpha+\lambda\cos^2\alpha)\begin{vmatrix}\sin\theta & \cos\theta\\ -\cos\theta & \sin\theta\end{vmatrix}=(1+\lambda)\cos^2\alpha$$

Therefore, the given determinant is independent of θ for all real values of λ. Also,

$\lambda=-1$, then it is independent of θ and α.

Hence, (A) and (C) are correct.

38. $AA^T=A^TA=I$. Also $A^T=A$, so $A^2=I \Rightarrow A$ is involutory matrix.

$\Rightarrow \quad |A^2|=|A|^2=1$ or, $|A|=\pm 1$.

But $|A|=\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=(a+b+c)\begin{vmatrix}1 & b & c\\ 1 & c & a\\ 1 & a & b\end{vmatrix}$

$$=(a+b+c)(ab+bc+ca-a^2-b^2-c^2)$$

$|A|=ab+bc+ca-a^2-b^2-c^2 \qquad (\because a+b+c=1)$

$\therefore \quad a^2+b^2+c^2-ab-bc-ca\geq 0$

So $\quad |A|=-1$. Hence $a^3+b^3+c^3-3abc=1$.

Again $a^2+b^2+c^2-ab-bc-ca=1$

$\Rightarrow \quad 1-3(ab+bc+ca)=1$, so $ab+bc+ca=0$,

$\Rightarrow$ atleast one of a, b, and c is negative.

Hence, (A), (B), (C) and (D) are correct.

Linked Comprehension Type Questions

1. $A^{-1}B^2+AB=0$

$\Rightarrow \quad A^{-1}B.BB^{-1}+AB.B^{-1}=0$

$\Rightarrow \quad AA^{-1}B+A.A=0$

$\Rightarrow \quad B=-A^2 \qquad \ldots(1)$

and $|A|=5 \qquad \ldots(2)$

$$A^6-2A^4B+A^2B^2=4A^6$$

$\therefore \quad |4A^6|=4^3\,|A|^6.=2^6\cdot 5^6=10^6$

Hence, the correct answer is (D).

2. $A^2\det(A^2)-\text{adj}(\text{adj } B)=25A^2-|B|B$

$$=25A^2-(-25)(-A^2)=0$$

Hence, the correct answer is (A).

3. $\because \quad BA^TB=I \qquad \ldots(1)$

taking transpose, we get $B^TAB^T=\text{I}$

multiply both sides by $A \Rightarrow AB^TAB^T=\text{A}$

$\Rightarrow \quad BB^T=A \qquad (\because AB^TA=B)$

$\Rightarrow \quad A$ is symmetric matrix.

$\Rightarrow \quad A^T=A$

From (1), we get $BAB=\text{I}$

$\Rightarrow \quad B^2=A^{-1} \qquad (\because |B|, |A|\neq 0)$

Hence, the correct answer is (B).

4. $|B^T-B|=|B^{-1}|\;|BB^T-B^2|=\dfrac{1}{|B|}|A-A^{-1}|$

$$=\frac{1}{|B|}|A-B^2|$$

$\because \quad |A|=1, |B|\neq -1$ and $|B|>0 \Rightarrow |B|=1$

$\Rightarrow \quad |B^T-B|=|A-B^2|=\begin{vmatrix}4 & 4\\ 4 & -4\end{vmatrix}=-32$

Hence, the correct answer is (C).

5. $$AB=\begin{bmatrix}\frac{x^2+1}{x} & 0 & 0\\ 0 & x & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}\frac{5x}{x^2+1} & 0 & 0\\ 0 & \frac{3}{x} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}=\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$(AB)^{-1}=\begin{bmatrix}\frac{1}{5} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix},\ (\mathrm{AB})^{-2}=\begin{bmatrix}\frac{1}{5^2} & 0 & 0\\ 0 & \frac{1}{3^2} & 0\\ 0 & 0 & \frac{1}{4^2}\end{bmatrix}\text{ and so on.}$$

$$\therefore\quad X=\begin{bmatrix}\frac{1}{5}+\frac{1}{5^2}+\ldots\frac{1}{5^n} & 0 & 0\\ 0 & \frac{1}{3}+\frac{1}{3^2}+\ldots\frac{1}{3^n} & 0\\ 0 & 0 & \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots\frac{1}{4^n}\end{bmatrix}$$

$$Y=\lim_{x\to\infty}X=\begin{bmatrix}\frac{\frac{1}{5}}{1-\frac{1}{5}} & 0 & 0\\ 0 & \frac{\frac{1}{3}}{1-\frac{1}{3}} & 0\\ 0 & 0 & \frac{\frac{1}{4}}{1-\frac{1}{4}}\end{bmatrix}$$

$$\Rightarrow\quad \mathrm{Y}=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}\qquad \therefore\ \mathrm{Y}^{-1}=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$$

$$|\operatorname{adj.}(\sqrt{5}\,Y^{-1})|=|(\sqrt{5})^2\operatorname{adj.}(Y^{-1})|$$
$$=5^3|\operatorname{adj.}(Y^{-1})|=5^3\times|\mathrm{Y}^{-1}|^2$$
$$=5^3\cdot(4\cdot2\cdot3)^2=5\,(5!)^2$$

Hence, the correct answer is (C).

6. $$AY=\begin{bmatrix}x+\frac{1}{x} & 0 & 0\\ 0 & \frac{1}{x} & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{4}\left(x+\frac{1}{x}\right) & 0 & 0\\ 0 & \frac{1}{2x} & 0\\ 0 & 0 & \frac{16}{3}\end{bmatrix}$$

$$\therefore\quad t_r(AY)=\frac{1}{4}\mathrm{x}+\frac{1}{4x}+\frac{1}{2x}+\frac{16}{3}$$
$$=\frac{1}{4}x+\frac{3}{4x}+\frac{16}{3}\geq 2\cdot\sqrt{\frac{1}{4}\cdot x\cdot\frac{3}{4x}}+\frac{16}{3}\geq 1\cdot732+5.333\geq 7.066$$

$\therefore$ Least integral value of $t_r(AY)=8$.

Hence, the correct answer is (A).

7. $Z=Y^{-1}-2I$

$$=\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}-2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$Z^2=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1^2\end{bmatrix}$$

$$\therefore\quad t_r(Z+Z^2+Z^3+\ldots Z^{10})=2+2^2+2^3+\ldots+2^{10}+10$$
$$=2\cdot\left(\frac{2^{10}-1}{2-1}\right)+10=2^{11}+8=2^a+b$$

$\therefore\quad a+b=11+8=19$

Hence, the correct answer is (D).

8. The characteristic equation $\begin{vmatrix}(8-\lambda) & -6 & 2\\ -6 & (7-\lambda) & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$is

which simplifies as $\lambda^3-18\lambda^2+45\lambda=0$

$\lambda(\lambda-3)(\lambda-15)=0$

$\therefore\ \lambda=0,3,15$

Hence, the correct answer is (B).

9. Since the eigenvalues are 0,1,1, the characteristic equation is $x(x-1)^2=0$

or $\quad x^3-2x^2+x=0$

the matrix A satisfies $x^3-2x^2+x=0$

Hence, the correct answer is (B).

10. If λ is eigenvalue of A,

then $AX=\lambda X$

Premultiplying by A^{-1}, $A^{-1}AX=A^{-1}\ \lambda X$

(i.e.) $IX=\lambda\left(A^{-1}X\right)$

$\therefore\quad A^{-1}X=\frac{1}{\lambda}X$

$\therefore\quad \frac{1}{\lambda}$ is an eigenvalue of A^{-1}

Hence, the correct answer is (B).

11. $$\begin{pmatrix}0 & . & .\\ . & 0 & .\\ . & . & 0\end{pmatrix}$$

$A=$ matrix skew symmetric

$B=$ matrix scalar

$n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$$= 7^3 + 7.1 - 1 = 7^3 + 6$$

Hence, the correct answer is (C).

12. A = Matrix lower diagonal
B = Matrix upper diagonal

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$= 7^6 + 7^6 - 7^3 = 2.7^6 - 7^3 = 7^3(2.7^3 - 1)$$

Hence, the correct answer is (C).

13. If (a, b, c) are trace elements

Trace	Trace elements
9	3,3,3
8	3,3,2
7	3,2,2(or)3,3,1

Required no. of matrices

$$= 1.7^6 + 3\left(7^6\right) + 6\left(7^6\right) = 10.7^6$$

Hence, the correct answer is (C).

14. Given $|A| = 8$ …(1)

$$A^3 - 6A^2 + 7A - 8I = 0$$

$\therefore \quad (A - 2I)^3 = 5A$ …(2)

$B = A - 2I$ …(3)

From (2) and (3)

$$|A - 2I|^3 = |5A| = |5I_3A| = 125|A|$$

$$\therefore \quad |B|^3 = 125 \times 8 = 1000 \therefore |B| = 10$$

Now, $B = A - 2I$

$$\therefore \quad A^{-1}B = I - 2A^{-1}$$

$$|\text{adj. }(I - 2A^{-1})| = |I - 2A^{-1}|^2 = |A^{-1}B|^2 = \left(\frac{|B|}{|A|}\right)^2$$

$$= \left(\frac{5}{4}\right)^2 = \frac{25}{16}$$

Hence, the correct answer is (A).

15. $$\text{adj.}\left(\left(\frac{B}{2}\right)^{-1}\right) = \left(adj.\left(\frac{B}{2}\right)^{-1}\right)$$

$$= \frac{\text{adj.}\left(\text{adj.}\left(\frac{B}{2}\right)\right)}{\left|\text{adj.}\frac{B}{2}\right|} = \frac{\left|\frac{B}{2}\right|\frac{B}{2}}{\left|\frac{B}{2}\right|^2} = \frac{4B}{|B|} = \frac{2B}{5}$$

$$\therefore \quad p = 2, q = 5 \Rightarrow (p + q) = 7$$

Hence, the correct answer is (A).

16. Use $|AB| = |A|\ |B|$ and $|A_0| = 0$. $\because\ |A_0| = 0$

$$|A_0 + A_0^2B_0^2 + A_0^3 + A_0^4B_0^4 + \ldots|$$

$$= |A_0|\ |I + A_0B^2 + A_0^2 + \ldots|$$

$$= 0$$

Hence, the correct answer is (C).

17. $$B_1 = \text{adj}B_0 = \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & +4 \\ -3 & +1 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = B_0$$

$B_1 = B_2 = B_3 \ldots B_{49} = B_0$ $\because\ |B_0| = 1$

Hence, the correct answer is (D).

18. As 2nd row of all the options is same, we are to look at the elements of the first row.

Let left inverse be $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad a + b + 2c = 1$$

$$-a + b + 3c = 0 \text{ i.e. } b = \frac{1 - 5c}{2}, a = \frac{1 + c}{2}$$

Thus matrices in the options A, B and D are the inverses and matrix in option C is not the left inverse.
Hence, the correct answer is (C).

19. Let right inverse is

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now $a - c + 2e = 1$
$b - d + 2f = 0$
$2a - c + e = 0$
$2b - d + f = 1$
Infinite solution
Hence, the correct answer is (D).

20. Only six places are to filled $\Rightarrow 2^6$ matrices
Hence, the correct answer is (C).

21. $$\begin{bmatrix} \Box & \Box \\ \Box & \Box \end{bmatrix}$$

Only four places, shown above, are to be filled $\Rightarrow 2^6$ matrices
Hence, the correct answer is (B).

22. A matrix is invertible iff its determinant $\neq O$
Matrices, with determinant $= O$, are

$$\underset{\text{(one)}}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{\text{(one)}}{\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}}, \underset{\text{(six)}}{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}} \Rightarrow 8 \text{ in all, out of a total of 16.}$$

Hence, the correct answer is (D).

23. $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 20 \\ 29 \end{bmatrix} \Rightarrow 29^2 \neq 20^2 + 26^2$$

Similarly, $Q\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}$, and $R\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 17 \end{bmatrix}$

Hence, the correct answer is (B).

24. $\det P = 1$, $\det Q = 1$, $\det R = 1$
$\det(PQ) = 1$, $\det(QR) = 1$, $\det(RP) = 1$
$\det(PQR) = 1$
Hence, the correct answer is (D).

25. $T_r(P + Q^T + 2R) = 14$

Hence, the correct answer is (C).

26. $|A| = 1[1 - 0] = 1$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \text{adj}A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow U_1 = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$U_2 = A^{-1}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$$

$$U_3 = A^{-1}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \Rightarrow \det U = 3$$

Hence, the correct answer is (A).

27. $$U^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

Sum of the elements is 0.

Hence, the correct answer is (B).

28. $$\begin{bmatrix} 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ -8 \\ -5 \end{bmatrix} = 5$$

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1. A → (q); B → (r); C → (s); D → (s)

(A) $|M_r| = \frac{1}{r-1} - \frac{1}{r}$

$$\underset{n\to\infty}{Lt} \sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{1}{r}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$+ \ldots\ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - 0 = 1$$

(B) $1\left(1 - \cos^2\gamma\right) - \cos\alpha\left(\cos\alpha - \cos\beta\cos\gamma\right)$

$$+ \cos\beta\left(\cos\alpha\cos\gamma - \cos\beta\right) = 2\cos\alpha\cos\beta\cos\gamma$$

$$\Rightarrow 1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

(C) $K^2 = |c| = \left(\det A\right)^2 = 4^2 \Rightarrow K = 4$

(D) $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$$A^4 = A^2A^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$A^4 = -4I \Rightarrow \lambda = 4$$

2. A → (r, s); B → (p, q); C → (p, q); D → (p, q)

As $A^T = A$ Row vectors and column vectors taken in pairs are orthogonal $A.A^T = A^T.A = 9I \Rightarrow A^2 = 9I \Rightarrow \left(\frac{1}{3}A\right)^2 = I$

$\Rightarrow \frac{1}{3}A.\frac{1}{3}A^T = I \Rightarrow \frac{1}{3}A$ is orthogonal

$\frac{1}{3}A$ is involutory. $\Rightarrow B$ and B^T is also involutory matrix.

B is orthogonal and involutory

$\therefore B^T$ & B^{-1} are orthogonal

3. A → (p, q, t); B → (s); C → (p, r); D → (r)

Here 24 matrices are possible.

Values of determinants corresponding to these matrices are as follows :

$\begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2$(4 matrices), $\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4$(4 matrices),

$\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8$(4 matrices)

And 12 more matrices are there, value of whose determinants are – 2, – 4, – 8.

(A) Possible non-negative values of det.(A) are 2, 4, 8.

(B) Sum of these 24 determinants is 0.

(C) Mod. (det.(A)) is least $\therefore\ |A| = \pm 2$

$\Rightarrow |\text{adj(adj(adj.}(A)))| = |A|^{(n-1)^3} = \pm 2 = |A|$

(D) Least value of det.(A) is – 8.

Now, $|4A^{-1}| = 16\frac{1}{|A|} = \frac{16}{-8} = -2$

4. A → (q); B → (p); C → (s); D → (r)

(A) $|2A^{-1}| = 2^3 \cdot \frac{1}{|A|} = \frac{2^3}{2} = 4$

(B) $|\text{adj}(\text{adj}(2A))| = |2A|^4 = 2^{12} \cdot \left(\frac{1}{8}\right)^4 = 1$

(C) $AB + BA = 0 \Rightarrow AB = -BA$

$\Rightarrow |A|\ |B| = -|B|\ |A|$

$\Rightarrow 2|A|\ |B| = 0$

$\Rightarrow |B| = 0 \quad \because\ |A| \neq 0$

(D) Matrix multiplication of A, B and C is not possible

5. A → (r); B → (p); C → (s); D → (q)

$8 + a + b = k,\ 13 + e + f = k,\ 10 + c = k,\ 11 + d = k$

$\Rightarrow c = 9, d = 8, a + b = 6 + 5$ or $5 + 6, e + f = 2 + 4$ or $4 + 2.$

$k = 19,\ a + b + c = 20,$

$2ef = 16$

$c + d = 17$

6. A → (s); B → (p); C → (r); D → (q)

$A^2 = O, A^k = O \ \forall k \geq 2$

$(A+I)^{50} = {}^{50}C_0 I + {}^{50}C_1 A + {}^{50}C_2 A^2 + \ldots + {}^{50}C_{50} A^{50}$

$(A+I)^{50} = I + 50\,A$

$(A+I)^{50} - 50\,A = I$

$\Rightarrow \quad a = 1, b = 0, c = 0, d = 1$

Now verify matching between two columns

7. A → (r); B → (s); C → (q); D → (p)

(A) $(I+A)^8 = {}^8C_0 I + {}^8C_0 IA + {}^8C_2 IA^2 + \ldots + {}^8C_8 IA^8$

$= {}^8C_0 I + {}^8C_1 A + {}^8C_2 A + \ldots + {}^8C_8 A$

$= I + A({}^8C_1 + {}^8C_2 + \ldots + {}^8C_8)$

$= I + A(2^8 - 1) \Rightarrow \lambda = 2^8 - 1$

(B) $|\operatorname{adj}(A^{-1})| = |A^{-1}|^2 = \dfrac{1}{|A|^2}$

$|(\operatorname{adj}(A^{-1}))^{-1}| = \dfrac{1}{|adjA^{-1}|} = |A|^2 = 2^2 = 4$

(C) $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$\Rightarrow |B| = \begin{vmatrix} a_{11} & \lambda^{-1}a_{12} & \lambda^{-2}a_{13} \\ \lambda a_{21} & a_{22} & \lambda^{-1}a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix}$$

$$= \frac{1}{\lambda^3}\begin{vmatrix} \lambda^2 a_{11} & \lambda a_{12} & a_{13} \\ \lambda^2 a_{21} & \lambda a_{22} & a_{23} \\ \lambda^2 a_{31} & \lambda a_{32} & a_{33} \end{vmatrix} = |A|$$

Hence, $|A| = |B| \Rightarrow \lambda = 1$.

(D) A diagonal matrix is commutative with every square matrix, if it is a scalar matrix.
So every diagonal element is 4.
$\therefore \ |A| = 64$.

8. A → (q, s); B → (p, r); C → (r, s); D → (r)

(A) $AB = O \Rightarrow |A| \neq 0 \Rightarrow A^{-1}$ exist. $A^{-1}AB = 0 \Rightarrow IB = 0$
$\Rightarrow B = O$

(B) $AB = 0,\ |B| \neq 0 \Rightarrow B^{-1}$ exist
$ABB^{-1} = O \Rightarrow AI = O \Rightarrow A = O$

(C) $AB = O \Rightarrow |A|\,|B| = 0 \Rightarrow$ three cases

CASE-1: $|A| = 0,\ |B| \neq 0$ then B^{-1} exist
$ABB^{-1} = O \Rightarrow AI = O \Rightarrow A = O$ but A is non zero matrix, not possible

CASE-2: $|A| \neq 0\ |B| = 0$ then A^{-1} exist
$A^{-1}AB = O \Rightarrow IB = O \Rightarrow B = O$ but B is non zero matrix, not possible

CASE-3: $|A| = 0,\ |B| = 0,\ A^{-1}, B^{-1}$ do not exist, hence it is possible

(D) $A^n = 0 \Rightarrow |A^n| = 0 \Rightarrow |A| = 0$

9. A → (q); B → (s, t); C → (r); D → (p)

(A) $A^2 = -I \ \Rightarrow |A| \neq 0 \quad \therefore$ A is of even order

(B) $(I+A)^n = C_0 I^n + C_1 IA + C_2 IA^2 + \ldots + C_n IA^n$

$= C_0 I + C_1 A + C_2 A + \ldots + C_n A$

$= C_0 I + A(C_1 + C_2 + \ldots + C_n)$

$= C_0 I + A(2^n - 1)$

$\therefore \lambda = 2^n - 1$

(C) $A^2 = A$ and $B = I - A$

$AB + BA + I - (I + A^2 - 2A)$

$= AB + BA + A$

$= A\,(I - A) + (I - A)\,A + A$

$= A - A + A - A + A = A$

(D) $A^* = A,\ B^* = B$

$(AB - BA)^* = B^*A^* - A^*B^* = BA - AB$

Integer/Numerical Answer Type Questions

1. $x^2 + y^2 + z^2 = 2x + 2y + 2z - 3$

$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0 \Rightarrow x + y + z = 3$

Hence, the correct answer is (3).

2. We have five entries 1 and the remaining four entries are 0. Since the matrix is symmetric, we must have an even number of zeros for $i \neq j$.
Hence, we have two cases :

(i) Two entries in the diagonal are zero.
We can select two places from three (in diagonal) in ${}^3C_2$ ways.
Now we have to select elements for the upper triangle.
For the upper triangle, we have three places of which one entry is 0 and two entries are 1.
One place from three can be selected in ${}^3C_1$ ways.
Hence, the number of matrices is ${}^3C_2 \times {}^3C_1 = 9$.

(ii) If all the entries in the principal diagonal is 1, we have two 0 and one 1 in the upper triangle.
Hence, the number of matrices is 3.

Thus, total matrices = 12.

Hence, the correct answer is (6).

3. $$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & -\lambda-1 & 1+\lambda \\ 2 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 9\lambda + 5$$

$|A - \lambda I| = 0$ gives $\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$

$A^3 - 3A^2 - 9A - 5I = 0$

Hence, the correct answer is (5).

4. $|A| = 16 \qquad \because\ |\text{adj adj } A| = |A|^{(n-1)^2}$

det (adj (adj$A$)) $= 16^9 = 2^{36}$

$\therefore \quad \alpha = 2^{36}$

$\alpha = (7+1)^{12} = 7M + 1$

Hence, the correct answer is (1).

5. The integer $\gamma\gamma\gamma = \gamma(111) = \gamma.3.37$ with γ equal to the final digit of β. Now 37 must divide one of $(\alpha\beta)$ and $(\delta\beta)$ say $(\alpha\beta)$. Therefore $\alpha\beta = 37$ or 74.

$\alpha\beta = 74$ is not possible because $(\alpha\beta)(\delta\beta) \ge 74.14 = 1036$

$\Rightarrow \quad \alpha\beta = 37, \delta\beta = 27, \gamma = 9$

$\Rightarrow$ trace of matrix $A = 3 + 7 + 9 + 2 = 21$.

Hence, the correct answer is (21).

6. $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & p & 1 \\ 1 & 1 & p \end{vmatrix} = \begin{vmatrix} 0 & 1-p & 0 \\ 0 & p-1 & 1-p \\ 1 & 1 & p \end{vmatrix} = (1-p)^2$$

$$|\text{adj.}(AB)| = |AB|^2 = (P-1)^2$$

$$|B^2 \text{adj.}A| = |B|^2\,|A|^2 = |AB|^2$$

$\Rightarrow \quad (p-1)^2 = p^2 - 3 \Rightarrow p = 2$

Hence, the correct answer is (2).

7. $$|A| = 1, \text{adj } A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}^T \text{ and } A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\therefore \quad B_1 = [1\ 1\ 1]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = [0\ 1\ 1]$$

$$B_2 = [1\ 2\ 0]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = [1\ 2\ -1]$$

$$B_3 = [2\ 1\ 1]\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = [1\ 1\ 0]$$

So, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ sum of elements of $B = 6$.

Hence, the correct answer is (6).

8. $$|z| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 4 & 1 & 0 \end{vmatrix} \qquad (R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 - 2R_2)$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -5 & -10 \end{vmatrix} = -10 + 5 = -5$$

$$|B^{-1}| = \frac{|Z|}{|A|} \quad \Rightarrow \quad |B| = \frac{|A|}{|Z|} = \frac{7}{-5}$$

$$Z = A\frac{1}{|B|}\text{adj. } B \Rightarrow Z|B| = A(\text{adj.}B)$$

$\therefore \quad t = \text{trace (adj.}B.A) = \text{trace } (A.\text{ adj. } B)$

$$= \text{trace } (Z|B|) = |B|.\text{ trace } (Z) = \frac{-28}{5}$$

Hence, the correct answer is (5).

9. Let $\left(\dfrac{A}{3}\right) = B$

$$|\text{adj. } B^{-1}| = |B^{-1}|^2 = \frac{1}{|B|^2} = \frac{1}{\left|\dfrac{A}{3}\right|^2}$$

$$= \frac{3^6}{|A|^2} = \frac{3^6}{9^2} = 3^2 = 9$$

Hence, the correct answer is (9).

10. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}$

$$\text{So, det.}(BA) = \begin{vmatrix} b_1a_1 + b_2a_4 & b_1a_2 + b_2a_5 & b_1a_3 + b_2a_6 \\ b_3a_1 + b_4a_4 & b_3a_2 + b_4a_5 & b_3a_3 + b_4a_6 \\ b_5a_1 + b_6a_4 & b_5a_2 + b_6a_5 & b_5a_3 + b_6a_6 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 0 \end{vmatrix} \times \begin{vmatrix} b_1 & b_2 & 0 \\ b_3 & b_4 & 0 \\ b_5 & b_6 & 0 \end{vmatrix} = 0$$

(column by row)

Hence, the correct answer is (0).

11. $$|\text{adj } A| = \begin{vmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{vmatrix}$$

$$|A|^2 = (+6)\begin{vmatrix} 1 & 1 & -2 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 0 \\ -1 & -6 & -9 \\ -7 & 12 & 9 \end{vmatrix}$$

$$|A|^2 = 6[-54 + 108) = 324$$

$$|A| = -18$$

$$A \text{ adj. } A = -18\text{I}$$

$$\Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix} = \begin{bmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{bmatrix}$$

$$\Rightarrow \quad -3\sum a_{ij} = -3(18) \Rightarrow \frac{1}{3}\sum a_{ij} = 6$$

Hence, the correct answer is (6).

12. $$|\text{A}| = \frac{|I+B|}{|I-B|} = 1 \qquad (\because\ (I+B)^T = I - B)$$

$\therefore \quad$ det.$(2A)$ – det.(adj. A)

$$= 8|A| - |A|^2 = 8 - 1 = 7$$

Hence, the correct answer is (7).

13. $$\left(x_1x_2 - x_1x_4\right)^2 + \left(x_2x_3 - x_2x_5\right)^2 + \ldots + \left(x_5x_2 - x_5x_4\right)^2 \le 0$$

$\Rightarrow \quad x_1 = x_2 = x_3 = x_4 = x_5 = 3$

Hence $|A| = 3^5 = 243$.

Hence, the correct answer is (9).

14. adj(adj$A$) $= |A|^{n-2}$ A where $n = 3$ (here)

$$= |A|\ A$$

$$\text{det(adj(adj}A)) = |A|^3 \text{ det}A$$

$$= |A|^4 = 23^4$$

$\Rightarrow \quad |A| = 23$

Now $|A| = 3u + 11 = 23 \Rightarrow u = 4$.

Hence, the correct answer is (4).

15. $A\,.\,\text{adj } A = |A| I \Rightarrow |A| = 2$

$$\because \frac{|adj\,(adj\,A)|}{|adjA|} = \frac{|A|^{(n-1)^2}}{|A|^{n-1}} = |A|^{n-1} = 2^{3-1} = 2^2 = 4$$

Hence, the correct answer is (4).

16. $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $|B| = -1$; $B^2 = I$, hence $B^3 = B$, $B^4 = B^2 = I$

$$\sum_{n=1}^{100} B^n = 50B + 50I$$

$$= 50\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 50\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \ |C| = 1$$

also $C^2 = -I$ hence $C^3 = -C$, $C^4 = I$

$C^{4n+2} = -I$, $n = 0$ to 24 so sum $= -25I$

$C^{4n+3} = -C$, $n = 0$ to 24 so sum $= -25C$

$C^{4n+4} = I$, $n = 0$ to 24 so sum $= 25I$

$C^{4n+1} = C$, $n = 0$ to 24 so sum $= 25C$

sum of all terms of matrix $C = 0$

$$A = \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix} + 0; \ |A| = 0$$

Hence, the correct answer is (0).

17. $|M^{-1}| = \dfrac{1}{|M|} = -36$ and $M^{-1} = \dfrac{adj\,M}{|M|}$

$\Rightarrow$ Trace (adj. M) $= |M|\ \text{Tr}(M^{-1}) = \dfrac{-1}{36} \times 12$

$\therefore$ $|15 \text{ tr (adj. } M)| = 5$

Hence, the correct answer is (5).

18. adj. adj. $A = |A|^{n-2} A$

adj. adj. adj. $A = |\text{adj. } A|^{n-2}$ adj. A

$= (|A|^{n-1})^{n-2}$ adj.A

Put $n = 3$

$$\Rightarrow \quad |A|^2 \text{ adj } A = \begin{bmatrix} 16 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

taking determinant both side $\Rightarrow |A|^6\ |\text{adj. } A| = 256$

$|A|^6\ |A|^2 = 2^8$

$|A| = 2$

$$\text{adj. } A = \frac{1}{4}\begin{bmatrix} 16 & 0 & -3 \\ 0 & 4 & 0 \\ 0 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix}$$

$$A^{-1} = \frac{adj.A}{|A|} = \begin{bmatrix} 2 & 0 & -\frac{3}{8} \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

$\therefore$ $\text{Tr}(A^{-1}) = 3$

Hence, the correct answer is (3).

19. det. $(MM - M'M) = $ dt. $(M - M')$ .det.M

$\because$ $M - M'$ 3 order matrix

$\Rightarrow$ $M - M'$ is skew-symmetric matrix

$\Rightarrow$ det.$(M - M') = 0$

Hence, the correct answer is (0).

20. $|A| = \begin{bmatrix} a & 2 & 1 \\ 0 & b & 0 \\ 0 & -3 & c \end{bmatrix} = abc = 1$

where $x^3 - 3x^2 + 2x - 1$

$= (x - a)(x - b)(x - c)$ ...(1)

given $A^2 - B^2 = 4I - 4B$

$A^2 = B^2 - 4B + 4I$

$\Rightarrow$ $A^2 = (B - 2I)^2$ $\Rightarrow A^2 - (B - 2I)^2 = 0$

$(A + B - 2I)(A - B + 2I) = 0$

$\because$ $A + B - 2I \neq 0$

$\therefore$ $A - B + 2I = 0$

$$B = A + 2I = \begin{vmatrix} a+2 & 2 & 1 \\ 0 & b+2 & 0 \\ 0 & -3 & c+2 \end{vmatrix}$$

$|B| = (a + 2)(b + 2)(c + 2)$

put $x = -2$ in (1)

$-8 - 12 - 4 - 1 \equiv (-2 - a)(-2 - b)(-2 - c)$

$25 = (a + 2)(b + 2)(c + 2)$

$\therefore$ $|B| = 25$ $\therefore$ $\sqrt{|B|} = 5$

Hence, the correct answer is (5).

21. $$\frac{|adj\,B|}{|C|} = \frac{|adj(adjA)|}{|5A|} = \frac{|A|^{(3-1)^2}}{5^3|A|} = \frac{|A|^3}{125}$$

Now $|A| = 5$

$$\therefore \frac{|adj\,B|}{|C|} = 1$$

Hence, the correct answer is (1).

22. Adj. $P = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 1 \\ 4 & 7 & 3 \end{bmatrix}$

$|\text{adj. } P| = 4$

$\Rightarrow$ $|P|^2 = 4 \Rightarrow |P| = \pm 2$

Hence, the correct answer is (8).

23. $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -12 \\ 3 & -5 \end{bmatrix}$$

Observing A, A^2, A^3 we can conclude that

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$$

$$\det(A^n) = \begin{vmatrix} 1+2n & -4n \\ n & 1-2n \end{vmatrix} = 1-4n^2+4n^2 = 1$$

$\therefore$ $\det(A^{2005}) = 1$

Hence, the correct answer is (1).

24. Here $|A-\lambda| = 0$

$$\begin{vmatrix} 1-\lambda & 0 \\ -1 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(7-\lambda) = 0 \Rightarrow \lambda^2 - 8\lambda + 7 = 0$$

$\Rightarrow$ $A^2 - 8A + 7I_2 = 0 \Rightarrow A^2 = 8A - 7I_2$

$\Rightarrow$ $k = -7 \Rightarrow |k| = 7$

Hence, the correct answer is (7).

25. Since $\begin{bmatrix} 12 & \alpha^2 \\ -16 & -12 \end{bmatrix}$ is always nilpotent,

$$\begin{bmatrix} 12 & \alpha^2 \\ -16 & -12 \end{bmatrix}\begin{bmatrix} 12 & \alpha^2 \\ -16 & -12 \end{bmatrix} = 0$$

$$\alpha^2 = 9 \Rightarrow \alpha = 3$$

Hence, the correct answer is (3).

26. $n(s) = 5^9$

$$n(A) = 5^3$$

$$P(A) = \frac{5^3}{5^9} = \frac{1}{5^6} = \frac{1}{(P-1)^P} \Rightarrow P = 6$$

Hence, the correct answer is (6).

27. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Hence, the correct answer is (6).

28. Let $A = \frac{1}{2}$

$$\left|\frac{1}{81} adj\left(2\, adj(3A)\right)\right| = \left|\frac{1}{81} adj\left(2 \cdot 3^2\, adj\, A\right)\right|$$

$$= \left|\frac{1}{81} 18^2 adj\left(adj\, A\right)\right| = |4\, \text{adj}(\text{adj}\, A)| = 64\, |\text{adj}(\text{adj}A)|$$

$$= 64(\det A)^4 = 64\left(\frac{1}{2}\right)^4 = 64.\frac{1}{16} = 4$$

Hence, the correct answer is (4).

29. $A^2 = A$. Number of idempotent diagonal matrices = 16
Hence, the correct answer is (2).

30. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 16 \\ 0 & 1 \end{bmatrix}$$

If n is no of matrices that are multiplied then product is $\begin{bmatrix} 1 & n^2 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $r = 2500$

Hence, the correct answer is (5).

31. A can be $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$ $A^{10} = I_{4,}$ hence $n = 4$

Now $\sin^{-1} x + \cos^{-1} x^2 = \frac{\pi}{2} \Rightarrow x = x^2 \Rightarrow x = 0$ or $x = 1$

So answer is 2.
Hence, the correct answer is (2).

32. $|A - I| = |A - A'A|$

$$= |A|\ |I - A'| \quad \because |A' - I| = |A - I|$$
$$= -|A - I|$$

$\Rightarrow$ $|A - I| = 0$

so $1 + |A - I| = 1$

Hence, the correct answer is (1).

33. $|A.\ \text{Adj}\ (A)|$

$$= |A|\ |A|^2 = 125$$

Hence, the correct answer is (5).

34. Clearly $f'(x) = 0$

$\therefore$ $f(x) = -\frac{5}{4}$

$$\sum_{r=1}^{12}\left(3 + \frac{5}{4}\right) = \frac{17}{4} \times 12 = 51$$

Hence, the correct answer is (5).

35. $x + 5 = y, x + y = 11 \Rightarrow y - x = 5$

Hence, the correct answer is (5).

36. Here $|A| = 1$

$\Rightarrow$ $\text{Adj (AdjA)} = |\text{A}|^{3-2}.\text{A} = \text{A}$

trace $(A) = 3 - 3 + 1 = 1$

(Where trace of a matrix is the sum of the elements in the principal diagonal)
Hence, the correct answer is (1).

37. $\because \Delta = 1(1 - 3\sin\theta\cos\theta) - 3\cos\theta(\sin\theta - 3\cos\theta) + 1(\sin^2\theta - 1)$

$$= 1 - 6\sin\theta\cos\theta + 9\cos^2\theta + \sin^2\theta - 1$$
$$= (\sin\theta - 3\cos\theta)^2$$

$\because -\sqrt{10} \le \sin\theta - 3\cos\theta \le \sqrt{10}$

$\therefore 0 \le (\sin\theta - 3\cos\theta)^2 \le 10$

$\Rightarrow 0 \le \Delta \le 10$

$|\text{Maximum value of } \Delta - \text{Minimum value of } \Delta|^3$

$$= |10 - 0|^3 = 1000$$

$$\therefore \frac{|\text{Maximum value of } \Delta - \text{Minimum value of } \Delta|^3}{250}$$

$$=\frac{1000}{250}=4$$

Hence, the correct answer is (4).

38. For non-diagonal entries, we required even no. of 1, even no. of –1 and even no. of 0, for diagonal three entries are remained, –1, 0, 1. So no. of cases in which trace = 0 are 3! And no. of symmetric matrices for each arrangement of 1, –1, 0 in diagonal = 3!

Total such matrices = 3! × 3! = 36

Hence, the correct answer is (6).

39. $a_{ij}=0\ \forall\ i\neq j$ and $a_{ij}=(n-1)^2+i\ \forall\ i=j$

$$\text{Sum of all the element of } A_n=\sum_{i=1}^{2n-1}[(n-1)^2+i]$$

$$=(2n-1)\,(n-1)^2+(2n-1)n=2n^3-3n^2+3n-1$$

$$=n^3+(n-1)^3$$

$$\text{So, } T_n=(-1)^n\,[n^3+(n-1)^3]=(-1)^n\,n^3-(-1)^{n-1}\,(n-1)^3$$

$$=V_n-V_{n-1}$$

$$\Rightarrow\ \sum_{n=1}^{102}T_n=\sum_{n=1}^{102}(V_n-V_{n-1})=V_{102}-V_0=(102)^3$$

$$\left[\frac{\sum\limits_{n=1}^{102}T_n}{520200}\right]=2.$$

Hence, the correct answer is (2).

40. $$PP^T=\begin{bmatrix}a&b&c\\c&a&b\\b&c&a\end{bmatrix}\begin{bmatrix}a&c&b\\b&a&c\\c&b&a\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow\ \begin{bmatrix}a^2+b^2+c^2&ab+bc+ca&ab+bc+ca\\ab+bc+ca&a^2+b^2+c^2&ab+bc+ca\\ab+bc+ca&ab+bc+ca&a^2+b^2+c^2\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$(a+b+c)^2=1+2\times 0$$

$$(a+b+c)^2=1$$

$$|a+b+c|=1$$

Hence, the correct answer is (1).

41. $$I+A\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&2\\1&1\end{pmatrix}=\begin{pmatrix}2&2\\1&2\end{pmatrix}\text{ and}$$

$$I-A=\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&2\\1&1\end{pmatrix}=\begin{pmatrix}0&-2\\-1&0\end{pmatrix}$$

$$\text{Now, } |I-A|=\begin{vmatrix}0&-2\\-1&0\end{vmatrix}=0-2=-2$$

$$\text{adj }(I-A)=\begin{pmatrix}0&2\\1&0\end{pmatrix};\ (I-A)^{-1}=\begin{pmatrix}0&1\\-\frac{1}{2}&0\end{pmatrix}$$

$$\therefore\ \phi(A)=(I+A)(I-A)^{-1}=\begin{pmatrix}2&2\\1&2\end{pmatrix}\begin{pmatrix}0&-1\\-\frac{1}{2}&0\end{pmatrix}$$

$$=\begin{pmatrix}-1&-2\\-1&-1\end{pmatrix}=-A;\ k=1$$

Hence, the correct answer is (1).

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1. $$AA^T=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{bmatrix}a_{11}&a_{21}&a_{31}\\a_{12}&a_{22}&a_{32}\\a_{13}&a_{23}&a_{33}\end{bmatrix}$$

$$=\begin{bmatrix}a^2{}_{11}+a^2{}_{12}+a^2{}_{13}&-&-\\-&a^2{}_{21}+a^2{}_{22}+a^2{}_{23}&-\\-&-&a^2{}_{31}+a^2{}_{32}+a^2{}_{33}\end{bmatrix}$$

$$\text{Trace }(AA^T)=\sum a_{ij}^2=a^2{}_{11}+a^2{}_{12}+a^2{}_{13}+a^2{}_{21}+a^2{}_{22}+a^2{}_{23}$$

$$+a^2{}_{31}+a^2{}_{32}+a^2{}_{33}=3$$

Any three elements are –1 or 1 rest 6 elements are zero

Hence, number of such matrices $= {}^9C_3\times 2^3=672.00$

Hence, the correct answer is (672).

2. $$A^2=\begin{bmatrix}x&1\\1&0\end{bmatrix}\begin{bmatrix}x&1\\1&0\end{bmatrix}=\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}$$

$$A^4=\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}$$

$$=\begin{bmatrix}(x^2+1)^2+x^2&x(x^2+1)+x\\x(x^2+1)+x&x^2+1\end{bmatrix}$$

$$a_{11}=(x^2+1)^2+x^2=109\Rightarrow(x^2-9)(x^2+12)=0$$

$$\Rightarrow\ x=\pm 3$$

$$a_{22}=x^2+1=10$$

Hence, the correct answer is (10).

3. $x^2+x+1=0$ has roots as $\alpha=\omega,\ \alpha^2=\omega^2$

$$A=\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow A^4 = A^2 . A^2 = I_3$$

Now $A^{31} = A^{28}. A^3 = A^3$

Hence, the correct answer is (D).

4. $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$; $|A| = 8 - 18 = -10$

$$A^{-1} = \frac{adjA}{|A|} = \frac{\begin{pmatrix} 4 & -2 \\ -9 & 2 \end{pmatrix}}{-10} \Rightarrow 10A^{-1} = \begin{pmatrix} -4 & 2 \\ 9 & -2 \end{pmatrix} = A - 6I$$

Hence, the correct answer is (A).

5. $b_{ij} = (3)^{(i+j-2)} a_{ij}$

$$B = \begin{bmatrix} a_{11} & 3a_{12} & 3^2 a_{13} \\ 3a_{21} & 3^2 a_{22} & 3^3 a_{23} \\ 3^2 a_{31} & 3^3 a_{32} & 3^4 a_{33} \end{bmatrix}$$

$$\Rightarrow |B| = 3 \times 3^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3^2 a_{31} & 3^2 a_{32} & 3^2 a_{33} \end{vmatrix} = 3^6 |A|$$

$$\Rightarrow |A| = \frac{81}{27 \times 27} = \frac{1}{9}$$

Hence, the correct answer is (D).

6. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix} \Rightarrow |A| = 6$

$$\frac{|adjB|}{|c|} = \frac{|adj(adjA)|}{|3A|} = \frac{|A|^4}{3^3 |A|} = \frac{|A|^3}{3^3} = \frac{(6)^3}{(3)^3} = 8$$

Hence, the correct answer is (A).

7. $A^T A = I \Rightarrow \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

$$= \begin{pmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$

Now, $(a + b + c)^2 = 1 \quad \Rightarrow \quad a + b + c = \pm 1$

So, $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \pm 1(1 - 0) = \pm 1$

$\Rightarrow \quad 3\,abc = 2 \pm 1 = 3, 1$

$\Rightarrow \quad abc = 1, \dfrac{1}{3}$

Hence, the correct answer is (D).

8. Given $P = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & -4 \\ 1 & 9 & -1 \end{bmatrix}$, Here $|P| = 0$ and also given $PX = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & -4 \\ 1 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} x + 2y + z = 0 \\ -2x + 3y - 4z = 0 \\ x + 9y - z = 0 \end{aligned}\right\}$$

$D = 0$, so system have infinite many solutions,

By solving these equation

we get $x = \dfrac{-11\lambda}{2}, y = \lambda, z = \dfrac{7\lambda}{2}$

Also given, $x^2 + y^2 + z^2 = 1 \Rightarrow \lambda = \pm \dfrac{1}{\sqrt{\dfrac{121}{4} + 1 + \dfrac{49}{4}}} = \pm\sqrt{\dfrac{2}{87}}$

so, there are 2 values of λ.

$\therefore$ so, there are 2 solution sets of (x, y, z).

Hence, the correct answer is (B).

9. $|A| \neq 0$

For $(P): A \neq I_2$

So, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$|A|$ can be -1 or 1 So (P) is false.

For (Q); $|A| = 1$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad \text{tr}(A) = 2$ so Q is true

Hence, the correct answer is (D).

10. Let $A = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$AX_3 = b_3 \Rightarrow \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \ell_3 \\ m_3 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow l_3 = 0, m_3 = 0, n_3 = 2;$$

$$AX_2 = b_2 \Rightarrow \begin{bmatrix} \ell_1 & \ell_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\ell_2 \\ 2m_2 \\ 2n_2 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow l_2 = 0, m_2 = 1, n_2 = -1$$

$$\text{Now } AX_3 = b_3 \Rightarrow \begin{bmatrix} \ell_1 & 0 & 0 \\ m_1 & 1 & 0 \\ n_1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \ell_1 = 1, m_1 = -1, n_1 = -1$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 2$$

Hence, the correct answer is (D).

11. $C = \text{adj } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{bmatrix}$

$|C| = |\text{adj } A| = 9 \Rightarrow |\text{adj } A| = |A|^2 = 9$

$\lambda = |A| = \pm 3 \Rightarrow |\lambda| = 3$

$B = \text{adj } C \Rightarrow |B| = |\text{adj } C| = |C|^2 = 81$

$\mu = |(B^{-1})^T| = |B|^{-1} = \dfrac{1}{81}$

We get $(|\lambda|, \mu) = \left(3, \dfrac{1}{81}\right)$

Hence, the correct answer is (C).

12. $A = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, A^2 = \begin{bmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{bmatrix}$

Similarly, $A^5 = \begin{pmatrix} \cos 5\theta & i\sin 5\theta \\ i\sin 5\theta & \cos 5\theta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(A) $a^2 + b^2 = \cos^2 5\theta - \sin^2 5\theta = \cos 10\theta = \cos 75°$

hence $0 \le a^2 + b^2 \le 1$ True

(B) $a^2 - d^2 = \cos^2 5\theta - \cos^2 5\theta = 0$ True

(C) $a^2 - b^2 = \cos^2 5\theta + \sin^2 5\theta = 1$ False

(D) $a^2 - c^2 = \cos^2 5\theta + \sin^2 5\theta = 1$ True

Hence, the correct answer is (C).

13. $A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

$B = A + A^4 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{bmatrix}$

$= \begin{bmatrix} (\cos\theta + \cos 4\theta) & (\sin\theta + \sin 4\theta) \\ -(\sin\theta + \sin 4\theta) & (\cos\theta + \cos 4\theta) \end{bmatrix}$

$|B| = (\cos\theta + \cos 4\theta)^2 + (\sin\theta + \sin 4\theta)^2$

$|B| = 2 + 2\cos 3\theta$, when $\theta = \dfrac{\pi}{5}$

$|B| = 2 + 2\cos\dfrac{3\pi}{5} = 2(1 - \sin 18^\circ) = 2\left(1 - \dfrac{\sqrt{5} - 1}{4}\right)$

$= 2\left(\dfrac{5 - \sqrt{5}}{4}\right) = \dfrac{5 - \sqrt{5}}{2}$

Hence, the correct answer is (B).

14. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}; A^{50} = \begin{bmatrix} \cos 50\theta & -\sin 50\theta \\ \sin 50\theta & \cos 50\theta \end{bmatrix}$

$(A^{-1})^{50} = \begin{bmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{bmatrix}$

At $\theta = \dfrac{\pi}{12}; (A^{-1})^{50} = \begin{bmatrix} \cos\dfrac{50\pi}{12} & \sin\dfrac{50\pi}{12} \\ -\sin\dfrac{50\pi}{12} & \cos\dfrac{50\pi}{12} \end{bmatrix}$

$= \begin{bmatrix} \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix}$

Hence, the correct answer is (C).

15. $A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$

$|A| = e^t \,.\, e^{-t} \,.\, e^{-t} \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t - \sin t & -\sin t + \cos t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$

Apply operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$|A| = e^{-t} \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t - 2\cos t & -2\sin t + \cos t \\ 0 & 2\sin t - \cos t & -2\cos t - \sin t \end{vmatrix}$

Open the determinant along C_1

$|A| = 5e^{-t}$ Invertible for all $t \in R$

Hence, the correct answer is (B).

16. $R_3 \to R_3 + R_1 - 2R_2$

$\det(A) = \begin{vmatrix} -2 & 4 + d & \sin\theta - 2 \\ 1 & 2 + \sin\theta & d \\ 1 & 0 & 0 \end{vmatrix} = d^2 + 4d + 4 - \sin^2\theta$

$\det(A) = (d + 2)^2 - \sin^2\theta$

min of $\det(A) = 8 = (d + 2)^2 - 1$

$(d + 2)^2 = 9 \Rightarrow d + 2 = \pm 3$ We get $d = 1, -5$

Hence, the correct answer is (C).

17. $\det(A) = 2(2b^2 + 2 - b^2) - b(2b - b) + 1(b^2 - b^2 - 1)$

$= 2(b^2 + 2) - b^2 - 1 = b^2 + 3$

$\dfrac{\det(A)}{b} = \dfrac{b^2 + 3}{b} = b + \dfrac{3}{b}$

Minimum value $= 2\sqrt{3}$

Hence, the correct answer is (D).

18. $AA^T = I_3$

$$\begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix} \begin{pmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4q^2 + r^2 & 2q^2 - r^2 & -2q^2 + r^2 \\ 2q^2 - r^2 & p^2 + q^2 + r^2 & p^2 - q^2 - r^2 \\ -2q^2 + r^2 & p^2 - q^2 - r^2 & p^2 + q^2 + r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compare

$p^2 + q^2 + r^2 = 1; 4q^2 + r^2 = 1; 2q^2 = r^2$

$$r^2 = 2p^2 \qquad p^2 = q^2 + r^2 p^2 = \frac{r^2}{2} + r^2 = \frac{3r^2}{2}$$

$$p^2 = \frac{3r^2}{2} = 1$$

$$p^2 + p^2 = 1 \quad \Rightarrow \quad |p| = \frac{1}{\sqrt{2}}$$

Hence, the correct answer is (A).

19. $|A|\,|B|\,|A^T| = 8$

$$|A|^2\,|B| = 8 \qquad \ldots(1)$$

$$|A|\,|B^{-1}| = 8$$

$$|A| = 8|B| \qquad \ldots(2)$$

From (1) and (2)

$$64|B|^2\,|B| = 8 \Rightarrow |B|^3 = \frac{1}{8} \Rightarrow |B| = \frac{1}{2} \text{ and } |A| = 4$$

$$\det(BA^{-1}B^T) = |B|\,\frac{1}{|A|}\,|\mathrm{B}| = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Hence, the correct answer is (C).

20. $$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 3 & 1 \end{bmatrix}$$

Similarly $P^5 = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 135 & 15 & 1 \end{bmatrix}$

$a_{21} = 15, a_{31} = 135, a_{32} = 15$

Now, $\dfrac{q_{21} + q_{31}}{q_{32}} = \dfrac{15 + 135}{15} = 10$

Hence, the correct answer is (D).

21. $|A| = 2(1 + \sin^2\theta)$; $\theta \in \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right)$ then $\sin^2\theta \in \left[0, \dfrac{1}{2}\right)$

$$|A| = 2(1 + \sin^2\theta) \in [2, 3)$$

Hence, the correct answer is (A).

22. $$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Similarly

$$A^{32} = \begin{bmatrix} \cos 32\alpha & -\sin 32\alpha \\ \sin 32\alpha & \cos 32\alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$32\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{64}$$

Hence, the correct answer is (A).

23. $|A| = (2 - b)(b - c)(c - 2)$

2, b, c, are in A.P. we have $b = \dfrac{2 + c}{2}$

$$\text{Det }(A) = \frac{1}{4}(c - 2)^3$$

$$2 \le \frac{1}{4}(c - 2)^3 \le 16$$

$$\Rightarrow \quad 8 \le (c - 2)^3 \le 64 \Rightarrow 2 \le c - 2 \le 4 \Rightarrow 4 \le c \le 6$$

Hence, the correct answer is (B).

24. $$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 + 2 + 3 + \ldots . + n - 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 1 + 2 + 3 + \ldots + (n - 1) = 78 \Rightarrow \frac{n(n-1)}{2} = 78 \Rightarrow n = 13$$

Now inverse of $\begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

Hence, the correct answer is (D).

25. $(A^T)(A) = 3I_3$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow \quad 8x^2 = 3 \Rightarrow x = \pm\sqrt{\frac{3}{8}} \text{ and } 6y^2 = 3 \Rightarrow y = \pm\sqrt{\frac{1}{2}}$$

4 materices are possible
Hence, the correct answer is (B).

26. A is symmetric then $A^T = A$; B is skew 8 symmetry then $B^T = -B$

$$A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \qquad \ldots(1)$$

Take Transpose we get $A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} = A - B$ …(2)

From (1) + (2), we get $A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$

From (1) – (2), we get $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

Hence, the correct answer is (D).

27. If B is inverse of A then $AB = \mathrm{I}$

$$\det(AB) = \det(I)$$

$$\det(A).\det(B) = 1$$

Given det $(A) = -1$ then det $(B) = -1$

$$\begin{vmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{vmatrix} = -1 \text{ or } (\alpha - 4)(\alpha + 3) = 0;\ \alpha = 4, -3$$

Sum of value of $\alpha = 4 - 3 = 1$
Hence, the correct answer is (D).

28. We have $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

$$\therefore 3A^2+12A=\begin{bmatrix}48 & -27\\ -36 & 39\end{bmatrix}+\begin{bmatrix}24 & -36\\ -48 & 12\end{bmatrix}=\begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$$

$$\text{Adj }(3A^2+12A)=\begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$$

Hence, the correct answer is (C).

29. $A(\text{adj } A)=A\,A^T$

$$\Rightarrow \quad A^{-1}A\,(\text{adj } A)=A^{-1}\,A\,A^T$$

$$\text{Adj } A=A^T$$

$$\Rightarrow \quad \begin{bmatrix}2 & b\\ -3 & 5a\end{bmatrix}=\begin{bmatrix}5a & 3\\ -b & 2\end{bmatrix}$$

$$\Rightarrow \quad a=\frac{2}{5}=\text{and } b=3$$

$$\Rightarrow \quad 5a+b=5$$

Hence, the correct answer is (C).

30. $$\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & 2\\ 2 & -2 & b\end{bmatrix}=\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1+4+4 & 2+2-4 & a+4+2b\\ 2+2-4 & 4+1+4 & 2a+2-2b\\ a+4+2b & 2a+2-2b & a^2+4+b^2\end{bmatrix}=\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$\Rightarrow \quad a+4+2b=0 \Rightarrow a+2\,b=-4 \qquad \ldots(1)$$

$$2a+2-2b=0 \Rightarrow 2a-2b=-2$$

$$\Rightarrow \quad a-b=-1 \qquad \ldots(2)$$

On solving (1) and (2) we get

$$-1+b+2b=-4$$

$$b=-1 \text{ and } a=-2$$

$$(a, b)=(-2, -1)$$

Hence, the correct answer is (B).

31. $BB'=B(A^{-1}A')'=B(A')'(A^{-1})'=BA(A^{-1})'$

$$=(A^{-1}A')(A(A^{-1})')$$

$$=A^{-1}A.\,A'.(A^{-1})' \;\{\text{as } AA'=A'A\}$$

$$=I(A^{-1}A)'$$

$$=I.I=I^2=I$$

Hence, the correct answer is (B).

32. $|P|=1(12-12)-\alpha(4-6)+3(4-6)=2\alpha-6$

Now, adj $A=P \qquad \Rightarrow |\text{adj } A|=|P|$

$$\Rightarrow \quad |A|^2=|P| \qquad \Rightarrow |A|^2=16$$

$$\Rightarrow \quad 2\alpha-6=16 \qquad \Rightarrow \alpha=11$$

Hence, the correct answer is (B).

33. Let $Au_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $Au_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$ then,

$$Au_1+Au_2=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow \quad A(u_1+u_2)=\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \ldots(1)$$

Also $A=\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix} \Rightarrow |A|=1$

We know, $A^{-1}=\dfrac{1}{|A|}adjA \Rightarrow A^{-1}=adj(A)\;\left(\because |A|=1\right)$

Now, from equation (1), we have $u_1+u_2=A^{-1}\begin{pmatrix}1\\1\\0\end{pmatrix}$

$$\Rightarrow \quad u_1+u_2=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 1 & -2 & 0\end{bmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

Hence, the correct answer is (D).

34. $\therefore \quad A'=A,\; B'=B$

Now $(A(BA))'=(BA)\,'A'=(A\,'B')A'=(AB)A=A(BA)$

Similarly $((AB)A)'=(AB)A$

So, $A(BA)$ and $(AB)A$ are symmetric matrices.

Again $(AB)'=B'A'=BA$

Now if $BA=AB$, then AB is symmetric matrix.

Hence, the correct answer is (A).

35. $\begin{bmatrix}1 & \ldots & \ldots\\ \ldots & 1 & \ldots\\ \ldots & \ldots & 1\end{bmatrix}$ are 6 non-singular matrices because 6

Blanks will be filled by 5 zeros and 1 one.

Similarly, $\begin{bmatrix}\ldots & \ldots & 1\\ \ldots & 1 & \ldots\\ 1 & \ldots & \ldots\end{bmatrix}$ are 6 non-singular matrices.

So, required cases are more than 7, non-singular 3 × 3 matrices.

Hence, the correct answer is (D).

36. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ where $a, b, c, d \neq 0$

$$A^2=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow A^2=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}$$

$$\Rightarrow \quad a^2+bc=1,\; bc+d^2=1$$

$$ab+bd=ac+cd=0$$

$$c\neq 0 \text{ and } b^1\,0 \Rightarrow a+d=0 \Rightarrow Tr(A)=0$$

$$|A|=ad-bc=-a^2-bc=-1$$

Hence, the correct answer is (B).

37. We know that $|\text{adj }(\text{adj } A)|=|\text{adj } A|^{2-1}=|A|^{2-1}=|A|$

∴ Both the statements are true and statement-2 is a correct explanation for statement-1.

Hence, the correct answer is (C).

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Single Correct Choice Type Problems

1. $$M=\alpha\begin{bmatrix}1&0\\0&1\end{bmatrix}+\frac{\beta}{|M|}\begin{bmatrix}\cos^4\theta & 1+\sin^2\theta\\ -1-\cos^2\theta & \sin^4\theta\end{bmatrix}$$

on comparing we have

$$-1-\sin^2\theta=\frac{\beta(1+\sin^2\theta)}{|M|}\Rightarrow\beta=-|M|$$

$$\sin^4\theta=\alpha+\frac{\beta\cos^4\theta}{|M|}\Rightarrow\alpha=\sin^4\theta+\cos^4\theta$$

Now $\alpha=(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta=1-\dfrac{\sin^2 2\theta}{2}$

$\Rightarrow\alpha^*=\dfrac{1}{2}$

We have,

$$|M|=\sin^4\theta\cos^4\theta+(1+\sin^2\theta)(1+\cos^2\theta)$$
$$=2+\sin^2\theta\cos^2\theta+\sin^4\theta\cos^4\theta$$
$$=(\sin^2\theta\cos^2\theta+1/2)^2+7/4$$

$$\Rightarrow\beta=-|M|=-\frac{7}{4}-\left(\frac{\sin^2 2\theta}{4}+\frac{1}{2}\right)^2$$

$$\Rightarrow\beta^*=-\frac{7}{4}-\left(\frac{1}{4}+\frac{1}{2}\right)^2=-\frac{7}{4}-\frac{9}{16}=-\frac{37}{16}$$

$$\alpha^*+\beta^*=-\frac{29}{16}.$$

Hence, the correct answer is (C).

2. Let matrix $M=\begin{bmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{bmatrix}$

Now $M^TM=\begin{bmatrix}m_{11}&m_{21}&m_{31}\\m_{12}&m_{22}&m_{32}\\m_{13}&m_{23}&m_{33}\end{bmatrix}\begin{bmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{bmatrix}$

$$=\begin{bmatrix}{m_{11}}^2+{m_{21}}^2+{m_{31}}^2 & \dots & \dots\\ \dots & {m_{12}}^2+{m_{22}}^2+{m_{32}}^2 & \dots\\ \dots & \dots & {m_{13}}^2+{m_{23}}^2+{m_{33}}^2\end{bmatrix}$$

Then sum of diagonal entries $=\sum {m_{ij}}^2=5$

where m_{ij} are from {0, 1, 2}

Total number of matrices

= no. of matrices with 5 one and 4 zeroes + no. of matrices with 1 one ,1 two and 7 zeroes

$$={}^9C_5+{}^9C_1.{}^8C_1=198$$

Hence, the correct answer is (B).

3. $$P=\begin{bmatrix}1&0&0\\4&1&0\\16&4&1\end{bmatrix}\Rightarrow P^2=\begin{bmatrix}1&0&0\\4.2&1&0\\16(1+2)&4.2&1\end{bmatrix},$$

$$P^3=\begin{bmatrix}1&0&0\\4.3&1&0\\16(1+2+3)&4.3&1\end{bmatrix}$$

$$P^{50}=\begin{bmatrix}1&0&0\\4.50&1&0\\16(1+2+\ldots+50)&4.50&1\end{bmatrix}=\begin{bmatrix}1&0&0\\200&1&0\\20400&200&1\end{bmatrix}$$

$\because P^{50}-Q=I$

$$\Rightarrow\begin{bmatrix}1&0&0\\200&1&0\\20400&200&1\end{bmatrix}-\begin{bmatrix}q_{11}&q_{12}&q_{13}\\q_{21}&q_{22}&q_{23}\\q_{31}&q_{32}&q_{33}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$q_{21}=200$; $q_{31}=20400$ and $q_{32}=200$

$$\therefore\ \frac{q_{31}+q_{32}}{q_{21}}=\frac{20400+200}{200}=103$$

Hence, the correct answer is (B).

4. We have $|Q|=\begin{vmatrix}2^2a_{11}&2^3a_{12}&2^4a_{13}\\2^3a_{21}&2^4a_{22}&2^5a_{23}\\2^4a_{31}&2^5a_{32}&2^6a_{33}\end{vmatrix}$

$$=2^2.2^3.2^4\begin{vmatrix}a_{11}&a_{12}&a_{13}\\2a_{21}&2a_{22}&2a_{23}\\2^2a_{31}&2^2a_{32}&2^2a_{33}\end{vmatrix}$$

$$=2^9.2.2^2\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=2^{12}\times|P|=2^{12}\times2=2^{13}$$

Hence, the correct answer is (D).

5. We have $P^T=2P+I\Rightarrow P=2P^T+I\Rightarrow P=2(2P+I)+I$

$\Rightarrow\quad P=4P+3I\Rightarrow P+I=0\Rightarrow PX=X=0\Rightarrow PX=-X$

Hence, the correct answer is (D).

6. Given $MN=NM$

$$M^2N^2(M^TN)^{-1}(MN^{-1})^T$$
$$=M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T.M^T$$
$$=M^2N.(M^T)^{-1}(N^{-1})^TM^T$$
$$=-M^2N(M)^{-1}(N^T)^{-1}M^T$$
$$=+M^2NM^{-1}N^{-1}M^T$$
$$=-MNMM^{-1}N^{-1}M\qquad(\because MN=NM)$$
$$=-MNN^{-1}M=-M^2$$

Hence, the correct answer is (C).

7. For being non-singular,

$$\begin{vmatrix}1&a&b\\\omega&1&c\\\omega^2&\omega&1\end{vmatrix}\neq0$$

or $\quad ac\omega^2-(a+c)\omega+1\neq0\quad\Rightarrow\quad(1-c\omega)(1+a\omega)\neq0$

$\Rightarrow\quad a\neq\dfrac{1}{\omega},\ c\neq\dfrac{1}{\omega}$

Hence, number of possible triplets of (a, b, c) is 2.

i.e., (ω,ω^2,ω) and (ω,ω,ω).

Hence, the correct answer is (A).

Multiple Correct Choice Type Problems

1. $M.\text{adj}(M) = |M|I$ replace M by $\text{adj}(M)$

$\text{adj}(M).\text{adj}(\text{adj}(M)) = |\text{adj}(M)|I$

$(\because\ |\text{adj}(M)| = |M|^{n-1} = |M|^2)$

$M\text{adj}(M).\text{adj}(\text{adj}(M)) = |M|^2 MI$

$|M|\ I\ .\ \text{adj}(\text{adj}(M)) = |M|^2M$

$\text{adj}(\text{adj}(M)) = |M|M = M^{-1}$

[using given $M^{-1} = \text{adj}\ (\text{adj}\ M)$]

$|M|MM = M^{-1}M \Rightarrow |M|\ M^2 = I$

$||M|\ M^2| = |I| \Rightarrow |M|^3|M^2| = 1$ we get $|M| = \det M = 1$ option B is correct

now $|M|\ M^2 = I$ we get $M^2 = I$ option C is correct

we know $\dfrac{adj(M)}{|M|} = M^{-1}$ and $|M|M = M^{-1} \Rightarrow M = M^{-1}$

$\Rightarrow (\text{adj}(M))^2 = (M^{-1})^2 = M^2 = I$ option D is also correct

Hence, (B), (C) and (D) are correct.

2. $M\ \text{adj}\ M = |M|\ I$

$$\Rightarrow \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-5a & -6+3a & 2-a \\ 0 & -2 & 0 \\ -8+8b & 6-6b & -4+2b \end{bmatrix} = |M|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow\quad a = 2, b = 1$ and $|M| = -2$, now $a + b = 3$

$$M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$|\text{adj}\ M^2| = |M^2|^2 = |M|^4 = 16$

$$(\text{adj}M)^{-1} + (\text{adj}M^{-1}) = (|M|M^{-1})^{-1} + |M^{-1}|M$$

$$= \frac{M}{|M|} + \frac{M}{|M|} = \frac{2M}{|M|} = -M$$

$$M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

We get $\alpha = 1, \beta = -1, \gamma = 1 \Rightarrow \alpha - \beta + \gamma = 3$

Hence, (A), (B) and (D) are correct.

3. $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$

$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$X = \sum P_k \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_k^T$$

Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ and $A = A^T$.

$$X^T = \left(P_1AP_1^T + P_2AP_2^T + \ldots + P_6AP_6^T\right)^T = X$$

So X is symmetric matrix

Let $Q = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; AQ = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$$XQ = P_1AP_1{}^TQ + P_2AP_2{}^TQ + \ldots + P_6AP_6{}^TQ$$

$$= P_1AQ + P_2AQ + \ldots + P_6AQ$$

$$XQ = (P_1 + P_2 + \ldots + P_6)AQ = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$$

$$= 30Q, \alpha = 30$$

$XQ = 30Q \Rightarrow (X - 30I)\ Q = 0$

So, $|X - 30I| = 0$, has non-trivial solution so $X - 30I$ not invertible.

trace $(P_k\ A\ P'_k) = 3$

Trace $X = 3 \times 6 = 18$

Hence, (A), (B) and (C) are correct.

4. (A) $PQ = QP$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix} = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

If we equate a_{12} from both

$x + 4 + x = 2 + 2x \Rightarrow 4 = 2 \quad\Rightarrow\quad x \in \phi$, no value exists.

(B) $R = PQP^{-1}$

$\det(R) = \det(PQP^{-1})$

$|R| = |PQP^{-1}| = |P| \cdot |Q| \cdot |P^{-1}|$

$$= |P| \cdot |Q| \cdot \frac{1}{|P|} = |Q|$$

$$\det(Q) = \begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{vmatrix} = 4\begin{vmatrix} 2 & x \\ x & 6 \end{vmatrix} = 48 - 4x^2.$$

$\det(R) = \det(Q) = 48 - 4x^2$

$$\det\begin{vmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{vmatrix} + 8 = 4\begin{vmatrix} 2 & x \\ x & 5 \end{vmatrix} + 8 = 48 - 4x^2.$$

(C) For $x = 0$

$$PQ = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{bmatrix}$$

$$p^{-1} = \frac{1}{|P|}(\text{Adj}P) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$R = PQP^{-1} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 12 \\ 0 & 0 & 18 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 12 & 6 & 4 \\ 0 & 24 & 8 \\ 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$$

Given, $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \Rightarrow (R - 6I)\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} -4 & 1 & \frac{2}{3} \\ 0 & -2 & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 0$$

$$\Rightarrow -4 + a + \frac{2}{3}b = 0 \ \ \& -2a + \frac{4}{3}b = 0$$

Solving we get $a = 2$, $b = 3$ so $a + b = 5$.

(D) For $x = 1$

$\det(R) = 48 - 4x^2 = 48 - 4 = 44 \neq 0$

$$R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = \gamma = 0$$

Hence, $\alpha\hat{i} + \beta\hat{j} + r\hat{k}$ cannot a unit vector. It is a null vector

Hence, (B) and (C) are correct.

5. $$\Delta = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 0$$

Given equation represent family of planes

$\Rightarrow$ $2x - 4y + 3z - b_2 + \lambda(-x + 2y + 5z - b_1) = 0$ is same as $x - 2y + 2z - b_3 = 0$ for same λ

$$\frac{2-\lambda}{1} = \frac{2\lambda - 4}{-2} = \frac{5\lambda + 3}{2} = \frac{-(b_1\lambda + b_2)}{-b_3}$$

$$\lambda = \frac{1}{7} \text{ and } 13b_3 = b_1 + 7b_2$$

(A) $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 2 & 6 \end{vmatrix} \neq 0$ unique solution exists

(B) $\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ 2 & 1 & 3 \end{vmatrix} = 0$ as planes are non parallel is must represent family of planes for solution to exist

$P_1 + \lambda P_2 = P_3$

$(5\mu + 1)x + (2\mu + 1)y + (6\mu + 3)z = b_1 + \mu b_2$

Must be third plane $2x + y + 3z + b_3 = 0$

which gives $\mu = 1, b_1 + b_2 + 3b_3 = 0$ which is not true always

(C) for consistent system, given planes must be coincident planes when $-b_1 = \frac{b_2}{2} = b_3$, not true

(D) $\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} \neq 0$ unique solution exists

Hence, (A) and (D) are correct.

6. Matrices written in options are square of some matrices as know that square of any real number is always non negative, If they are square of an real matrices then its determinant can not be negative.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1; \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1; \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1;$$

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$$

Hence, (B) and (D) are correct.

7. As $PQ = kI \Rightarrow Q = kP^{-1}I = kP^{-1}$

now $Q = \frac{k}{|P|}(adjP)$

$$\Rightarrow \quad Q = \frac{k}{(20 + 12\alpha)}\begin{bmatrix} - & - & - \\ - & - & (-3\alpha - 4) \\ - & - & - \end{bmatrix}$$

$$\because \quad q_{23} = \frac{-k}{8} \quad \Rightarrow \quad \frac{k}{(20 + 12\alpha)}(-3\alpha - 4) = \frac{-k}{8}$$

$$\Rightarrow \quad 2(3\alpha + 4) = 5 + 3\alpha \quad \Rightarrow \quad \alpha = -1$$

also $|P|\,|Q| = k^3|I| \quad \Rightarrow \quad (20 + 12\alpha)\frac{k^2}{2} = k^3$

$\Rightarrow \quad k = 4$ we get $4\alpha - k + 8 = 0$

$|P| = 20 + 12\alpha = 8$ and $|Q| = 8$

$|P(\text{adj}Q)| = |P||\text{adj}Q| = |P|\,|Q|^2 = 2^3(2^3)^2 = 2^9$

$|Q(\text{adj}P)| = |Q||\text{adj}P| = |Q|\,|P|^2 = 2^3(2^3)^2 = 2^9$

Hence, (B) and (C) are correct.

8. $\left(Y^3Z^4 - Z^4Y^3\right)^T = \left(Z^T\right)^4\left(Y^T\right)^3 - \left(Y^T\right)^3\left(Z^T\right)^4$

$= -Z^4Y^3 + Y^3Z^4 \quad \Rightarrow$ symmetric matrices

Similarly we will identify for other options as well

$X^{44} + Y^{44}$ is symmetric

$X^4Z^3 - Z^3X^4$ skew symmetric

$X^{23} + Y^{23}$ skew symmetric

Hence, (C) and (D) are correct.

9. Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $|M| = ac - b^2$

(A) the first column of M is the transpose of the second row of M then $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ $\therefore a = b = c$, we get $|M| = 0$, M is not invertible

(B) the second row of M is the transpose of the first column of M

$[b\ c] = [a\ b]$ $\therefore$ $a = b = c$, we get $|M| = 0$, M is not invertible

(C) M is a diagonal matrix with nonzero entries in the main diagonal

Now $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$

$\therefore$ M is invertible

(D) the product of entries in the main diagonal of M is not the square of an integer

$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $|M| = ac - b^2 \neq 0$ $\therefore$ M is invertible

Hence, (C) and (D) are correct.

10. Given $MN = NM$, $M^2 - N^4 = O$ and $M \neq N^2$ where O is zero matrix

$(M + N^2)(M - N^2) = M^2 - MN^2 + N^2M - N^4 = -MN^2 + N^2M$

$= -(MN)N + N(NM) = -(NM)N + N(MN) = 0$

$(M + N^2)(M - N^2) = O$

Let $M + N^2$ is invertible, $|M + N^2| \neq 0$

now $(M + N^2)^{-1}(M + N^2)(M - N^2) = (M + N^2)^{-1}.O = O$

$I(M - N^2) = O \Rightarrow M = N^2$ but it is not true

It means $M + N^2$ is not invertible, $|M + N^2| = 0$

$|M^2 + MN^2| = |M|\ |M + N^2| = 0$

$M(M + N^2)(M - N^2) = MO = O$

$(M^2 + MN^2)(M - N^2) = O$

Clearly here we can take $M - N^2 = U$ and U is not zero matrix (given $M \neq N^2$)

Hence, (A) and (B) are correct.

11. (A) $(N^TMN)^T = N^TM^TN = N^TMN$ if M is symmetric and is $-N^TMN$ if M is skew symmetric

(B) $(MN - NM)^T = N^TM^T - M^TN^T = NM - MN = -(MN - NM)$. So, $(MN - NM)$ is skew symmetric

(C) $(MN)^T = N^TM^T = NM \neq MN$ if M and N are symmetric. So, MN is not symmetric

(D) (adj. M) (adj. N) = adj$(NM) \neq$ adj (MN).

Hence, (C) and (D) are correct.

12. We know for a third order matrix P, $|\text{Adj } P| = |P|^2$

Where $|\text{Adj } P| = 1(3 - 7) - 4(6 - 7) + 4(2 - 1) = 4$

$\therefore \quad |P|^2 = 4 \Rightarrow |P| = 2$ or -2

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. From equation (E), we get

$a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$a + b + c = 0$

Here $\begin{vmatrix} 1 & 8 & 7 \\ 9 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$

Therefore system has infinite many solutions. Solving these, we get $b = 6a$ and $c = -7a$

Now (a, b, c) lies on $2x + y + z = 1 \Rightarrow b = 6, c = -7$

$\therefore \quad 2a + 6a - 7a = 1 \Rightarrow a = 1$

$\therefore \quad 7a + b + c = 7 + 6 - 7 = 6$

Hence, the correct answer is (D).

2. If $a = 2$ then $b = 12$, $c = -14$

$$\therefore \frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = 3\omega + 1 + 3\omega^2$$

$= 1 - 3 = -2$

Hence, the correct answer is (A).

3. If $b = 6$ then $a = 1$, $c = -7$

$\therefore$ Equation becomes $x^2 + 6x - 7 = 0$ or $(x + 7)(x - 1) = 0$

Whose roots are 1 and –7.

Let $\alpha = 1$ and $\beta = -7$

$$\therefore \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1 - \frac{6}{7}} = 7$$

Hence, the correct answer is (B).

4. If A is symmetric then $b = c$

$\Rightarrow \quad |A| = a^2 - b^2 = (a + b)(a - b)$

Which is divisible by p if $(a + b)$ is divisible by p or $(a - b)$ is divisible by p.

Now $(a + b)$ is divisible by p if (a, b) can take values $(1, p - 1)$, $(2, p - 2)$, $(3, p - 3)$,…$(p-1, 1)$

$\therefore \quad (p - 1)$ ways.

Also $(a - b)$ is divisible by p only when $a - b = 0$

i.e. $a = b$, then (a, b) can take values (0, 0) (1, 1), (2, 2)… $(pY, p - 2)$

$\therefore \quad p$ ways.

If A is skew symmetric, then $a = 0$ and $b = -c$ or $b + c = 0$

Which gives $|A| = 0$ when $b^2 \Rightarrow b = 0, c = 0$

But this possibility is already included when A is symmetric and $(a, b) = (0, 0)$

Again if A is both symmetric and skew symmetric, then clearly A is null matrix which case is already included.

Hence total number of ways $= p + (p - 1) = 2p - 1$

Hence, the correct answer is (D).

5. Trace $A = a + a = 2a$ is not divisible by p

$\Rightarrow \quad a$ is not divisible by $p \Rightarrow a \neq 0$

But $|A|$ is divisible by $p \Rightarrow a^2 - bc$ is divisible by p

It will be so if no dividing a^2 by p suppose we get $m \dfrac{\ell}{p}$

Then on dividing bc by p we should get $n\dfrac{\ell}{p}$ for some integral value of m, n and l.

i.e. the remainder should be same in each case, so that

$$\frac{a^2 - by}{p} = \left(m + \frac{\ell}{2}\right) - \left(n + \frac{\ell}{p}\right) = (m - n) = \text{an integer}$$

For this to happen a can take any value for 1 to $p - 1$, also if b takes any value from 1 to $p - 1$ then c should take only that value corresponding to which the remainder is same.

$\therefore \quad$ No. of ways $= (p - 1) \times (p - 1) \times 1 = (p - 1)^2$.

Hence, the correct answer is (C).

6. Total number of matrices = total number of ways a, b, c can be selected $= p \times p \times p = p^3$.

Number of ways which det (A) is divisible by p and trace $(A) \neq 0$ are equal to number of ways det (A) is divisible by p and track (A) is not divisible by $p = (p-1)^2$

Also number of ways when det (A) is divisible by p and trace $A = 0$ are the ways when bc is multiple of p

$\Rightarrow \quad b = 0$ or $c = 0$

for $b = 0$, c can take values $0, 1, 2, \ldots, p-1$

For $c = 0$, b can take values $0, 1, 2, \ldots, p-1$

Here $(b, c) = (0, 0)$ is coming twice.

$\therefore$ Total ways of selecting b and $c = p + p - 1 = 2p - 1$

$\therefore$ Total number of ways when det (A) is divisible by $p = (p-1)^2 + 2p - 1 = p^2$

Hence the number of ways when det (A) is not divisible by $p = p^3 - p^2$.

Hence, the correct answer is (D).

7. Each element of set A is 3×3 symmetric matrix with five of its entries as 1 and four of its entries as 0, we can keep in diagonal either 2 zero and one 1 or no zero and three 1 so that the left over zeros and one's are even in number.

Hence taking 2 zeros and one 1 in diagonal the possible cases are $\frac{3!}{2!} \times \frac{3!}{2!} = 9$

and taking 3 ones in diagonal the possible cases are $1 \times \frac{3!}{2!} = 3$

$\therefore$ Total elements A can have $= 9 + 3 = 12$

Hence, the correct answer is (A).

8. The given system will have unique solution if $|A| \neq 0$ which is so for the matrices.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

which are 6 in number.

Hence, the correct answer is (B).

9. For the given system to be inconsistent $|A| = 0$. The matrices for which $|A| = 0$ are

$$\underset{\text{(i)}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}, \underset{\text{(ii)}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}, \underset{\text{(iii)}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}, \text{and}$$

$$\underset{\text{(iv)}}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}, \underset{\text{(v)}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}, \underset{\text{(vi)}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}},$$

On solving $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We find for $A =$ (i)

By Cramer's rule $D_1 = 0 = D_2 = D_3$

$\therefore$ infinite many solution

For A (ii)

By Cramer's rule $D_1 \neq 0$

$\Rightarrow$ no solution i.e. inconsistent.

Similarly we find the system as inconsistent in cases (iii), (v) and (vi)

Hence for four cases system is inconsistent.

Hence, the correct answer is (B).

Integer/Numerical Answer Type Questions

1. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \text{tr}(A) = a + d = 3$

$$A^3 = A^2 A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot A$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\text{tr}(A^3) = (a^2 + bc)a + (ab + bd)c + (ac + cd)b + (bc + d^2)d = -18$

$\Rightarrow \quad a^3 + abc + abc + bdc + abc + bdc + bcd + d^3 = -18$

$\Rightarrow \quad a^3 + d^3 + 3abc + 3bdc = -18$

$\Rightarrow \quad (a + d)\,[a^2 + d^2 - ad] + 3bc(a + d) = -18$

$\Rightarrow \quad 3\,[(a + d)^2 - 3ad] + 3bc(3) = -18$

$\Rightarrow \quad 9 - 3ad + 3bc = -6 \Rightarrow 9 + 6 = 3(ad - bc)$

$\Rightarrow \quad ad - bc = 15/3 = 5 = |A|$

Hence, the correct answer is (5).

2. $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$

All numbers can be from $\{-1, 0, 1\}$

For Δ maximum let suppose it as 6, or it first three terms should be +1 and last three terms should be –1. But it not possible for any combination of given nine numbers

Similarly Δ can not be 5 as well

$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$ it is the maximum value of determinant possible

Hence, the correct answer is (4).

3. Here, $\Delta = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$

$1[1 - \alpha^2] - \alpha[\alpha - \alpha^3] + \alpha^2[\alpha^2 - \alpha^2] = 0$

$\Rightarrow \quad 1 - \alpha^2 - \alpha^2 + \alpha^4 + 0 = 0$

$\Rightarrow \quad (\alpha^2 - 1)^2 = 0$

$\Rightarrow \quad \alpha = -1$ or $+1$

For $\alpha = 1$, the equations become

$$x + y + z = 1, x + y + z = -1 \text{ and } x + y + z = 1$$

which give no solution (among these two are parallel planes), $\alpha = 1$ is rejected

For $\alpha = -1$ the equations become

$$x - y + z = 1, -x + y - z = -1, x - y + z = 1$$

which are all three coincident planes and hence infinitely any solutions,

$$\alpha = -1 \text{ is acceptable}$$

Now $1 + \alpha + \alpha^2 = 1$

Hence, the correct answer is (1).

4. $z = \omega$ (where ω is cube root of unity) $P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$

given $P^2 = -I$

$$\Rightarrow \quad P^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{2r} + \omega^{4s} & (-1)^r \omega^{2s+r} + \omega^{r+2s} \\ (-1)^r \omega^{2s+r} + \omega^{r+2s} & \omega^{2r} + \omega^{4s} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow \quad \omega^{2r} + \omega^{4s} = -1, \ ((-1)^r + 1)\omega^{2s+r} = 0; \ r, s \in \{1, 2, 3\}$

$\Rightarrow \quad$ second equation represent $r = 1, 3$

If $r = 1$ then $\omega^{4s} = -1 - \omega^2 = \omega \Rightarrow s = 1$

If $r = 3$ then $\omega^{4s} = -1 - 1 = -2$

$\Rightarrow \quad$ No value of s is possible

Only one ordered pairs (r, s) is acceptable

Hence, the correct answer is (1).

5. Let $M = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} b_1 = -1 \\ b_2 = 2 \\ b_3 = 3 \end{matrix}$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & c_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} a_1 - b_1 = 1 \\ a_2 - b_2 = 1 \\ a_3 - b_3 = -1 \end{matrix}$$

$\Rightarrow \quad a_1 = 0, a_2 = 3, a_3 = 2$

and $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & c_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow \begin{matrix} a_3 + b_3 + c_3 = 12 \\ \Rightarrow c_3 = 7 \end{matrix}$

$\therefore$ Sum of diagonal elements $= a_1 + b_2 + c_3 = 0 + 2 + 7 = 9$

Hence, the correct answer is (9).

6. $|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 2\sqrt{k} & 1+2k & -2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix}, C_2 \to C_2 - C_3$$

$$= \begin{vmatrix} 2k-1 & 0 & 2\sqrt{k} \\ 4\sqrt{k} & 0 & 1-2k \\ -2\sqrt{k} & 1+2k & -1 \end{vmatrix}, R_2 \to R_2 - R_3$$

$$= (1 + 2k)(8k - 4k + 4k^2 + 1) = (2k + 1)^3$$

Also $|B| = 0$ as B is skew symmetric of odd order.

$\therefore \quad |\text{Adj } A| + |\text{Adj } B| = |A|^2 + |B|^2 = 10^6$

$\Rightarrow (2k+1)^6 = 10^6 \Rightarrow 2k + 1 = 10 \Rightarrow k = 4.5$

$\therefore \quad [k] = 4$

Hence, the correct answer is (4).